

CONFIGURATIONS OF POINTS ON DEGENERATE VARIETIES AND PROPERNESS OF MODULI SPACES

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CONTENTS

Introduction	1
1. Stable expanded configurations	2
2. Properness of the stack of predeformable maps	9
References	15

INTRODUCTION

Consider a smooth variety X and a smooth divisor $D \subset X$. Kim and Sato in [KS09, 1.1, esp. Theorem 1] define a natural compactification of $(X \setminus D)^n$, denoted $X_D^{[n]}$, which is a moduli space of stable configurations of n points lying on expansions of (X, D) in the sense of [Li01].

The purpose of this note is to generalize [KS09, Theorem 1] to the case where X is an algebraic stack; and to construct an analogous moduli stack $W_\pi^{[n]}$ for a degeneration $\pi : W \rightarrow B$ of schemes or algebraic stacks. We construct $X_D^{[n]}$ and $W_\pi^{[n]}$ and prove their properness using a universal construction introduced in [ACFW11]:

Theorem 0.0.1 (See Propositions 1.2.5 and 1.5.4). *The stacks $W_\pi^{[n]}$ and $X_D^{[n]}$ are smooth and algebraic, and the morphisms $W_\pi^{[n]} \rightarrow W_\pi^n$ and $X_D^{[n]} \rightarrow X^n$ are projective.*

In particular, if W is a scheme, or a Deligne–Mumford stack, so is $W_\pi^{[n]}$. Similarly if X is a scheme, or a Deligne–Mumford stack, so is $X_D^{[n]}$.

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We then use these spaces for a concrete application. In [AF11], a degeneration formula for Gromov–Witten invariants of schemes and stacks is developed, generalizing the approach of Jun Li [Li01, Li02]. This in particular requires proving properness of Li’s stack of pre-deformable expanded stable maps in the case where the target (X, D) or $W \rightarrow B$ is a Deligne–Mumford stack. One could simply adapt Li’s proof, or follow the age-old tradition of imposing such endeavor as an exercise on “the interested reader”.

Instead, we prefer to provide a different proof here, which uses the properness of $X_D^{[n]}$ and $W_\pi^{[n]}$. We denote by $K_{pd}(W/B)$ and $K_{pd}(X, D)$ the stacks of pre-deformable expanded stable maps in the sense of Jun Li, and by $K(W/B)$ and $K(X)$ the stacks of usual stable maps, see Section 2.1.

Theorem 0.0.2 (See Theorem 2.2.1). *The morphism $K_{pd}(W/B) \rightarrow K(W/B)$ and $K_{pd}(X, D) \rightarrow K(X)$ are proper.*

Similar ideas are used in [KKO11] to prove the properness of their space of ramified maps.

Convention. To keep ideas simple we work over an algebraically closed base field k ; in Section 2 we also assume $\text{char } k = 0$. A scheme will be a scheme of finite type over k ; a curve will be a purely one-dimensional scheme; we will use the words *algebraic stack* for an algebraic stack in the sense of Artin, and assume locally finite type over k ; we will write DM for Deligne–Mumford stacks. A point in an algebraic stack, and in particular in a scheme, will be a k -valued point.

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1. STABLE EXPANDED CONFIGURATIONS

Suppose X is a smooth algebraic stack. Then X^n is a parameter space for n ordered points in X . If $D \subset X$ is a smooth divisor, we may wish to consider a space of n ordered points on X where points are not allowed to land in D , but rather X is replaced by an expansion. We construct in Section 1.5 such a compactification, denoted $X_D^{[n]}$: it is an immediate generalization of the first construction in [KS09] to algebraic stacks, though our method is different.

Similarly, suppose $\pi : W \rightarrow B$ is a flat morphism with W a smooth algebraic stack, B a smooth curve and all fibers W_b smooth except for $W_0 := W_{b_0}$ which is the union of two smooth stacks X_1 and X_2

intersecting transversely along a divisor D . Then the self fibered product W_B^n can be viewed as a parameter space of n ordered points on fibers of π . We construct in Section 1.2 a space of n -tuples of points on the fibers which are not allowed to land in the singular locus of π but rather W is replaced by an expansion.

1.1. Notation for degenerations.

Convention 1.1.1. In this section we fix $\pi : W \rightarrow B$, a flat morphism such that B is a smooth curve, W is a smooth algebraic stack, and $b_0 \in B$ is the unique critical value of π ; we set $W_0 := \pi^{-1}(b_0)$ and assume $W_0 = X_1 \sqcup_D X_2$ is the union of two smooth closed substacks X_1 and X_2 intersecting transversally along D , a smooth divisor in each X_i . This implies that W_0 is first-order smoothable along its singular locus D , i.e., the normal bundle N_{D/X_1} is dual to N_{D/X_2} .

We use the notation of [ACFW11, Section 2.3]. The *expansion of length $\ell \geq 0$ of W_0* is

$$W(\ell) := X_1 \sqcup_{D=D_1^-} P_1 \sqcup_{D_1^+=D_2^-} \cdots \sqcup_{D_{\ell-1}^+=D_\ell^-} P_\ell \sqcup_{D_\ell^+=D} X_2$$

where the *exceptional components* P_j are all isomorphic to the \mathbb{P}^1 -bundle $\mathbb{P}(N_{D/X_1} \oplus \mathcal{O}_D) = \mathbb{P}(\mathcal{O}_D \oplus N_{D/X_2})$ and D_j^- , D_j^+ are the zero and infinity sections in P_j (see [ACFW11, Definition 2.3.1] for details). The automorphism group of $W(\ell)$ is defined to be \mathbb{G}_m^ℓ where the j -th factor acts on the fibers of P_j fixing D_j^- and D_j^+ .

