Rend. Sem. Mat. Univ. Padova, DRAFT, 1–14

Generating sets of Galois equivariant Krasner analytic functions

Victor Alexandru (*) – Marian Vâjâitu (**) –
Alexandru Zaharescu (***)

Abstract – Given a prime number p and x an element of the Tate field $\mathbb{C}_p$, the main goal of the present paper is to provide an explicit generating set, which is given by the trace function of x and all its derivatives, for the $\mathbb{C}_p$-Banach algebra of the Galois equivariant Krasner analytic functions defined on the complement in $\mathbb{P}^1(\mathbb{C}_p)$ of the orbit of x with values in $\mathbb{C}_p$.

Mathematics Subject Classification (2010). Primary: 11S99; Secondary: 11S20, 11S80.

Keywords. Generating sets, generating degree, Krasner analytic functions, trace functions.

1. Introduction

Let p be a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic valuation.

(*) *Indirizzo dell'A.*: Faculty of Mathematics, University of Bucharest, 14 Academiei Street, RO-010014 Bucharest, Romania.
E-mail: vralexandru@yahoo.com

(**) *Indirizzo dell'A.*: Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania.
E-mail: Marian.Vajaitu@imar.ro

(***) *Indirizzo dell'A.*: Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania, and Department of Mathematics, University of Illinois at Urbana-Champaign, Altgeld Hall, 1409 W. Green Street, Urbana, IL, 61801, USA.
E-mail: zaharesc@math.uiuc.edu

Let $A \subseteq B$ be topological commutative rings and M an arbitrary subset of B. We denote by $A[M]$ the smallest A-subalgebra of B that contains M. Such a set M is said to be a *generating set* of B over A if the ring $A[M]$ is dense in B. The concept of generating degree was introduced by Ioviţă and Zaharescu [11]. The *generating degree* of B/A is defined to be

$$\operatorname{gdeg}(B/A) := \min\{|M|, \text{ where } M \text{ is a generating set of } B/A\}.$$

(Here $|M|$ denotes the number of elements of M if M is finite and ∞ if M is not finite.)

The generating degree of B over $\mathbb{Z}$, if $\operatorname{char} B = 0$, or over $\mathbb{F}_p$, if $\operatorname{char} B = p$, will be denoted by $\operatorname{gdeg}(B)$ and will be called shortly *the absolute generating degree* of B.

An important result [4, Theorem 1] says that $\operatorname{gdeg}(E) = 1$ for any closed subfield E of $\mathbb{C}_p$. In particular $\operatorname{gdeg}(\mathbb{C}_p) = 1$, which means that $\widetilde{\mathbb{Z}[x]} = \mathbb{C}_p$, that is, there exists an element $x \in \mathbb{C}_p$, called *generic*, such that the topological closure of the ring $\mathbb{Z}[x]$ in $\mathbb{C}_p$ coincides with $\mathbb{C}_p$.

Let us denote $\mathbb{P} = \mathbb{P}^1(\mathbb{C}_p)$. Let $U \subset \mathbb{P}$ be an *affinoid*, by which we mean that $U = \mathbb{P} \setminus \cup_{i=1}^{g} B_i$, where each ball B_i is an open ball in $\mathbb{P}$. Such a subset $U \subset \mathbb{P}$ is called a *wide open set in the sense of Coleman*, if at least one ball B_i is a closed ball. We denote by $\mathcal{A}(U, \mathbb{C}_p)$ the set of rigid analytic functions defined on U with values in $\mathbb{C}_p$, that is, the functions that are uniform limit of rational functions with poles outside U. Another important result [4, Theorem 1] is that $\operatorname{gdeg}(\mathcal{A}(U, \mathbb{C}_p)) = \infty$, if U is a wide open set; $\operatorname{gdeg}(\mathcal{A}(U, \mathbb{C}_p)) \leq g + 1$, if U is an affinoid of the form $\mathbb{P} \setminus \cup_{i=1}^{g} B_i$; and $\operatorname{gdeg}(\mathcal{A}(U, \mathbb{C}_p)) = 2$, if U is a closed ball in $\mathbb{P}$.

The notions of trace, trace function and trace series that are associated with an element x of $\mathbb{C}_p$ were introduced and investigated in a series of articles [3, 13, 17]. Given an element $x \in \mathbb{C}_p$, the *trace* of x is defined by the equality

$$Tr(x) = \int_{O(x)} t \, \mathrm{d}\pi_x(t)\,, \tag{1}$$

provided that the integral with respect to the Haar distribution π_x on the right side of (1) is well defined. This is the case, for example, when π_x is bounded, that is, when π_x is a measure. Here, $O(x) = \{\sigma(x) : \sigma \in Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)\}$ is the orbit of x, which is a compact subset of $\mathbb{C}_p$. The integral is also well defined when x is a Lipschitz element, as shown in [3].

The *trace function* $F(z)$ of x is defined by

$$F(z) = \int_{O(x)} \frac{1}{z-t} \, \mathrm{d}\pi_x(t),$$

for all those $z \in \mathbb{C}_p$ for which the integral is well defined. This is an analytic object that embodies a significant amount of algebraic data.

The main goal of the present paper is to provide an explicit generating set given by the trace function and all its derivatives. Furthermore, we also describe the generating degree of the $\mathbb{C}_p$-Banach algebra of Galois equivariant Krasner analytic functions [9, 14, 17], defined on the complement in $\mathbb{P}$ of the Galois orbits of an element of $\mathbb{C}_p$. After the presentation of the necessary definitions and notations in Section 2, our main results, stated in Theorem 3.2 and Theorem 4.2, are proved in Sections 3 and 4, which are dedicated to the algebraic case and the transcendental case, respectively.

