On finite $p$-groups minimally of class greater than two

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Abstract – Let $G$ be a finite nilpotent group of class three whose proper subgroups and proper quotients are nilpotent of class at most two. We show that $G$ is either a 2-generated $p$-group or a 3-generated 3-group. In the first case the groups of maximal order with respect to a given exponent are all isomorphic except in the cases where $p = 2$ and $\exp(G) = 2^r$, $r \geq 4$. If $G$ is 3-generated, then we show that there is a unique group of maximal order and exponent 3; but a similar result is not valid for exponent 9.

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1. Introduction

Let $\mathcal{K}$ be a class of finite groups. The finite group $G$ is called a minimal non-$\mathcal{K}$-group (we write $G \in Min(\mathcal{K})$), if $G \not\in \mathcal{K}$ but every proper subgroup and every proper quotient of $G$ belongs to $\mathcal{K}$.

For the class $\mathcal{K} = A$ of all abelian groups, the structure of the groups in $Min(A)$ can easily be derived from results of Miller-Moreno and Rédei (see [1, p. 281] and [1, p. 309]) and Lemma 2.1 below. Indeed, it is easy to see that such a group $G$ is either a semidirect product $G = [N]Q$ of a minimal normal subgroup $N$ by a complement $Q$ of prime order, or it is one of the following groups:
Preliminaries

The quaternion group

Suppose that $A$ is a finite nilpotent group of class $c$, $c$ normal subgroups, then $G$ is of order $2^c$.

Proof. Suppose that $G$ has class $c$ and that all proper quotients of $G$ are of class

There exists precisely one $p$-group $G \in Min(A)$ of exponent $p^r$. If $p^r = 4$ we get two groups: the dihedral group $D_4$ and the quaternion group $Q_8$, while the case $p^r = 2$ does not allow any such group.

In this paper, we discuss the minimal non-$N_2$-groups, where $N_2$ denotes the class of all nilpotent groups of class $\leq 2$. The structure of non-nilpotent groups $G \in Min(N_2)$ follows immediately from the aforementioned results of Miller-Moreno and Rédei. Hence we will restrict attention to finite $p$-groups.

We prove that the $p$-groups in $Min(N_2)$ are either 2-generated or 3-generated 2-Engel. In order to give information on the $p$-groups in $Min(N_2)$ we determine the structure of the 2-generated free groups in the variety $W$ of all nilpotent groups of exponent $p^r$ $(r \geq 2)$ and class three, satisfying the law $[x, y, z]_p = 1$, and the structure of the 3-generated free groups in the variety $V$ of all 2-Engel groups of exponent $3^r$. We prove that there is a unique 2-generated group of exponent $p$ in $Min(N_2)$: its order is $p^3$ with $p \geq 5$. If $G$ is a 2-generated group in $Min(N_2)$ of exponent $p^r$ with $r \geq 2$ and $p$ odd we see that $|G| \leq p^{p^r}$; if $p = 2$, $r \geq 3$ then $|G| \leq 2^{3(r-1)}$; and if $p = 2$ and $r = 2$ then $|G| \leq 2^{3^{r-1}}$. We give an explicit construction of the groups in $Min(N_2)$ of exponent $p^r$ and maximal order and we show that such groups are all isomorphic except in the case $p = 2$ and $r \geq 4$. If $G$ is a 3-generated group of exponent 3 in $Min(N_2)$, we show that $|G| = 3^7$ and $G$ is isomorphic to the 3-generated relatively free group in the variety of all groups of exponent 3 but the groups of of exponent 9 of maximal order in $Min(N_2)$ are not isomorphic.

In the following the notation is standard. $G = [N]Q$ indicates the semidirect product of the normal subgroup $N$ by the subgroup $Q$, and $d(G)$ indicates the minimal number of generators of $G$. Moreover $o(x)$ is the order of the element $x$. If $V$ is a variety, $Fr_n(V)$ denotes the relatively free group of rank $n$ in $V$.

All groups considered in this paper are finite.

2. Preliminaries

Lemma 2.1. A finite nilpotent group of class $c (c \geq 2)$ has all of its proper quotients of class at most $c - 1$ if and only if $Z(G)$ is cyclic and the $c$-th term of the lower central series $c_c(G)$ is of order $p$.

Proof. Suppose that $G$ has class $c$ and that all proper quotients of $G$ are of class at most $c - 1$. Then $G$ is monolithic. Indeed, if $N_1$ and $N_2$ are two distinct minimal normal subgroups, then $G = G/N_1 \cap N_2$ is embedded in $G/N_1 \times G/N_2$ which is nilpotent of class at most $c - 1$. As $G$ is monolithic, $Z(G)$ is cyclic. If $N$ is the
minimal normal subgroup of $G$, then $G/N$ is nilpotent of class at most $c - 1$. So $\Gamma_c(G) = N$.