Let \mathcal{A} be the stack quotient $[\mathbb{A}^1/\mathbb{G}_m]$, so that morphisms $S \rightarrow \mathcal{A}$ are pairs (L, s) consisting of a line bundle and section on S , see e.g. [Cad07, Lemma 2.1.1]. We denote by \mathfrak{T} the universal stack of expansions of the degeneration $\varpi : \mathcal{A}^2 \rightarrow \mathcal{A}$ induced by $t = xy$, with universal expansion $(\mathcal{A}^2)' \rightarrow \mathcal{A}^2$.

The *moduli stack of expansions* of $W \rightarrow B$ is $\mathfrak{T}_B = \mathfrak{T} \times_{\mathcal{A}} B$ where $B \rightarrow \mathcal{A}$ is the morphism associated to the Cartier divisor $\{b_0\} \subset B$; the universal expansion is $W' = (\mathcal{A}^2)' \times_{\mathcal{A}^2} W \rightarrow W$, where $W \rightarrow \mathcal{A}^2$ is the smooth morphism induced by the divisors X_1 and X_2 : see [ACFW11, Definition 2.3.6] for details.

1.2. Configurations on degenerations.

Definition 1.2.1. A *stable expanded configuration* (\mathcal{W}, σ_i) of degree n on $W \rightarrow B$ consists of

- (1) a point \mathcal{W} of \mathfrak{T}_B , hence either a smooth fiber W_b or an expansion of W_0 , and
- (2) an ordered collection of n smooth points $\sigma_i \in \mathcal{W}^{\text{sm}}$,

such that, in case \mathcal{W} is an expansion, the following stability condition holds:

- each exceptional component P_j contains at least one σ_i .

An *isomorphism* $\rho : (\mathcal{W}, \sigma_i) \rightarrow (\mathcal{W}', \sigma'_i)$ is an isomorphism of expanded degenerations $\rho : \mathcal{W} \rightarrow \mathcal{W}'$ such that $\rho \circ \sigma_i = \sigma'_i$. Note that the stability condition is equivalent to requiring that the only automorphism of \mathcal{W} fixing all the σ_i 's is the identity automorphism.

Definition 1.2.2. A family of stable expanded configurations of degree n over a B -scheme S is given by

- (1) a family $\mathcal{W}_S \rightarrow W \times_B S$ of expanded degenerations over S , i.e. an object of $\mathfrak{T}_B(S)$;
- (2) n sections $\sigma_i : S \rightarrow \mathcal{W}_S$;

such that for every $s \in S$ the fiber $(\mathcal{W}_s, \sigma_{i,s})$ over s is a stable expanded configuration of degree n .

Morphisms of families of stable expanded configurations are defined in the obvious way; the resulting fibered category $W_\pi^{[n]}$ is clearly a stack, and indeed a sheaf if W is a scheme or algebraic space. Composing σ_i with $\mathcal{W} \rightarrow W$, we obtain an object of the fibered power W_π^n of W over B , giving a morphism $W_\pi^{[n]} \rightarrow W_\pi^n$.

The notation should not be confused with the notation for the degree- n Hilbert scheme of a surface.

The following lemma is an immediate consequence of the definition:

Lemma 1.2.3. *Assume $V \subset W$ is open and ϖ is the restriction of π . Then $V_\varpi^{[n]} = V_\varpi^n \times_{W_\pi^n} W_\pi^{[n]}$.*

We construct $W_\pi^{[n]}$ starting with a special case, described in the following lemma:

Lemma 1.2.4. *Assume $B = \mathbb{A}^1$ and $W = W_{\mathbb{P}} \subset \mathbb{P}^2 \times \mathbb{A}^1$ is the pencil $tZ^2 = XY$. Then for every $n \geq 1$ the stack $W_\pi^{[n]}$ is naturally isomorphic to the moduli space of stable weighted n -pointed genus 0 maps $\overline{\mathcal{M}}_{0,(\epsilon, \dots, \epsilon)}(W, \beta_F)$ in the sense of Hassett, where β_F is the class of the fiber and ϵ is a real number in $(0, 1/n)$.*

In particular in this case $W_\pi^{[n]}$ is a smooth variety, projective over B and hence over W_π^n .

Proof. Fix an integer $n > 0$, and ϵ as in the statement, and write $M := \overline{\mathcal{M}}_{0,(\epsilon, \dots, \epsilon)}(W, \beta_F)$ for brevity.

Let S be a scheme, and (\mathcal{W}, σ_i) a family of stable expanded configurations over S . Then every fiber \mathcal{W}_s of \mathcal{W} is a nodal curve of arithmetic

genus zero; moreover, if a component \bar{C} of \mathcal{W}_s is contracted by the natural morphism $f : \mathcal{W} \rightarrow W$, then \bar{C} is rational and contains at least two nodes and a marked point, hence is stable since $1 + 1 + \epsilon > 2$. This defines a natural map $W_\pi^{[n]} \rightarrow M$ as sheaves.

To define the inverse, let S be a scheme and $(C_S, \sigma_{i,S}, f_S) \in M(S)$; we want to show that it is a stable expanded configuration. Let (C, x_i, f) be a point in M ; since the arithmetic genus of the nodal curve C is zero, every irreducible component of C is smooth and rational. Moreover, no contracted component can be a tail, i.e., intersect the rest of the curve in only one point, since if such a component contains r marked points we have $1 + r\epsilon \leq 1 + n\epsilon < 2$, contradicting stability.

