A Morse lemma at infinity for nonlinear elliptic fractional equations

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Abstract – In this paper, we consider the following nonlinear fractional critical equation with zero Dirichlet boundary condition

\[ A_s u = K u^{\frac{n+2}{n-2s}}, \quad u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega, \]

where \( K \) is a positive function, \( \Omega \) is a regular bounded domain of \( \mathbb{R}^n, n \geq 2 \) and \( A_s, s \in (0, 1) \) represents the spectral fractional Laplacian operator \( (-\Delta)^s \) in \( \Omega \) with zero Dirichlet boundary condition. We prove a version of Morse lemmas at infinity for this problem. We also exhibit a relevant application of our novel result. More precisely, we characterize the critical points at infinity of the associated variational problem and we prove an existence result for \( s = \frac{1}{2} \) and \( n = 3 \).

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Contents

1. Introduction ........................................ 2

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In this paper, we consider the following problem

\[
\begin{cases}
A_s u = Ku^p, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^n, n \geq 2 \) is a regular bounded domain, \( K \) is a \( C^3 \)-positive function on \( \overline{\Omega} \), \( p = \frac{n+2s}{n-2s}, s \in (0,1) \) and \( A_s \) represents the fractional Dirichlet Laplacian operator \((-\Delta)^s\) in \( \Omega \), defined by using the spectrum of the Laplacian \(-\Delta\) in \( \Omega \) with zero Dirichlet boundary condition.

In the last years, fractional Laplacian has attracted the attention of many researchers. It appears in numerous applications in diverse domains including biology, physics, modeling populations and mathematical finance. The nonlocal character of the fractional Laplacian makes more challenging to handle as many results of the classical Laplacian do not extend to the fractional setting. After the innovative work of Caffarelli-Silvestre [13] who provided to the fractional Laplacian on \( \mathbb{R}^n, n \geq 2 \), a local interpretation in one more dimension, a large amount of studies were developed on problems involving the fractional Laplacian and fractional operators. In the setting of the spectral operator \( A_s \) on a bounded domain there exists many interesting papers addressed various aspects of the problem. In [12], Cabré and Tan studied the subcritical case; that is the equation (1) with subcritical nonlinearities \( p < \frac{n+2s}{n-2s} \). In the particular case \( s = 1/2 \), they established a similar extension to the one of Caffarelli-Silvestre for a nonlocal elliptic equation on bounded domains with the zero Dirichlet boundary condition and proved an existence result. For similar extensions we refer to [11] and [22]. In the same case of subcritical nonlinearities \( p < \frac{n+2s}{n-2s} \) and \( s = 1/2 \), K. Sharaf [19] proved that problem (1) has infinitely distinct solutions. In the setting of critical exponent \( p = \frac{n+2s}{n-2s} \), J. Tan [21] proved that if \( \Omega \) is a
starshaped domain, the equation (1) has no solutions in the case \( s = 1/2 \) and \( K \equiv 1 \). Later, Abdelhedi-Chtioui-Hajaiej [3] investigated the effect of the topology of \( \Omega \) on the existence of solutions of the critical problem (1) for \( s \in (0, 1) \) and \( p = \frac{n+2s}{n-2s} \). They proved that (1.1) has a solution when \( K \equiv 1 \) and \( \Omega \) admits a nontrivial group of homology. In [4], the authors constructed singular solutions for fractional Yamabe Problem \((K \equiv 1)\). They proved that there exists a positive solution that blow-up at a given sub-manifold \( \Sigma \) for fractional Yamabe problem \((-\Delta)^s u = u^{\frac{n+2s}{n-2s}}\), in \( \mathbb{R}^n \setminus \Sigma \). In [18], the authors extended the classical result of Brezis-Nirenberg to the case of non-local fractional operators. They proved the existence of solution for the following problem \((-\Delta)^s u = \lambda u + u^{\frac{n+2s}{n-2s}}\) in \( \Omega \) and \( u = 0 \) in \( \mathbb{R}^n \setminus \Omega \), in the case \( 2s < n < 4s \). For more existence results on problems involving fractional operator, one can see [14] and the references therein.

In this paper, our aim is to give sufficient conditions on the function \( K \neq 1 \) ensuring the existence of a solution for problem (1). Unlike the classical setting, this study presents many challenges. More precisely the non-locality of \((-\Delta)^s\) requires different and much more complicated estimates to prove. This difficulty is partially overcome thanks to an extension in the spirit of Caffarelli-Silvestre. However, the extra complexity comes from the boundary since the regular part of the Green’s function goes to infinity and therefore it dominates \( \Delta K \) (see Proposition 3.1). Contrary to the subcritical case \( p < \frac{n+2s}{n-2s} \), an additional difficulty lies in that problem (1) is a critical one from a variational viewpoint because of the failure of the Palais-Smale condition (P.S.). This means that there exist sequences along which the Palais-Smale sequences are bounded, their gradient goes to zero, and which do not converge [5]. To overcome such a difficulty, we prove a version of Morse Lemma at infinity in the neighborhood of highly concentrated functions; the so called critical points at infinity following the terminology of A. Bahri [5]. These are the noncompact flow lines of \( J \); the Euler-Lagrange functional associated to (1), along which \( J \) is bounded and its gradient goes to zero. Such a Morse lemma at infinity which is a centerpiece in our analysis is obtained through the construction of an appropriate pseudo-gradient for which the Palais-Smale condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter the neighborhood of a finite numbers of critical points of \( K \) such that the related matrix (see (3)) is positive definite. In order to prove this Morse Lemma, we need to perform a refined expansion of the Euler-Lagrange functional associated to (1) and its gradient near a potential critical points.
“at infinity”. These expansions, which are of independent interest, are highly nontrivial and use the techniques developed by A. Bahri [5] and O. Rey [16] in the framework of the "Theory of critical points at infinity". As an application of this Morse Lemma at infinity, we will characterize the critical points at infinity associated to problem (1) and we will prove an existence result in the case $s = \frac{1}{2}$ and $n = 3$.

Our choice of the case $s = \frac{1}{2}$ and $n = 3$ is motivated by the special features of the noncompactness in this dimension and for this choice of the fractional exponent. In fact, when looking for possible formation of blow up points, it turns out that the strong interaction of the bubbles in the case $s < \frac{n-2}{2}$ forces all blow up points to be single while in the case $s > \frac{n-2}{2}$ such interaction of two bubbles is negligible with respect to the self interaction, while in the case $s = \frac{n-2}{2}$, there is a phenomenon of balance, (see Proposition 3.1). Since $s \in (0, 1)$, the only case where $s = \frac{n-2}{2}$ is $n = 3$ and $s = 1/2$.

Throughout this paper, we assume that $K$ has only non-degenerate critical points $y_1, \ldots, y_m$ such that

$$(2) \quad -c_2 \frac{2}{3} \Delta K(y_i) + c_1 \tilde{H}((y_i, 0), y_i) \neq 0, \forall i = 1, \ldots, m,$$

where $\tilde{H}$ is the regular part of the Green function defined in (9). Let $\mathcal{K}$ be the set of all critical points of $K(x)$ satisfying

$$-c_2 \frac{2}{3} \Delta K(y_i) + c_1 \tilde{H}((y_i, 0), y_i) > 0, \forall i = 1, \ldots, m.$$

For any $\tau_p = (y_{\ell_1}, \ldots, y_{\ell_p}) \in \mathcal{K}^p, 1 \leq p \leq \# \mathcal{K}$ such that $y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j$, we define a $p \times p$ symmetric matrix $M(\tau_p) = (m_{ij})_{1 \leq i, j \leq p}$ by,

$$(3) \quad m_{ii} = -c_2 \frac{2}{3} \Delta K(y_{\ell_i}) + c_1 \frac{\tilde{H}((y_{\ell_i}, 0), y_{\ell_i})}{K(y_{\ell_i})^3},
\quad m_{ij} = -c_1 \frac{\tilde{G}((y_{\ell_i}, 0), y_{\ell_i})}{K(y_{\ell_i}) K(y_{\ell_j})}, \forall j \neq i,$$

where $c_1$ and $c_2$ are a positive constants and $\tilde{G}$ is the Green function defined in (8).

$(H_1)$: Assume that $\rho(\tau_p)$ the least eigenvalue of $M(\tau_p)$ is non zero, for any $1 \leq p \leq \# \mathcal{K}$. 
(H2): Assume that for each \( x \in \partial \Omega \), \( \frac{\partial K(x)}{\partial \nu} < 0 \), where \( \nu \) is the outward normal to \( \Omega \).

Our main results are the following.

Let

\[
C^\infty := \{ \tau_p = (y_{l1}, ..., y_{lp}) \in K^p, 1 \leq p \leq 2K, \text{s.t. } y_{li} \neq y_{lj} \forall 1 \leq i \neq j \leq p \text{ and } \rho(\tau_p) > 0 \}.
\]

For \( a \in \Omega \) and \( \lambda > 0 \), we denote \( \tilde{P}_\delta(\alpha_i, \lambda_i) \) the almost solution of problem (1).

**Theorem 1.1.** Let \( K \) be a positive function on \( \Omega \). Assume that \( n = 3 \) and \( s = \frac{1}{2} \). Under conditions (H1) and (H2), let \( \varepsilon > 0 \) sufficiently small. There exists a change of variables, such that for any \( u = \sum_{i=1}^{p} \alpha_i \tilde{P}_\delta(\alpha_i, \lambda_i) + v \in V(p, \varepsilon) \),

\[
(a_i, \lambda_i, v) \rightarrow (a'_i, \lambda'_i, V)
\]

where \( V \) belongs to a neighborhood of zero in a fixed Hilbert space such that the Euler-Lagrange functional associated to (1) is given by

\[
J\left( \sum_{i=1}^{p} \alpha_i \tilde{P}_\delta(\alpha_i, \lambda_i) + v \right) = J\left( \sum_{i=1}^{p} \alpha_i \tilde{P}_\delta(a'_i, \lambda'_i) \right) + \|V\|^2.
\]

Furthermore, if each \( a_i \) belongs to a neighborhood of \( y_{li} \in C^\infty \), then we can find another change of variables \( (a_i, \lambda_i) \rightarrow (a'_i, \lambda'_i) \) such that for \( \eta \) a fixed small constant

\[
J\left( \sum_{i=1}^{p} \alpha_i \tilde{P}_\delta(\alpha_i, \lambda_i) \right) = \frac{S^{1/3}}{\left( \sum_{i=1}^{p} \alpha_i^2 K(a'_i) \right)^{\frac{1}{3}}} \left( 1 + (c'_1 - \eta)^T \Lambda' \Lambda' \right),
\]

where \( \Lambda' = (1/\lambda'_1, ..., 1/\lambda'_p) \), \( c'_1 \) is a positive constant and \( V(p, \varepsilon) \) is the set defined in (7).

The following results illustrate the usefulness of the above theorem. In fact, using the above version of Morse Lemma at infinity for this problem, we characterize the critical points at infinity of the associated functional.
Theorem 1.2. Let $K$ be a positive function on $\Omega$. Assume that $n = 3$ and $s = \frac{1}{2}$. Under conditions $(H_1)$ and $(H_2)$, the only critical points at infinity associated to problem (1) in $V(p, \varepsilon)$ correspond to

$$(y_{\ell_1}, \ldots, y_{\ell_p})_\infty := \sum_{i=1}^{p} \frac{1}{K(y_{\ell_i})^{\frac{3}{2}}} P\delta(y_{\ell_i}, \infty),$$

where $(y_{\ell_1}, \ldots, y_{\ell_p}) \in C^\infty$.

Using the above characterization of critical points at infinity, we can prove the following existence result.

Theorem 1.3. Let $K$ be a positive function on $\Omega$. Assume that $n = 3$ and $s = \frac{1}{2}$. Under conditions $(H_1)$ and $(H_2)$, if

$$\sum_{\tau_\tau = (y_{\ell_1}, \ldots, y_{\ell_p}) \in C^\infty} (-1)^{4p - 1 - \sum_{j=1}^{p} \text{ind}(K, (y_{\ell_j}))} \neq 1,$$

then (1) has a solution, where $\text{ind}(K, (y_{\ell_j}))$ denotes the Morse index of $K$ at $y_{\ell_j}$.

We point out that the result of Theorem 1.3 can be seen as the fractional counterpart of Theorem 1.4 of [10], (for $s = 1$).

We mention also that there exists another fractional integro-differential operator widely studied defined by

$$(-\Delta)^{s}u(x) = \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{n+2s}} dy.$$ 

In [17], a comparison between the two operators, (the integral one and the spectral one), is given. Using the spectrum of the two operators, the authors of [17] proved that the eigenvalues of the two operators are different and their eigenfunctions are also different. For more similarities and differences between the two operators, see section 2.3 of [1], see also [15].