2. Notation and definitions

Let p be a prime number and $\mathbb{Q}_p$ the field of p-adic numbers endowed with the p-adic absolute value $|\cdot|$, normalized such that $|p| = 1/p$. We denote by v_p the p-adic valuation. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and denote by the same symbol $|\cdot|$ the unique extension of $|\cdot|$ to $\overline{\mathbb{Q}}_p$. Further, denote by $(\mathbb{C}_p, |\cdot|)$ the completion of $(\overline{\mathbb{Q}}_p, |\cdot|)$ and by $O_{\mathbb{C}_p}$ the ring of integers of $\mathbb{C}_p$ (see the monographs by Amice [6] and by Artin [7] for the definitions of the fundamental concepts).

Consider the Galois group $\mathcal{G} = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. As we observed in [3], the group $\mathcal{G}$ is canonically isomorphic with the group $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$ of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. We shall identify these two groups. For any $x \in \mathbb{C}_p$ denote $O(x) = \{\sigma(x) : \sigma \in \mathcal{G}\}$ the orbit of x, and let $\widetilde{\mathbb{Q}_p[x]}$ be the topological closure of the ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$.

Let $\mathbb{Q}_p \subseteq K$ be a closed subfield of $\mathbb{C}_p$. An element x of $\mathbb{C}_p$ is said to be a *generating element* (or a *generic element*) for K provided $\widetilde{\mathbb{Q}_p[x]} = K$, notion introduced in the works [2] and [10].

By Galois theory in $\mathbb{C}_p$, as developed, chronologically, by Tate [16], Sen [15] and Ax [8], the closed subgroups of the Galois group $\mathcal{G}$ are in one-to-one correspondence with the closed subfields of $\mathbb{C}_p$. For any closed subgroup H of $\mathcal{G}$ denote

$$Fix(H) = \{x \in \mathbb{C}_p : \sigma(x) = x \text{ for all } \sigma \in H\}.$$

Then $Fix(H)$ is a closed subfield of $\mathbb{C}_p$. Denote $H(x) = \{\sigma \in \mathcal{G} : \sigma(x) = x\}$. Then $H(x)$ is a closed subgroup of $\mathcal{G}$. Also, $Fix(H(x)) = \widetilde{\mathbb{Q}_p[x]}$. For any $\varepsilon > 0$, $H(x,\varepsilon) := \{\sigma \in \mathcal{G} : |\sigma(x) - x| < \varepsilon\}$ is an open subgroup of $\mathcal{G}$ of finite index and $[\mathcal{G} : H(x,\varepsilon)] = N(x,\varepsilon)$ is the number of open balls of radius ε that cover $O(x)$. From the article [2], we know that the map $\sigma \rightsquigarrow \sigma(x)$ from $\mathcal{G}$ to $O(x)$ is continuous, and it defines a homeomorphism from $\mathcal{G}/H(x)$ (endowed with the quotient topology) to $O(x)$ (endowed with the induced topology from $\mathbb{C}_p$). In such a way $O(x)$ is a closed compact and totally disconnected subset of $\mathbb{C}_p$, and the group $\mathcal{G}$ acts continuously on $O(x)$: if $\sigma \in \mathcal{G}$ and $\tau(x) \in O(x)$ then $\sigma \star \tau(x) := (\sigma\tau)(x)$.

For any real number $\varepsilon > 0$ denote $B(x,\varepsilon) = \{y \in \mathbb{C}_p : |y - x| < \varepsilon\}$ and $B[x,\varepsilon] = \{y \in \mathbb{C}_p : |y - x| \leq \varepsilon\}$. Denote $\mathbb{P} = \mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ and $E(x,\varepsilon) = \{y \in \mathbb{P} : |y - t| \geq \varepsilon, \text{ for all } t \in O(x)\}$. The complement of $E(x,\varepsilon)$ in $\mathbb{P}$ is denoted by $V(x,\varepsilon)$. Both sets $E(x,\varepsilon)$ and $V(x,\varepsilon)$ are open and closed, and one has $\cap_\varepsilon V(x,\varepsilon) = O(x)$. Denote $E(x) = \cup_\varepsilon E(x,\varepsilon) = \mathbb{P} \setminus O(x)$. Let S_ε be a complete system of representatives for the right cosets of $\mathcal{G}$ with respect to $H(x,\varepsilon)$. We shall assume that the neutral element e of $\mathcal{G}$ is in S_ε. We know from [3] that for any $0 < \varepsilon' < \varepsilon$, $|S_\varepsilon|$ divides $|S_{\varepsilon'}|$. Then $V(x,\varepsilon) = \cup_{\sigma \in S_\varepsilon} B(\sigma(x),\varepsilon)$.

Next, if X is a compact subset of $\mathbb{C}_p$, then by an open ball in X we mean a subset of the form $B^*(x,\varepsilon) = B(x,\varepsilon) \cap X$ where $x \in \mathbb{C}_p$ and $\varepsilon > 0$. Let us denote by $\Omega(X)$ the set of subsets of X that are open and compact. It is easy to see that any $D \in \Omega(X)$ can be written as a finite union of open balls in X, any two disjoint.

Definition 2.1 (Mazur and Swinnerton-Dyer [12]). By a distribution on X with values in $\mathbb{C}_p$ we mean a map $\mu : \Omega(X) \to \mathbb{C}_p$ that is finitely additive, that is, if $D = \cup_{i=1}^n D_i$ with $D_i \in \Omega(X)$ for $1 \leq i \leq n$ and $D_i \cap D_j = \emptyset$ for $1 \leq i \neq j \leq n$, then $\mu(D) = \sum_{i=1}^n \mu(D_i)$.

The norm of μ is defined by $\|\mu\| := \sup\{|\mu(D)| : D \in \Omega(X)\}$. If $\|\mu\| < \infty$ we say that μ is a measure on X.

