A notion analogous to the discriminant for transcendental elements in certain extensions of local fields

Sever Achimescu (∗) – Victor Alexandru (∗∗) – Corneliu Stelian Andronescu (∗∗∗)

Abstract – Let $(K, |·|)$ be a local field. In this paper we define an invariant analogous to the discriminant over $K$ for certain transcendental elements over $K$.

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1. Introduction

Let $(K, |·|)$ be a local field. Let $\overline{K}$ be a separable algebraic closure of $K$ and let $\overline{K} = \Omega$ be the completion of $K$ with respect to the unique extension of $|·|$ also denoted $|·|$ such that $K \subseteq \overline{K} \subseteq \Omega = \overline{K}$. The unique extension by continuity of $|·|$ to $\Omega$ will also be denoted by $|·|$. In particular we may consider $K = \mathbb{Q}_p$ and $|·| = |·|_p$ the $p$–adic absolute value normalised such that $|p| = \frac{1}{p}$. We also denote by $|·|_p$ the unique extension of $|·|_p$ to $\mathbb{Q}_p$.

E-mail: sachimescu@yahoo.com

(**) Indirizzo dell’A.: Department of Mathematics, University of Bucharest, str. Academiei 14, postal code 010014 Bucharest, Romania.
E-mail: vralexandru@yahoo.com

(***) Indirizzo dell’A.: University of Pitesti, Department of Mathematics and Computer Science, 110040 Pitesti str.Targu din Vale 1, Arges, Romania.
E-mail: corneliuandronescu@yahoo.com
and further by continuity to $C_p$, the completion of $\mathbb{Q}_p$ with respect to $|\cdot|_p$, usually called the $p$-adic complex field. Generalizations of all of these may be found in [2].

In this paper we define an invariant analogous to the discriminant over $K$ for transcendental elements of $\overline{K} = \Omega$ over $K$.

2. Background material

By a local field $(K, |\cdot|)$ we understand a complete field with respect to a discrete absolute value $|\cdot|$ with a finite residue field. For example, if $\text{char}K = 0$ then $K$ is isomorphic to a finite extension of the $p$-adic field $\mathbb{Q}_p$ and $|\cdot|$ is the unique extension to $K$ of the $p$-adic absolute value $\mathbb{Q}_p$, normalized such that $|p| = \frac{1}{p}$. It is well known (see [2]) that $|\cdot|$ uniquely extends to a fixed algebraic closure $K$ of $K$ and further by continuity to $\overline{K} = \Omega$, the completion of $K$ with respect to $|\cdot|$.

Since any $\sigma \in \text{Gal}(\overline{K}/K)$ is an isometry with respect to $|\cdot|$, it follows that $\sigma$ uniquely extends to a continuous $K$-automorphism of $\Omega$. Let us denote $G_K = \text{Gal}_{cont}(\Omega/K) \cong \text{Gal}(\overline{K}/K)$. Let $T \in \Omega$. Let $C_K := \{\sigma(T), \sigma \in G_K\}$ be the orbit of $T$ with respect to the action of $G_K$ on $\Omega$. Recall from [3], Theorem 3.5 that if $T$ is transcendental over $K$ then $C_K$ is an infinite compact subset of $\Omega$.

In [4] at page 29 it is associated a chain to each $T \in \Omega$ defined in terms of the distances from $T$ to its conjugates over $K$, that is $\{\sigma(T), \sigma \in G_K\}$. The orbit of $T$ with respect to the action of the group $G_K$ is $C_K(T) := \{\sigma(T), \sigma \in G_K\}$. $C_K(T)$ is always a compact set and it is a finite set if and only if $T \in \overline{K}$. Let us denote $B[T, \epsilon] = \{\beta \in \Omega, |\beta - T| \leq \epsilon\}$

Let $N(K, T, \epsilon)$ the number of disjoint such closed balls of radius $\epsilon$ covering $C_K(T)$. The function

$$(0, \infty) \rightarrow N - \{0\}$$

$$\epsilon \rightarrow N(K, T, \epsilon)$$

is a decreasing step function. It is bounded if and only if $T \in \overline{K}$. Its image is an increasing sequence $1 = N_1 < N_2 < ...$ which is infinite if and only if $T \in \Omega - \overline{K}$.

