Rend. Sem. Mat. Univ. Padova, DRAFT, 1–15

A natural fibration for rings

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Abstract – A ringed partially ordered set with zero is a pair (L, F), where L is a partially ordered set with a least element 0_L and $F\colon L \to \mathbf{Ring}$ is a covariant functor. Here the partially ordered set L is given a category structure in the usual way and **Ring** denotes the category of associative rings with identity. Let $\mathbf{RingedParOrd}_0$ be the category of ringed partially ordered sets with zero. There is a functor $\mathcal{H}\colon \mathbf{Ring} \to \mathbf{RingedParOrd}_0$ that associates to any ring R a ringed partially ordered set with zero $(\mathrm{Hom}(R), F_R)$. The functor $\mathcal{H}$ has a left inverse $Z\colon \mathbf{RingedParOrd}_0 \to \mathbf{Ring}$. The category $\mathbf{RingedParOrd}_0$ is a fibred category.

Mathematics Subject Classification (2010). Primary 18D30.

Keywords. Ringed space, Fibred category, Zariski spectrum of a ring.

Contents

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1. Introduction

This paper is devoted to an extension of the classical construction $R \mapsto (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ of the ringed space, from the case of a commutative ring R to the case of R an arbitrary, not-necessarily commutative, ring.

The second author and L. Heidari Zadeh introduced in [4] a contravariant functor $\mathrm{Hom}(-)\colon \mathbf{Ring} \to \mathbf{ParOrd}_0$ from the category **Ring** of rings to the category $\mathbf{ParOrd}_0$ of partially ordered sets with a least element 0. It associates to any (associative) ring R with identity the partially ordered set $\mathrm{Hom}(R)$ of all pairs $(\mathfrak{a}, M)$, where $\mathfrak{a}$ is the kernel of any ring morphism $\varphi\colon R \to S$, S any ring, and $M := \varphi^{-1}(U(S))$. Here $U(S)$ denotes the group of units of S. The partial order on $\mathrm{Hom}(R)$ is defined, for every $(\mathfrak{a}, M), (\mathfrak{a}', M') \in \mathrm{Hom}(R)$, by $(\mathfrak{a}, M) \leq (\mathfrak{a}', M')$ if $\mathfrak{a} \subseteq \mathfrak{a}'$ and $M \subseteq M'$. The least element of $\mathrm{Hom}(R)$ is the pair $(0, U(R))$. The category $\mathbf{ParOrd}_0$ is a full subcategory of the category **ParOrd** of all partially ordered sets.

Several properties of the functor $\mathrm{Hom}(-)$ are studied in [4]. For instance, the subset $\mathrm{Max}(R)$ of all maximal elements of the partially ordered set $\mathrm{Hom}(R)$ plays an important role. For example, when the ring R is commutative, the set $\mathrm{Max}(R)$ is in one-to-one correspondence with the Zarisky spectrum $\mathrm{Spec}(R)$ of R. Notice that the functor $\mathrm{Spec}\colon \mathbf{CRing} \to \mathbf{ParOrd}$ is also contravariant, like $\mathrm{Hom}(-)\colon \mathbf{Ring} \to \mathbf{ParOrd}_0$. Here **CRing** denotes the full subcategory of **Ring** of all commutative rings. Hence $\mathrm{Max}(R)$ can be used as a good substitute for the spectrum of a possibly non-commutative ring R, though the assignment $R \mapsto \mathrm{Max}(R)$ is not a contravariant functor [4, Theorem 5.7]. This is not quite surprising, because, in the commutative case, the maximal spectrum, i.e., the topological subspace of $\mathrm{Spec}(R)$ whose elements are all maximal ideals of the commutative ring R, is not a functor either.

For any commutative ring R, the Zariski spectrum $\mathrm{Spec}(R)$ is a topological space and it is possible to construct the ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$, where the stalk of the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ at any point P in $\mathrm{Spec}(R)$ is the localization R_P of R at the prime ideal P. In this paper, we adapt the construction of ringed space from the functor $\mathrm{Spec}\colon \mathbf{CRing} \to \mathbf{ParOrd}$ to the functor $\mathrm{Hom}(-)\colon \mathbf{Ring} \to \mathbf{ParOrd}_0$. The idea is to associate to every point $(\mathfrak{a}, M)$ in the set $\mathrm{Hom}(R)$ the "localization" $S_{(R/\mathfrak{a}, M/\mathfrak{a})}$ of the ring R at the point $(\mathfrak{a}, M)$. To this end, we introduce ringed partially ordered sets with zero. A *ringed partially ordered set with zero* is a pair (L, F), where L is a partially ordered set with zero and $F\colon L \to \mathbf{Ring}$ is a covariant functor. The partially ordered set L is given a category structure in the usual way (there is at most one morphism $\ell \to \ell'$ for every $\ell, \ell' \in L$, and such a morphism exists if and only if $\ell \leq \ell'$ in L).