Definition 2.2. We say that a distribution μ on X is Lipschitz if

$$\lim_{\varepsilon \to 0} \varepsilon \max |\mu(B^*(a,\varepsilon))| = 0,$$

where the "max" is taken over all the balls $B^*(a,\varepsilon)$ from $\Omega(X)$.

Remark 2.3. Any measure on X is a Lipschitz distribution and any Lipschitz function on X is Riemann integrable with respect to any Lipschitz distribution.

Definition 2.4. An element $x \in \mathbb{C}_p$ is called Lipschitz if and only if $\lim_{\varepsilon\to 0} \frac{\varepsilon}{|N(x,\varepsilon)|} = 0$, where $N(x,\varepsilon)$ is the number of balls of radius ε that cover the orbit of x.

Remark 2.5. In case $x \in \mathbb{C}_p$ is Lipschitz, then the Haar distribution π_x on the orbit of x is also Lipschitz.

Definition 2.6. An element $x \in \mathbb{C}_p$ is called p-bounded if there exists an $s \in \mathbb{N}$ such that p^s does not divide the number $N(x,\varepsilon)$, for any $\varepsilon > 0$.

In this case π_x is a measure. It is clear that a p-bounded element of $\mathbb{C}_p$ is also Lipschitz.

Definition 2.7 ([14]). A subset D of $\mathbb{P}$ is called infraconnected if its diameter δ is positive and for each $a \in D$ the set $\{|x-a| : x \in D\}$ is dense in $[0,\delta]$. Here $\delta = \delta(D) = \sup_{x,y\in D} |x-y| \leq \infty$.

Definition 2.8 ([9], [14]). Let D be a closed infraconnected subset of $\mathbb{P}$. A function $f : D \to \mathbb{C}_p$ is said to be Krasner analytic on D provided that f is a uniform limit of rational functions having no poles in D. We denote by $\mathcal{A}(D,\mathbb{C}_p)$ the set of all Krasner analytic functions defined on D with values in $\mathbb{C}_p$.

Remark 2.9. A key example of a closed infraconnected subset of $\mathbb{P}$ is $E(x)$. If x is Lipschitz (see, for example, [1], [3] and [5]), the trace function F of x is Krasner analytic on $E(x)$ with values in $\mathbb{C}_p$.

Remark 2.10. Let X be a compact subset of $\mathbb{C}_p$ and, for any $\delta > 0$ let X_δ be a finite covering of X with open balls of radius δ. One has that $\mathbb{P}\setminus X_\delta$ is a connected affinoid in the sense of [9] and it is also a infraconnected set in the sense of [14]. A function f of $\mathcal{A}(\mathbb{P}\setminus X_\delta, \mathbb{C}_p)$ is a holomorphic map in the sense of the Rigid Analytic Spaces, see [9].

The set $X \subset \mathbb{C}_p$ is said to be $\mathcal{G}$-equivariant provided that $\sigma(x) \in X$ for any $x \in X$ and any $\sigma \in \mathcal{G}$. (Such an example is $X = O(x)$.)

Definition 2.11. Let X be a $\mathcal{G}$-equivariant compact subset of $\mathbb{C}_p$ and let μ be a distribution on X with values in $\mathbb{C}_p$. We say that μ is $\mathcal{G}$-equivariant if $\mu(\sigma(B)) = \sigma(\mu(B))$, for any ball B in X and any $\sigma \in \mathcal{G}$.

Remark 2.12. On a Galois orbit $O(x)$ there exists a unique $\mathcal{G}$-equivariant probability distribution with values in $\mathbb{Q}_p$, namely the Haar distribution π_x.

A Krasner analytic function defined on a $\mathcal{G}$-equivariant closed infraconnected subset D of $\mathbb{P}$ is called *equivariant* if for any $z \in D$ one has $O(z) \subset D$ and $f(\sigma(z)) = \sigma(f(z))$ for all $\sigma \in \mathcal{G}$.

Let $\mathcal{A}^{\mathcal{G}}(D, \mathbb{C}_p)$ be the set of Galois equivariant Krasner analytic functions defined on a subset D of $\mathbb{P}$ as above with values in $\mathbb{C}_p$, and let $\mathcal{A}_0^{\mathcal{G}}(D, \mathbb{C}_p)$ be its subset consisting only of those functions that vanish at ∞.

Remark 2.13. The trace function of a Lipschitz element x of $\mathbb{C}_p$, as in Remark 2.9, is a Galois equivariant Krasner analytic functions defined on $E(x)$ with values in $\mathbb{C}_p$ that vanishes at ∞.

3. The algebraic case

We begin this section with the following lemma that characterizes a function field generated by the derivatives of a certain precisely defined function.

Lemma 3.1. *Let K be a field of characteristic zero and let $f \in K[X]$ be a monic polynomial of degree $n \geq 2$. Denote $F = f'/f$ and $A = K[F, F', \ldots, F^{(n-1)}]$. Then*

$$A = \left\{ \frac{G}{f^m} : m \geq 1,\ G \in K[X],\ \deg G \leq mn \right\}.$$

Proof. Let $\overline{K}$ be an algebraic closure of K and let

$$f(X) = \prod_{i=1}^{n} (X - \alpha_i), \quad \alpha_i \in \overline{K},\ 1 \leq i \leq n$$

be the decomposition of f in $\overline{K}$. Then, the derivatives of $F = f'/f$ can be written as

$$F = \sum_{i=1}^{n} \frac{1}{X - \alpha_i},\ F' = \sum_{i=1}^{n} \frac{-1}{(X - \alpha_i)^2}, \ldots, F^{(n-1)} = \frac{(-1)^{n-1}}{(n-1)!} \sum_{i=1}^{n} \frac{1}{(X - \alpha_i)^n}.$$

By Newton's identities it follows that $1/f = \prod_{i=1}^{n} \frac{1}{X-\alpha_i} \in A$.