Let $\epsilon_j = \inf\{\epsilon > 0, N(K, T, \epsilon) = N_j\}$. Since each $\sigma \in G_K$ is an isometry, each of the $N_j$ balls of radius $\epsilon_j$ covering $C_K(T)$ is covered by
the same number of balls of radius $\epsilon_{j+1}$ which intersect $C_K(T)$. It follows that $N_j|N_{j+1}$, for all $j$. If $T \in \Omega - \overline{K}$ we obtain the infinite chain

$$\mathcal{N}_K(T) = \left( \begin{array}{c} \epsilon_1 \\ N_1 \\ \epsilon_2 \\ N_2 \\ \ldots \end{array} \right)$$

Note that $\epsilon_1 = \sup\{|T - \sigma(T)|, \sigma \in G_K\}$ is the diameter of $C_K$ and $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_n > \ldots$ and $\lim_{n \to \infty} \epsilon_n = 0$. The sequence $(\epsilon_n)_{n \geq 1}$ is said to be the fundamental sequence associated to $T$. For $T = \alpha \in \overline{K}$ one obtains a finite chain

$$\mathcal{N}_K(\alpha) = \left( \begin{array}{c} \epsilon_1 \\ N_1 \\ \epsilon_2 \\ N_2 \\ \ldots \end{array} \right)$$

where $\epsilon_k(\alpha) = 0$ and $N_k(\alpha) = \deg_K(\alpha)$.

3. Main Result

Let $B_1^m, B_2^m, \ldots, B_{N_m}^m$ be a partition of $C_K(T)$ with balls of radius $\epsilon_m$, that is $B_1^m = \{z \in C_K(T), |z - T| \leq \epsilon_m\}$. Any $\sigma \in G_K$ permutes $B_1^m, B_2^m, \ldots, B_{N_m}^m$, and if we denote $H_m = \{\sigma \in G_K, \sigma(B_1^m) = B_1^m\} \subseteq G_K$ we have $N_m = [G_K : H_m]$. Let $T_i \in B_i^m$ arbitrarily fixed with $T_1 = T$. Then $|T - T_i| > \epsilon_m$ for all $i \geq 2$. For example, if $R_m$ is a set of representatives for $(G_K/H_m)_{left}$ then we can pick $T_i = \sigma_i T$, $\sigma_i \in R_m$. Let us notice that the number $x_m := \prod_{i=2}^{N_m} |T - T_i|$ does not depend on the choice of $T_i \in B_i^m$, it depends on the choice of $T$ (in fact, of $C_K(T)$) only. Indeed, for $T_i' \in B_i^m$ we have $|T - T_i'| = |T - T_1 + T_1 - T_i'| = |T - T_1|$ since $|T_1 - T_i'| < |T - T_1|$.

**Theorem 3.1.** For $m_0$ large enough we have $x_m^{1/m+1} < x_m^{1/m}$ for all $m \geq m_0$.

**Proof**

Since $N_m|N_{m+1}$ we have $N_{m+1} = N_m \cdot E_m$ that is each ball $B_i^m$ of diameter (radius) $\epsilon_m$ has a partition consisting of $E_m$ balls $B_{ij}^{m+1}$ of diameter $\epsilon_{m+1}$ satisfying $E_m = \{H_m : H_{m+1}\}$.