Ringed partially ordered sets with zero are the objects of a category $\mathbf{RingedParOrd}_0$. There are canonical functors

$$\mathbf{Ring} \underset{Z}{\overset{\mathcal{H}}{\rightleftarrows}} \mathbf{RingedParOrd}_0 .$$

The functor $\mathcal{H}$ acts on a ring R sending it to the pair $(\mathrm{Hom}(R), F_R)$, where $F_R\colon \mathrm{Hom}(R) \to \mathbf{Ring}$ is the covariant functor that maps a pair $(\mathfrak{a}, M)$ to the stalk $S_{(R/\mathfrak{a}, M/\mathfrak{a})}$. Here we follow the notation in [4], so that $S_{(R/\mathfrak{a}, M/\mathfrak{a})}$ is the localization obtained from the factor ring $R/\mathfrak{a}$ formally inverting the elements in the cancellative submonoid $M/\mathfrak{a}$ of $R/\mathfrak{a}$.

The main aim of this paper is to study the ringed partially ordered set with zero associated to a ring R and the functors $\mathcal{H}$ and Z. The functor Z acts on objects sending a ringed partially ordered set (L, F) with zero to the ring $F(0_L)$. We get that $Z \circ \mathcal{H} = 1_{\mathbf{Ring}}$. The contravariant functors $\mathcal{H}$ and Z are not a pair of adjoint functors neither on the right nor on the left.

In the last section of the paper, we consider fibrations (fibred categories). There is a natural fibration

$$\begin{array}{rccc} p\colon & \mathbf{RingedParOrd}_0 & \longrightarrow & \mathbf{ParOrd}_0 \\ & (L, F) & \longmapsto & L. \end{array}$$

To prove it, we give a sufficient condition on a morphism in $\mathbf{RingedParOrd}_0$ to be cartesian with respect to the functor p, and show the existence of cartesian liftings in $\mathbf{RingedParOrd}_0$ with respect to p.

In this paper, all rings are associative rings R with identity $1_R \neq 0_R$. For any ring R, $U(R)$ denotes the group of invertible elements of R.

2. Ringed partially ordered sets

As we have already said in the Introduction, we will denote by $\mathbf{ParOrd}_0$ the full subcategory of $\mathbf{ParOrd}$ whose objects are all partially ordered sets L with a least element 0_L. If L, M are partially ordered sets, the morphisms in $\mathbf{ParOrd}$ from L to M are the mappings $f\colon L \to M$, such that $\ell, \ell' \in L$ and $\ell \leq \ell'$ implies $f(\ell) \leq f(\ell')$. Partially ordered sets can be viewed as categories with at most one arrow between any two objects, and in which any two isomorphic objects are equal. Morphisms $L \to M$ in $\mathbf{ParOrd}$ correspond to functors $L \to M$.

A *ringed partially ordered set with zero* is a pair (L, F), where L is a partially ordered set with zero and $F\colon L \to \mathbf{Ring}$ is a covariant functor. Ringed partially ordered sets with zero are the objects of a category $\mathbf{RingedParOrd}_0$. Given any two objects $(L, F), (M, G)$ in $\mathbf{RingedParOrd}_0$, a morphism $(f, \eta)\colon (L, F) \to (M, G)$ is a pair (f, η), where $f\colon L \to M$ is a morphism in $\mathbf{ParOrd}$ and $\eta\colon G \circ f \to F$ is a natural transformation of functors:

$$L \underset{F}{\overset{G\circ f}{\rightrightarrows}} \; \Downarrow \eta \; \mathbf{Ring}\,.$$

Thus η consists of an indexed family $(G(f(a)) \xrightarrow{\eta_a} F(a))_{a \in L}$ of morphisms in $\mathbf{Ring}$ such that, for every pair of elements $a, b \in L$ with $a \le b$, we have that $\eta_b \circ G(f(\le_{a,b})) = F(\le_{a,b}) \circ \eta_a$, i.e., the diagrams

$$\begin{array}{ccc}
G(f(a)) & \xrightarrow{G(f(\le_{a,b}))} & G(f(b)) \\
{\scriptstyle \eta_a}\downarrow & & \downarrow{\scriptstyle \eta_b} \\
F(a) & \xrightarrow[F(\le_{a,b})]{} & F(b)
\end{array}$$

commute.

Morphisms of ringed partially ordered sets compose as follows. Given morphisms

$$(f, \eta)\colon (L, F) \to (M, G) \quad \text{and} \quad (g, \varepsilon)\colon (M, G) \to (N, H),$$

of ringed partially ordered sets, the composite morphism $(g, \varepsilon) \circ (f, \eta)$ is the ordered pair $(g \circ f, \eta \circ \varepsilon')$, where ε' is the natural transformation between the functors $H \circ (g \circ f), G \circ f\colon L \to \mathbf{Ring}$ defined by the position $\varepsilon'_l = \varepsilon_{f(l)}$, for every $l \in L$.

The identity morphism of any ringed partially ordered set (L, F) is the pair $(1_L, 1_F)$, where 1_L is the identity of the partially ordered set L and 1_F is the identity natural transformation of the functor F.