Now, let us show that any fraction g/f, where $g \in K[X]$ is a polynomial of degree $\leq n$, is an element of A. We have $F' = (f'/f)' = f''/f - (f'/f)^2$, therefore $f''/f \in A$. Again, by using Newton's identities, we find that A is closed to formal derivation. Then $(f''/f)' = f^{(3)}/f - (f^{(2)}/f) \cdot (f'/f)$, so $f^{(3)}/f \in A$. Next, inductively we obtain $f^{(i)}/f$ for any $1 \leq i \leq n-1$ and by this we see that $1/f$, $f^{(n-1)}/f, \ldots, f^{'}/f \in A$. Because the polynomials $f', f^{(2)}, \ldots, f^{(n-1)}, 1$ have distinct degrees, it follows that $g/f \in A$ for any polynomial $g \in K[X]$ of degree $< n$, and it is immediately seen that this also holds true if $\deg g = n$.

Let us consider $G \in K[X]$ a polynomial of degree $\leq mn$, as in the hypothesis of Lemma 3.1. Writing G as $G = g_0 + g_1 f + \cdots + g_m f^m$, with $g_i \in K[X]$ polynomials of degree $< n$ for $1 \leq i \leq m$, it follows that

$$\frac{G}{f^m} = \frac{g_0}{f^m} + \frac{g_1}{f^{m-1}} + \cdots + \frac{g_{m-1}}{f} + g_m \in A,$$

which completes the proof of the lemma. □

Now, let $\alpha \in \overline{\mathbb{Q}}_p$ be an algebraic element of $\mathbb{C}_p$ of degree d and let $F(z)$ be the trace function of α. Let $g \in \mathcal{A}_0^{\mathcal{G}}(\mathbb{P} \setminus O(\alpha), \mathbb{C}_p)$ be a Galois equivariant Krasner analytic function that vanishes at ∞. By Mittag-Leffler Theorem one has

$$g(z) = \sum_{i=1}^{d} \sum_{k \geq 1} \frac{a_{i,k}}{(z-\alpha_i)^k},$$

where $\alpha_i = \sigma_i(\alpha)$, $1 \leq i \leq d$, are all the conjugates of $\alpha_1 = \alpha$ and $a_{i,k} = \sigma_i(a_{1,k}) \in \mathbb{Q}_p(\alpha)$, for any $k \geq 1$ and any $1 \leq i \leq d$, which results from the fact that g is Galois equivariant, see [1] and [5]. By Mittag-Leffler's condition of convergence and [1], it follows that the norm of g is $\|g\| = \sup_{k \geq 1} |a_{1,k}|^{1/k}$.

Now, we will show that g is in the closure of $\mathbb{Q}_p[F, F', \ldots, F^{(d-1)}]$ with respect to this norm. We have that $F = \frac{1}{d} \cdot \frac{f'}{f}$, where $f = Irr(\alpha/\mathbb{Q}_p)$ is the minimal polynomials of α over $\mathbb{Q}_p$. Denote $F_k = \sum_{i=1}^{d} \frac{a_{i,k}}{(z-\alpha_i)^k}$, for any $k \geq 1$. It is easy to see that $F_k = \frac{G_k}{f^k}$ where $G_k \in \mathbb{Q}_p[X]$ is a polynomial of degree $\leq kd$. By Lemma 3.1 one has that $F_k \in \mathbb{Q}_p[F, F', \ldots, F^{(d-1)}]$, for any $k \geq 1$. Because $g = \sum_{k \geq 1} F_k$, by summing up we obtain the following result.

Theorem 3.2. *Let $\alpha \in \overline{\mathbb{Q}}_p$ be an algebraic element of $\mathbb{C}_p$ of degree d and let F be the trace function of α. Then the closure of $\mathbb{Q}_p[F, F', \ldots, F^{(d-1)}]$, with respect to the above norm, in $\widetilde{\mathcal{M}}$ where $\mathcal{M} = \mathbb{Q}_p(\alpha)\Big(\frac{1}{z-\alpha_1}, \ldots, \frac{1}{z-\alpha_d}\Big)$, coincides with $\mathcal{A}_0^{\mathcal{G}}(\mathbb{P} \setminus O(\alpha), \mathbb{C}_p)$.*

Remark 3.3. In the terminology of the generating sets, as in the Introduction, from the above theorem one has that the set $\{F, F', \ldots, F^{(d-1)}\}$ is a generating set of $\mathcal{A}_0^{\mathcal{G}}(\mathbb{P} \setminus O(\alpha), \mathbb{C}_p)$ over $\mathbb{Q}_p$, which implies that $\operatorname{gdeg}(\mathcal{A}_0^{\mathcal{G}}(\mathbb{P} \setminus O(\alpha), \mathbb{C}_p)/\mathbb{Q}_p) \le d$.