Now let $i \geq 2$, $T_i \in B_i^m = \bigcup_{j=1}^{E_m} B_{ij}^{m+1}$. From each $B_{ij}^{m+1}$ we pick a conjugate $T_{ij}$ of $T = T_1$ with $T_{11} = T_1 = T$. Then $|T - T_i| = |T - T_{ij}|$
for all \( 1 \leq j \leq E_m \) thus \( |T - T_i|^{E_m} = \prod_{j=1}^{E_m} |T - T_{ij}| \) therefore \( |T - T_i|^{\frac{1}{N_m}} = \prod_{j=1}^{E_m} |T - T_{ij}|^{\frac{1}{N_{m+1}}} \) since \( N_{m+1} = N_mE_m \).

From \( \epsilon_m \to 0 \) it follows that \( \epsilon_m < 1 \) for \( m \geq m_0, m_0 \) large enough. Thus \( |T - T_{1j}| = |T_1 - T_{1j}| \leq \epsilon_m < 1 \) for all \( j \geq 2 \) therefore \( 1 > \prod_{j=2}^{E_m} |T - T_{1j}| \).

Finally,

\[
\frac{1}{x_m^n} = \prod_{i=2}^{N_m} |T - T_i|^{\frac{1}{N_m}} = \prod_{i=2}^{N_m} \prod_{j=1}^{E_m} |T - T_{ij}|^{\frac{1}{N_{m+1}}} > \frac{N_m E_m}{\prod_{i=2}^{N_m} \prod_{j=1}^{E_m} |T - T_{ij}|^{\frac{1}{N_{m+1}}}} \cdot \prod_{j=2}^{E_m} |T - T_{1j}|^{\frac{1}{N_{m+1}}} = \frac{x_{m+1}}{x_m^{n-1}}
\]

q.e.d.

**Corollary 3.2.** The sequence \( (\frac{1}{x_m^n})_m \) converges and its limit is an invariant of \( T \) (and of \( C_K(T) \)).

Let us denote \( \Delta_K(T) := \lim_{m \to \infty} \frac{1}{x_m^n} \)

Then \( \Delta_K(T) \) can be considered an analogous of the discriminant of \( T = \alpha \in \overline{K} \) according to the following observation: \( x_m^{N_m} = \prod_{i \neq j} |T_j - T_i| \) since we can take \( T_i = \sigma_iT \), for a suitable \( \sigma_i \in R_m \) and the automorphisms of \( G_K \) are also isometries with respect to \( |\cdot| \).

Now we give a method for computing the numerical invariants \( \Delta_K(T) \) in certain cases. For this we make use of the following results from [3]:

Let \( K \) be a complete valued field of rank one valuation \( v \). Let \( \overline{K} \) be a fixed algebraic closure such that \( K \subset \overline{K} \) is a countably generated extension and let \( \Omega \) be the completion of \( \overline{K} \) with respect to the unique prolongation (also denoted by \( v \)) of the valuation \( v \) to \( \overline{K} \), we denote by \( |\cdot|_v \) the absolute value associated to \( v \) on \( \Omega \), that is \( |x| = c^v(x) \) with \( c \in (0,1) \). From [3] there exists a one-to-one correspondence between the set of closed subfields \( L \) satisfying \( K \subseteq L \subseteq \Omega \) and the subfields \( l \) satisfying \( K \subseteq l \subseteq \overline{K} \) and \( L = \overline{l} \) (the closure with respect to \( |\cdot|_v \)) and \( l = L \cap \overline{K} \).

Let us recall that \( T \in \Omega \) is said to be a generic element for the closed subfield \( L \) if \( L = K(T) \). It is proved that if \( \Omega = \mathbb{C}_p \) endowed with the usual \( p \)-adic absolute value and \( T \in \mathbb{C}_p - \mathbb{Q}_p \) then \( \mathbb{Q}_p(T) = \mathbb{Q}_p[\overline{T}] \) (see [1]).
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Theorem 2 from [3] implies the following:
Let \( L \subset \Omega \) be a closed subfield such that \( K \subseteq L \) is a transcendental extension. There is an element \( T \in \Omega \) such that \( L = K(T) \). Such an element \( T \) can be obtained as follows: let \( l = L \cap \overline{K} \). We construct a sequence \((\alpha_n)_{n \geq 0}, \alpha_n \in l \) satisfying:

1. For all \( n \) we have \( |\alpha_{n+1} - \alpha_n| < \min\{|\sigma(a_n) - a_n|, \sigma \in G_K, \sigma(\alpha_n) \neq \alpha_n\} \)
2. \( |\alpha_{n+1} - \alpha_n| \to 0 \)
3. \( \bigcup_n K(\alpha_n) = l \)

Let \( d_n = [K(\alpha_n) : K] = \deg \alpha_n \) and let \( T = \lim \alpha_n \in \Omega - \overline{K} \). From (1) it follows that in each of the \( R_n \) balls of radius \( \epsilon_n = |T - \alpha_n| = |\alpha_{n+1} - \alpha_n| \) which gives a partition of the orbit of \( T \) with respect to the action of \( G_K \) there exists exactly one conjugate of \( \alpha_n \) over \( K \). According to the notation of the previous paragraph it follows that \( \prod_{\sigma_i \in R_n} |T - \sigma_i(T)| = \prod_{\sigma_i \in R_n} |\alpha_n - \sigma_i(\alpha_n)| \)

and \( |R_n| = d_n \). Since each \( \sigma \in G_K \) is an isometry it follows that \( x_n^{d_n} = \prod_{\sigma_i \in R_n, \sigma_i \neq \sigma_j} |\sigma_i(\alpha_n) - \sigma_j(\alpha_n)| = \Delta_K(\alpha_n) \).

In conclusion

**Theorem 3.3.** Let \( L \) be a closed subfield of \( \Omega \). Then there exists a generic element \( T \) for \( L \), that is \( L = K(T) \) satisfying \( T = \lim \alpha_n, \alpha_n \in l = L \cap \overline{K} \) and \( \Delta_K(T) = \lim \text{disc}_K(\alpha_n) \overline{a_{n+1}} = d_n = \deg K(\alpha_n) \).

Now let \( K = \mathbb{Q}_p \) thus \( \overline{\mathbb{Q}_p} = C_p \) (the completion with respect to \(|\cdot|_p\)). Let \( T, U \in C_p - \overline{\mathbb{Q}_p} \) satisfying \(|T|_p \leq 1, |U|_p \leq 1 \) and \( \overline{\mathbb{Z}_p}[T] = \overline{\mathbb{Z}_p}[U] \). We want to prove that \( \Delta_{C_p}(T) = \Delta_{C_p}(U) \).

From \( \overline{\mathbb{Z}_p}[T] = \overline{\mathbb{Z}_p}[U] \) it follows that there exist sequences of polynomials \( P_n, R_n \in \mathbb{Z}_p[X] \) such that \( U = \lim P_n(T) \) and \( T = \lim R_n(U) \). For each \( \sigma \in G = \text{Gal}_{cont}(C_p/\mathbb{Q}_p) \) and for each positive integer \( n \) we have

\[ |P_n(T) - \sigma P_n(T)| = |P_n(T) - \sigma P_n(T)| \leq |T - \sigma T| \]

since \( P_n \in \mathbb{Z}_p[X] \). Thus

\[ |U - \sigma U| = \lim_{n \to \infty} P_n(T) - \sigma \lim_{n \to \infty} P_n(T) = \lim_{n \to \infty} (P_n(T) - \sigma P_n(T)) \leq |T - \sigma T| \]

for all \( \sigma \in G \). Since \( U - \sigma U | \leq |T - \sigma T| \), for all \( \sigma \in G \) thus \( |T - \sigma T| = |U - \sigma U| \), for all \( \sigma \in G \).