3. The contravariant functor $\mathcal{H}\colon \mathbf{Ring} \to \mathbf{RingedParOrd_0}$.

We will now define a contravariant functor $\mathcal{H}\colon \mathbf{Ring} \to \mathbf{RingedParOrd}_0$ similar to the construction, for every commutative ring R, of the ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. We need some results proved in [4].

Let R be any ring and N be a subset of R. Let $X := \{ x_n \mid n \in N \}$ be a set of non-commuting indeterminates in one-to-one correspondence with the set N. Let $R\{X\}$ be the free R-ring over X ([1] and [5, Example 1.9.20 on Page 124]). There are a canonical ring morphism $\varphi\colon R \to R\{X\}$ and a mapping $\varepsilon\colon X \to R\{X\}$ such that, for every ring S, every ring morphism $\psi\colon R \to S$ and every mapping $\zeta\colon X \to S$, there is a unique ring morphism $\widetilde{\psi}\colon R\{X\} \to S$ with $\psi = \widetilde{\psi}\varphi$ and $\zeta = \widetilde{\psi}\varepsilon$.

Let I be the two-sided ideal of $R\{X\}$ generated by the subset $\{ x_n n - 1 \mid n \in N \} \cup \{ n x_n - 1 \mid n \in N \}$. Set $S_{(R,N)} := R\{X\}/I$. In general, I could clearly be the improper ideal of $R\{X\}$ and $S_{(R,N)}$ could be the zero ring. There is a canonical mapping $\chi_{(R,N)}\colon R \to S_{(R,N)}$, composite mapping of $\varphi\colon R \to R\{X\}$ and the canonical projection $R\{X\} \to R\{X\}/I$. The R-ring $R\{X\}/I$ is the universal N-inverting R-ring in the sense of [3, Proposition 1.3.1].

Theorem 3.1. [4, Theorem 3.3] *Let R be a ring and $(\mathfrak{a}, M)$ be an element of* $\mathrm{Hom}(R)$. *Then $S_{(R/\mathfrak{a},M/\mathfrak{a})}$ is a non-zero ring, and if $\psi\colon R \to S_{(R/\mathfrak{a},M/\mathfrak{a})}$ denotes the composite mapping of the canonical projection $\pi\colon R \to R/\mathfrak{a}$ and $\chi_{(R/\mathfrak{a},M/\mathfrak{a})}\colon R/\mathfrak{a} \to S_{(R/\mathfrak{a},M/\mathfrak{a})}$, then*

$$\ker(\psi) = \mathfrak{a} \qquad \text{and} \qquad \psi^{-1}(U(S_{(R/\mathfrak{a},M/\mathfrak{a})})) = M.$$

Moreover, for any ring morphism $f\colon R \to S$ such that $\ker(f) \supseteq \mathfrak{a}$ and $f^{-1}(U(S)) \supseteq M$, there is a unique ring morphism $g\colon S_{(R/\mathfrak{a},M/\mathfrak{a})} \to S$ such that $g\psi = f$.

Theorem 3.1 shows that, for any pair $(\mathfrak{a}, M)$ in $\mathrm{Hom}(R)$, there is a canonical ring morphism $\psi\colon R \to S_{(R/\mathfrak{a},M/\mathfrak{a})}$ that realizes the pair $(\mathfrak{a}, M)$. It is easily seen that:

Theorem 3.2. *There is a covariant functor $F_R\colon$* $\mathrm{Hom}(R) \to$ **Ring** *that maps any object $(\mathfrak{a}, M)$ of* $\mathrm{Hom}(R)$ *to the localization $S_{(R/\mathfrak{a},M/\mathfrak{a})}$ (notation as in Theorem 3.1) and any morphism $(\mathfrak{a}, M) \to (\mathfrak{a}', M')$ in* $\mathrm{Hom}(R)$ *to the unique ring morphism $h\colon S_{(R/\mathfrak{a},M/\mathfrak{a})} \to S_{(R/\mathfrak{a}',M'/\mathfrak{a}')}$ making the diagram*

$$\begin{array}{ccc} & R & \\ {\scriptstyle\psi}\swarrow & & \searrow{\scriptstyle\psi'} \\ S_{(R/\mathfrak{a},M/\mathfrak{a})} & \xrightarrow[h]{} & S_{(R/\mathfrak{a}',M'/\mathfrak{a}')} \end{array} \tag{1}$$

commute.

We will now define a contravariant functor $\mathcal{H}\colon\ \mathbf{Ring} \to \mathbf{RingedParOrd}_0$. It associates to any object R of **Ring** the ringed partially ordered set with zero $(\mathrm{Hom}(R), F_R)$. In order to define how $\mathcal{H}$ acts on morphisms, recall that $\mathrm{Hom}(-)\colon\ \mathbf{Ring} \to \mathbf{ParOrd}$ is a contravariant functor [4]. For any ring morphism $f\colon R \to T$, the mapping $\mathrm{Hom}(f)\colon \mathrm{Hom}(T) \to \mathrm{Hom}(R)$ is the partially ordered set morphism defined by $\mathrm{Hom}(f)(\mathfrak{b}, N) = (f^{-1}(\mathfrak{b}), f^{-1}(N))$ for every $(\mathfrak{b}, N) \in \mathrm{Hom}(T)$. The following theorem shows that we have a canonically defined ringed partially ordered set morphism $(\mathrm{Hom}(f), \eta(f))$.