4. The transcendental case

We preserve the notation and definitions from the previous sections. Let $\alpha \in \overline{\mathbb{Q}}_p$ be an algebraic element of $\mathbb{C}_p$ of degree d and let $\alpha_1 = \alpha = \sigma_1(\alpha), \alpha_2 = \sigma_2(\alpha), \ldots, \alpha_d = \sigma_d(\alpha)$ be all the conjugates of α over $\mathbb{Q}_p$. We denote by $f = Irr(\alpha/\mathbb{Q}_p)$ the minimal polynomials of α over $\mathbb{Q}_p$. Let F be the trace function of α. It is clear that $F = \frac{1}{d} \cdot \frac{f'}{f} = \frac{1}{d}\sum_{i=1}^{d} \frac{1}{z-\alpha_i}$. Denote $F^{[k+1]} = \frac{(-1)^k}{k!} \cdot F^{(k)}$, for any $k \ge 0$. We have seen in the previous section that a rational function of the form

$$G(z) = \sum_{i=1}^{d} \sum_{k} \frac{a_{i,k}}{(z-\alpha_i)^k} \in \overline{\mathbb{Q}}_p(z) \tag{2}$$

that is Galois equivariant is of the form $G(z) = \sum_k \frac{G_k(z)}{f^k(z)} \in \mathbb{Q}_p(z)$ and it is an element of the ring $\mathbb{Q}_p[F, F', \ldots, F^{(d-1)}] = \mathbb{Q}_p[F^{[1]}, \ldots, F^{[d]}]$.

For the sake of simplicity, in what follows, we consider α such that $|\alpha| \le 1$ and $a_{i.k}$ such that $|a_{i.k}| \le 1$, for any i, k. These facts imply that $G_k(z) \in \mathbb{Z}_p[z]$.

Let m be a positive integer. Let $Y_1, Y_2, \ldots, Y_m$ be variables and denote $p_k = \sum_{i=1}^{m} Y_i^k$, for any $k \ge 1$. Also, we denote by $s_k(Y_1, \ldots, Y_m)$ the elementary symmetric polynomials in variables $Y_1, Y_2, \ldots, Y_m$. By Newton's identities

$$p_k - p_{k-1}s_1 + p_{k-2}s_2 - \cdots + (-1)^{k-1}p_1 s_{k-1} + (-1)^k k s_k = 0,$$

for any $1 \le k \le m$, one has

$$s_k \in \frac{1}{m!}\mathbb{Z}[p_1, p_2, \ldots, p_d]. \tag{3}$$

In what follows we take $m = d$ and $Y_1 = \frac{1}{z-\alpha_1}, \ldots, Y_d = \frac{1}{z-\alpha_d}$. With this notation, it is clear that $F^{[k]} = \frac{1}{d}p_k$, for any $k \ge 1$.

The rational function $\frac{1}{z-\alpha}$ is a root of the following equation

$$\left(\frac{1}{z-\alpha}\right)^d - s_1 \cdot \left(\frac{1}{z-\alpha}\right)^{d-1} + \cdots + (-1)^d s_d = 0,$$

and the same equation is verified by each $\frac{1}{z-\alpha_i}$, for any $1 \leq i \leq d$. By (3) it follows that $\frac{d!}{z-\alpha_1}, \ldots, \frac{d!}{z-\alpha_d}$ are algebraic elements that are integral over the ring

$$\mathbb{Z}_p\Big[p_1\Big(\tfrac{1}{z-\alpha_1}, \ldots, \tfrac{1}{z-\alpha_d}\Big), \ldots, p_d\Big(\tfrac{1}{z-\alpha_1}, \ldots, \tfrac{1}{z-\alpha_d}\Big)\Big].$$

A similar conclusion is obtained if instead of $1/(z-\alpha_i)$ we take $1/(z-\alpha_i)^r$, for any $r \geq 1$. More precisely, we find that $d!/(z-\alpha_1)^r, \ldots, d!/(z-\alpha_d)^r$ are algebraic elements that are integral over the ring

$$\mathbb{Z}_p\Big[p_1\Big(\tfrac{1}{(z-\alpha_1)^r}, \ldots, \tfrac{1}{(z-\alpha_d)^r}\Big), \ldots, p_d\Big(\tfrac{1}{(z-\alpha_1)^r}, \ldots, \tfrac{1}{(z-\alpha_d)^r}\Big)\Big],$$

for any $r \geq 1$. Denote

$$\Lambda := \mathbb{Z}_p\Big[p_1\Big(\tfrac{1}{(z-\alpha_1)^r}, \ldots, \tfrac{1}{(z-\alpha_d)^r}\Big), \ldots, p_d\Big(\tfrac{1}{(z-\alpha_1)^r}, \ldots, \tfrac{1}{(z-\alpha_d)^r}\Big);\ \text{for all } r \geq 1\Big]$$

and

$$\Omega := \mathbb{Q}_p\Big[p_1\Big(\tfrac{1}{(z-\alpha_1)^r}, \ldots, \tfrac{1}{(z-\alpha_d)^r}\Big), \ldots, p_d\Big(\tfrac{1}{(z-\alpha_1)^r}, \ldots, \tfrac{1}{(z-\alpha_d)^r}\Big);\ \text{for all } r \geq 1\Big].$$

One has the following result.

LEMMA 4.1. *The ring Λ is integrally closed in Ω.*

PROOF. First of all, we mention that the elements of these rings are considered as functions defined on the complement of a neighborhood of the orbit of α over $\mathbb{Q}_p$ and the convergence is given by the uniform convergence norm on that domain. By this, a function from Λ is divisible by p if and only if each of the coefficients of the monomials of the function is divisible by p. With this in mind, we see that a function $H \in \Omega \setminus \Lambda$ has a unique representation of the form $H = \frac{\overline{H}}{p^s}$, where $\overline{H} \in \Lambda \setminus p\Lambda$ and s is a positive integer. Now, if H were a a root of a monic polynomial with coefficients in Λ, then p would divide $\overline{H}$ in Λ. Since this is a contradiction, the proof of the lemma is completed. □

By this, if we fix a neighborhood of the orbit of α and consider G a rational Galois equivariant function as in (2) with all the coefficients $a_{i.k}$ integers over $\mathbb{Z}_p$, on one hand we have that $d!G$ is an element of Ω and on the other hand it is an algebraic integer over Λ. By Lemma 4.1 we know that $d!G \in \Lambda$, which means that

$$G \in \frac{1}{(d-1)!}\mathbb{Z}_p[F^{[1]}, \ldots, F^{[n]}, \ldots]. \tag{4}$$