Let $t \in B$ such that $f(C) = W_t$. If $t \neq 0$, W_t is integral, hence there is a unique component C_0 of C mapping isomorphically to W_t . If C had other components, than at least one of them must be a tail since every tree with at least two vertices has at least two end vertices; since all other components are contracted, this yields a contradiction. Therefore $C \rightarrow W_t$ must be an isomorphism.

If $t = 0$, then W_t is the union of two irreducible components X_1 and X_2 meeting in one point p , and C has two non-contracted components C_1 and C_2 such that f maps C_i to X_i isomorphically. Any other irreducible component must be contracted, hence C_1 and C_2 are the only tails of C , which implies that C is a chain of rational curves, with C_1 and C_2 as extremes. Moreover, the stability condition implies that each irreducible component must contain at least one marked point x_i . This concludes the proof, since other conditions (flatness of $C_S \rightarrow S$, properness of $W_S \rightarrow W \times_T S$) are part of the definition of stable maps.

It follows that $W_\pi^{[n]}$ is a smooth variety, projective over B and hence over W_π^n (see [AG08, Theorem 1.9], [BM09, Theorem 1.1.4], [MM08, Theorem 1.7], see also [Has03, Theorem 1]). ♣

We now return to the general case.

Proposition 1.2.5. *The stack $W_\pi^{[n]}$ is smooth and algebraic, and the morphism $W_\pi^{[n]} \rightarrow W_\pi^n$ is projective. In particular, if W is a scheme, or a Deligne–Mumford stack, so is $W_\pi^{[n]}$.*

Proof. CASE 1: $B = \mathbb{A}^1$ and $W = W_{\mathbb{A}} = \mathbb{A}^2$ mapping via $t = xy$. This case is the restriction of the case treated in Lemma 1.2.4 to an open set $W_{\mathbb{A}} \subset W_{\mathbb{P}}$, so it follows by Lemma 1.2.3.

Note that in this case there is an action of \mathbb{G}_m on B and a compatible action of \mathbb{G}_m^2 on W ; if we write (s_1, s_2) for a point on \mathbb{G}_m^2 , it acts on W via $(x, y) \mapsto (s_1x, s_2y)$, and $s = s_1s_2 \in \mathbb{G}_m$ acts on B via $t \mapsto st$.

This induces an action of the n -th fiber product \mathbb{G}_m^{n+1} of \mathbb{G}_m^2 over \mathbb{G}_m on $W_\pi^{[n]} \rightarrow W_\pi^n$ equivariant with respect to the \mathbb{G}_m action on B .

CASE 2: $B = [\mathbb{A}^1/\mathbb{G}_m] = \mathcal{A}$ and $W = W_{\mathcal{A}} = [\mathbb{A}^2/\mathbb{G}_m^2] = \mathcal{A}^2$. This is the quotient of the previous case by the action of \mathbb{G}_m^{n+1} . Projectivity is preserved, since the relatively ample line bundle admits an equivariant structure; indeed, every line bundle on a toric variety admits such structure, see Section 3.4 of [Ful93]. For reference below we denote the morphism described in this case $\varpi : W_{\mathcal{A}} \rightarrow \mathcal{A}$.

GENERAL CASE: We use the notation in Section 1.1.

Consider the smooth morphisms $B \rightarrow \mathcal{A}$ given by the divisor b_0 and $W \rightarrow W_{\mathcal{A}} = \mathcal{A}^2$ given by the divisors X_1 and X_2 : Composing any family of stable expanded configurations of degree n on W with these morphisms gives a family of stable configurations in the fibers of $W_{\mathcal{A}}$, hence we get a natural morphism

$$W_\pi^{[n]} \rightarrow W_\pi^n \times_{(W_{\mathcal{A}})_{\varpi}^n} (W_{\mathcal{A}})_{\varpi}^{[n]}.$$

We construct an inverse of this morphism as follows. Let S be a B -scheme, and $S \rightarrow (W_{\mathcal{A}})_{\varpi}^{[n]}$ a morphism over \mathcal{A} given by a family of stable expanded configurations $\tilde{\sigma}_i : S \rightarrow (W_{\mathcal{A}})'$; then to every B -morphism $g := (g_1, \dots, g_n) : S \rightarrow W_\pi^n$ we can associate a family of stable expanded configurations on W by letting $\sigma_i : S \rightarrow W$ be the morphisms induced by $\tilde{\sigma}_i$ and $g_i : S \rightarrow W$.

It is easy to see that these two constructions are inverse of each other. ♣

1.3. Other degenerate cases. The construction given in the general case of Proposition 1.2.5 apply in all cases introduced in [ACFW11, Section 2.2]. In particular one can consider the situation of having a degenerate fiber with no chosen smoothing as introduced in [ACFW11, 2.2.2]: here B is a point $B = \{b_0\}$ and W is first order smoothable, i.e. the union of two smooth components X_1 and X_2 meeting transversally along a smooth divisor D , with N_{D/X_1} dual to N_{D/X_2} . The result is the following:

Proposition 1.3.1. *The stack $W_\pi^{[n]}$ is algebraic, and the morphism $W_\pi^{[n]} \rightarrow W_\pi^n$ is projective. In particular, if W is a scheme, or a Deligne–Mumford stack, so is $W_\pi^{[n]}$.*

Proof. We can modify the previous proof as follows. First, consider $B \rightarrow \mathcal{A}$ induced by the unique morphism $B \rightarrow 0 \subset \mathbb{A}^1$; the fiber product $W_B := W_{\mathcal{A}} \times_{\mathcal{A}} B$ is the transversal union of two smooth irreducible components $W_{B,i}$, each isomorphic to \mathcal{A} , meeting transversally along

\mathbb{G}_m . The pairs (X_i, D) define smooth morphisms $X_i \rightarrow W_{B,i}$ and the condition on the normal bundles ensures that they glue to define a smooth morphism $W \rightarrow W_B$. The rest of the proof is the same as the general case above. \clubsuit

Note that in this case $W_\pi^{[n]}$ is not smooth. Our construction gives a smooth map to $\{b_0\} \times_{\mathcal{A}} (W_{\mathcal{A}})^{[n]}$; its singularities are therefore modeled on the boundary of $\overline{\mathcal{M}}_{0,(\epsilon, \dots, \epsilon)}(W, \beta_F)$, which is a normal crossings divisor.