Theorem 3.3. *For any ring morphism $f\colon R \to T$, there is a natural transformation of functors $\eta(f)\colon F_R \circ \mathrm{Hom}(f) \to F_T$. It assigns to every $(\mathfrak{b}, N) \in \mathrm{Hom}(T)$ the ring morphism*

$$\eta(f)_{(\mathfrak{b},N)}\colon F_R(\mathrm{Hom}(f)(\mathfrak{b}, N)) \to F_T(\mathfrak{b}, N)$$

that is the canonical mapping $\eta(f)_{(\mathfrak{b},N)}\colon S_{(R/f^{-1}(\mathfrak{b}),f^{-1}(N)/f^{-1}(\mathfrak{b}))} \to S_{(T/\mathfrak{b},N/\mathfrak{b})}$:

$$\mathrm{Hom}(T) \underset{F_T}{\overset{F_R\circ\mathrm{Hom}(f)}{\rightrightarrows}} \Downarrow \eta(f)\ \mathbf{Ring}\,.$$

Proof. It suffices to verify that, for every $(\mathfrak{b}, N) \leq (\mathfrak{b}', N') \in \mathrm{Hom}(T)$, the diagram

$$\begin{array}{ccc} S_{(R/f^{-1}(\mathfrak{b}),\, f^{-1}(N)/f^{-1}(\mathfrak{b}))} & \longrightarrow & S_{(R/f^{-1}(\mathfrak{b}'),\, f^{-1}(N')/f^{-1}(\mathfrak{b}'))} \\ {\scriptstyle \eta(f)_{(\mathfrak{b},N)}}\big\downarrow & & \big\downarrow{\scriptstyle \eta(f)_{(\mathfrak{b}',N')}} \\ S_{(T/\mathfrak{b},\, N/\mathfrak{b})} & \xrightarrow[F_T(\leq_{(\mathfrak{b},N),(\mathfrak{b}',N')})]{} & S_{(T/\mathfrak{b}',\, N'/\mathfrak{b}')} \end{array}$$

commutes. The verification is straightforward from the universal properties of the universal inverting R-rings. □

The functor $\mathcal{H}$ associates to any ring morphism $f\colon R \to T$ the ordered pair

$$(\mathrm{Hom}(T) \xrightarrow{\mathrm{Hom}(f)} \mathrm{Hom}(R), F_R \circ \mathrm{Hom}(f) \xrightarrow{\eta(f)} F_T).$$

In order to check that in this way we have well defined a functor $\mathcal{H}$, we only verify here in detail that, given rings R, T and V and ring morphisms $f\colon R \to T$, $g\colon T \to V$, we have that

$$\mathcal{H}(g \circ f) = \mathcal{H}(f) \circ \mathcal{H}(g). \tag{2}$$

Now $\mathcal{H}(g \circ f) = (\mathrm{Hom}(g \circ f), \eta(g \circ f))$ and

$$\begin{aligned}\mathcal{H}(f) \circ \mathcal{H}(g) &= (\mathrm{Hom}(f), \eta(f)) \circ (\mathrm{Hom}(g), \eta(g)) \\ &= (\mathrm{Hom}(f) \circ \mathrm{Hom}(g), \eta(g) \circ \eta(f)'). \end{aligned} \tag{3}$$

As $\mathrm{Hom}(-)\colon$ **Ring ParOrd** is a contravariant functor, in order to verify (2), we only need to check that

$$\eta(g \circ f) = \eta(g) \circ \eta(f)'. \tag{4}$$

Here the natural transformation of functors $\eta(f)'$ is defined by the position

$$\eta(f)'_{(\mathfrak{c},O)} = \eta(f)_{\mathrm{Hom}(g)(\mathfrak{c},O)}$$

for every $(\mathfrak{c}, O) \in \mathrm{Hom}(V)$. Hence, we have the natural transformations $\eta(g \circ f)\colon F_R \circ \mathrm{Hom}(g \circ f) \to F_V$ (these are functors $\mathrm{Hom}(V) \to$ **Ring**), $\eta(f)\colon F_R \circ \mathrm{Hom}(f) \to F_T$ (two functors $\mathrm{Hom}(T) \to$ **Ring**) and $\eta(g)\colon F_T \circ \mathrm{Hom}(g) \to F_V$ (functors $\mathrm{Hom}(V) \to$ **Ring**). If we compose with $\mathrm{Hom}(g)$ the second of these three natural tranformations, we get the natural transformation $\eta(f)'\colon F_R \circ \mathrm{Hom}(f) \circ \mathrm{Hom}(g) \to F_T \circ \mathrm{Hom}(g)$ (functors $\mathrm{Hom}(V) \to$ **Ring**). Hence we have that $\eta(g \circ f) = \eta(g) \circ \eta(f)'$.