The remainder of the present paper is organized as follow. The next section will be devoted to the variational structure of the problem (1). In Section three we will give the expansion of the functional $J$ and its gradient. In Section 4 we will study the $v$-part of $u$. In Section 5 we will construct a suitable pseudo-gradient to characterize the critical points associated to problem (1). Lastly, in Section 6 we will prove our main results.
2. Variational structure of problem

In order to state the variational structure of problem (1), we need to introduce the following notations. Following the results of [13] for $\Omega = \mathbb{R}^n$ and [12] for bounded domain $\Omega$, see also [11], and [22], we consider the equivalent local problem to (1) on the half cylinder with base $\Omega$:

$$C = \Omega \times [0, \infty) = \{(x, t), \text{s.t. } x \in \Omega \text{ and } t \in [0, \infty)\}.$$ 

Let

$$C_{0L}^{\infty}(C) = \{v \in C^{\infty}(\overline{C}), \text{s.t. } v = 0 \text{ on } \partial_L C\},$$

where $\partial_L C$ denotes the lateral boundary of $C$. It is defined by $\partial \Omega \times [0, \infty)$. Let $H_{0L}^s(\Omega)$ be the Hilbert Sobolev space defined as the closure of $C_{0L}^{\infty}(C)$ with respect to

$$|v| = \left(\int_C t^{1-2s} |\nabla v|^2\right)^{\frac{1}{2}},$$

and equipped with the following inner product:

$$<v, w>_{H^s_{0L}(C)} = \int_C t^{1-2s} \nabla v \nabla w, \forall v, w \in H^s_{0L}(C).$$

Let

$$H_0^s(\Omega) = \{u = \text{tr}(v), v \in H^s_{0L}(C), \text{ with } \text{div}(t^{1-2s}v) = 0 \text{ in } C\}.$$

Following [11] and [22], we associate to any $u \in H_0^s(\Omega)$ the unique $s$-harmonic function denoted $h_u$ in $H^s_{0L}(C)$, the unique solution of the following problem:

$$\begin{cases} 
\text{div } (t^{1-2s} \nabla v) = 0 & \text{in } C, \\
v = 0 & \text{on } \partial_L C, \\
v = u & \text{on } \Omega \times \{0\}. 
\end{cases}$$

(See [11] and [22] for the uniqueness and the explicit expression of $h_u$).

It follows that $A_s$ is expressed by the following map:

$$u = \sum_{k=1}^{\infty} b_k e_k \mapsto A_s(u) = \partial^s_\nu(h_u)/\nu_{\Omega \times \{0\}},$$

where $\nu$ denotes the unit outward normal vector to $C$ on $\Omega \times \{0\}$ and for any $v \in H^s_{0L}(C)$ and any $x \in \Omega$, we have:

$$\partial^s_\nu(v)(x, 0) = -c_s \lim_{t \to 0^+} t^{1-2s} \frac{\partial v}{\partial t}(x, t) \quad \text{and} \quad c_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}.$$
Therefore, problem (1) is equivalent to the following local problem

\[
\begin{cases}
\text{div} \ (t^{1-2s} \nabla v) = 0 \text{ in } C, \\
v > 0 \text{ in } C, \\
v = 0 \text{ on } \partial_L C, \\
\partial_s^* (v) = v^{\frac{n+2s}{n-2s}} \text{ on } \Omega \times \{0\}.
\end{cases}
\]  

(4)

Consequently, if \( v \) satisfies (4), then \( u(x) = v(x,0) := \text{tr}(v)(x), \forall x \in \Omega \) is a solution of (1).

In order to present the variational structure associated to (4), we introduce the following Hilbert space constructed by all \( s \)-harmonic functions in \( H_{0L}^s(C) \). More precisely, let

\[
H = \{ v \in H_{0L}^s(C), \text{s.t. div}(t^{1-2s} \nabla v) = 0 \text{ in } C \}.
\]

The Euler-Lagrange functional associated to (4) is

\[
J(v) = \frac{1}{\left( \int_{\Omega} K(x)v(x,0)^{\frac{2n}{n-2s}} \, dx \right)^{\frac{n-2s}{n}}} \quad v \in \Sigma,
\]

where \( \Sigma = \{ v \in H, \| v \| = c_s^{-1/2} \} \).

The critical points of \( J \) in \( \Sigma^+ := \{ u \in \Sigma, u > 0 \} \) correspond to the solutions of (4), see [3].

Since \( p + 1 = \frac{2n}{n-2s} \) is the critical Sobolev exponent of the Sobolev trace embedding \( v \in H \mapsto \text{tr}(v) \in L^{p^*+1}(\Omega) \) which is continuous but not compact for \( p = \frac{n+2s}{n-2s} \), the functional \( J \) is of class \( C^1 \) and fails the Palais-Smale condition. This means that there exist sequences along which \( J \) is bounded but its gradient goes to zero and which do not converge.

For \( a \in \Omega \) and \( \lambda > 0 \), define

\[
\delta_{(a,\lambda)}(x) = c_n \frac{\lambda^{\frac{n-2s}{2}}}{\left( 1 + \lambda^2 |x-a|^2 \right)^{\frac{n-2s}{2}}}, \quad x \in \mathbb{R}^n.
\]

Here \( c_n \) is a fixed positive constant chosen so that the \( s \)-harmonic extension \( \tilde{\delta}_{(a,\lambda)} \) of \( \delta_{(a,\lambda)} \) satisfies

\[
\begin{cases}
\text{div} \ (t^{1-2s} \nabla \tilde{\delta}_{(a,\lambda)}) = 0 \text{ in } \mathbb{R}^{n+1}, \\
\partial_s^* \tilde{\delta}_{(a,\lambda)} = \frac{\delta_{(a,\lambda)}^{\frac{n+2s}{n-2s}}}{\delta_{(a,\lambda)}} \text{ on } \mathbb{R}^n \times \{0\}.
\end{cases}
\]  

(5)
Next, we denote by $S$ the following integral which is independent of $a$ and $\lambda$,

$$S = \int_{\mathbb{R}^n} \delta_{(a,\lambda)}^{\frac{2s}{n}} \, dx.$$ 

We also define $P\tilde{\delta}_{(a,\lambda)}$ the unique solution in $H^s_0(C)$ of the equation

$$\begin{cases}
\text{div} \left( t^{1-2s} \nabla v \right) &= 0 \quad \text{in } C, \\
v &= 0 \quad \text{on } \partial L C, \\
\partial^s_n v &= \tilde{\delta}_{(a,y)}^{\frac{n-2s}{n}} \quad \text{on } \Omega \times \{0\},
\end{cases}$$

where $\tilde{\delta}_{(a,\lambda)}$ is the solution of (5).

Arguing as [20], the following proposition describes the failure of the Palais-Smale sequences for the functional $J$.

**Proposition 2.1.** Assume that (4) has no solution. Let $(v_k)_k$ be a sequence in $\Sigma^+ := \{v \in \Sigma, v \geq 0\}$ such that $J(v_k) \to c$ and $\partial J(v_k) \to 0$. There exists $p \in \mathbb{N}^*$ and a subsequence of $(v_k)_k$ denoted again $(v_k)_k$ such that

$$V(p, \varepsilon) = \left\{ u \in \Sigma^+, \exists a_1, \ldots, a_p \in \Omega, \exists \lambda_1, \ldots, \lambda_p > \frac{1}{\varepsilon} \text{ and} \right\}$$

$$\alpha_1, \ldots, \alpha_p > 0, s.t. \left\| u - \sum_{i=1}^p \alpha_i P\tilde{\delta}_{(a_i,\lambda_i)} \right\| < \varepsilon \text{ with } \lambda_i d(a_i, \partial \Omega) > \varepsilon^{-1},$$

$$\left| \frac{\alpha_i \frac{4s}{n} K(a_i)}{\alpha_j \frac{4s}{n} K(a_j)} - 1 \right| < \varepsilon, \text{ and } \varepsilon_{ij} < \varepsilon, \forall i \neq j \right\}.$$ 

Here $\varepsilon_{ij} = \frac{1}{\left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{n-2s}{n}}}.$

Arguing as ([7], proposition 7), we can find the following optimal representation of $u \in V(p, \varepsilon)$.

**Proposition 2.2.** Let $p \in \mathbb{N}^*$. There exists $\varepsilon > 0$ such that for all $u \in V(p, \varepsilon)$, the following minimization problem

$$\inf_{\alpha_i, a_i, \lambda_i} \left\| u - \sum_{i=1}^p \alpha_i P\tilde{\delta}_{(a_i,\lambda_i)} \right\|$$
admits a unique solution \((\bar{\alpha}, \bar{\lambda})\) modulo a parametrization on the indices set. Therefore, we write \(u = \sum_{i=1}^{p} \bar{\alpha}_i P\bar{\delta}_{(\bar{\alpha}, \bar{\lambda})} + v\), for any \(u \in V(p, \varepsilon)\). Here \(v \in \mathcal{H}\) and satisfies

\[(V_0): \left\langle v, \psi \right\rangle = 0 \text{ for } \psi \in \{P\bar{\delta}_{(\bar{\alpha}, \bar{\lambda})}, \frac{\partial P\bar{\delta}_{(\bar{\alpha}, \bar{\lambda})}}{\partial \bar{\lambda}_i}, \frac{\partial P\bar{\delta}_{(\bar{\alpha}, \bar{\lambda})}}{\partial \alpha_i}, i = 1, ..., p\}.\]

The estimate of \(P\bar{\delta}_{(\alpha, \lambda)} - \bar{\delta}_{(\alpha, \lambda)}\) involves the Green’s function and its regular part. For \(x, y \in \Omega\) and \(t \geq 0\), let \(\tilde{G}(x, t, y)\) be the \(s\)-harmonic extension of the Green’s function of \((-\Delta)^s\). So that

\[
\begin{align*}
\text{(8)} & \quad \begin{cases} 
\text{div} \left(t^{1-2s} \nabla \tilde{G}(\cdot, y)\right) = 0 & \text{in } C, \\
\tilde{G}(\cdot, y) = 0 & \text{on } \partial_L C, \\
\partial_y \tilde{G}(\cdot, y) = \delta_y & \text{on } \Omega \times \{0\},
\end{cases} \\
\text{div} \left(t^{1-2s} \nabla \tilde{H}(\cdot, y)\right) = 0 & \text{in } C, \\
\tilde{H}(x, t, y) = \frac{\hat{c}}{\|x - y, t\|^n_{\mathbb{R}^{n+1}}} & \text{on } \partial_L C, \\
\partial_y \tilde{H}(\cdot, y) = 0 & \text{on } \Omega \times \{0\},
\end{align*}
\]

where \(\delta_y\) is the Dirac mass at \(y\). Let \(\tilde{H}\) be the regular part of \(\tilde{G}\). It satisfies

3. Expansion of the Functional and its Gradient near potential critical points at Infinity

In the first part of this section, we deal with the functional \(J\) associated to problem (1).

**Proposition 3.1.** Let \(n \geq 2\) and \(s \in (0, 1)\). For \(\varepsilon > 0\) small enough and \(u = \sum_{i=1}^{p} \alpha_i P\bar{\delta}_{(\bar{\alpha}, \bar{\lambda})} + v \in V(p, \varepsilon)\), we have the following expansion

\[
J(u) = \frac{S^{2s/n} (\sum_{i=1}^{p} \alpha_i^2)}{\left(\sum_{i=1}^{p} \alpha_i^{2s/n} K(a_i)\right)^{\frac{n-2s}{n}}} \left[ 1 - \frac{n-2s}{nS(\sum_{i=1}^{p} K(a_i))^{\frac{n}{n-2s}}} c_2 \sum_{i=1}^{p} \frac{\Delta K(a_i)}{K(a_i) \pi_i^2} \right. \\
+ \frac{c_1}{S} \sum_{i=1}^{p} \frac{\tilde{H}(a_i, 0, a_i)}{K(a_i)^{\frac{n-2s}{n}}} \left( \varepsilon_{ij} + \frac{1}{(K(a_i) K(a_j))^{\frac{n-2s}{n}}} (\xi_{ij} - \frac{\tilde{H}(a_i, 0, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2s}{2s}}}) \right) \\
+ \frac{Q(v, v)}{S \sum_{j=1}^{p} \alpha_j^2} - f(v) + o\left( \sum_{j \neq i} \varepsilon_{ij} \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-2s}} + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} + \|v\|^2 \right),
\]

where \(d_k\) are the distances between the critical points at infinity.
where

\[ c_1 = \int_{\mathbb{R}^n} \frac{dz}{(1 + |z|^2)^{\frac{n+2s}{2}}}, \quad c_2 = \frac{1}{n} \int_{\mathbb{R}^n} |z|^2 \frac{|z|^2 - 1}{(1 + |z|^2)^{n+1}} dz, \quad S = \int_{\mathbb{R}^n} \frac{dz}{(1 + |z|^2)^n}, \]

\[ Q(v, v) = \|v\|^2 - \frac{(n + 2s)}{(n - 2s)} \frac{\sum_j \alpha_j^2}{\sum_j \frac{n+2s}{n-2s}} \int K(\sum_i \alpha_i P\delta_i)^{\frac{4s}{n-2s}} v^2, \]

\[ f(v) = \frac{2}{S \sum_j \alpha_j^{\frac{n+2s}{n-2s}}} \int K(\sum_i \alpha_i P\delta_i)^{\frac{n+2s}{n-2s}} v. \]

**Proof.** Let us recall that

\[ J(u) = \|u\|^2. \]

We need to estimate

\[ N(u) = \sum_{i=1}^{p} \alpha_i^2 \|P\delta_i\|^2 + \sum_{i \neq j} \alpha_i \alpha_j < P\delta_i, P\delta_j > + \|v\|^2. \]