Now, let x be a transcendental element of $\mathbb{C}_p$ and let $0 < \varepsilon < 1$ be a positive real number. Denote $K_x = \overline{\mathbb{Q}}_p \cap \widetilde{\mathbb{Q}_p[x]}$. Let α be an algebraic element of $\mathbb{C}_p$ such that $\alpha \in B(x, \varepsilon) \cap K_x$. By [5, Proposition 4.1], there exists a sequence $\{\alpha_n\}_{n\geq 1}$ of elements of K_x and a sequence $\{\varepsilon_n\}_{n\geq 1}$ of positive real numbers such that:

i) $\varepsilon_1 = \varepsilon$, $\alpha_1 = \alpha$,

ii) for any $n \geq 1$ one has $\varepsilon_{n+1} \leq \inf\{\varepsilon_n/2, |x - \alpha_n|\}$,

iii) $|x - \alpha_n| < \varepsilon_n$, $n \geq 1$, and $\deg \alpha_n$ is smallest with this property.

It is clear that $x = \lim_{n\to\infty} \alpha_n$. Denote $d_n = \deg \alpha_n = \deg f_n$, where $f_n = \mathrm{Irr}(\alpha_n/\mathbb{Q}_p)$ is the minimal polynomial of α_n over $\mathbb{Q}_p$, for any $n \geq 1$. Also, for the sake of simplicity, we can suppose that $|x| = |\alpha| = |\alpha_n| \leq 1$, for any $n \geq 1$, and that the sequence $\{\alpha_n\}_{n\geq 1}$ satisfies the following condition

$$\lim_{n\to\infty} \frac{\varepsilon_n}{|d_n!| \cdot |d_{n+1}|} = 0. \tag{5}$$

Denote $F_n(z) = \frac{1}{d_n} \cdot \frac{f_n'(z)}{f_n(z)}$ and $F(z) = \lim_{n\to\infty} F_n(z)$. It is easy to see that F_n is the trace function of α_n, for any $n \geq 1$, and F is the trace function of x, which does exist because x is a Lipschitz element. The trace functions F_n and F are considered defined on $E(x)$ and the convergence is given by the uniform convergence norm.

For any $\sigma \in S_\varepsilon$, denote $O_\sigma(x, \varepsilon) = \{\tau(x) : \tau \in \sigma H(x, \varepsilon)\}$, which is a compact subset of $\mathbb{C}_p$. In fact, one has $O_\sigma(x, \varepsilon) = B(\sigma(x), \varepsilon) \cap O(x)$. For any integers $m, n \geq 0$, denote

$$F_{m,n}(z) = \int_{O(x)} \frac{t^m}{(z-t)^n}\, \mathrm{d}\pi_x(t) \quad \text{and} \quad F^\sigma_{m,n}(z) = \int_{O_\sigma(x,\varepsilon)} \frac{t^m}{(z-t)^n}\, \mathrm{d}\pi_x(t).$$

It is clear that

$$F_{m,n}(z) = \sum_{\sigma \mathcal{S}_\varepsilon} F^\sigma_{m,n}(z),$$

which is in fact the Mittag-Leffler decomposition of $F_{m,n}(z)$ on $E(x, \varepsilon)$.

Let $G \in \mathcal{A}_0^{\mathcal{G}}(E(x, \varepsilon), \mathbb{C}_p)$ be a Galois equivariant Krasner analytic function that vanishes at ∞. By Mittag-Leffler decomposition we find that

$$G(z) = \sum_{\sigma \in S_\varepsilon} G_\sigma(z),$$

where

$$G_\sigma(z) = \sum_{m\geq 1} \frac{a_{\sigma,m}}{(z-\sigma(\alpha))^m} \quad \text{and} \quad \lim_{m\to\infty} \frac{|a_{\sigma,m}|}{\varepsilon^m} = 0, \ \sigma \in S_\varepsilon. \tag{6}$$

From Galois equivariance of G (see [1] and [5]), we find that $a_{\sigma,m} = \sigma(a_{e,m})$, for any $m \geq 1$. Also, by the condition of convergence of (6), we may suppose that $|a_{e,m}| \leq 1$, for any $m \geq 1$.

For any $m \geq 1$ denote

$$h_m(\alpha) = a_{e,1}\alpha^{m-1} + \binom{m-1}{1} a_{e,2}\alpha^{m-2} + \cdots + \binom{m-1}{m-1} a_{e,m}.$$

Let us recall recall the following result.

Theorem ([5, Theorem 4.2]). *Let x be a Lipschitz element of $\mathbb{C}_p$, $\varepsilon > 0$, $\alpha \in B(x,\varepsilon) \cap K_x$ and $G \in \mathcal{A}_0^{\mathcal{G}}(E(x,\varepsilon), \mathbb{C}_p)$. Then for any $z \in E(x,\varepsilon)$ one has*

$$G(z) = \sum_{\sigma\in S_\varepsilon} \sum_{m\geq 1} \sum_{0\leq j<m} \frac{(-1)^j}{j!} \sigma(h_m^{(j)}(\alpha)) F_{j,m}^\sigma(z). \tag{7}$$

Next, for any $m \geq 1$ and any $0 \leq j < m$, denote

$$G_m(z) = \sum_{\sigma\in S_\varepsilon} \sum_{0\leq j<m} \frac{(-1)^j}{j!} \sigma(h_m^{(j)}(\alpha)) F_{j,m}^\sigma(z)$$

and

$$G_{m,j}(z) = \sum_{\sigma\in S_\varepsilon} \frac{(-1)^j}{j!} \sigma(h_m^{(j)}(\alpha)) \int_{O_\sigma(x,\varepsilon)} \frac{t^j}{(z-t)^m} \, \mathrm{d}\pi_x(t).$$