1.4. Notation for pairs.

Convention 1.4.1. In this section we fix a pair (X, D) , where X is a smooth algebraic stack and D a smooth divisor in X .

The *expansion of length* $\ell \geq 0$ of (X, D) is the pair $(X(\ell), D(\ell))$ with

$$X(\ell) := X_1 \sqcup_{D=D_1^-} P_1 \sqcup_{D_1^+=D_2^-} \cdots \sqcup_{D_{\ell-1}^+=D_\ell^-} P_\ell$$

and $D(\ell) := D_\ell^- \subset P_\ell$. As in 1.1.1 the *exceptional components* P_j are all isomorphic to $\mathbb{P}(N_{D/X_1} \oplus \mathcal{O}_D)$ and D_j^-, D_j^+ are their zero and infinity sections (see [ACFW11, Definition 2.3.1]). Its automorphism group is again defined to be \mathbb{G}_m^ℓ acting componentwise on the P_j 's.

Following the notation of [ACFW11, Section 2.1] we let again $\mathcal{A} := [\mathbb{A}^1/\mathbb{G}_m]$ and denote by $\mathcal{D} \subset \mathcal{A}$ the smooth divisor $[0/\mathbb{G}_m]$. We denote by \mathcal{T} the universal stack of expansions of pairs: it parametrizes expansions of $(\mathcal{A}, \mathcal{D})$, with universal expansion denoted by $(\mathcal{A}', \mathcal{D}')$. The stack \mathcal{T} also parametrizes expansions of any pair (X, D) , with universal family (X', D') where $X' = \mathcal{A}' \times_{\mathcal{A}} X$ and $D' = \mathcal{D}' \times_{\mathcal{D}} D$, see [ACFW11, Definition 2.1.6].

1.5. Configurations on pairs.

Definition 1.5.1. A *stable expanded configuration* (X', D', σ_i) of degree n on (X, D) consists of

- (1) a point (X', D') of \mathcal{T} , that is an expansion of the pair (X, D) and
- (2) an ordered collection of n smooth points $\sigma_i \in (X')^{\text{sm}} \setminus D'$,

such that the following stability condition holds:

- each exceptional component \mathbb{P}_j contains at least one σ_i .

An *isomorphism* $\rho : (X', D', \sigma_i) \rightarrow (\bar{X}', \bar{D}', \bar{\sigma}_i)$ is an isomorphism of expanded pairs $\rho : (X', D') \rightarrow (\bar{X}', \bar{D}')$ such that $\rho \circ \sigma_i = \bar{\sigma}_i$.

Definition 1.5.2. A family of stable expanded configurations of degree n over a scheme S is given by

- (1) a family of expansions $(X'_S, D'_S) \rightarrow (X, D) \times S$ of the pair (X, D) parametrized by S , that is an object of $\mathcal{T}(S)$;
- (2) n sections $\sigma_i : S \rightarrow X'_S$;

such that for every $s \in S$ the fiber over s is a stable expanded configuration of degree n .

Morphisms of families of stable expanded configurations are defined as before; we denote the resulting category $X_D^{[n]}$, and again it is a stack with a natural map to X^n .

Lemma 1.5.3. *Assume $X = \mathbb{P}^1$ and $D = \{0\}$. Consider the moduli stack of weighted $(n+1)$ -pointed stable maps $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{0;(\epsilon, \dots, \epsilon, 1)}(\mathbb{P}^1, 1)$ of degree 1, with its last evaluation map $ev := ev_{n+1} : \overline{\mathcal{M}} \rightarrow \mathbb{P}^1$. Then $X_D^{[n]} = ev^{-1}(\{0\})$ and hence is a smooth projective variety, therefore the morphism $X_D^{[n]} \rightarrow X^n$ is projective.*

Proof. We briefly review the argument which is very similar to that of Lemma 1.2.4. The morphism $X_D^{[n]} \rightarrow \overline{\mathcal{M}}$ is defined by using D as $(n+1)$ -st section; it clearly maps to $ev^{-1}(0)$. To define the opposite morphism, note that if (C, x_i, f) is a point in $\overline{\mathcal{M}}$, the curve C can have only two tails, one mapping isomorphically to \mathbb{P}^1 and the other containing the $(n+1)$ -st section, so it is a chain of rational curves.

As in Lemma 1.2.4, smoothness and projectivity follow from [AG08, Theorem 1.9], [BM09, Theorem 1.1.4], [MM08, Theorem 1.7], [Has03, Theorem 1]). ♣

Proposition 1.5.4. *The stack $X_D^{[n]}$ is smooth and algebraic, and the morphism $X_D^{[n]} \rightarrow X^n$ is projective. In particular, if X is a scheme, or a Deligne–Mumford stack, so is $X_D^{[n]}$.*

Proof. CASE 1: $X = \mathbb{A}^1$ and $D = \{0\}$. This case is the restriction of Lemma 1.5.3 to the open subscheme $(\mathbb{A}^1)^n \subset (\mathbb{P}^1)^n$, and the analog of Lemma 1.2.3 holds.

Note that in this case there is an action of \mathbb{G}_m on \mathbb{A}^1 fixing $\{0\}$. This induces an action of \mathbb{G}_m^n on $X_D^{[n]} \rightarrow X^n$.