Let \( F : C_{\mathbb{Q}_p}(T) \to C_{\mathbb{Q}_p}(U) \) defined as follows: \( F(x) = \lim_{n \to \infty} P_n(x), \forall x = \sigma T \in C_{\mathbb{Q}_p}(T) \). Its inverse is \( F^{-1} : C_{\mathbb{Q}_p}(U) \to C_{\mathbb{Q}_p}(T), F^{-1}(y) = \lim_{n \to \infty} R_n(y) \), \( \forall y = \sigma U \in C_{\mathbb{Q}_p}(U) \). For each ball \( B \) of radius \( \epsilon \) in \( C_{\mathbb{Q}_p}(T) \) its image \( F(B) \) is a ball of radius \( \epsilon \) in \( C_{\mathbb{Q}_p}(U) \). Therefore \( T \) and \( U \) have the same fundamental associated sequence \( \epsilon_1 > \epsilon_2 > \ldots > \epsilon_m > \ldots \) with \( \epsilon_m \to 0 \). For each \( m \geq 1 \)
we consider the unique partition \( B_1^{(m)}, B_2^{(m)}, \ldots, B_N^{(m)} \) with \( \epsilon_m \)-radius closed balls of \( C_{\mathbb{Q}_p}(U) \). It follows that \( F(B_1^{(m)}), F(B_2^{(m)}), \ldots, F(B_N^{(m)}) \) is the unique partition with \( \epsilon_m \)-radius closed balls of \( C_{\mathbb{Q}_p}(U) \).

As above, let \( H_m \) be both the \( G \)-stabilizer of \( B_1^{(m)} \) and the \( G \)-stabilizer of \( F(B_1^{(m)}) \). Let \( R_m = \{ \sigma_1 \}_{i=1}^{N_m} \) be a complete system of representatives of \( (G/H_m)_{\text{left}} \). Since \( |T - \sigma T| = |U - \sigma U| \), for all \( \sigma \in G \), it follows that

\[
x_m(T) = \prod_{i=2}^{N_m} \frac{|T - \sigma_i T|}{|U - \sigma_i U|} = x_m(U)
\]

thus

\[
\Delta_{\mathbb{Q}_p} (T) = \lim_{m \to \infty} x_m(T)^{1/N_m} = \lim_{m \to \infty} x_m(U)^{1/N_m} = \Delta_{\mathbb{Q}_p} (U)
\]

We proved the following

**Theorem 3.4.** Let \( T, U \in \mathbb{C}_p - \overline{\mathbb{Q}_p} \) satisfying \( |T|_p \leq 1, |U|_p \leq 1 \) and \( \overline{Z}_p[T] = \mathbb{Z}_p[U] \). Then \( \Delta_{\mathbb{Q}_p} (T) = \Delta_{\mathbb{Q}_p} (U) \) is an invariant of the closed ring \( \overline{Z}_p[T] \).

**Remark 3.5.** The above theorem is the transcendental analogue for the following well-known result (see [6]): let \( T, U \) be algebraic over \( \mathbb{Q}_p \) satisfying \( |T|_p \leq 1, |U|_p \leq 1 \) and \( Z_p[T] = Z_p[U] \). Then \( \text{disc}_{\mathbb{Q}_p} (T) = \text{disc}_{\mathbb{Q}_p} (U) \).

### 4. A notable example

In this section we give an example of a non-zero \( \Delta_K(T) \).

First we recall notions and properties from [1].