A computation similar to the one performed in [3], (precisely page 399 of [3]), show that for \( \lambda_i d_i \) large enough and, we have the following estimates

\[ \|P\delta_i\|^2 = S - c_1 \frac{\bar{H}((a_i, 0), a_i)}{\lambda_i^{n+2s}} + O(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{n+2-2s}}). \]

\[ < P\delta_i, P\delta_j > = c_1(\varepsilon_{ij} - \frac{\bar{H}((a_i, 0), a_i)}{\lambda_i \lambda_j^{n+2s}}) + O(\frac{\varepsilon_{ij}^{n+2s}}{\lambda_i \lambda_j \log(\lambda_i \lambda_j)} + \frac{\sum_{k=1}^{n+2-2s} \log(\lambda_i d_k)}{(\lambda_i d_k)^{n+2-2s}}). \]

Using (10), (11) and (12), we derive that

\[ N(u) = \sum_{i=1}^{p} \alpha_i^2 \left(S - c_1 \frac{\bar{H}((a_i, 0), a_i)}{\lambda_i^{n+2s}} \right) + \sum_{j \neq i} \alpha_i \alpha_j c_1 \left(\varepsilon_{ij} - \frac{\bar{H}((a_i, 0), a_i)}{\lambda_i \lambda_j^{n+2s}} \right) + \|v\|^2 + R, \]
where
\[ R = O\left(\sum_{j \neq i} \varepsilon_{ij}^{-\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=1}^{p} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right). \]

For the denominator, we have
\[
\int_{\Omega} K\left(\sum_{i=1}^{p} \alpha_i P\tilde{\delta}_i + v\right)^{2n/(n-2s)} = \int_{\Omega} K\left(\sum_{i=1}^{p} \alpha_i P\tilde{\delta}_i\right)^{2n/(n-2s)} + \frac{2n}{n-2s} \int_{\Omega} K\left(\sum_{i=1}^{p} \alpha_i P\tilde{\delta}_i\right)^{n+2s/(n-2s)} v
\]
\[
\int_{\Omega} K\left(\sum_{i=1}^{p} \alpha_i P\tilde{\delta}_i\right)^{2n/(n-2s)} = \sum_{i=1}^{p} \alpha_i^{2n/(n-2s)} \int_{\Omega} K P\tilde{\delta}_i^{-\frac{n}{n-2s}}
\]
\[
\frac{2n}{n-2s} \sum_{i \neq j} \alpha_i^{n+2s/(n-2s)} \alpha_j \int_{\Omega} K P\tilde{\delta}_i^{-\frac{n}{n-2s}} P\tilde{\delta}_j + O\left(\sum_{i \neq j} \int_{\Omega} P\tilde{\delta}_i^{-\frac{4s}{n-2s}} \inf(P\tilde{\delta}_i, P\tilde{\delta}_j)^{2}\right).
\]

A computation similar to the one performed in [5], (see the proof of the estimate E4 of [5] page 4), shows that
\[
\int P\tilde{\delta}_i^{-\frac{4s}{n-2s}} \inf(P\tilde{\delta}_i, P\tilde{\delta}_j)^{2} = O\left(\sum_{j \neq i} \varepsilon_{ij}^{-\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right),
\]
\[
\int K P\tilde{\delta}_i^{-\frac{n+2s}{n-2s}} P\tilde{\delta}_j = c_1 K(a_i) \left(\varepsilon_{ij} - \frac{\bar{H}((a_i, 0), a_j)}{(\lambda_i \lambda_j)^{n-2s}}\right) + O\left(\sum_{j \neq i} \varepsilon_{ij}^{-\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right),
\]
\[
\int K P\tilde{\delta}_i^{-\frac{2s}{n-2s}} = K(a_i) S + c_2 \frac{\Delta K(a_i)}{\lambda_i^2} - c_1 K(a_i) \frac{2n}{n-2s} \frac{\bar{H}((a_i, 0), a_i)}{\lambda_i^{n-2s}} + O\left(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{n+2-2s}}\right).
\]

Using (16), (17) and (18) we derive that
\[
\int \left(\sum_{i=1}^{p} \alpha_i P\tilde{\delta}_i\right)^{2n/(n-2s)} = S \sum_{i=1}^{p} \alpha_i^{2n/(n-2s)} K(a_i) + c_1 \sum_{i=1}^{p} \alpha_i^{n+2s} \Delta K(a_i) \frac{\lambda_i^2}{\lambda_i^2},
\]
Observe that
\[-c_1 \frac{2n}{n-2s} \sum_{i=1}^{p} \alpha_i \chi_i \mathcal{K}(a_i) \frac{\tilde{H}((a_i, 0), a_i)}{\lambda_i^{n-2s}} + \frac{2n}{n-2s} c_1 \sum_{i \neq j} \alpha_i \chi_i \beta_j \bigg( \varepsilon_{ij} \bigg) \frac{\tilde{H}((a_i, 0), a_j)}{(\lambda_i \lambda_j)^{2-2s}} \bigg)
+ O \left( \sum_{j \neq i} \varepsilon_{ij}^{n-2s} \log \varepsilon_{ij}^{-1} + \sum_{k=1}^{n} \log \left( \lambda_k \delta_k \right)^{n+2} \right).
\]

Combining (13), (14) and (19) and the fact that \( J(n) \alpha_i \chi \mathcal{K}(a_i) = 1 + o(1) \), for each \( i \), the result follows.

**Proposition 3.2.** Let \( n = 3 \) and \( s = \frac{1}{2} \). For \( u = \sum_{i=1}^{p} \alpha_i \mathcal{P}_{(a_i, \lambda_i)} \in V(p, \varepsilon) \), we have the following expansion

\[
\left< \partial J(u), \lambda_i \frac{\partial \mathcal{P}_{(a_i, \lambda_i)}}{\partial \lambda_i} \right> = 2J(u) \left[ \frac{2}{3} \alpha_i c_2 \frac{\Delta \mathcal{K}(a_i)}{\mathcal{K}(a_i) \chi_i^2} - c_1 \alpha_i \frac{\tilde{H}((a_i, 0), a_i)}{\lambda_i^2} \right.
- c_1 \sum_{j \neq i} \alpha_j \left( \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{\tilde{H}((a_i, 0), a_i)}{(\lambda_i \lambda_j)} \right) + o \left( \frac{1}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \right) + R.
\]

**Proof.** We have

\[
< \partial J(u), h > = 2J(u) \left( < u, h > - J(u)^{\frac{1}{2}} \int_{\Omega} \mathcal{K} u^2 h \right).
\]

Thus,

\[
< \partial J(u), \lambda_i \frac{\partial \mathcal{P}_{(a_i, \lambda_i)}}{\partial \lambda_i} > = 2J(u) \left[ \sum_{j=1}^{p} \alpha_j \mathcal{P}_{(a_j, \lambda_j)} \lambda_i \frac{\partial \mathcal{P}_{(a_j, \lambda_j)}}{\partial \lambda_i} > - J(u)^{\frac{1}{2}} \int_{\Omega} K \left( \sum_{j=1}^{p} \alpha_j \mathcal{P}_{(a_j, \lambda_j)} \right)^{\frac{1}{2}} \lambda_i \frac{\partial \mathcal{P}_{(a_i, \lambda_i)}}{\partial \lambda_i} \right].
\]

Observe that

\[
\int_{\Omega} K \left( \sum_{j=1}^{p} \alpha_j \mathcal{P}_{(a_j, \lambda_j)} \right)^{\frac{1}{2}} \lambda_i \frac{\partial \mathcal{P}_{(a_j, \lambda_j)}}{\partial \lambda_i} = \sum_{j=1}^{p} \alpha_j \mathcal{P}_{(a_j, \lambda_j)} \int_{\Omega} K \mathcal{P}_{(a_j, \lambda_j)} \frac{\partial \mathcal{P}_{(a_j, \lambda_j)}}{\partial \lambda_i},
\]

\[
+ 2 \sum_{i \neq j} \int_{\Omega} K \left( \alpha_i \mathcal{P}_{(a_i, \lambda_i)} \right) \lambda_i \frac{\partial \mathcal{P}_{(a_j, \lambda_j)}}{\partial \lambda_i} + O \left( \sum_{j \neq i} \int_{\Omega} \mathcal{P}_{(a_j, \lambda_j)} \inf (\delta_j, \delta_i) \right)^2.
\]

A computation similar to the one performed in [5], (see precisely the proof of the estimate (2.5) of [5] page 15), shows that

\[
< \mathcal{P}_{(a_i, \lambda_i)} \lambda_i \frac{\partial \mathcal{P}_{(a_i, \lambda_i)}}{\partial \lambda_i} = c_1 \frac{\tilde{H}((a_i, 0), a_i)}{\lambda_i^2} + O \left( \log \left( \lambda_i \delta_i \right) \lambda_i^2 \right).
\]
Thus, \( (21) \)

We have

\[
\int_\Omega KP^j_i \frac{\partial P^j_i}{\partial \lambda_i} = -c_2 \frac{\Delta K(a_i)}{\lambda_i} + 2c_1 K(a_i) \frac{\tilde{H}((a_i, 0), a_i)}{\lambda_i^2} + \mathcal{O}\left(1 \right)
\]

\[
\int_\Omega KP^j_i \frac{\partial P^j_i}{\partial \lambda_i} = K(a_j) < P^j_i, \lambda_i \frac{\partial P^j_i}{\partial \lambda_i}> + \mathcal{O}\left(1 \right)
\]

Using the above estimates and the fact that \( J(u) \alpha_i K(a_i) = 1 + o(1) \), for each \( i \), the proposition follows. \( \square \)

**Proposition 3.3.** Let \( n = 3 \) and \( s = \frac{1}{2} \). For \( u = \sum_{i=1}^p \alpha_i P^j_\alpha(a_i, \lambda_i) \in V(p, \varepsilon) \), we have the following expansion

\[
\left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial P^j_\alpha(a_i, \lambda_i)}{\partial a_i} \right\rangle = J(u) \left[ -\alpha_i^2 c_3 J(u) \frac{\nabla K(a_i)}{\lambda_i} \left(1 + o(1)\right) \right.
\]

\[
\left(21 \right) \left[ c_1 \alpha_i \frac{\partial \tilde{H}((a_i, 0), a_i)}{\partial a_i} \left(1 + o(1)\right) \right.
\]

\[
+ 2c_1 \sum_{j \neq i} \alpha_j \left( \frac{1}{\lambda_i} \frac{\partial \tilde{H}((a_i, 0), a_i)}{\partial a_i} - \frac{\partial \tilde{H}((a_i, 0), a_i)}{\lambda_i (\lambda_i, \lambda_j)} \right) \left(1 - J(u) \frac{1}{2} (\alpha_j K(a_j) + \alpha_i K(a_i)) \right)
\]

\[
+ R + O\left(1 \right) + \sum_{j \neq i} \lambda_i |a_i - a_j| \varepsilon_{ij}^2.
\]

**Proof.** We have

\[
\left\langle \partial J(u), h \right\rangle = 2 J(u) \left( < u, h > - J(u) \frac{1}{2} \int_{\Omega} K u^2 h \right).
\]

Thus,

\[
\left\langle \partial J(u), \frac{1}{\lambda_j} \frac{\partial P^j_\alpha}{\partial a_i} \right\rangle = 2 J(u) \left[ < \sum_{j=1}^p \alpha_j P^j_\alpha, \frac{1}{\lambda_j} \frac{\partial P^j_\alpha}{\partial a_i} > - J(u) \frac{1}{2} \int_{\Omega} K \left( \sum_{j=1}^p \alpha_j P^j_\alpha \right)^2 \frac{1}{\lambda_i} \frac{\partial P^j_\alpha}{\partial a_i} \right].
\]
Observe that

\begin{equation}
\int_{\Omega} K \left( \sum_{j=1}^{p} \alpha_j P \delta_j \right)^2 \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} = \sum_{j=1}^{p} \alpha_j^2 \int_{\Omega} K P \delta_j^2 \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i}
\end{equation}

\begin{equation}
+ 2 \sum_{i \neq j} \int_{\Omega} K (\alpha_i P \delta_i) \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} (\alpha_j P \delta_j) + O \left( \sum_{j \neq i} \int_{\Omega} P \delta_j \inf (\delta_j, \delta_i)^2 \right).
\end{equation}

Using similar computation to the one of [5] (precisely page 125 of [5]), we get the following estimates

\begin{equation}
<P \delta_i, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} >= -\frac{1}{2} c_1 \frac{\partial \tilde{H}((a_i, 0), a_i)}{\partial a_i} + O \left( \frac{1}{(\lambda_i d_i)^4} \right).
\end{equation}

\begin{equation}
<P \delta_j, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} >= \frac{c_1}{\lambda_i} \left( \frac{\partial \tilde{\epsilon}_{ij}}{\partial a_i} - \frac{1}{(\lambda_i \lambda_j)} \frac{\partial \tilde{H}((a_i, 0), a_j)}{\partial a_i} \right)
+ O \left( \sum_{k=i,j} \frac{\lambda_i |a_i - a_j| \tilde{\epsilon}_{ij}}{\lambda_i d_k^4} + \sum_{i \neq j} \lambda_i |a_i - a_j| \tilde{\epsilon}_{ij}^2 \right).
\end{equation}

\begin{equation}
\int_{\Omega} K P \delta_i^2 \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} = c_3 \frac{\nabla K(a_i)}{\lambda_i} \left( 1 + o(1) \right) - \frac{c_1 K(a_i)}{\lambda_i^3} \frac{\partial \tilde{H}((a_i, 0), a_i)}{\partial a_i}
+ O \left( \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} + \frac{1}{\lambda_i^4} \right).
\end{equation}

\begin{equation}
\int_{\Omega} K P \delta_j P \delta_i \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} = K(a_j) < P \delta_j, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} > + O \left( \sum_{k=i,j} \frac{1}{(\lambda_k d_k)^4} + \sum_{j \neq i} \tilde{\epsilon}_{ij}^2 \log \tilde{\epsilon}_{ij}^{-1} \right).
\end{equation}

\begin{equation}
2 \int_{\Omega} K P \delta_j P \delta_i \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} = K(a_j) < P \delta_j, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} >
+ O \left( \sum_{k=i,j} \frac{1}{(\lambda_k d_k)^4} + \sum_{j \neq i} \tilde{\epsilon}_{ij}^2 \log \tilde{\epsilon}_{ij}^{-1} \right).
\end{equation}

Using (22), . . . (27) and the fact that $J(u)^2 \alpha_i K(a_i) = 1 + o(1)$, for each $i$, the proposition follows. \qed
4. The \( v \)-part of \( u \)

From Proposition 2.2, for any \( u \in V(p, \varepsilon) \), we have
\[
u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}(a_i, \lambda_i) + v,
\]
where \( v \in H \) and satisfies \((V_0)\). In this section we deal with the \( v \)-part of \( u \), in order to show that it is negligible to the concentration phenomenon in the case \( n = 3 \) and \( s = \frac{1}{2} \).