The goal of this section is to show that a Galois equivariant Krasner analytic function $G \in \mathcal{A}_0^{\mathcal{G}}(E(x,\varepsilon), \mathbb{C}_p)$ is in the adherence of the ring $\mathbb{Q}_p[F^{[1]}, F^{[2]}, \ldots, F^{[k]}, \ldots]$ with respect to the convergence uniform norm on $E(x,\varepsilon)$. For this it is enough to show that the functions $G_{m,j}$ are in the adherence of this ring, for any $m \geq 1$ and any $0 \leq j < m$. We have seen that all the numbers $\frac{(-1)^j}{j!}\sigma(h_m^{(j)}(\alpha))$ are algebraic integers over $\mathbb{Q}_p$, for any $m \geq 1$, any $0 \leq j < m$ and any $\sigma \in S_\varepsilon$. Denote

$$H_{m,j}(z) = \sum_{\sigma\in S_\varepsilon} \frac{(-1)^j}{j!} \sigma(h_m^{(j)}(\alpha)) \int_{O_\sigma(\alpha_n,\varepsilon)} \frac{u^j}{(z-u)^m} \, \mathrm{d}\pi_{\alpha_n}(u), \tag{8}$$

for any $m \geq 1$, any $0 \leq j < m$ and any $\sigma \in S_\varepsilon$. Here $O_\sigma(\alpha_n,\varepsilon) = \{\tau(\alpha_n) : \tau \in \sigma H(\alpha_n,\varepsilon)\}$ and π_{α_n} is the Haar distribution on the orbit $O(\alpha_n)$. By (4) we know that

$$H_{m,j} \in \frac{1}{(d_n-1)!} \mathbb{Z}_p[F_n^{[1]}, \ldots, F_n^{[m]}]. \tag{9}$$

In what follows we evaluate $|G_{m,j}(z)-H_{m,j}(z)|$. Because $\frac{(-1)^j}{j!}\sigma(h_m^{(j)}(\alpha))$ are algebraic integers for any $\sigma\in S_\varepsilon$, we have

$$\begin{aligned}&|G_{m,j}(z)-H_{m,j}(z)|\le\\ &\le\max_{\sigma\in S_\varepsilon}\left|\int_{O_\sigma(x,\varepsilon)}\frac{t^j}{(z-t)^m}\,\mathrm{d}\pi_x(t)-\int_{O_\sigma(\alpha_n,\varepsilon)}\frac{u^j}{(z-u)^m}\,\mathrm{d}\pi_{\alpha_n}(u)\right|.\end{aligned}\tag{10}$$

For the sake of simplicity we evaluate the module on the right-hand side of (10) for $\sigma=e$, which is the neutral element of $\mathcal{G}$. By [3, Corollary 3.7 and Remark 3.8], we infer that $N(x,\varepsilon_n)$ is a divisor of d_n and, moreover, $N(x,\varepsilon_n)$ divides $N(x,\varepsilon_{n+1})$ for any $n\ge 1$.

Denote $I=\int_{O_\sigma(x,\varepsilon)}\frac{t^j}{(z-t)^m}\,\mathrm{d}\pi_x(t)$ and $J=\int_{O_\sigma(\alpha_n,\varepsilon)}\frac{u^j}{(z-u)^m}\,\mathrm{d}\pi_{\alpha_n}(u)$. One has $J=\frac{N(x,\varepsilon)}{d_n}\sum_i\frac{(\alpha_n^{(i)})^j}{(z-\alpha_n^{(i)})^m}$, where the sum is made over all $\frac{d_n}{N(x,\varepsilon)}$ conjugates of α_n in $O_e(\alpha_n,\varepsilon)$.

Denote by I_n the Riemann sum that corresponds to the integral I for a cover of $O_e(x,\varepsilon)$ with balls of radius ε_n and with the intermediate points given by $\frac{N(x,\varepsilon)}{N(x,\varepsilon_n)}$ conjugates of x. After a simple calculation, we obtain

$$|I_n-J|\le\frac{1}{\varepsilon^{m+1}}\cdot\frac{\varepsilon_n}{|d_n|},$$

and

$$|I_{i+1}-I_i|\le\frac{1}{\varepsilon^{m+1}}\cdot\frac{\varepsilon_i}{|d_{i+1}|},$$

for any $i\ge n$. Then

$$\begin{aligned}|I-J|&=\big|(I-I_{n+k+1})+(I_{n+k+1}-I_{n+k})+\cdots\\ &\qquad\qquad+(I_{n+1}-I_n)+(I_n-J)\big|\\ &\le\frac{1}{\varepsilon^{m+1}}\cdot\max_{i\ge n}\frac{\varepsilon_i}{\inf\{|d_i|,|d_{i+1}|\}},\end{aligned}\tag{11}$$

for k large enough. By (11) and (10) we infer that

$$|G_{m,j}(z)-H_{m,j}(z)|\le\frac{1}{\varepsilon^{m+1}}\cdot\max_{i\ge n}\frac{\varepsilon_i}{\inf\{|d_i|,|d_{i+1}|\}},\tag{12}$$

for any $z\in E(x,\varepsilon)$, any $m\ge 1$ and any $0\le j<m$. The right-hand side of (12) exists by condition (5).