For \( K = \mathbb{Q}_p \) and \( \Omega = \mathbb{C}_p \) the absolute value \( | \cdot | = | \cdot |_p \) is associated to the \( p \)-adic valuation \( v \) via \( |x|_p = (\frac{1}{p})^v(x) \) for all \( x \in \mathbb{C}_p \). According to Proposition 2.2 page 135 for all \( T \in \mathbb{C}_p \), \( T \) transcendental over \( \mathbb{Q}_p \), there exists a so called *distinguished sequence* \( (\alpha_n)_{n \geq 0} \) with \( \alpha_n \in \overline{\mathbb{Q}_p} \) such that \( T = \lim \alpha_n \), where \( \alpha_0 \in \mathbb{Q}_p \) and \( |T - \alpha_0|_p = \min \{|T - \alpha|_p, \alpha \in \mathbb{Q}_p \} \). The distinguished sequence \( (\alpha_n)_{n \geq 0} \) satisfies the following conditions:

1. \( D_n = \deg \alpha_n > D_m = \deg \alpha_m \) for all \( m < n \), furthermore \( D_m \) divides \( D_n \);
2. \( |T - \alpha_n|_p < |T - \alpha_{n-1}|_p \);
3. If \( \gamma \in \mathbb{Q}_p \) and \( \deg \gamma < \deg \alpha_n \) then \( |T - \gamma|_p \leq |T - \alpha_n|_p \).

We also quote from [1] the following: if we denote \( f_n \) the minimal polynomial of \( \alpha_n \) over \( \mathbb{Q}_p \), \( n \geq 0 \); and if we also denote \( \gamma_n = v_p(f_n(\alpha_{n+1})) \)
we have $\gamma_n > \gamma_{n-1}$ and $\frac{\gamma_n}{B_n} > \frac{\gamma_{n-1}}{B_{n-1}}$ for all $n \geq 1$. Thus there exists $l(T) = \lim_{n} \frac{f_n}{B_n} \in \mathbb{R} \cup \{+\infty\}$. Also recall that the numbers $D_n$, $|T - \alpha_n|$, $|\gamma_n|$ depend on $T$ only; they do not depend on the distinguished sequence associated to $T$. The above statements are equivalent to: $|f_n(T)|_{p}^{1/p} < |f_{n-1}(T)|_{p}^{1/p-1}$ since $|f_n(T)|_{p} = (\frac{1}{p})^{\gamma_n}$, thus there exists $((\frac{1}{p})^{l(T)} = \lim_{n} |f_n(T)|_{p}^{1/p} \in [0, +\infty)$.

Now, in order to construct the example, we need the lemma below.

Let $T \in C_{p} - Q_{p}$ and let $(\alpha_m)_{m \geq 0}$ be a distinguished sequence converging to $T$. Put $\overline{\alpha}_{m} := |T - \alpha_m|_{p}$. Let $B_{1}^{\alpha_{m}}, B_{2}^{\alpha_{m}}, ..., B_{N_{m}}^{\alpha_{m}}$ be closed balls in $C_{p}$ of radius $\overline{\alpha}_{m}$ such that their intersections to the orbit $C_{Q_{p}}(T)$ give a partition of $C_{Q_{p}}(T)$. For this it suffices that the balls are conjugated and they give a partition of the orbit of $\alpha_m$. More precisely, $B_{i}^{\alpha_{m}} = \{x \in C_{p}, |T - x| \leq \overline{\alpha}_{m}\}$ and $B_{i}^{\alpha_{m}} = \sigma_{i} B_{1}^{\alpha_{m}}$ for some $\sigma_{i} \in G_{Q_{p}}$. Since each ball $B_{i}^{\alpha_{m}}$ contains the same number $F_{m}$ of conjugates of $\alpha_m$ we have $\deg\alpha_{m} = D_{m} = N_{m} F_{m}$. Let $T_{i} \in B_{i}^{\alpha_{m}} \cap C_{Q_{p}}(T)$, $i \geq 2$ and let $\overline{x}_{m} := \prod_{i=2}^{N_{m}} |T - T_{i}|$. Note that the product which defines $\overline{x}_{m}$ does not depend on the choice of $T_{i}$. Let us denote by $\epsilon_m$ the radius of the ball $\overline{B}_{i}^{\alpha_{m}} = B_{i}^{\alpha_{m}} \cap C_{Q_{p}}(T)$. Notice that $\epsilon_m$ is a term of the fundamental sequence associated to $T$. We have that $\epsilon_{m} \leq \epsilon_{m'}$. The balls $\overline{B}_{i}^{\alpha_{m}}$, $i = 1, N_{m}$ are conjugated and they give a partition of the orbit $C_{Q_{p}}(T)$ and we also have $N_{m'} = N_{m}$ and $x_{m'} = \overline{x}_{m}$, with $x_{m}$ and $N_{m}$ defined as in Theorem 1. From $\overline{x}_{m} \to 0$ it follows that $\epsilon_{m'} \to 0$ thus $N_{m'} \to \infty$. Therefore the sequence $(\overline{x}_{m}^{\frac{1}{N_{m}}})_{m \geq 0}$ is a subsequence of $(x_{m}^{\frac{1}{N_{m}}})_{m \geq 0}$ studied in Theorem 3.1.