**Proposition 4.1.** There is a \( C^1 \)-map which to each \((\alpha_i, a_i, \lambda_i)\) such that
\[
\sum_{i=1}^{p} \alpha_i \tilde{\delta}(a_i, \lambda_i) \text{ belongs to } V(p, \varepsilon)
\]
associates \( v = v(\alpha, a, \lambda) \) such that \( v \) is unique and satisfies:
\[
J\left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}(a_i, \lambda_i) + v \right) = \min_{\tilde{v} \in (V_0)} \left\{ J\left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}(a_i, \lambda_i) + \tilde{v} \right) \right\}.
\]
Furthermore, we have the following estimate:
\[
\|v\| \leq \alpha \|f\| = O\left( \left( \sum_{i=1}^{p} \frac{n+2s}{(n-2s)} \right) \sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{3}{2}} \right).
\]

**Proof.** First let us observe that
\[
\int K\left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}(a_i) \right) v^2 = \sum_{i=1}^{p} \alpha_i \frac{4s}{n-2s} K(a_i) \int \tilde{\delta}(a_i) v^2 + o(\|v\|^2).
\]
Using the fact that
\[
\frac{\alpha_i \frac{4s}{n-2s} K(a_i)}{\alpha_j \frac{4s}{n-2s} K(a_j)} = 1 + o(1),
\]
we derive that
\[
Q(v, v) = \|v\|^2 - \frac{(n+2s)}{(n-2s)} \sum_{j} \alpha_j \frac{4s}{n-2s} \int K\left( \sum_{i} \alpha_i \tilde{\delta}_i \right) \frac{4s}{n-2s} v^2
\]
(28)
\[
= \|v\|^2 - 2 \sum_{i} \int P\tilde{\delta}_i v^2 + o(\|v\|^2),
\]
where \( Q(v, v) \) is the quadratic form defined in Proposition 3.1. Using Lemma 3.5 of [2], we can deduce that it is a positive quadratic form on \( v \), hence the existence of such \( \bar{v} \). Since \( \bar{v} \) minimize \( J \) in the \( v \)-space, it is easy to check that

\[
\exists \alpha > 0, s.t., \quad \alpha \|\bar{v}\|^2 \leq |(f, \bar{v})| \leq \alpha \|f\| \|\bar{v}\|,
\]
(29)
where $f$ is defined in Proposition 3.1. Thus, it is sufficient to estimate $|f|$. We have

$$f(v) = \frac{2}{S} \sum_j \sum_{i=1}^n \alpha_i^2 \int K P \delta_i^2 v + O\left( \sum_{j \neq i} \int_{P \delta_i \leq P \delta_j} P \delta_j \delta_i \|v\| \right).$$

Observe that

$$\int K P \delta_i^2 v = \int K \tilde{\delta}_i^2 v + O\left( \int_{B_i \cup B_i^c} \delta_i \theta_i \|v\| \right),$$

where $B_i = \{ x, |x - a_i| < d_i \}$ and $\theta_i = \tilde{\delta}_i - P \tilde{\delta}_i$. Arguing as in [5], we have

$$\int K \tilde{\delta}_i^2 v \leq c \|v\| \left( \frac{\nabla K(a_i)}{\lambda_i} + \frac{1}{\lambda_i^2} \right).$$

Using the Hölder’s inequality, we need to estimate:

$$\int_{B_i^c} (\tilde{\delta}_i \theta_i)^3 \leq \int_{B_i^c} \tilde{\delta}_i^3 = O\left( \frac{1}{(\lambda_i d_i)^2} \right),$$

$$|\theta_i|_\infty \int_{B_i} (\tilde{\delta}_i^3)^{\frac{2}{3}} = O\left( \frac{1}{(\lambda_i d_i)^2} \right).$$

Also, we have

$$\int_{P \delta_i \leq P \delta_j} P \delta_i P \delta_j |v| \leq \left| \int_{P \delta_i \leq P \delta_j} \left( P \delta_i P \delta_j \right)^{\frac{3}{2}} \right|,$$

$$\int_{P \delta_i \leq P \delta_j} \left( P \delta_i P \delta_j \right)^{\frac{3}{2}} \leq \left[ \int \left( \tilde{\delta}_i \tilde{\delta}_j \right)^{\frac{3}{2}} \right] = O\left( \varepsilon_{ij}^2 (\log \varepsilon_{ij}^{-1}) \right).$$

Thus, the estimate of $|f|$ and the result follows.

5. Construction of the pseudogradient

This section is devoted to the construction of a suitable pseudo-gradient of $J$ for which the Palais-Smale condition is satisfied along the decreasing flow lines as long as these flow lines do not enter in some neighborhood which allows us to identify the critical points at infinity of the variational structure associated to (1) for $n = 3$ and $s = \frac{1}{2}$. This construction is more difficult when there is a boundary, since the regular part of the Green’s function is unbounded near the boundary and may blow-up in this region. To overcome this difficulties, a careful study of the behavior of the Green’s
function and its regular part near the boundary is required. Observe that for \( u = \sum_{i=1}^{p} \alpha_i P \tilde{\delta}_{a_i, \lambda_i} + v \in V(p, \varepsilon) \) using Proposition 4.1 and after a change of variables

\[ v - \bar{v} \rightarrow V, \ v \in H, \text{ satisfying } (V_0), \]

we can write

\[ J(u) = J(\sum_{i=1}^{p} \alpha_i P \tilde{\delta}_{a_i, \lambda_i} + \bar{v}) + \| V \|^2. \]

As in ([6], Lemma 2), we can use the variables \((\alpha_i, a_i, \lambda_i, \bar{v}, V \in H)\) as a variables of \( u \) instead of \((\alpha_i, a_i, \lambda_i, v)\). Moving \( V \) according to the differential equation \( \frac{\partial V}{\partial s} = -\mu V \), where \( \mu \) is a very large constant. Then at \( s = 1, V(s) = e^{-\mu s}V(0) \) will be very small as we wish. Thus, in order to define our deformation, we can work as if \( V \) was zero. The deformation will extend immediately with the same properties to a neighborhood of zero in the \( V \) variable. Therefore, we need to define a vector field in \( \{ \sum_{i=1}^{p} \alpha_i P \tilde{\delta}_{a_i, \lambda_i} + \bar{v} \in V(p, \varepsilon) \} \). We begin by giving the following crucial result.

**Theorem 5.1.** Let \( n = 3 \) and \( s = \frac{1}{2} \). For any \( p \geq 1 \) and \( \varepsilon > 0 \) small enough, there exists a pseudo-gradient \( W \) in \( V(p, \varepsilon) \) satisfying the following:

There exists a constant \( c > 0 \) such that for any \( u = \sum_{i=1}^{p} \alpha_i P \tilde{\delta}_{a_i, \lambda_i} \in V(p, \varepsilon) \), we have

\[ (i) \quad < -\partial J(u), W(u) > \geq c \sum_{i=1}^{p} \left( \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_i d_i)^3} + \sum_{j \neq i}^{\frac{2}{3}} \varepsilon_{ij}^2 \right), \]

\[ (ii) \quad < -\partial J(u + \bar{v}), W(u) + \frac{\partial \bar{v}}{\partial (\alpha, a, \lambda)}(W(u)) > \geq c \sum_{i=1}^{p} \left( \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_i d_i)^3} + \sum_{j \neq i}^{\frac{2}{3}} \varepsilon_{ij}^2 \right). \]

\( |W| \) is bounded and satisfies the Palais-Smale condition along its flow lines as long as these flow lines do not enter in a small neighborhood of

\[ (y_{t_1}, \ldots, y_{t_p})_{\infty} := \sum_{i=1}^{p} \frac{1}{K(y_{t_i})^{\frac{3}{2}}} P \tilde{\delta}_{y_{t_i}, \infty}, \]
where \((y_{\ell_1}, \ldots, y_{\ell_p}) \in C^\infty\).

Before giving the proof of Theorem 5.1, we need to state two results which deal with two specific cases of Theorem 5.1. The proof of these results will be given later (pages 18 and 28 respectively). Let \(\eta > 0\) be a constant small enough such that:

- \(\eta < (1/4) \min d(y_i, y_j)\) for \(i \neq j\) and \(y_k\)'s are the critical points of \(K\).
- \(\eta < (1/4) \min d(y_i, \partial \Omega)\)
- \(|c_2 \frac{\Delta K(a)}{K(a)} - c_1 \overline{H}((a,0), a)| > c > 0\) for any \(a \in \cup B(y_i, 2\eta)\) (one can see that this condition occurs under assumption (2)).

**Proposition 5.2.** In \(V_1(p, \varepsilon) := \{u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i, \lambda_i} \in V(p, \varepsilon), a_i \notin \cup_k B(y_k, \eta), \forall 1 \leq i \leq p\}\), there exists a pseudo-gradient \(W_1\) so that the following holds: There is a constant \(c > 0\) independent of \(u \in V_1(p, \varepsilon)\) so that:

\[
< -\partial J(u), W_1(u) >\geq c \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} + \frac{1}{\lambda_i d_i^3} + \sum_{j \neq i} \varepsilon_{ij}^3 \right).
\]

**Proposition 5.3.** In \(V_2(p, \varepsilon) := \{u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i, \lambda_i} \in V(p, \varepsilon), a_i \in \cup_k B(y_k, 2\eta), \forall 1 \leq i \leq p\}\), there exists a pseudo-gradient \(W_2\) so that the following holds: There is a constant \(c > 0\) independent of \(u \in V_2(p, \varepsilon)\) so that:

\[
< -\partial J(u), W_2(u) >\geq c \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} + \frac{\|\nabla K(a_i)\|}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \right).
\]

\(W_2\) is bounded and the only case where \(\lambda_i(s), i = 1, \ldots, p, s \geq 0, \) tend to \(\infty\) is when each point \(a_i(t), i = 1, \ldots, p,\) goes to \(y_{\ell_i}\) with \((y_{\ell_1}, \ldots, y_{\ell_p}) \in C^\infty.\)

**Proof of Theorem 5.1.** We divide the set \(\{1, \ldots, p\}\) into two sets. The first contains the indices of the points near the \(\partial \Omega\) and the second contains the indices of the points far away from \(\partial \Omega\). Let us define

\[
B := \{i, 1 \leq i \leq p, s.t., d_i \geq 2d_0\},
\]

\[
B_1 := B \cup \{i \notin B, \text{ such that there exist } (i_1, \ldots, i_r) \text{ with } i_1 = i, i_r \in B,\}
\]
and $|a_{ik-1} - a_{ik}| < \frac{d_0}{p}$, $\forall k \leq r$,

$$B_2 := \{1, \ldots, p\} \setminus B_1.$$

Observe that:

(32) $d_i := d(a_i, \partial \Omega) \leq 2d_0, \forall i \in B_2,$

(33) $|a_i - a_j| \geq \frac{d_0}{p}, \forall i \in B_1, j \in B_2.$

No we write $u := u_1 + u_2 = \sum_{i \in B_1} \alpha_i P\overline{\delta}_{a_i, \lambda_i} + \sum_{i \in B_2} \alpha_i P\overline{\delta}_{a_i, \lambda_i}$.