4. A contravariant functor $Z\colon$ RingedParOrd$_0$ → Ring

We now define a contravariant functor

$$Z\colon \mathbf{RingedParOrd}_0 \to \mathbf{Ring}\,.$$

The functor Z maps any ringed partially ordered set with zero (L, F) to the ring $F(0_L)$ and maps a morphism of ringed partially ordered sets with zero $(f, \eta)\colon (L, F) \to (M, G)$ to the composite ring morphism

$$G(0_M) \xrightarrow{G(\leq_{0_M, f(0_L)})} G(f(0_L)) \xrightarrow{\eta_{0_L}} F(0_L).$$

To check that Z is a functor, let $(L, F), (M, G)$ and (N, H) be ringed partially ordered sets with zero and $(f, \eta)\colon (L, F) \to (M, G), (g, \varepsilon)\colon (M, G) \to (N, H)$ be morphisms. In order to verify that

$$Z((g, \varepsilon) \circ (f, \eta)) = Z(f, \eta) \circ Z(g, \varepsilon), \tag{5}$$

notice that

$$
\begin{aligned}
Z((g,\varepsilon)\circ(f,\eta)) &= Z(g\circ f,\eta\circ\varepsilon')\\
&= (\eta\circ\varepsilon')_{0_L}\circ H(\leq_{0_N,g(f(0_L))})\\
&= \eta_{0_L}\circ\varepsilon_{f(0_L)}\circ H(\leq_{0_N,g(f(0_L))}). \qquad (6)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
Z(f,\eta)\circ Z(g,\varepsilon) &= (\eta_{0_L}\circ G(\leq_{0_M,f(0_L)}))\circ(\varepsilon_{0_M}\circ H(\leq_{0_N,g(0_M)}))\\
&= \eta_{0_L}\circ G(\leq_{0_M,f(0_L)})\circ\varepsilon_{0_M}\circ H(\leq_{0_N,g(0_M)}). \qquad (7)
\end{aligned}
$$

Thus the equality in (5) holds, because

$$
\begin{aligned}
&\eta_{0_L}\circ\varepsilon_{f(0_L)}\circ H(\leq_{0_N,g(f(0_L))}) = && (8)\\
&= \eta_{0_L}\circ\varepsilon_{f(0_L)}\circ H(\leq_{g(0_M),g(f(0_L))}\circ\leq_{0_N,g(0_M)}) && (9)\\
&= \eta_{0_L}\circ\varepsilon_{f(0_L)}\circ H(\leq_{g(0_M),g(f(0_L))})\circ H(\leq_{0_N,g(0_M)}) && (10)\\
&= \eta_{0_L}\circ G(\leq_{0_M,f(0_L)})\circ\varepsilon_{0_M}\circ H(\leq_{0_N,g(0_M)}). && (11)
\end{aligned}
$$

The equality in (9) is due to the fact that $0_N\leq g(0_M)$, and $0_M\leq f(0_L)$ implies $g(0_M)\leq g(f(0_L))$ because g is monotone. The equalities in (10) and (11) are due to the fact that H respects composition and $\varepsilon\colon H\circ g\to G$ is a natural transformation, respectively. Hence we get that the diagram

$$
\begin{array}{ccccccc}
H(0_N) & \xrightarrow{H(\leq_{0_N,g(0_M)})} & H(g(0_M)) & \xrightarrow{\varepsilon_{0_M}} & G(0_M) & & \\
& \searrow^{H(\leq_{0_N,g(f(0_L))})} & \downarrow{\scriptstyle H(\leq_{g(0_M),g(f(0_L))})} & & \downarrow{\scriptstyle G(\leq_{0_M,f(0_L)})} & & \\
& & H(g(f(0_L))) & \xrightarrow[\varepsilon_{f(0_L)}]{} & G(f(0_L)) & \xrightarrow[\eta_{0_L}]{} & F(0_L).
\end{array}
$$

commutes. Therefore Z is a functor.

PROPOSITION 4.1. $Z\circ\mathcal{H}=1_{\mathbf{Ring}}$.

PROOF. On objects, we have that

$$
\begin{aligned}
Z\circ\mathcal{H}(R) &= Z(\mathrm{Hom}(R),F_R)\\
&= F_R(0_{\mathrm{Hom}(R)})\\
&= F_R(0,U(R))\\
&= S_{(R/0,U(R)/0)} = R.
\end{aligned}
$$

On morphisms, we have

$$
\begin{aligned}
&Z \circ \mathcal{H}\,(R \xrightarrow{f} T) = Z((\mathrm{Hom}(T), F_T) \xrightarrow{(\mathrm{Hom}(f),\eta(f))} (\mathrm{Hom}(R), F_R)) = \\
&= (F_R(0_{\mathrm{Hom}(R)}) \xrightarrow{F_R(\leq)} F_R(\mathrm{Hom}(f)(0_{\mathrm{Hom}(T)})) \xrightarrow{\eta(f)_{0_{\mathrm{Hom}(T)}}} F_T(0_{\mathrm{Hom}(T)})) \\
&= (F_R(0_R, U(R)) \to F_R(\ker(f), f^{-1}(U(T))) \to F_T(0_T, U(T))) \\
&= (R \to S_{(R/\ker(f), f^{-1}(U(T))/\ker(f))} \to T) \\
&= (R \xrightarrow{f} T).
\end{aligned}
$$

□

We will now show that the contravariant functors $\mathcal{H}$ and Z do not form a pair of adjoint functors, neither on the right nor on the left. Recall that if $\mathcal{C}$ and $\mathcal{D}$ are categories and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ are contravariant functors

$$\mathcal{D} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{C},$$

then F and G are said to be *adjoint on the right* if $\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(Y, F(X))$, naturally in X and Y, for every $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$. They are said to be *adjoint on the left* if $\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(G(Y), X)$, naturally in X and Y, for every $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.