CASE 2: $X = [\mathbb{A}^1/\mathbb{G}_m] = \mathcal{A}$ and $D = [\{0\}/\mathbb{G}_m]$. As in the proof of Proposition 1.2.5, this is the quotient of the previous case by the action of \mathbb{G}_m^n . Projectivity is preserved, since again the relatively ample line bundle admits an equivariant structure.

This is a universal case, the resulting configuration stacks are denoted by $\mathcal{A}_{\mathcal{D}}^{[n]} \rightarrow \mathcal{A}^n$.

GENERAL CASE: The pair (X, D) defines a smooth morphism $X \rightarrow \mathcal{A}$. Composing any family of stable expanded configurations of degree n on (X, D) with this structure morphism gives a family of stable configurations of $(\mathcal{A}, \mathcal{D})$, hence we get a natural morphism

$$X_D^{[n]} \rightarrow X^n \times_{\mathcal{A}^n} \mathcal{A}_{\mathcal{D}}^{[n]}.$$

This is an isomorphism exactly as in Proposition 1.2.5. ♣

2. PROPERNESS OF THE STACK OF PREDEFORMABLE MAPS

In this section we assume the characteristic char k of the base field is 0, as we rely on work making this assumption.

Defining numerical invariants via virtual classes requires constructing a suitable moduli stack of DM type, proving its properness and constructing a perfect obstruction theory.

For instance if $\pi : W \rightarrow B$ is a smooth projective morphism, with B an affine scheme, then the connected components of the stack $K(W)$ of stable maps to the fibers of π are proper over B and carry a relative perfect obstruction theory; this allows to define GW invariants and prove their invariance under deformations. Properness of $K(W) \rightarrow B$ holds under the weaker assumption that $W \rightarrow B$ be flat and projective; however, in this case the natural obstruction theory is not perfect.

Both the properness results and the construction of perfect obstruction theories are the key technical core of Jun Li's statement and proof of the degeneration formula for GW invariants in the language of algebraic geometry. In this section we want to give an alternative proof of the properness part, extending it to the case of DM stacks; we describe the set-up following the notation of [AF11, Definitions 3.1.4, 3.2.9, C.1.6], where we combine the cases of pairs and degenerations.

2.1. Notation and background. We let $W \rightarrow B$ be either one of the cases of interest: (A) a degeneration in the sense of Convention 1.1.1, or (B) a first-order smoothable variety $W \rightarrow B = \{b_0\}$ as in Section 1.3, or (C) the trivial map from X to a point in the context of Convention 1.4.1; the notation $K(W)$ denotes the algebraic stack of stable maps to the fibers of $W \rightarrow B$. The *special locus* \mathcal{W}^{sp} of \mathcal{W} is either $\mathcal{W}^{\text{sp}} = \mathcal{W}^{\text{sing}}$ the singular locus for a degeneration, or $\mathcal{W}^{\text{sp}} = \mathcal{W}^{\text{sing}} \cup D$ in the case of a pair.

We use T to denote the corresponding stack of expansions: either the stack of expanded degenerations \mathfrak{T}_B in cases (A) and (B), or the stack

of expanded pairs \mathcal{T} in case (C); we have a natural morphism $T \rightarrow B$ and a universal expansion $W' \rightarrow W \times_B T$. If P_j is an exceptional component in an expansion, we call its *interior* the complement P_j^o of $D_j^+ \cup D_j^-$.

We denote by \mathfrak{K} the stack of twisted stable maps to the fibers of $W' \rightarrow T$; while our target is allowed to be a Deligne–Mumford stack, we keep the terminology lighter by always using the shorter term *stable maps* to refer to either stable maps in case W is a scheme, or *twisted stable maps* as in [AV02] in case it is a stack.

A point of \mathfrak{K} is a tuple $(C, \Sigma, \mathcal{W}, f)$ where $\mathcal{W} \rightarrow W$ is a point in T , and (C, Σ, f) is a stable map to \mathcal{W} , where Σ is the ordered set of marked points. Such a stable map is called *degenerate* if there is an irreducible component X of C mapping to \mathcal{W}^{sp} ; otherwise it is called *non-degenerate*. A non-degenerate map is called *predeformable* if all points on C mapping to the divisor $D \subset \mathcal{W}$ are marked, and if near each point mapping to $\mathcal{W}^{\text{sing}}$ the map is given on strict henselizations by

$$(k[x, y, z_1, \dots, z_d]/xy)^{\text{sh}} \rightarrow (k[u, v]/uv)^{\text{sh}} \quad x \mapsto u^r, y \mapsto v^r$$

for some $r > 0$. Here and later the superscript *sh* means *strict henselization* and may be thought of as a choice of local analytic coordinates.

We denote by \mathfrak{K}_{nd} the open substack of \mathfrak{K} whose points are non-degenerate maps, and by $\mathfrak{K}_{\text{pd}} \subset \mathfrak{K}_{\text{nd}}$ the locally closed substack of predeformable maps, with its reduced substack structure; while in [Li01, Pages 541–547] the locus \mathfrak{K}_{pd} is endowed with a delicate schematic structure, we do not need it since all we are interested in is properness. Finally, we denote by $K \subset \mathfrak{K}$ the maximal open DM substack, and we let $K_{\text{nd}} := K \cap \mathfrak{K}_{\text{nd}}$ and $K_{\text{pd}} := K \cap \mathfrak{K}_{\text{pd}}$.