Observe that $u_1 \in V_1(\text{card } B_1, \varepsilon_1)$. Then we apply the previous construction of Propositions 5.2 to the subpack of functions $u = \sum_{i \in B_1} \alpha_i P\overline{\delta}_{a_i, \lambda_i}$ forgetting the indices $i \notin B_1$. Let $W_1(u_1)$ be the vector field thus defined. The same argument can be repeated for $u_2$ which is in $V_2(\text{card } B_2, \varepsilon_2)$ and we will denote by $W_2(u_2)$ the vector field thus defined. Define $W$ as $W(u) = W_1(u_1) + W_2(u_2)$. Thus, we have

$$< -\partial J(u), W(u) > = < -\partial J(u), W_1(u_1) + W_2(u_2) >$$

$$\geq c \sum_{i \in B_1} \left( \frac{1}{\lambda_i} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j \in B_1} \varepsilon_{ij} + O\left( \sum_{j \in B_2} \varepsilon_{ij} \right) \right)$$

$$+ c \sum_{i \in B_2} \left( \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_i d_i)^3} + \sum_{j \in B_2} \varepsilon_{ij}^2 + O\left( \sum_{j \in B_1} \varepsilon_{ij} \right) \right)$$

Observe that for $i \in B_1, j \in B_2$ and using (33), we have $\varepsilon_{ij} = o\left( \frac{1}{\lambda_i^2} \right)$. Thus claim (i) of Theorem 5.1 follows. The proof of claim (ii) is similar to the proof given in Appendix 2 of [6]. By the construction of the vector field $W$, we can deduce that the Palais-Smale condition is satisfied along the decreasing flow lines of the pseudo-gradient $W$ as long as the concentration points of the flow do not enter in some neighborhood of $(y_{t_1}, \ldots, y_{t_p}) \in C^\infty$ since $\max_{1 \leq i \leq p} \lambda_i(t)$ remains bounded in this region. However, if the concentration points are near critical points $(y_{t_1}, \ldots, y_{t_p}) \in C^\infty$, $\lambda_i(s)$ increases on the flow line and goes to $+\infty$. Thus, we obtain a critical point at infinity. This finishes the proof of Theorem 5.1. \(\square\)
Proof of Proposition 5.2. In order to construct the required pseudo-gradient, we need to introduce the following notations:

For each $i \in \{1, \ldots, p\}$, let

$$I_1 = \{i, \text{s.t.} \ 1 \leq \sum_{k \neq i} \varepsilon_{ki} < \frac{p}{2p+1} \sum_{j=1}^{p} \frac{\bar{H}(a_i, 0), a_j}{(\lambda_i \lambda_j)}\},$$

$$I_2 = \{i, \text{s.t.} \ 1 \leq \sum_{k \neq i} \varepsilon_{ki} > \frac{p}{2p+1} \sum_{j=1}^{p} \frac{\bar{H}(a_i, 0), a_j}{(\lambda_i \lambda_j)}\}.$$

Without loss of generality, we can order the $\lambda_i$'s: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$. Let us define

$$I = \{1\} \cup \{i, \text{s.t.} \ \forall k \leq i, c \lambda_k d_k \leq \lambda_k - d_k - 1 \leq \lambda_k d_k\},$$

where $c$ is a constant chosen small enough.

Case 1: $I \cap I_2 \neq \emptyset$ and $I \neq \{1, \ldots, p\}$.

In $I_2$, we order the $\lambda_i$'s: $\lambda_{i_1} \leq \lambda_{i_2} \leq \cdots \leq \lambda_{i_r}$. Let

$$X_1 = -\sum_{k=1}^{r} 2^k \alpha_{ik} \lambda_i \frac{\partial P_{\bar{\delta}}}{\partial \lambda_{ik}} + m \sum_{i \in I_2} \frac{\alpha_i \partial P_{\bar{\delta}}}{\partial a_i} \nabla K(a_i),$$

where $m$ is a small constant. We claim that

$$-\partial J(u), X_1(u) \geq c \sum_{i \in I_2} \left( \frac{1}{\lambda_i} \right) + \frac{1}{(\lambda_i d_i)^{n+1-2s}} + \sum_{j \neq i} \varepsilon_{ij} + R.$$

Indeed, using Proposition 3.2, we derive that

$$\langle -\partial J(u), -\sum_{k=1}^{r} 2^k \alpha_{ik} \lambda_i \frac{\partial P_{\bar{\delta}}}{\partial \lambda_{ik}} \rangle = 2 \sum_{i=1}^{r} J(u) \sum_{k=1}^{r} \left[ -\sum_{j \neq ik} 2^k \alpha_{jk} \alpha_{ik} \lambda_i \frac{\partial}{\partial \lambda_{ik}} \nabla K(a_i) \right] + (1 + o(1)),$$

$$= \frac{n-2s}{2} \sum_{j=1}^{p} 2^k \alpha_{jk} \lambda_j \frac{\bar{H}((a_j, 0), a_i)}{(\lambda_j \lambda_{ik})} \left(1 + o(1)\right) + R + O\left(\frac{1}{\lambda_i^{2s}}\right).$$

Observe that

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \varepsilon_{ij} \left(1 - 2 \lambda_j \frac{\lambda_j}{\lambda_i} \varepsilon_{ij}\right).$$

Thus, for $\lambda_i \geq \lambda_j$,

$$-2 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq -2 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \varepsilon_{ij} + O\left(\varepsilon_{ij}^2\right).$$
Furthermore, using Lemma 5.17 of [10] page 445, the regular part of the Green’s function satisfies

\[ \tilde{H}((a_i, 0), a_j) \leq \max(d_i, d_j)^{-2}. \]

For \( j \in I_1 \) and \( i \neq j \), if \( d_j/2 \leq d_i \leq 2d_j \), using (38) we obtain

\[ \frac{\lambda_j}{\lambda_i} \varepsilon_{ij} = o(1). \]

In the other case (i.e. \( d_i \leq d_j/2 \) or \( d_i \geq 2d_j \)), we use the inequality \( |a_i - a_j| \geq \frac{1}{2} \max(d_i, d_j) \) to obtain (39). Thus,

\[ \langle -\partial J(u), -\sum_{k=1}^r 2^k \alpha_{ik} \lambda_k \frac{\partial \tilde{P}_{\delta_k}}{\partial \lambda_k} \rangle \geq c \sum_{i \in I_2} \left( \sum_{i \neq j} \varepsilon_{ij} \left( 1 + o(1) \right) - 2^{p} \sum_{j=1}^{p} \frac{H((a_i, 0), a_j)}{(\lambda_i \lambda_j)} \right) + R. \]

Using Lemma 5.17 of [10] page 445, we get

\[ \tilde{H}((a_i, 0), a_i) = (2d_i)^{2s-n} + o(d_i^{-2}), \]

for each point \( a_i \) near the boundary. From another part, for each \( a_i \) in a compact set \( K \) of \( \Omega \), we have \( \tilde{H}((a_i, 0)a_i) \geq c \), we derive that

\[ \tilde{H}((a_i, 0), a_i) \geq c(d_i)^{-2}, \]

for each \( a_i \in \Omega \). Since \( i \in I_2 \), we obtain

\[ \langle -\partial J(u), \sum_{i \in I_2}^r 2^k \alpha_{ik} \lambda_k \frac{\partial \tilde{P}_{\delta_k}}{\partial \lambda_k} \rangle \geq c \sum_{i \in I_2} \left( \sum_{i \neq j} \varepsilon_{ij} + O \left( \frac{1}{\lambda_i^2} \right) + o \left( \frac{1}{\lambda_i d_i} \right) \right) + R. \]

From another part, using (38), (41) and Proposition 3.3, we derive that

\[ \langle -\partial J(u), \sum_{i \in I_2} \frac{\alpha_i}{\lambda_i} \frac{\partial \tilde{P}_{\delta_i}}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|} \rangle \geq c \sum_{i \in I_2} \left( \frac{|\nabla K(a_i)|}{\lambda_i} \right) + O \left( \frac{1}{(\lambda_i d_i)^3} + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} \right). \]

Using the fact that \( a_i \) is far away from the critical points of \( K \) we derive that \( |\nabla K(a_i)| > c > 0 \). Combining (43), (44) and using the fact that \( m \) is
small, claim (34) follows.

Since \( \lambda_1 d_1 \leq \lambda_2 d_2 \leq \cdots \leq \lambda_p d_p \), we can make appear the term \( (\lambda_1 d_1)^{-2} \) in the upper bound of (34) and therefore we can also make all the \( (\lambda_i d_i)^{-2} \) appear in this formula. Thus, we derive

\[
(45) \quad < -\partial J(u), X_1(u) > \geq c \left( \sum_{i=1}^{p} \left( \frac{1}{(\lambda_i d_i)^2} \right) + \sum_{i \in I_2} \left( \frac{1}{\lambda_i} + \sum_{j \neq i} \epsilon_{ij} \right) \right) + R.
\]

For \( i \in I_1 \), we have

\[
(46) \quad \frac{1}{2p+1} \sum_{k \neq i} \epsilon_{kl} \leq \sum_{j=1}^{p} \frac{H(a_{i}, 0), a_{j}}{(\lambda_i \lambda_j)} \leq \frac{p}{(\lambda_1 d_1)^2}.
\]

Therefore, from \( (\lambda_1 d_1)^{-2} \) we can make appear the term \( \sum_{i \in I_1, j \neq i} \epsilon_{ij} \) in the upper bound of (45). Hence

\[
(47) \quad < -\partial J(u), X_1(u) > \geq c \left( \sum_{i=1}^{p} \left( \frac{1}{(\lambda_i d_i)^2} + \sum_{j \neq i} \epsilon_{ij} \right) + \sum_{i \in I_2} \left( \frac{1}{\lambda_i} \right) \right) + R.
\]

Furthermore, using Proposition 3.3 and the fact that \( |\nabla K(a_i)| > c > 0 \), we derive that

\[
(48) \quad < -\partial J(u), \sum_{i \in I_1} \frac{\alpha_i}{\lambda_i} \partial P \delta_i \nabla K(a_i) > \geq c \left( \sum_{i \in I_1} \left( \frac{1}{\lambda_i} + O \left( \frac{1}{(\lambda_i d_i)^3} + \sum_{j \neq i} \epsilon_{ij} \right) \right) \right).
\]

Combining (47) and (48), we derive that for \( X'_1 = X_1 + m \sum_{i \in I_1} \frac{\alpha_i}{\lambda_i} \partial P \delta_i \nabla K(a_i) \)
and \( m \) small enough, we have

\[
(49) \quad < -\partial J(u), X'_1 > \geq c \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^3} + \sum_{j \neq i} \epsilon_{ij}^3 \right).
\]

Hence the estimates of Proposition 5.2 follows in this case.

**Case 2:** \( I \cap I_2 \neq \emptyset \) and \( I = \{1, \ldots, p\} \).

Let \( i_1 = \min\{i, s.t., i \in I_2\} \) and \( I_{i_1} = \{j \notin I_2, s.t., \lambda_i d_i \leq 2c_2 \lambda_j d_j\} \). Let

\[
X_2 = - \sum_{i \in I_2 \cup I_{i_1}} \frac{\lambda_i}{\lambda_i} \frac{\partial P \delta_i}{\partial \lambda_i}
\]
Using Proposition 3.2, we obtain
\[
\left\langle -f_{\mathbf{d}(u)}, X_2(u) \right\rangle = 2c_1 J(u) \sum_{i \in I_{2 \cup I_1}} \left[ - \sum_{j \neq i} \alpha_j \frac{\partial \varepsilon_{ji}}{\partial \lambda_i} \left( 1 + o(1) \right) - \sum_{j=1}^{p} \alpha_j \frac{\tilde{H}((a_j, 0), a_i)}{\lambda_j \lambda_i} \left( 1 + o(1) \right) \right] + R + O\left( \sum_{i \in I_{2 \cup I_1}} \frac{1}{\lambda_i^2} \right).
\]

Since \( I = \{1, \ldots, p\} \), we have for each \( i \neq j \),
\[
\varepsilon_{ij} = \left( \frac{1}{\lambda_i \lambda_j |a_i - a_j|^2} \right) \left( 1 + O\left( \frac{1}{\lambda_i^2 |a_i - a_j|^2} + \frac{1}{\lambda_j^2 |a_j - a_j|^2} \right) \right)
\]
\[
= \left( \frac{1}{\lambda_i \lambda_j |a_i - a_j|^2} \right) + O\left( \frac{1}{(\lambda_i d_i)^4} + \varepsilon_{ij}^2 \right).
\](51)

Indeed, if \( d_i \leq d_j/2 \) or \( d_i \geq 2d_j \), we have \( |a_i - a_j| \geq \frac{1}{2} \max(d_i, d_j) \) and the result follows. In the other case, if \( d_j/2 \leq d_i \leq 2d_j \), using that \( i, j \in I \), we derive that \( \frac{\lambda_i}{d_i} \) and \( \frac{\lambda_j}{d_j} \) are bounded. Therefore, \( (\lambda_i |a_i - a_j|)^{-2} = O(\varepsilon_{ij}) \) for \( k = i, j \). Thus, for \( i \in I_1 \), using (36) and (51), we get
\[
\sum_{j \neq i} \left( - \lambda_i \frac{\partial \varepsilon_{ji}}{\partial \lambda_i} \frac{\tilde{H}((a_i, 0), a_j)}{\lambda_i \lambda_j} \right) = \sum_{j \neq i} \frac{\tilde{G}((a_i, 0), a_j)}{(\lambda_i \lambda_j)}
\]
\[
+ O\left( \frac{1}{(\lambda_i d_i)^4} + \varepsilon_{ij}^2 \right).
\]

Furthermore, for \( i \in I_{11} \),
\[
\frac{\tilde{H}((a_i, 0), a_i)}{(\lambda_i)^2} = O\left( \frac{1}{(\lambda_i d_i)^2} \right) = O\left( \frac{2c_2}{(\lambda_i d_i)^2} \right) = o\left( \frac{1}{(\lambda_i d_i)^2} \right).
\]

For \( i \in I_2 \), using (36) and (51), we get
\[
- \sum_{j \neq i} \lambda_i \frac{\partial \varepsilon_{ji}}{\partial \lambda_i} - \sum_{j=1}^{p} \frac{\tilde{H}((a_i, 0), a_j)}{(\lambda_i \lambda_j)} \geq c \left( \sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} \right).
\]

Therefore, using the fact that the Green’s function is positive, we derive
\[
\left\langle -\partial J(u), X_2(u) \right\rangle \geq c \sum_{i \in I_2} \left( \sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} \right) + R + O\left( \sum_{i \in I_{2 \cup I_1}} \frac{1}{\lambda_i^2} \right).
\]

Using the fact that \( I \cap I_2 \neq \emptyset \) and arguing as in the Case 1, we can obtain (45) and we can make appear the term \( \sum_{i \in I_1, j \neq i} \varepsilon_{ij} \) in the upper bound of
Thus, for $X_2' = X_2 + \sum_{i \in I_2} \alpha_i \partial P \frac{\tilde{\delta}_i}{\lambda_i} \nabla K(a_i) + m \sum_{i \in I_1} \alpha_i \partial P \frac{\tilde{\delta}_i}{\lambda_i} \nabla K(a_i)$ and $m$ small enough, we have

$$\langle -\partial J(u), X_2' \rangle \geq c \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_{i})^3} + \sum_{j \neq i} \varepsilon_{ij}^2 \right).$$

Hence the estimates of Proposition 5.2 follows in this case.