Proposition 4.2. *The functors* $\mathbf{Ring} \underset{Z}{\overset{\mathcal{H}}{\rightleftarrows}} \mathbf{RingedParOrd}_0$ *are not adjoint on the right.*

Proof. It suffices to exhibit a ring R and a ringed partially ordered set (L, F) for which there is not a bijection between the sets

(12)

$$\mathrm{Hom}_{\mathbf{Ring}}(R, F(0_L)) \quad \text{and} \quad \mathrm{Hom}_{\mathbf{RingedParOrd}_0}((L, F), (\mathrm{Hom}(R), F_R)).$$

Set $R := \mathbb{Z}$. Let $L := \{*\}$ be the partially ordered set with one element $*$ and

$$(L, F) := (\{*\}, \{*\} \xrightarrow{F} \mathbf{Ring}),$$

where $F\colon \{*\} \to \mathbf{Ring}$ sends the element $*$ of L to $\mathbb{Q}$. Then the left hand side of (12) is

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{Ring}}(R, F(0_L)) &= \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}, F(*)) \\
&= \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}, \mathbb{Q}) = \{\mathbb{Z} \hookrightarrow \mathbb{Q}\},
\end{aligned}
$$

i.e., the left hand side of (12) is in this case a set of cardinality one. For the right hand side of (12), we have that

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{RingedParOrd}_0}((L,F),(\mathrm{Hom}(R),F_R)) = \\ &\qquad = \mathrm{Hom}_{\mathbf{RingedParOrd}_0}((\{*\}, * \mapsto \mathbb{Q}),(\mathrm{Hom}(\mathbb{Z}),F_{\mathbb{Z}})).\end{aligned}$$

Recall that an element of $\mathrm{Hom}_{\mathbf{RingedParOrd}_0}((\{*\}, * \mapsto \mathbb{Q}),(\mathrm{Hom}(\mathbb{Z}),F_{\mathbb{Z}}))$ is an ordered pair (f,η), where $f\colon \{*\} \to \mathrm{Hom}(\mathbb{Z})$ is a partially ordered set morphism and $\eta\colon F_{\mathbb{Z}} \circ f \to F$ is a natural transformation:

$$\{*\} \underset{F}{\overset{F_{\mathbb{Z}}\circ f}{\rightrightarrows}} \Downarrow \eta \ \mathbf{Ring}\,.$$

Such a natural transformation is exactly a ring morphism

$$\eta_*\colon F_{\mathbb{Z}}(\mathfrak{a}, M) \to \mathbb{Q},$$

where $(\mathfrak{a}, M) = f(*)$. Now for every prime $p \in \mathbb{Z}$, let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at its maximal ideal (p). The pair $(0, \mathbb{Z} \setminus (p))$ is in $\mathrm{Hom}(\mathbb{Z})$, because it is the pair that corresponds to the ring morphism $\mathbb{Z} \hookrightarrow \mathbb{Z}_{(p)}$. In this point $(0, \mathbb{Z} \setminus (p))$ of $\mathrm{Hom}(\mathbb{Z})$, we have that

$$F_{\mathbb{Z}}(\mathfrak{a}, M) = F_{\mathbb{Z}}(0, \mathbb{Z} \setminus (p)) = S_{(\mathbb{Z}, \mathbb{Z}\setminus(p))} = \mathbb{Z}_{(p)},$$

so that there is exactly one morphism $\eta_*\colon F_{\mathbb{Z}}(\mathfrak{a}, M) \to \mathbb{Q}$. Hence, in

$$\mathrm{Hom}_{\mathbf{RingedParOrd}_0}((L,F),(\mathrm{Hom}(R),F_R)),$$

there is at least one element (f,η) for each prime p. Thus

$$\mathrm{Hom}_{\mathbf{RingedParOrd}_0}((L,F),(\mathrm{Hom}(R),F_R))$$

is an infinite set. Therefore there cannot be a bijection between the two sets in (12). □

Remark 4.3. Note that the homomorphisms from $(L,F) = (\{*\}, * \mapsto \mathbb{Q})$ to $(\mathrm{Hom}(\mathbb{Z}), F_{\mathbb{Z}})$ are in one-to-one correspondence with pairs $(\mathbf{0}, M)$, i.e., localizations of $\mathbb{Z}$.

To conclude this section, we show that the functors $\mathcal{H}$ and Z do not form a pair of adjoint functors on the left.