Let $(C, \Sigma, \mathcal{W}, f)$ be a point of \mathfrak{K} ; we call a smooth rational irreducible component X of C *semistable*, if it maps to a fiber of an exceptional component P_j of \mathcal{W} , with exactly one point of X meeting each of D_j^+ and D_j^- , and no other marked or singular points of C on X . (The terminology in [Li01, Proof of Lemma 3.2] is *trivial component*.)

Semistable components can be used to characterize points of \mathfrak{K} belonging to the Deligne–Mumford locus K , see [Li01, Lemma 3.2], [AF11, Definition 3.1.5].

Lemma 2.1.1. *A point $(C, \Sigma, \mathcal{W}, f)$ of \mathfrak{K} is not in the Deligne–Mumford locus K if and only if there is an exceptional component P_j such that every component of C whose image meets the interior $P_j^o = P_j \setminus (D_j^+ \cup D_j^-)$ is a semistable component.*

Proof. The following argument can be found in the proof of [Li01, Lemma 3.2]. A point is in K if and only if no positive dimensional subgroup of the group of automorphisms of the expansion \mathcal{W} lifts to an automorphism of C . The Deligne–Mumford condition is equivalent to ensuring that for each P_j there is at least one component X mapping to P_j to which no covering of the \mathbb{G}_m action lifts. It is easy to see that the only components whose image meets P_j^o to which the action lifts are exactly the semistable components. ♣

Notation 2.1.2. When we use the valuative criterion, we always take $\Delta = \text{Spec } R$ for R a discrete valuation ring, with generic point η and closed point s . Given a commutative diagram

$$\begin{array}{ccc} \eta & \xrightarrow{f_\eta} & X \\ \downarrow & & \downarrow p \\ \Delta & \xrightarrow{g} & Y \end{array}$$

we will view liftings of g to X as a groupoid whose objects are the two-commutative diagrams

$$\begin{array}{ccc} \eta & \xrightarrow{f_\eta} & X \\ \downarrow & \nearrow f & \downarrow p \\ \Delta & \xrightarrow{g} & Y \end{array}$$

and whose morphisms $f \Rightarrow f'$ are those sending the two-commutative diagram of f to that of f' . We will sometimes write “there is a unique lifting with certain properties” to mean “there exists a lifting with certain properties, and it is unique up to unique isomorphism”.

2.2. Statement and proof. The aim of this section is to prove the following theorem:

Theorem 2.2.1. *The morphism $K_{pd} \rightarrow K(W)$ induced by $K_{pd} \subset \mathfrak{K}$ is proper.*

Lemma 2.2.2. *The substack of predeformable maps is closed in $\mathfrak{K}_{nd} \subset \mathfrak{K}$, the open locus of nondegenerate maps.*

Proof. This is a local statement in the étale topology; hence we can use the proof given by Jun Li, see [Li01, Lemma 2.7]. ♣

Lemma 2.2.3. *Let $(C, \Sigma, f : C \rightarrow \mathcal{W}) \in \mathfrak{K}$ be a point in the closure of \mathfrak{K}_{pd} ; denote $(\bar{C}, \bar{\Sigma}, \bar{f})$ its image in $K(W/T)$, i.e. the stabilisation of the composition of f with the structure morphism $\mathcal{W} \rightarrow W$. Then any*

connected component Z of the locus in C contracted in \bar{C} is a chain of rational curves which maps to either a marked or singular point of \bar{C} .

Proof. Each contracted component must be rational and map to a fiber of \mathcal{W} to W , hence the second part of the statement is obvious. For the first we must prove that Z has no an end component X , i.e. one having only one node and no marked point.

We argue by contradiction: such an end component cannot be contracted in \mathcal{W} since f is stable, so it must map nontrivially to some fiber in an exceptional component of \mathcal{W} ; since each fiber intersects \mathcal{W}^{sp} in two points, there is a non-marked point x on X which is smooth on C but maps to \mathcal{W}^{sp} .

We use again Jun Li's argument, since this is an étale local computation. In case $x \in D$, then in a neighborhood of x the divisor D intersects every nearby fiber. So x is in the closure of the locus of marked points and hence it is marked, contradicting the assumption. Now consider the case when x maps to $\mathcal{W}^{\text{sing}}$. By assumption there exist a family

$$(f_{\Delta} : C_{\Delta} \rightarrow \mathcal{W}_{\Delta}, \Sigma_{\Delta}) \in \mathfrak{K}(\Delta)$$

such that $(f_{\eta} : C_{\eta} \rightarrow \mathcal{W}_{\eta}, \Sigma_{\eta}) \in \mathfrak{K}_{\text{pd}}$ and

$$(f_s : C_s \rightarrow \mathcal{W}_s, \Sigma_s) = (f : C \rightarrow \mathcal{W}, \Sigma).$$

The problem is local at z so we may assume

$$\mathcal{W}_{\Delta}^{\text{sh}} = (\text{Spec } R[u, v, w_1, \dots, w_m]/(uv - a))^{\text{sh}}$$

for some $a \in R$,

$$C_{\Delta}^{\text{sh}} = (\text{Spec } R[x])^{\text{sh}},$$

and on the central fiber

$$f^*u = 0, \quad f^*v = x^r.$$

Consider the homomorphism $f_{\Delta}^* : R[u, v, w_i]/(uv - a) \rightarrow R[x]$ and let u_{Δ}, v_{Δ} be the images of u and v ; they must satisfy $u_{\Delta}v_{\Delta} = a$, and be equal to $(0, x^r)$ modulo m_s . In particular v_{Δ} , viewed as a polynomial in x , has positive degree; therefore the only possibility that its product with u_{Δ} has degree zero is that $u_{\Delta} = a = 0$, which means that f_{η} is not predeformable. \clubsuit

Lemma 2.2.4. *The commutative diagram*

$$\begin{array}{ccc} \mathfrak{K} & \longrightarrow & T \\ \downarrow & & \downarrow \\ K(W) & \longrightarrow & B \end{array}$$

induces a proper morphism $\mathfrak{K} \rightarrow K(W) \times_B T$.