Proceeding in the same manner as in (12), we find that

$$|F_n^{[k]}(z)-F^{[k]}(z)|\ll\frac{1}{\varepsilon^{m+1}}\cdot\max_{i\ge n}\frac{\varepsilon_i}{\inf\{|d_i|,|d_{i+1}|\}},\tag{13}$$

uniformly in $z \in E(x, \varepsilon)$, for any $k \leq m$, where "$\ll$" means that the right-hand side of (12) is multiplied by an absolute constant. By (9), (12) and (13), we infer that

$$\left| G_{m,j}(z) - \frac{1}{(d_n - 1)!} P(F^{[1]}, F^{[2]}, \ldots, F^{[m]}) \right| \ll \frac{1}{\varepsilon^{m+1}} \cdot \max_{i \geq n} \frac{\varepsilon_i}{|d_i!| \cdot |d_{i+1}|},$$

which, by (5), tends to zero if $n \to \infty$. Here P is a multivariate polynomial with coefficients in $\mathbb{Z}_p$. In conclusion, we have obtained the following result.

THEOREM 4.2. *Let x be a transcendental element of $\mathbb{C}_p$ and $\{\alpha_n\}_{n \geq 1}$ a sequence of algebraic elements of $\overline{\mathbb{Q}}_p$ such that*

$$\lim_{n \to \infty} \frac{|x - \alpha_n|}{|d_n!| \cdot |d_{n+1}|} = 0, \tag{14}$$

where d_n is the degree of α_n, for any $n \geq 1$. Let F be the trace function of x. Then any Galois equivariant Krasner analytic function of $\mathcal{A}_0^{\mathcal{G}}(E(x, \varepsilon), \mathbb{C}_p)$ is in the closure of the ring $\mathbb{Q}_p[F, F', \ldots, F^{(n)}, \ldots]$, for any positive real number ε.

REMARK 4.3. For any sequence of algebraic numbers $\beta_1, \beta_2, \ldots, \beta_n, \ldots$, we can find a sequence of natural numbers $m_1, m_2, \ldots, m_n, \ldots$ such that the series $p^{m_1}\beta_1 + p^{m_2}\beta_2 + \cdots + p^{m_n}\beta_n + \cdots$ converges and such that the equality (14) holds true. Here we denote by x the sum of this series and by $\alpha_n = \sum_{i=1}^{n} p^{m_i}\beta_i$ the sum of the truncated series. The degrees $d_1, d_2, \ldots, d_n, \ldots$ of $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ are bounded by some expressions that depend on the degrees of $\beta_1, \beta_2, \ldots, \beta_n, \ldots$. So we are done, by taking the sequence $m_1, m_2, \ldots, m_n, \ldots$ to increase rapidly enough.

REMARK 4.4. From Theorem 4.2, it follows that for any transcendental element x of $\mathbb{C}_p$ that satisfies condition (14) and for any positive real number ε, the ensemble formed by the trace function of x and all its derivatives is a generating set over $\mathbb{Q}_p$ for the set $\mathcal{A}_0^{\mathcal{G}}(E(x, \varepsilon), \mathbb{C}_p)$, that is, for the set of Galois equivariant Krasner analytic functions defined on $E(x, \varepsilon)$ with values in $\mathbb{C}_p$.

REFERENCES

[1] V. ALEXANDRU – N. POPESCU – M. VÂJÂITU – A. ZAHARESCU, *Representation results for equivariant rigid analytic functions*, Algebr. Represent. Theory **15** (2012) pp. 137–145.

[2] V. Alexandru – N. Popescu – A. Zaharescu, *On the closed subfields of* $\mathbb{C}_p$, J. Number Theory **68** (1998), no. 2, pp. 131–150.

[3] V. Alexandru – N. Popescu – A. Zaharescu, *Trace on* $\mathbb{C}_p$, J. Number Theory **88** (2001), no. 1, pp. 13–48.

[4] V. Alexandru – N. Popescu – A. Zaharescu, *The generating degree of* $\mathbb{C}_p$, Canad. Math. Bull. **44** (2001), no. 1, pp. 3–11.

[5] V. Alexandru – N. Popescu – A. Zaharescu, *A representation theorem for a class of rigid analytic functions*, J. Théor Nombres Bordeaux **15** (2003), pp. 639–650.

[6] Y. Amice, *Les nombres p-adiques*, Presse Univ. de France, Collection Sup., 1975.

[7] E. Artin, *Algebraic Numbers and Algebraic Functions*, Gordon and Breach, New York, 1967.

[8] J. Ax, *Zeros of polynomials over local fields—The Galois action*, J. Algebra **15** (1970), pp. 417–428.

[9] J. Fresnel – M. van der Put, *Rigid Analytic Geometry and its Applications*, Birkhauser, 2004.

[10] A. Ioviţă – A. Zaharescu, *Completions of r.a.t.-Valued Fields of Rational Functions*, J. Number Theory **50** (1995), no. 2, pp. 202–205.

[11] A. Ioviţă – A. Zaharescu, *Generating elements for* B_{dR}^{+}, Kyoto J. Math. **39** (1999), no. 2, pp. 233–248.

[12] B. Mazur – P. Swinnerton-Dyer, *Arithmetic of Weil curves*, Invent. Math. **25** (1974), pp. 1–61.

[13] N. Popescu – M. Vâjâitu – A. Zaharescu, *On the existence of trace for elements of* $\mathbb{C}_p$, Algebr. Represent. Theory **9** (2006), pp. 47–66.

[14] A. M. Robert, *A course in p-adic analysis*, Springer-Verlag, New-York, 2000.

[15] S. Sen, *On automorphisms of local fields*, Ann. of Math. (2) **90** (1969), pp. 33–46.

[16] J. T. Tate, *p-divisible groups*, Proc. Conf. Local Fields (Driebergen, 1966) pp. 158–183 Springer, Berlin, 1967.

[17] M. Vâjâitu – A. Zaharescu, *Trace functions and Galois invariant p-adic measures*, Publ. Mat. **50** (2006), pp. 43–55.