For $c_3$ a fixed small constant, let us define

$$L = \{ j \in I_1, s.t. \exists i \in I_1, s.t. c_3 \max(d_i, d_j) \geq |a_i - a_j| \}.$$

For $i \in L$, let $i_0$ the index such that

$$c_3 \max(d_i, d_{i_0}) \geq |a_i - a_{i_0}|.$$

**Case 3:** $I \cap I_2 = \emptyset$ and there exists $i, i_0 \in I$ satisfying (56).

Let

$$X_3 = -\alpha_i \lambda_i \frac{\partial \tilde{P}}{\partial \lambda_i}.$$

We have

$$\langle -\partial J(u), X_3(u) \rangle = 2c_1 J(u) \left[ -\sum_{j \neq i} \alpha_j \alpha_i \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \left( 1 + o(1) \right) \right. - \sum_{j=1}^{p} \alpha_j \alpha_i \frac{\tilde{H}(a_j, 0), a_i}{(\lambda_j \lambda_i)} \left( 1 + o(1) \right) \] + R + O(\frac{1}{\lambda_i^2}).$$

Arguing as in Case 1, the terms with $j \in I_2$ can be seen like $O(\varepsilon_{ij})$. Next, we interest with the indices $j \in I_1$. Observe that for $i, k \in I_1$, we have (51).

Indeed, if $d_i \leq d_k/2$ or $d_i \geq 2d_k$, we have $|a_i - a_k| \geq \frac{1}{2} \max(d_i, d_k)$ and the result follows. In the other case, if $d_k/2 \leq d_i \leq 2d_k$, using that $i, k \in I_1$, we have as in (39)

$$\varepsilon_{ij} = o\left( \left( \frac{\lambda_i}{\lambda_k} \right) + \left( \frac{\lambda_k}{\lambda_i} \right) \right).$$

Therefore,

$$\frac{1}{\lambda_i^2}|a_i - a_k|^2 = \frac{1}{\lambda_k} \frac{1}{\lambda_j \lambda_k} |a_i - a_k|^2 \leq \frac{\lambda_i}{\lambda_k} \varepsilon_{ij} \leq \frac{1}{\lambda_k} \frac{1}{(\lambda_i d_i)(\lambda_i d_k)} = O\left( \frac{1}{(\lambda_i d_i)^2} \right), \forall r = i, k.$$
Thus we obtain (52) with the indices \( j \in I_1 \). Using (42), and the fact that the Green’s function is positive, we derive that

\[
\langle -\partial J(u), X_3(u) \rangle \geq c \left( \frac{1}{(\lambda_i d_i)^2} + \frac{\tilde{G}((a_i,0), a_{i_0})}{(\lambda_i \lambda_{i_0})} \right) + R + O \left( \sum_{j \neq i, j \in I_2} \varepsilon_{ij} \right).
\]

Since \( i, i_0 \in I \) satisfying (56), we can assume that \( \lambda_i \geq \lambda_{i_0} \) and thus

\[
\frac{1}{(\lambda_i d_i)^2} + \frac{\tilde{H}((a_i,0), a_{i_0})}{(\lambda_i \lambda_{i_0})} \leq \left( \frac{c_3^2}{\lambda_i \lambda_{i_0} |a_i - a_{i_0}|^2} \right).
\]

Using (51), we derive that

\[
\langle -\partial J(u), X_3(u) \rangle \geq c \left( \varepsilon_{ii_0} + \frac{1}{(\lambda_i d_i)^2} \right) + R + O \left( \sum_{j \neq i, j \in I_2} \varepsilon_{ij} \right).
\]

Since \( i \in I \) and the term \((\lambda_i d_i)^{-2}\) appears in the lower-bound of the above estimate, arguing as in the Case 1, we can make appear all the \((\lambda_k d_k)^{-2}\) and \(\sum_{k \neq j, k \in I_1} \varepsilon_{kj}\) appear in this lower-bound. For \(m_1\) a fixed large constant and a small constant \(m_2\), the pseudo-gradient \(X_3 + m_1 X_1 + m_2 \sum_{i \in I_1} \alpha_i \frac{\partial P\bar{\delta}_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|}\) satisfies the estimates of Proposition 5.2 in this case.

**Case 4:** \( I \cap I_2 = \emptyset \) and \( \forall i, i_0 \in I \), such that \( c_3 \max(d_i, d_{i_0}) < |a_i - a_{i_0}| \).

For \( d_0 \) a fixed small constant, we introduce the following sets

\[
I' = \{ i \in I, d_i < d_0 \}.
\]

\[
L_i = \{ j \in L, s.t., i \text{ and } j \text{ satisfy (56)} \}.
\]

**Case 4.1:** \( I \cap I_2 = \emptyset \) and \( \forall i, i_0 \in I \), such that \( c_3 \max(d_i, d_{i_0}) < |a_i - a_{i_0}| \). If \( I' \neq \emptyset \), let

\[
X_4 = \sum_{i \in I'} \alpha_i \frac{\partial P\bar{\delta}_i}{\partial a_i} \left( -\frac{\eta_i}{\lambda_{j_0}} \right),
\]

where \( \lambda_{j_0} = \max\{\lambda_i, i \in I'\} \) and \( \eta_i \) is the outward normal to \( \partial \Omega_{d_i} = \{ x \in \Omega, s.t., d(x, \partial \Omega) = d_i \} \) at \( a_i \).

Using Proposition 3.3, we derive

\[
\langle -\partial J(u), X_4 \rangle = \frac{J(u) c_1}{\lambda_{j_0}} \sum_{i \in I'} \left[ -c_3 \alpha_i \frac{\partial K(a_i)}{\partial \eta_i} (1 + o(1)) \right]
\]
Observe that for each 
\[ \partial \bar{H}(i, a, j) \leq \frac{c}{d_i(\lambda_i d_i, \lambda_j d_j)} = o\left(\frac{1}{d_i(\lambda_i, \lambda_j)^2}\right). \]
For \( i \) and \( j \) in \( I' \), if \( \frac{d_i}{d_j}, \frac{d_j}{d_i} \) and \( |a_i - a_j| \) are bounded and arguing as in the Appendix of [8] we derive that \( \frac{\partial \bar{H}}{\partial \eta}(i, a, j) > 0 \). In the other case, we have
\[ \frac{\partial \bar{H}}{\partial \eta}(i, a, j) \leq \frac{1}{d_i \max(d_i, d_j, |a_i - a_j|)^2} = o\left(\frac{1}{(d_i d_j)^2}\right). \]
Thus,
\[ \frac{1}{(\lambda_i, \lambda_j)} \frac{\partial \bar{H}}{\partial \eta}(i, a, j) = o\left(\frac{1}{d_i(\lambda_i d_j)^2} + \frac{1}{d_j(\lambda_i d_i)^2}\right). \]
Observe that for each \( i \in I' \), using (41) and arguing as [5] and [16], we have
\[ \frac{\partial \bar{H}}{\partial \eta}(i, a, j) = \frac{1}{2^{2-1} d_i^2(1 + o(1))}. \]
Also, for \( i, j \in I' \) we have \( \eta_i - \eta_j = O(|a_i - a_j|) \). Therefore,
\[ \frac{\partial \varepsilon_{ij}}{\partial \eta_i} + \frac{\partial \varepsilon_{ij}}{\partial \eta_j} = \lambda_i \lambda_j (a_i - a_j) \varepsilon_{ij}^2 (\eta_j - \eta_i) = O(\varepsilon_{ij}). \]
For \( i \in I' \) and \( j \in I_1 \setminus (I \cup L) \) we have \( c_3 \max(d_i, d_j) \leq |a_i - a_j| \), then
\[ \frac{\partial \varepsilon_{ij}}{\partial \eta_i} = (n - 2s) \lambda_i \lambda_j (a_i - a_j) \varepsilon_{ij}^2 \leq \frac{c}{(\lambda_i \lambda_j)^2 |a_i - a_j|^3} \]
\[ = O\left(\frac{1}{c_3(\lambda_i d_i, \lambda_j d_j)d_i}\right) = O\left(\frac{1}{c_3(\lambda_i d_i)^2 d_i}\right). \]
for $c_2$ and $c_3$ chosen such that $c_2 = o(c_3^3)$.

For $i \in I'$ and $j \in I \setminus I'$, we claim that

$$\frac{\partial H}{\partial \eta_i}((a_i, 0), a_j) \frac{1}{(\lambda_i \lambda_j)} \frac{\partial \varepsilon_{ij}}{\partial a_i} \eta_i = - \frac{\partial \tilde{G}((a_i, 0), a_j)}{\partial \eta_i} \frac{1}{(\lambda_i \lambda_j)} + o\left(\frac{\lambda_i}{(\lambda_1 \lambda_d)^5}\right).$$

Indeed, since $I \cap I_2 = \emptyset$ then $i$ and $j$ belong to $I_1$. Using (51) and the fact that (56) is not satisfied, we derive

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} = 2\lambda_i \lambda_j (a_i - a_j) \varepsilon_{ij}^2 = \frac{2(a_i - a_j)}{(\lambda_i \lambda_j) |a_i - a_j|^2} \left(1 + O\left(\frac{1}{(\lambda_1 \lambda_d)^2}\right)\right).$$

Therefore,

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} = \frac{1}{(\lambda_i \lambda_j)} \frac{\partial}{\partial a_i} \left(\frac{1}{|a_i - a_j|^2}\right) + O\left(\frac{\lambda_i}{c_3^3 (\lambda_1 \lambda_d)^5}\right),$$

and our claim follows. Thus,

$$\left\langle - \partial J(u), X_4 \right\rangle \geq \frac{c}{\lambda_{j_0}} \sum_{i \in I'} \left[ \frac{1}{(\lambda_i d_i)^{n-2s d_i}} - \sum_{j \in I \setminus I'} \frac{1}{(\lambda_i \lambda_j)} \frac{\partial \tilde{G}((a_i, 0), a_j)}{\partial \eta_i} \right]
+ O\left( \sum_{i \neq j, j \in I_1 \cup L_i} \varepsilon_{ij} \right) + O\left( \sum_{i \neq j} \varepsilon_{ij} \log \varepsilon_{ij}^{-1} + \sum_k \log(\lambda_k d_k) \right) + O\left( \sum_{i \neq j} \lambda_i |a_i - a_j| \varepsilon_{ij}^{-\frac{1}{3}} \right).$$

Observe that for $i \in I'$ and $j \in I \setminus I'$ we have $d_i \leq d_j$. Then, arguing as in [5] and [16], we can deduce that $- \frac{\partial \tilde{G}((a_i, 0), a_j)}{\partial \eta_i} > 0$. For $i, j \in I'$, using the fact that (56) is not satisfied, we derive

$$\varepsilon_{ij} = O\left(\frac{1}{(\lambda_i d_i)^2} + \frac{1}{(\lambda_j d_j)^2}\right).$$

Since $d_i$ and $d_j$ are small, we have

$$\varepsilon_{ij} = o\left(\frac{1}{d_i (\lambda_i d_i)^2} + \frac{1}{d_j (\lambda_j d_j)^2}\right).$$

Using the fact that $j_0 \in I'$ and that $\lambda_{j_0} d_{j_0}$ and $\lambda_1 d_1$ are of the same order, we can make all the $(\lambda_i d_i)^3$ for $i \in I_1$ appear in the lower bound. Thus,

$$\left\langle - \partial J(u), X_4 \right\rangle \geq c \sum_{i=1}^p \frac{1}{(\lambda_i d_i)^3} + O\left( \sum_{i \neq j, j \in I_2 \cup L_i} \varepsilon_{ij} \right)$$

(68)
\[ + O \left( \sum_{i \neq j} \varepsilon_{ij}^{-1} + \sum_k \log(\lambda_k d_k) \right) + O \left( \sum_{i \neq j} \lambda_i |a_i - a_j| \varepsilon_{ij}^{\frac{3}{2}} \right). \]

Let
\[ X_5 = - \sum_{i \in I'} \alpha_i \lambda_i \partial P \tilde{\delta}_i - \sum_{j \in I_1} \alpha_j \lambda_j \partial P \tilde{\delta}_j. \]

Using Case 3, we get
\[ (69) \quad \langle - \partial J(u), X_5(u) \rangle \geq c \left( \sum_{i \in I', j \in I_i} \varepsilon_{ij} \right) + O \left( \sum_{j \neq i, i' \in I', j \in I_2} \varepsilon_{ij} \right) + R. \]

For \( m_1 \) and \( m_2 \) two fixed large constants, using (45) (68) and (69) we get
\[ (70) \quad \langle - \partial J(u), X_4 + m_1 X_1 + m_2 X_5 \rangle \geq c \left( \sum_{i=1}^p \frac{1}{(\lambda_i d_i)^3} + \sum_{i \neq j \in I_2} \varepsilon_{ij} \right) + R \]
\[ + O \left( \sum_{i \neq j} \lambda_i |a_i - a_j| \varepsilon_{ij}^{\frac{3}{2}} \right). \]

As in the Case 1, we can make appear the term \( \sum_{i \in I_1, j \neq i} \varepsilon_{ij} \) in the upper bound of (45). Hence the estimates of Proposition 5.2 follows in this case.