**Lemma 4.1.** For $|T|_{p} \leq 1$ and $m \geq 1$ we have $\overline{x}_{m}^{\frac{1}{N_{m}}} > |f_{m}(T)|_{p}^{1/p}$. 

**Proof:** Since $D_{m} = N_{m} F_{m}$ we have to prove that $\overline{x}_{m}^{F_{m}} > |f_{m}(T)|_{p}$. We have $\overline{x}_{m} := \prod_{i=2}^{N_{m}} |T - T_{i}|$, $T_{i} \in B_{i}^{\alpha_{m}}$ and $|f_{m}(T)|_{p} = \prod_{s=1}^{D_{m}} |T - T_{\sigma_{s}(\alpha_{m})}|$, where $\{\sigma_{s}(\alpha_{m}), s = 1..D_{m}\}$ are the conjugates of $\alpha_{m}$ with $\alpha_{m} = \sigma_{1}(\alpha_{m})$. Let us denote $\{\sigma_{i}(\alpha_{m}), s = 1..F_{m}\}$ those conjugates of $\alpha_{m}$ belonging to the ball $\overline{B}_{i}^{\alpha_{m}}$. Then for all $i \geq 2$ we have $|T - T_{i}|_{p} > |T - \sigma_{i}(\alpha_{m})|_{p}$ thus $|T - T_{i}|_{p}^{F_{m}} = \prod_{s=1}^{F_{m}} |T - \sigma_{i}(\alpha_{m})|_{p}$. Therefore $|T - T_{i}|_{p}^{F_{m}} = \prod_{s=1}^{F_{m}} |T - \sigma_{i}(\alpha_{m})|_{p}$.

On the other hand in the ball $B_{i}^{\alpha_{m}}$ there is at least one conjugate of $\alpha_{m}$ (for example $\alpha_{m}$ itself) and for each of these conjugates we have
\[ |T - \sigma_{1, \alpha_m}|_p < 1. \] Therefore

\[
\prod_{s=1}^{F_m} |T - \sigma_{1, \alpha_m}|_p < 1
\]

and by multiplying the above two products we obtain \( \prod_{m} x_m^{F_m} > |f_m(T)|_p \), q.e.d.

Taking into account that the sequence \( (x_m^{\frac{1}{l(T)}})_{m \geq 0} \) is a subsequence of \( (x_m^m)_{m \geq 0} \) studied in Theorem 1 it follows by applying the above lemma that

\[
\Delta_{Q_p}(T) \geq \lim_{n} |f_n(T)|^{\frac{1}{p}} = \left( \frac{1}{p} \right)^{l(T)} \geq 0
\]

Remark 4.2. In [1] page 142, using an argument from [5], it is given an example of an element \( T \in C_2 - Q_2 \) with \( l(T) = 2 \). Therefore, for that \( T \), we have \( \Delta_{Q_2} \geq (\frac{1}{2})^2 > 0 \).

A thorough study of how \( \Delta_{Q_p} \) and \( (\frac{1}{p})^{l(T)} \) are related to each other is yet to be done.

References