Proof. The fibered product is $\mathfrak{K}(W_T/T)$ where $W_T := W \times_B T$. The structure map $\mathcal{W} \rightarrow W_T$ induces the morphism from \mathfrak{K} to the fiber product, and it is proper by [AV02, Corollary 9.1.3]. \clubsuit

Proof of Theorem 2.2.1. To prove properness of the morphism $K_{pd} \rightarrow K(W)$ we use the valuative criterion. We will fix from now on a commutative diagram

$$(1) \quad \begin{array}{ccccc} \eta & \longrightarrow & K_{pd} & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \longrightarrow & K(W) & \longrightarrow & B, \end{array}$$

and our aim will be showing that, after a base change $\tilde{\Delta} \rightarrow \Delta$, there is a unique lifting of $\Delta \rightarrow K(W)$ to $\tilde{\Delta} \rightarrow K_{pd}$. We will denote by $(\bar{C}, \bar{\Sigma}, \bar{f})$ the family of stable maps over Δ corresponding to $\Delta \rightarrow K(W)$ and by $(C_\eta, \Sigma_\eta, f_\eta : C_\eta \rightarrow \mathcal{W})$ the family corresponding to $\eta \rightarrow K_{pd}$.

From Lemma 2.2.4 it follows that there exists an equivalence, compatible with base change, between liftings of $\Delta \rightarrow B$ to T and liftings, up to base change $\tilde{\Delta} \rightarrow \Delta$, of $\Delta \rightarrow K(W)$ to \mathfrak{K} . Given a lifting $a : \tilde{\Delta} \rightarrow T$, we will denote by $(C^a, \Sigma^a, f^a : C^a \rightarrow \mathcal{W})$ the family of stable maps corresponding to the lifting $\tilde{\Delta} \rightarrow \mathfrak{K}$ induced by a .

The key to producing a lifting of $\Delta \rightarrow B$ to T is to use an auxiliary choice of a stable configuration of points as a guide.

Proposition 2.2.5. *After a base change $\tilde{\Delta} \rightarrow \Delta$, there exist a positive integer N and closed subschemes $(\bar{p}_1, \dots, \bar{p}_N) =: \bar{P} \subset \bar{C}$ such that*

- (1) *the induced morphisms $\bar{p}_i \rightarrow \tilde{\Delta}$ are isomorphisms;*
- (2) *each \bar{p}_i is contained in the smooth, unmarked locus of \bar{C} ;*
- (3) *\bar{P}_η lifts uniquely to $P_\eta \subset C_\eta$;*
- (4) *every irreducible component of \bar{C}_s intersects \bar{P}_s .*

Proof. Write \bar{C}^* for the open dense substack obtained by deleting the nodal and marked locus of \bar{C} . The inverse image of \bar{C}^* in C is isomorphic to \bar{C}^* , so once we construct P inside \bar{C}^* , condition (3) becomes a consequence of (2). Now \bar{C}^* is a smooth representable curve over Δ . For each irreducible component $X_i \subset \bar{C}_s^*$ let $\bar{p}_{i,s}$ be a closed point and $U_i \subset \bar{C}^*$ an affine neighborhood such that $U_i \cap \bar{C}_s^* \subset X_i$. There exists an element $h_s \in \Gamma(\mathcal{O}_{U_i \cap \bar{C}_s^*})$ vanishing to order 1 on $\bar{p}_{i,s}$. Let h be a lift of h_s in $\Gamma(\mathcal{O}_{U_i})$, and H its zero locus. Then $H \rightarrow \Delta$ is quasi-finite, and replacing Δ by the localization of H at $\bar{p}_{i,s}$ we get a section of $U_i \rightarrow \Delta$

meeting X_i . Denoting its image in \bar{C} by \bar{p}_i and repeating this for all components we obtain the required set of sections P .

♣

We now replace Δ by $\tilde{\Delta}$, so we may assume sections as in the proposition exist over Δ .

Proposition 2.2.6. (1) *The morphism $\eta \rightarrow \mathcal{W}^N$ induced by f_η and $P_\eta \subset C_\eta$ defines a stable configuration, and hence a morphism $\eta \rightarrow W_\pi^{[N]}$ which lifts the morphism $\Delta \rightarrow W^N$ given by \bar{f} and \bar{P} ;*

(2) *There exists a unique lifting of $\Delta \rightarrow B$ to $a_0 : \Delta \rightarrow T$ such that the induced map $s \rightarrow \mathcal{W}^{[N]}$ is a stable configuration.*

Proof. (1) If $X \subset C_\eta$ is an irreducible component then X contracts to a point in \bar{C}_η if and only if it contains no p_i , if and only if it is semistable. The result then follows from Lemma 2.1.1.

(2) This follows from the fact that $W_\pi^{[N]} \rightarrow W_B^N$ is projective by Proposition 1.2.5, hence proper and representable. ♣

Proposition 2.2.7. *The lifting $(C_s^{a_0}, \Sigma_s^{a_0}, f_s^{a_0}) \in \mathfrak{K}$ corresponding to a_0 lies in K_{pd} .*

Proof. Since K_{pd} is closed in K_{nd} it suffices to show that $(C_s^{a_0}, \Sigma_s^{a_0}, f_s^{a_0}) \in K_{\text{nd}}$, and since $K_{\text{nd}} = K \cap \mathfrak{K}_{\text{nd}}$ it suffices to show that is in both K and \mathfrak{K}_{nd} .