**Case 4.2:** \( I \cap I_2 = \emptyset \) and \( \forall i, i_0 \in I \), such that \( \varepsilon_3 \max(d_i, d_i_0) < |a_i - a_i_0| \).

If \( I' = \emptyset \). For each \( i \in I \) we have \( d_i \geq d_0 \) since \( I' = \emptyset \). We denote by \( M = (m_{ij}, a_i, a_j \in I) \) the matrix defined by (3), \( \rho \) its least eigenvalue and \( e \) the eigenvector associated to \( \rho \). Let \( \eta > 0 \) be such that for any \( x \) belongs to a neighborhood \( C(e, \eta) \) of \( e \) where
\[ C(e, \eta) = \{ y \in (\mathbb{R}^*_+)^r, r = \text{card} I, s.t. \left| \frac{y}{|y|} - e \right| < \eta \}, \]
we have:
\[ (71) \quad t_x M x - \rho |x|^2 \leq \frac{1}{2} |\rho| |x|^2 \]
and
\[ (72) \quad t_x \partial M \partial a_i x = \left( \frac{\partial \rho}{\partial a_i} + o(1) \right) |x|^2. \]

For each \( x \in C(e, \eta)^c \), we have
\[ t_x M x - \rho |x|^2 > c |x|^2. \]

Let \( \Lambda = \left( \frac{1}{\lambda_{j_1}}, \ldots, \frac{1}{\lambda_{j_r}} \right) \) for \( I = \{j_1, \ldots, j_r\} \).
If $\Lambda$ belongs to the set $C(e, \eta)^c$ then we move the vector $\Lambda$ to $C(e, \eta)$ as in [9] along
\[
\Lambda(t) = \frac{(1-t)\Lambda + t|\Lambda|e}{(1-t)\Lambda + t|\Lambda|e}.
\]

Using Proposition (3.2), we derive that there exists a pseudo-gradient $X_6$ such that
\[
\langle -\partial J(u), X_6 \rangle = c \left[ -\frac{d}{dt} \left( t_\Lambda M \Lambda(t) \right) + o \left( \sum_{j \not= i, i \in I} \varepsilon_{ij} + \sum_{i \in I} \frac{1}{\lambda_i^2} \right) \right]
\]
\[
+ O \left( \sum_{i \not= j, i \in I \cup I_1 \setminus I} \varepsilon_{ij} + R \right).
\]

As in [9], we have
\[
\frac{d}{dt} \left( t_\Lambda M \Lambda(t) \right) < -c|\Lambda|^2 = -c \sum_{i \in I} \frac{1}{\lambda_i^2}.
\]

For $i \in I, j \in I_1 \setminus I$, we have
\[
\varepsilon_{ij} = O \left( \left( \frac{1}{(\lambda_i d_i)(\lambda_j d_j)} \right) \right) = O \left( \frac{c_2}{d_0^2 \lambda_i^2} \right) = o \left( \frac{1}{\lambda_i^2} \right),
\]
for $c_2$ chosen such that $c_2 = o(d_0^2)$. Thus,
\[
\langle -\partial J(u), X_6 \rangle \geq c \sum_{i \in I} \frac{1}{\lambda_i^2} + o \left( \sum_{j \not= i, i \in I} \varepsilon_{ij} + \sum_{i \in I} \frac{1}{\lambda_i^{n-2s}} \right) + R.
\]

In the case where $\Lambda$ belongs to the set $C(e, \eta)$, the construction of the vector-field $W_6$ depends on the value of $\rho$ and $|\rho|$. Since zero is a regular value of $\rho$ then there exists a constant $\rho_0 > 0$ such that either $|\rho| > \rho_0$ or $|\rho| > \rho_0$.

If $\rho < -\rho_0$, we decrease all the $\lambda_i$'s for $i \in I$. If we assume that $c_2 = o(\rho_0 d_0^2)$ then using Proposition 3.2, (36), (38) and (74) we obtain (75) in this case.

If $|\rho| > \rho_0$ and $\rho < -\rho_0$, then we move the points $a_i$'s along $\lambda_{j_0} \dot{a}_i = -\frac{\partial \rho}{\partial a_i}$ for each $a_i \in I$ and $\lambda_{j_0} = \max \{ \lambda_i, i \in I \}$. Using Proposition 3.3, we derive
\[
\langle -\partial J(u), X_6 \rangle = \frac{1}{\lambda_{j_0}} \sum_{i \in I} \left[ -\partial J(u), \frac{\partial P \delta(a_i, \Lambda)}{\partial a_i} \right] \left( -\frac{\partial \rho}{\partial a_i} \right)
\]
\[
= \frac{1}{\lambda_{j_0}} \sum_{i \in I} \left[ -c \frac{\partial \rho}{\partial a_i} \left( t_\Lambda \frac{\partial M}{\partial a_i} \Lambda \right) + o \left( \sum_{j \in I \cup I_1} \lambda_i \varepsilon_{ij} \right) \right].
\]
sets: \(\nabla\|X\|\) combination of all previous cases. □

The vector field \(W\)

Proposition 5.2.

Observe that for \(i \in I\) and \(j \in I_1 \setminus (I \cup L_i)\), (56) is not satisfied. Thus,

\[
\frac{\partial \tilde{H}((a_i, 0), a_j)}{\partial a_i} \frac{1}{(\lambda_i \lambda_j)} \leq \frac{1}{d_i^2} \frac{1}{(\lambda_i \lambda_j)} \leq \frac{c_2 D}{\lambda_i^2 d_0^2},
\]

where \(D\) is the diameter of \(\Omega\). Arguing as in (66), we obtain

\[
\left| \frac{\partial \tilde{H}((a_i, 0), a_j)}{\partial a_i} \frac{1}{(\lambda_i \lambda_j)} \right| \leq \frac{c}{(\lambda_i \lambda_j)} |a_i - a_j|^3 \leq \frac{(c_2 D)}{c_3 \lambda_i^2 d_0^4}.
\]

Since \(\Lambda \in C(\epsilon, \eta)\), then using (71), (72), (76) and (79) we obtain

\[
\left\langle -\partial J(u), X_6 \right\rangle \geq \frac{c}{2\lambda_{j_0}} |\rho' \Lambda|^2 + O \left( \sum_{i \in I_1, j \in I_2 \cup L_i} \lambda_i \epsilon_{ij} \right) + R + O \left( \sum_{i \neq j} \lambda_i |a_i - a_j| \epsilon_{ij}^2 \right).
\]

Thus, in both cases, for \(m_1\) and \(m_2\) two fixed large constants, the pseudogradient \(X_6 + m_1 X_1 + m_2 X_5\) satisfies (70) and then the estimates of Proposition 5.2.

The vector field \(W_1\) required in Proposition 5.2 will be defined by a convex combination of all previous cases. □

Proof of Proposition 5.3. Let \(\eta > 0\) be a fixed constant small enough with \(|y_i - y_j| > \eta, \forall i \neq j\). We divide the set \(V_1(p, \epsilon)\) into the following sets:

\[
V_1^1(p, \epsilon) := \{ u = \sum_{i=1}^{p} \alpha_i P \tilde{\delta}_{(a_i, \lambda_i)} \in V_1(p, \epsilon), a_i \in B(y_{j_i}, \eta), \forall i = 1, \ldots, p, \forall k \neq i, y_{j_i} \neq y_{j_k} \}
\]

and \(\rho(y_{j_1}, \ldots, y_{j_p}) > 0\).

Thus, in both cases, \(m_1\) and \(m_2\) two fixed large constants, the pseudogradient \(X_6 + m_1 X_1 + m_2 X_5\) satisfies (70) and then the estimates of Proposition 5.2.

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\]

and \(\rho(y_{j_1}, \ldots, y_{j_p}) > 0\).

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The vector field \(W_1\) required in Proposition 5.2 will be defined by a convex combination of all previous cases. □
where Λ = \(81\)

Therefore

Furthermore, since \(u \in V_1^1(p, \varepsilon)\), for each \(j \neq i\), we have \(|a_i - a_j| \geq \eta\). Therefore

\[
\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\varepsilon_{ij}(1 + o(1)) = -\frac{1}{(\lambda_i \lambda_j)|a_i - a_j|^2}(1 + o(1)).
\]

Thus,

\[
\left\langle -\partial J(u), Z_1 \right\rangle = 2J(u) \left[ \sum_{i=1}^{p} \left( -c_2 \frac{\Delta K(a_i)}{K(a_i)^2} + c_1 \frac{\tilde{H}((a_i), 0, a_i)}{K(a_i)^2} \right) + c_1 \sum_{j \neq i} \frac{1}{(K(a_i)K(a_j))} \left( \frac{\tilde{G}((a_i), 0, a_j)}{(\lambda_i \lambda_j)} \right) \right] + o \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \right)
\]

\[
(81)= 2J(u)^T \Lambda M A + o \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \right) \geq c \sum_{i=1}^{p} \frac{1}{\lambda_i^2} \geq c \sum_{i=1}^{p} \left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} \right),
\]

where \(\Lambda = (\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_p})\). We let now:

\[
Z_a = \sum_{i=1}^{p} \varphi(\lambda_i | \nabla K(a_i)) \frac{a_i}{\lambda_i} \frac{\partial \tilde{P}(a_i, \lambda)}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|},
\]
where \( \varphi : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \varphi(t) = \begin{cases} 0, & \text{if } |t| \leq 1 \\ 1, & \text{if } |t| \geq 2 \end{cases} \).

Using Proposition 3.3, we obtain

\[
\left\langle -\partial J(u), Z_a \right\rangle \geq c \sum_{i=1}^p \varphi(\lambda_i |\nabla K(a_i)|) \left( \frac{|\nabla K(a_i)|}{\lambda_i} + O\left( \frac{1}{\lambda_i^2} \right) + O\left( \sum_{j \neq i} \varepsilon_{ij} \right) \right).
\]

For \( C > 0 \) a fixed constant large enough and \( \tilde{W}_1 = Z_a + CZ_1 \), we derive that

\[
\left\langle -\partial J(u), \tilde{W}_1 \right\rangle \geq c \left( \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]

\( \bullet \) If \( u \in V^2_{l}(p, \varepsilon) \), let \( \rho \) be the least eigenvalue of \( M \). Then there exists an eigenvector \( e \) associated to \( \rho \) such that \( |e| = 1 \) with \( \varepsilon_i > 0 \) for all \( i \) [8]. Let \( \gamma > 0 \) such that for any \( x \in B(e, \gamma) = \{ y \in S^{p-1}, \| y - e \| \leq \gamma \} \), we have \( ^t x M x < \frac{1}{2} \rho \). Two cases may occur. **Case 1:** If \( \frac{A}{|\lambda|} \in B(e, \gamma) \). In this case, we define \( \tilde{W}_2 = -CZ_1 + Z_a \). Using the estimates (81), (82) and the fact that \( \rho(\tau) < 0 \), we derive that

\[
\left\langle -\partial J(u), \tilde{W}_2 \right\rangle \geq c \left( \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]

**Case 2:** If \( \frac{A}{|\lambda|} \notin B(e, \gamma) \). In this case, let \( y(t) = (1 - t)\Lambda + t|\Lambda|e, \Lambda(t) = y(t)/\|y(t)\| \) and we define

\[
Z_2 = -\sum_{i=1}^p |\lambda_i| \lambda_i^2 \frac{\partial P_{\delta_i}}{\partial \lambda_i} \left[ \frac{|\lambda_i| e_i - \Lambda_i}{\|y(0)\|} - \frac{y_i(0)}{\|y(0)\|^2} (y(0), |\lambda|e - \Lambda) \right].
\]

Using Proposition 3.2, we derive that

\[
\left\langle -\partial J(u), Z_2 \right\rangle = -c|\lambda|^2 \frac{\partial}{\partial t} (^t \Lambda(t) M \Lambda(t)) + o\left( \sum_k \frac{1}{\lambda_k^2} \right).
\]

Since \( (^t \Lambda(t) M \Lambda(t)) = \rho + \frac{(1 - t)^2}{\|y(t)\|^2} (^t \Lambda M \Lambda - \rho |\lambda|^2) \), we derive that \( \frac{\partial}{\partial t} (^t \Lambda(t) M \Lambda(t)) < -c \). Therefore, using (82) and (84) for \( \tilde{W}_2 = CZ_2 + Z_a \), we derive

\[
\left\langle -\partial J(u), \tilde{W}_2 \right\rangle \geq c \left( \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]
The vector field $\widetilde{W}_2$ defined on $V^1_2(p, \varepsilon)$ will be a convex combination of $\widetilde{W}^1_1$ and $\widetilde{W}^2_2$.