Proposition 4.4. *The functors* $\mathbf{Ring} \underset{Z}{\overset{\mathcal{H}}{\rightleftarrows}} \mathbf{RingedParOrd}_0$ *are not adjoint on the left.*

Proof. Since $Z \circ \mathrm{Hom}(-) = 1_{\mathbf{Ring}}$, the natural mapping

$$\mathrm{Hom}_{\mathbf{RingedParOrd}_0}((\mathrm{Hom}(R), F_R), (L, F)) \to \mathrm{Hom}_{\mathbf{Ring}}(F(0_L), R) \tag{13}$$

is the map that sends any morphism

$$(f, \eta) \in \mathrm{Hom}_{\mathbf{RingedParOrd}_0}((\mathrm{Hom}(R), F_R), (L, F))$$

to $Z(f, \eta)$. It suffices to show that this map is not injective for suitable objects R and (L, F). Let $R := \mathbb{Z}/p^2\mathbb{Z}$ be the ring with p^2 elements of characteristic p^2, p a prime, and set $(L, F) := (\mathrm{Hom}(R), F_R)$. Then the map (13) is the canonical monoid antihomomorphism

$$\mathrm{End}_{\mathbf{RingedParOrd}_0}(\mathrm{Hom}(R), F_R) \to \mathrm{End}_{\mathbf{Ring}}(F(0_{\mathrm{Hom}(R)}))$$

induced by the contravariant functor Z, and it suffices to show that this canonical antihomomorphism is not injective. Since it sends 1 to 1, it suffices to prove that there is a morphism

$$(f, \eta) \in \mathrm{Hom}_{\mathbf{RingedParOrd}_0}((\mathrm{Hom}(R), F_R), (L, F)),$$

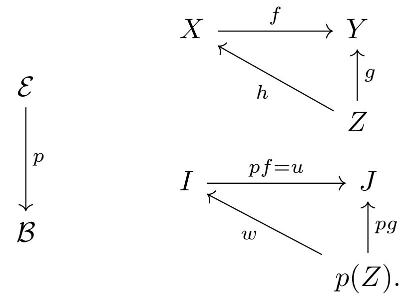
$(f, \eta) \neq 1$, with $Z(f, \eta) = 1$. For $R = \mathbb{Z}/p^2\mathbb{Z}$, the partially ordered set $\mathrm{Hom}(R)$ is the linearly ordered set $L = \{0, 1\}$ with two elements $0 = (0, U(R))$ and $1 = (pR, U(R))$. The structure functor $F_R \colon L \to \mathbf{Ring}$ maps 0 to R and 1 to R/pR. Let f be the partially ordered set endomorphism of L that maps both 0 and 1 to 0. Let $\eta \colon F_R \circ f \to F_R$ be the natural transformation defined by $\eta_0 \colon F_R \circ f(0) = R \to F_R(0) = R$, η_0 = "identity ring morphism $R \to R$", and $\eta_1 \colon F_R \circ f(1) = R \to F_R(1) = R/pR$, η_1 = "canonical projection $R \to R/pR$". Then $(f, \eta) \in \mathrm{Hom}_{\mathbf{RingedParOrd}_0}((\mathrm{Hom}(R), F_R), (L, F))$, $(f, \eta) \neq 1$ and $Z(f, \eta) = 1$. □

5. The fibration p: $\mathbf{RingedParOrd_0} \to \mathbf{ParOrd_0}$.

Let $p \colon \mathcal{E} \to \mathcal{B}$ be a functor. An object $X \in \mathrm{Ob}(\mathcal{E})$ such that $pX = I$ is said to *lie above* I. Similarly, a morphism f in $\mathcal{E}$ with $pf = u$ is said to *lie above* u. We now recall the two main definitions relative to this section.

Definition 5.1. Let $\mathcal{E}$ and $\mathcal{B}$ be categories and $p\colon \mathcal{E} \to \mathcal{B}$ a functor between them. A morphism $f\colon X \to Y$ in $\mathcal{E}$ is *cartesian* over a morphism $u\colon I \to J$ in $\mathcal{B}$ if $pf = u$ and for every $g\colon Z \to Y$ in $\mathcal{E}$ for which one has $pg = u \circ w$ for some $w\colon pZ \to I$, there exists a unique $h\colon Z \to X$ in $\mathcal{E}$ above w with $f \circ h = g$.

The situation is the following:



Definition 5.2. A functor $p\colon \mathcal{E} \to \mathcal{B}$ is a *fibration* if for every $Y \in \mathrm{Ob}(\mathcal{E})$ and every $u\colon I \to pY$ in $\mathcal{B}$, there is a cartesian morphism $f\colon X \to Y$ in $\mathcal{E}$ that lies above u.