STEP 1: $(C_s^{a_0}, \Sigma_s^{a_0}, f_s^{a_0}) \in \mathfrak{K}_{\text{nd}}$. We need to show that no component of C^{a_0} maps to \mathcal{W}^{sp} . These components come in two types: those contracted in \bar{C}_s and those that are not.

STEP 1A: components contracted in $C_s^{a_0} \rightarrow \bar{C}_s$. By Lemma 2.2.3, any connected component of the locus in $C_s^{a_0}$ contracted in \bar{C}_s is a chain of rational curves which maps to either a marked or singular point of \bar{C}_s . But in a chain of rational curves over a point in \bar{C} every component maps to a fiber of $\mathcal{W} \rightarrow W$, so a component mapping to \mathcal{W}^{sp} maps to a point in \mathcal{W} , contradicting stability.

STEP 1B: consider a component $X_i \subset \bar{C}_s^*$. Then X_i is canonically a component of $(C^{a_0})_s^*$ using notation as in the proof of Proposition 2.2.5. Since \bar{p}_i meets X_i and \bar{p}_i does not land in \mathcal{W}^{sp} , it follows that neither does X_i .

STEP 2: It remains to prove that $(C_s^{a_0}, \Sigma_s^{a_0}, f_s^{a_0}) \in K$. Since the map $s \rightarrow \mathcal{W}^{[N]}$ induced by a_0 is stable, the configuration $(p_{i,s})$ in \mathcal{W} is stable. So every exceptional component of \mathcal{W} has at least one $f_s^{a_0}(p_i)$

in its interior some component $X_i \subset \bar{C}_s^*$. Since these are not semistable components, we have that $(C_s^{a_0}, \Sigma_s^{a_0}, f_s^{a_0}) \in K$, as required. ♣

Proposition 2.2.8. *Let $a : \Delta \rightarrow T$ be any lifting of $\Delta \rightarrow B$. If $(C_s^a, \Sigma_s^a, f_s^a) \in K_{\text{pd}}$ then a is isomorphic to a_0 .*

Proof. By properness of $W_\pi^{[n]}$, it suffices to show that $f_s^a(P)$ is a stable configuration. This means that every exceptional component of \mathcal{W} contains a point of $f_s^a(P)$ in its interior, and that $f_s^a(P)$ is disjoint from \mathcal{W}^{sp} .

Since the inverse image of \mathcal{W}^{sp} is contained in its nodes and marked points, and since P is chosen disjoint from those, we have that $f_s^a(P)$ is disjoint from \mathcal{W}^{sp} . On the other hand, by Lemma 2.1.1, every exceptional component of \mathcal{W} contains in its interior the image of some component $X_i \subset \bar{C}_s^*$. Since X_i contains a point of P , every exceptional component of \mathcal{W} contains in its interior the image of a point of P , as required. ♣

To conclude the proof of Theorem 2.2.1 we use the valuative criterion of properness. It is enough to show that there exists a unique lifting a of Δ to T such that the induced point (C^a, Σ^a, f^a) is contained in K_{pd} . Hence it is enough to prove that a lifting $a : \Delta \rightarrow T$ induces a map to K_{pd} if and only if it is isomorphic to a_0 , where a_0 is defined in Part (2) of Proposition 2.2.6. This is given by Propositions 2.2.8 and 2.2.7. ♣

Corollary 2.2.9. *Under the assumptions for this section, assume moreover that $W \rightarrow B$ is proper and has projective coarse moduli space. Then K_{pd} is proper over B .*

Proof. This follows since by [AV02, Theorem 1.4.1] the stack $K(W)$ is proper. ♣

REFERENCES

- [ACFW11] D. Abramovich, C. Cadman, B. Fantechi, and J. Wise, *Expanded degenerations and pairs*, Comm. Algebra 41 (2013), no. 6, 2346–2386. MR 3225278
- [AF11] D. Abramovich and B. Fantechi, *Orbifold techniques in degeneration formulas*, Annali della Scuola Normale di Pisa, to appear. arXiv:1103.5132.
- [AG08] V. Alexeev and G. M. Guy, *Moduli of weighted stable maps and their gravitational descendants*, J. Inst. Math. Jussieu 7 (2008), no. 3, 425–456. MR MR2427420 (2009f:14112)

- [AV02] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75 (electronic). MR 1862797 (2002i:14030)
- [BM09] A. Bayer and Yu. I. Manin, *Stability conditions, wall-crossing and weighted Gromov-Witten invariants*, Mosc. Math. J. **9** (2009), no. 1, 3–32, backmatter. MR MR2567394
- [Cad07] C. Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), no. 2, 405–427. MR 2306040 (2008g:14016)
- [Ful93] William Fulton, *Introduction to toric varieties.(am-131)*, vol. 131, Princeton University Press, 1993.
- [Has03] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), no. 2, 316–352. MR 1957831 (2004b:14040)
- [KKO11] B. Kim, A. Kresch, and Y.-G. Oh, *A compactification of the space of maps from curves*, Trans. Amer. Math. Soc. **366** (2014), no. 1, 5174. MR 3118390
- [KS09] B. Kim and F. Sato, *A generalization of Fulton-MacPherson configuration spaces*, 2009, pp. 435–443. MR 2551189 (2010m:14068)
- [Li01] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), no. 3, 509–578. MR MR1882667 (2003d:14066)
- [Li02] ———, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293. MR 1938113 (2004k:14096)
- [MM08] Anca M. Mustața and Andrei Mustața, *The Chow ring of $\overline{M}_{0,m}(\mathbb{P}^n, d)$* , J. Reine Angew. Math. **615** (2008), 93–119. MR 2384333 (2009a:14004)

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