- If $u \in V^3_1(p, \varepsilon)$. In this case, we can assume without lose of generality, that $1, \ldots, q$ are the indices which satisfies $-c_2 \frac{\Delta K(a_i)}{K(a_i)} + c_1 H((a_i,0), a_i) < 0$. Let

$$I = \{i/\lambda_i < (1/10) \inf_{k \in \{1, \ldots, q\}} \lambda_k \}.$$ 

Let also, $M_I$ the matrix defined by the points $(a_i)_i \in I$ and $\rho_I$ be the least eigenvalue of $M_I$. We define

$$Z_3 = -\sum_{i=1}^q \alpha_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}.$$ 

Since for any $i \neq j$ we have $|a_i - a_j| > c$ and using (81), we derive that

$$\left\langle -\partial J(u), Z_3 \right\rangle \geq c \sum_{i=1}^q \left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \frac{G((a_i,0), a_j)}{\lambda_i \lambda_j} \right)$$

$$+ o\left(\frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}\right) + R.$$ (86)

(87)

$$\geq c \sum_{i \notin I} \left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}\right) + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}\right) + R.$$ 

Observe that in the case where $I \neq \emptyset$, we have to add an other vector field since only the indices such that $i \notin I$ appear. Then, if the matrix $M_I$ is positive definite, we define $Z'_3 = Z_1/I$, (i.e the action of $Z_1$ using only the indices of $I$). If $M_I$ is not positive definite, we define $Z'_3 = Z_2/I$. In both case, we have

(88) $\left\langle -\partial J(u), Z'_3 \right\rangle \geq c \sum_{i \notin I} \left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}\right) - c \sum_{i \in I, j \notin I} \varepsilon_{ij}.$

Using (82), (86) and (88) for $\widetilde{W}_3 = CZ_3 + Z'_3 + mZ_0$, where $C$ is a a large constant and $m$ is a small constant, we derive that

(89) $\left\langle -\partial J(u), \widetilde{W}_3 \right\rangle \geq c \left( \sum_{i=1}^p \frac{\left| \nabla K(a_i) \right|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}\right).$ 

- If $u \in V^1_1(p, \varepsilon)$. In this case, there is at least one $B_i = \{j/a_j \in B(y_i, \eta)\}$ which contains at least two indices. Without lose of generality, we can assume that $1, \ldots, q$ are the indices such that the set $B_i(1 \leq i \leq q)$ contains
We derive now, we can make appear in (91) the term $1/\lambda_i$. Using Proposition 3.2, we obtain

$$\bar{\chi}(\lambda_j) = \sum_{i \neq j, i, j \in B_k} \chi(\lambda_j/\lambda_i).$$

We define

$$Z_4 = -\sum_{k=1}^{q} \sum_{j \in B_k} \alpha_j \bar{\chi}(\lambda_j) \lambda_j \frac{\partial P_j}{\partial \lambda_j}.$$

Using Proposition 3.2, we obtain

$$\langle -\partial J(u), Z_4 \rangle = 2J(u) \sum_{k=1}^{q} \sum_{j \in B_k} \alpha_j \bar{\chi}(\lambda_j) \left[ -c \sum_{j \neq i} \alpha_i \left( \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \frac{\bar{H}((\alpha_i, 0), a_{ij})}{(\lambda_i \lambda_j)} \right) \right]$$

$$+ O \left( \frac{1}{\lambda_i^2} \right) + o \left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} \right).$$

If $j \in B_k$ with $k \leq q$, if $\bar{\chi}(\lambda_j) \neq 0$, then there exists $i \in B_k$ such that $\lambda_i^{-2} = o(\varepsilon_{ij})$ (for $\eta$ small enough). If $i \notin B_k$ or $i \in B_k$ with $\lambda_i$ and $\lambda_j$ are of the same order, then we have $\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\varepsilon_{ij} (1 + o(1))$, for $r = i, j$.

In the case where $i \in B_k$ with $\lambda_i < \lambda_j$, $\left( \frac{\lambda_i}{\lambda_j} < \gamma' \right)$ we have $\bar{\chi}(\lambda_j) - \bar{\chi}(\lambda_i) \geq 1$.

Thus,

$$-\bar{\chi}(\lambda_j) \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} - \bar{\chi}(\lambda_i) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \geq -\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} = \varepsilon_{ij} (1 + o(1)).$$

We derive now,

$$\langle -\partial J(u), Z_4 \rangle \geq c \sum_{k=1}^{q} \sum_{j \in B_k, \bar{\chi}(\lambda_j) \neq 0} \left( \frac{1}{\lambda_j^2} + \sum_{i \neq j} \varepsilon_{ij} \right).$$

Observe that we have to add some missing terms. Let $\lambda_{i_0} = \inf \{ \lambda_i/i = 1, \ldots, p \}$, two case may occur

**Case 1:** If there exists $j$ such that $\bar{\chi}(\lambda_j) \neq 0$ and $\frac{\lambda_{i_0}}{\lambda_j} > \gamma'$. In this case, we can make appear in (91) the term $1/\lambda_{i_0}^2$ and therefore all the $1/\lambda_i^2$ and the $\varepsilon_{ij}$. Thus, we define in this case $W'_4 = CZ_4 + Z_a$ where $C$ is a large constant.

**Case 2:** If for all $j$, we have $\bar{\chi}(\lambda_j) = 0$ or $\frac{\lambda_{i_0}}{\lambda_j} < \gamma'$. In this case, we define

$$D = \left( \{ i/\bar{\chi}(\lambda_j) = 0 \} \cup \bigcup_{k=1}^{q} B_k \right) \cap \{ i/\lambda_i \leq 1 < 1/\gamma' \}.$$

For each $i, r \in D$, such that $i \neq r$, we have $a_i \in B(y_{i_j}, \eta)$ and $a_r \in B(y_{r_j}, \eta)$ since the set $\{ i/\bar{\chi}(\lambda_j) = 0 \}$ contains at most one index from each $B_j$ for
Let \( u_1 = \sum_{i \in D} \alpha_i \tilde{\delta}_i \). This element has to satisfy one of the three above cases. Thus, we can apply the associated vector field which we will denote \( Z_4' \) and we have the estimate

\[
< -\partial J(u), Z_4' > \geq c \sum_{i \in D} \left( \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \leq j \in D} \varepsilon_{ij} \right) + O( \sum_{k \in D, r \in D} \sum_{i \notin D} \frac{1}{\lambda_i^2}).
\]

For \( k \in D \) and \( r \notin D \), we have either \( r \in B^q := \{ i/\bar{\chi}(\lambda_j) \neq 0 \} \cup (\cup_{j=1}^q B_j) \) or \( r \in (B^q)^c \). For the case where \( r \in B^q \), the term \( \varepsilon_{kr} \) appear in (91). If \( r \in (B^q)^c \) and since \( r \notin D \) we deduce that \( \lambda_{i_0}/\lambda_r < \gamma' \). Furthermore, we can prove that \( a_k \) and \( a_r \) are not in the same \( B(y, \eta) \) and therefore \( |a_k - a_r| > c \).

Thus,

\[
\varepsilon_{kr} \leq \frac{c \gamma'}{\lambda_k \lambda_r} = o(\varepsilon_{ki_0}).
\]

Since \( i_0 \in D \), then from \( 1/\lambda_{i_0}^2 \) we can make appear in (91) all the \( 1/\lambda_i^2 \) and \( \varepsilon_{ir} \) for \( i, r \in (B^q)^c \) (we have \( |a_i - a_r| > c \)). Thus, we derive that

\[
< -\partial J(u), Z_4' > \geq c \left( \sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \leq j \in (B^q)^c} \frac{1}{\lambda_i^2} + \sum_{i, j \neq i} \varepsilon_{ij} \right) + O( \sum_{k \in D, r \in B^q} \varepsilon_{kr}).
\]

Using (82), (91) and (93) on the vector field \( \tilde{W}_4'' = CZ_4 + Z_4' + mZ_n \) where \( C \) is a large constant and \( m \) is a small constant, we derive

\[
< -\partial J(u), \tilde{W}_4'' > \geq c \left( \sum_{i} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i} \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} \right).
\]

The vector field \( \tilde{W}_4 \) defined on \( V_4^4(p, \varepsilon) \) will be a convex combination of \( \tilde{W}_4' \) and \( \tilde{W}_4'' \).

Now, we define the pseudo-gradient \( W_2 \) as a convex combination of \( \tilde{W}_i \) for \( i = 1, \ldots, 4 \).

6. Proof of the main results

**Proof of Theorem 1.1.** Using Theorem 5.1, the proof of Theorem 1.1 is similar to the proof of Lemma 2 of [6] page 326 given in Appendix 2 of [6].

**Proof of Theorem 1.2.** First, we recall the definition of critical point at infinity (see [5]). Under the assumption that (1) has no solution, a critical point at infinity of \( J \) is a limit of a non-compact flow line \( u(s) \) of the gradient vector field \( (-\partial J) \). By Propositions 2.1 and 2.2, \( u(s) \) can be written as:
\[ u(s) = \sum_{i=1}^{p} \alpha_i(s) \tilde{P}_{a_i}(s) + v(s). \]

Denoting by \( y_i = \lim_{s \to +\infty} a_i(s) \) and \( \alpha_i = \lim_{s \to +\infty} \alpha_i(s) \), we then denote by
\[ \sum_{i=1}^{p} \alpha_i P\tilde{y}_i,\infty \text{ or } (y_1, \ldots, y_p)_{\infty} \]
such a critical point at infinity.

Using Theorem 5.1, the only case the \( \lambda_i \)'s are not bounded is when each point \( a_i(t), i = 1, \ldots, p \), goes to \( y_{\ell_i} \) with \( (y_{\ell_1}, \ldots, y_{\ell_p}) \in C^\infty \). In this case, using Theorem 1.1, the normal form of \( J \) allows us to split the variables, and it is easy to see that we have a critical point at infinity. The index of such a critical point at infinity is the same as the index of the critical point of
\[ g(\alpha_1, \ldots, \alpha_p, a_1, \ldots, a_p) = \frac{\sum_{i=1}^{p} \alpha_i^2}{(\sum_{i=1}^{p} \alpha_i^2 K(a_i))^2}. \]

In the \( \alpha_i \)'s variables we have a degenerated critical point \((\bar{\alpha}_1, \ldots, \bar{\alpha}_p)\) which satisfies \( \frac{\bar{\alpha}_i K(a_i)}{\bar{\alpha}_j K(a_j)} = 1 \). This critical point has a Morse index equal to \( p - 1 \). Thus,
\[ g(\bar{\alpha}_1, \ldots, \bar{\alpha}_p, a_1, \ldots, a_p) = \left( \sum_{i=1}^{p} \frac{1}{K(a_i)^2} \right)^{\frac{1}{2}} (1 - |X|^2), \]
where \( X \) belongs to \( \mathbb{R}^{p-1} \). Thus the Morse index is equal to
\[ 4p - 1 - \sum_{j=1}^{p} \text{ind}(K, (y_{\ell_j})). \]

\[ \square \]

**Proof of Theorem 1.3.** Under Theorem 1.2, the only critical points at infinity of the associated variational problem are in one to one correspondence with the elements \((y_1, \ldots, y_p)_{\infty}, (y_1, \ldots, y_p)_{\infty} \in C^\infty \). For each \((y_1, \ldots, y_p)_{\infty} \in C^\infty \), we denote by \( W^u_{\infty}(y_1, \ldots, y_p)_{\infty} \) the unstable manifold of the critical points at infinity \((y_1, \ldots, y_p)_{\infty} \). Recall that \( i(y_1, \ldots, y_p)_{\infty} \) the index of \((y_1, \ldots, y_p)_{\infty} \) is equal to the dimension of \( W^u_{\infty}(y_1, \ldots, y_p)_{\infty} \). Using now the gradient flow of \((-\partial J)\) to deform \( \Sigma^+ \). It follows then by deformation Lemma that
where \( \simeq \) denotes retracts by deformation. It follows from the above deformation retract that the problem (1) has necessary a solution \( w \). Otherwise, it follows from (95) that

\[
1 = \chi(\Sigma^+) = \sum_{(\tau_p) \in \mathcal{C}^\infty} (-1)^{i(\tau_p)},
\]

where \( \chi \) denotes the Euler-Poincaré Characteristic and \( i(\tau_p) = p - 1 + \sum_{j=1}^{p} (3 - \text{ind}(K, (y_\ell_j))) \). Such an equality contradicts the assumption of Theorem 1.3.

\[\square\]

References