The situation is the following:

$$\begin{array}{c} X \overset{f}{\dashrightarrow} Y \\ \\ I \xrightarrow{\ u\ } pY. \end{array}$$

Let us go back to our category $\mathbf{RingedParOrd}_0$. The projection functor

$$p\colon\ \mathbf{RingedParOrd}_0 \to \mathbf{ParOrd}_0$$

sends an object (L, F) of $\mathbf{RingedParOrd}_0$ to the partially ordered set L and a ringed partially ordered set morphism $(L, F) \xrightarrow{(f,\eta)} (M, G)$ to the partially ordered set morphism $f\colon L \to M$. We will show that p is a fibration.

First of all, we give a sufficient condition for a morphism in the category $\mathbf{RingedParOrd}_0$ to be cartesian with respect to the functor $p\colon\ \mathbf{RingedParOrd}_0 \to \mathbf{ParOrd}_0$.

Proposition 5.3. *Let $(f, \eta)\colon (L, F) \to (M, G)$ be a ringed partially ordered set morphism. If*

$$\eta_l\colon (G \circ f)(l) \to F(l)$$

is a ring isomorphism for every $l \in L$, then $(f, \eta)\colon (L, F) \to (M, G)$ is cartesian with respect to the functor $p\colon$ $\mathbf{RingedParOrd}_0 \to \mathbf{ParOrd}_0$.

Proof. Let $(f, \eta)\colon (L, F) \to (M, G)$ be a ringed partially ordered set morphism such that $\eta_l\colon (G \circ f)(l) \to F(l)$ is a ring isomorphism for every $l \in L$. We must prove that $(f, \eta)\colon (L, F) \to (M, G)$ is cartesian. Let $(N, H) \xrightarrow{(g,\varepsilon)} (M, G)$ be a ringed partially ordered set morphism such that g factors through f via $w\colon N \to L$, i.e., $g = f \circ w$:

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ & {}_{w}\nwarrow & \uparrow g \\ & & N. \end{array} \tag{14}$$

We must show that the factorization in (14) uniquely determines a ringed partially ordered set morphism $\bar{h} = (h_1, h_2)\colon (N, H) \to (L, F)$ that lies above w and makes the diagram

$$\begin{array}{ccc} (L, F) & \xrightarrow{(f,\eta)} & (M, G) \\ & {}_{\bar{h}=(h_1,h_2)}\nwarrow & \uparrow (g,\varepsilon) \\ & & (N, H). \end{array}$$

commute. The ringed partially ordered set morphism $\bar{h} = (h_1, h_2)$ lies above w if and only if $h_1 = w$. Since $g = f \circ w$, it suffices to show that the factorization in (14) uniquely determines a natural transformation of functors $h_2\colon F \circ w \to H$ such that

$$h_2 \circ \eta' = \varepsilon. \tag{15}$$

Equivalently, we must have $h_{2_n} \circ \eta'_n = \varepsilon_n$ for every $n \in N$. That is, the composite mapping

$$Gfwn \xrightarrow{\eta'_n = \eta_{wn}} Fwn \xrightarrow{h_{2_n}} Hn \tag{16}$$

must be equal to

$$Ggn \xrightarrow{\varepsilon_n} Hn \qquad \text{for every } n \in N. \tag{17}$$

Since $Gfwn = Ggn$, the ring morphisms in (16) and in (17) have the same domain and codomain. Moreover, $\eta'_n = \eta_{wn}$ is a ring isomorphism, so (16) and (17) are equivalent to

$$h_{2_n} = \varepsilon_n \circ (\eta'_n)^{-1}.$$

Therefore the ringed partially ordered set morphism $\bar{h} = (h_1, h_2)\colon (N, H) \to (L, F)$ is uniquely determined, and

$$\bar{h} = (w, \varepsilon \circ (\eta')^{-1}).$$

□

The functor $p\colon \mathbf{RingedParOrd}_0 \to \mathbf{ParOrd}_0$ admits cartesian liftings, as the following Proposition shows.

PROPOSITION 5.4. *Given any ringed partially ordered set (M, G) and any partially ordered set morphism $f\colon L \to p(M, G) = M$, there exists a canonical lifting of f to a cartesian ringed partially ordered set morphism $f^{\uparrow}$. The morphism $f^{\uparrow}\colon (L, G \circ f) \to (M, G)$ is the ordered pair $(f, 1_{G\circ f})$.*

The situation is the following:

$$(L, G \circ f) \overset{(f, 1_{G\circ f})}{\dashrightarrow} (M, G)$$

$$L \xrightarrow{\quad f \quad} p(M, G) = M.$$

PROOF. The morphism $(f, 1_{G\circ f})$ lies above f. By definition, for the natural transformation of functors $1_{G\circ f}$, we have that

$$1_{G\circ f_l} = 1_{G(f(l))}\colon G(f(l)) \longrightarrow G(f(l)).$$

□

From Proposition 5.4, we get that:

THEOREM 5.5. *The functor $p\colon \mathbf{RingedParOrd}_0 \to \mathbf{ParOrd}_0$ is a fibration.*

REMARKS 5.6. (1) The idea of our construction of ringed partially ordered sets was touched upon in the last paragraph of Section 4 in [4].

(2) An approach, somehow similar to ours, appears in [6].

(3) Further details of our proofs are available in [2]. The results in this paper are essentially contained in that Master Thesis.

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