Conversely let $G$ be a nilpotent group of class $c$ and assume that $Z(G)$ is cyclic and $\Gamma_c(G)$ is of order $p$. Then for every normal subgroup $K$ of $G$, we have $\Gamma_c(G) \subseteq K$. So

\[ \Gamma_c(G/K) = \Gamma_c(G)K/K = 1. \]

q.e.d.

**Lemma 2.2.** Let $G$ be a nilpotent group such that all of its proper subgroups have class at most $c$ but $G$ has not class $c$. Then $Z(G) = N$.

**Proof.** Let $M$ be a maximal subgroup of $G$. Then $M \leq G$. Suppose that $Z(G) \not\subseteq M$. Then $G = Z(G)M$ and so $G$ has class $c$, a contradiction. q.e.d.

**Lemma 2.3.** Let $G$ be a $p$-group in $\text{Min}(N_2)$. Then either $G$ can be generated by two elements, or $G$ is a 2-Engel 3-group generated by three elements.

**Proof.** Suppose that $G$ cannot be generated by two elements. Then for all $x, y \in G$ we have that $\langle x, y \rangle$ is a proper subgroup of $G$. So it is nilpotent of class 2. In particular $G$ satisfies the 2-Engel condition. If $p \neq 3$ then $G$ is nilpotent of class two ([1, p. 288]), a contradiction. So $p = 3$. Moreover $G$ is generated by three elements, otherwise all subgroups generated by three elements would be proper subgroups of $G$, and $G$ would be nilpotent of class two, a contradiction. q.e.d.

We now give a sufficient criterion for a $p$-group generated by two elements to have all of its proper subgroups of class two.

**Lemma 2.4.** Let $G$ be a $p$-group which can be generated by two elements. Assume that $[\Phi(G), G] \leq Z(G)$. Then every proper subgroup of $G$ is nilpotent of class two.

**Proof.** It suffices to show that every maximal subgroup $M$ of $G$ is of class two. As $G$ is generated by two elements, we have $G/\Phi(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. So $M = \langle \Phi(G), x \rangle$ for some $x$ in $M$. We get $M' = \Phi(G)' \cdot [\Phi(G), x]$. By hypothesis, both factors are contained in $Z(G)$, so that the class of $M$ is two. q.e.d.

### 3. $\text{Min}(N_2)$-groups with two generators

We start with the smallest case:

**Proposition 3.1.** Let $G \in \text{Min}(N_2)$ be a group of prime exponent $p$. If $d(G) = 2$, then $p \geq 5$, $|G| = p^4$ and $G \cong [N,u]$, where $N = \langle v_1 \rangle \times \langle v_2 \rangle \times \langle v_3 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and the action of $u$ on $N$ is given by

\[ v_1^u = v_1, \quad v_2^u = v_1v_2, \quad v_3^u = v_2v_3. \]
As every proper subgroup of $G$ with free generators $y \in F$ is cyclic of order $p^2$. Moreover $G'/Γ_3(G)$ is cyclic of order $p$. So we get $|G| = p^4$.

Proof. As $exp(G) = p$, we infer that $p \neq 2$ and $|G/G'| = p^2$. Moreover $G'/Γ_3(G)$ is cyclic of order $p$ and by Lemma 2.1, we have $|Γ_3(G)| = p$. An inspection of the groups of order $p^4$ (see [1, p. 346]) yields the result. \hspace{1cm} q.e.d.

A group $G$ in $Min(N_2)$ of exponent $p^r$ belongs to the variety $W$ of all groups of exponent $p^r$ and nilpotent of class three satisfying the law $[x, y, z]^p = 1$ (see Lemma 2.1).

We now collect some information of $Fr_2(W)$:

**Proposition 3.2.** Let $p^r$ be a power of a prime $p$ and $r \geq 2$. Let $F = Fr_2(W)$ with free generators $x, y$. Then:

a) We have $F/F' ≃ \mathbb{Z}_{p^r} × \mathbb{Z}_{p^r}$ and either $|F'/Γ_3(F)| = p^r$ if $p \geq 3$ or $|F'/Γ_3(F)| = 2^{r-1}$. Moreover $Γ_3(F) ≃ \mathbb{Z}_p × \mathbb{Z}_p$ and hence $|F| = p^{3r+2}$ for $p \geq 3$ and $|F| = 2^{3r+1}$ if $p = 2$.

b) $Z(F) ≃ \begin{cases} \mathbb{Z}_{p^r-1} × \mathbb{Z}_p × \mathbb{Z}_p & \text{for } p \geq 3 \\ \mathbb{Z}_{2^{r-2}} × \mathbb{Z}_2 × \mathbb{Z}_2 × \mathbb{Z}_2 & \text{if } p = 2 \text{ and } r \geq 3 \\ \mathbb{Z}_2 × \mathbb{Z}_2 & \text{if } p = 2 \text{ and } r = 2 \end{cases}$

c) $|F^p, F| ≤ Z(F)$.

d) Every proper subgroup of $F$ is nilpotent of class two.

Proof. a) As $exp(F) = p^r$, we infer that $|F/F'| ≤ p^{2r}$. Moreover $F'/Γ_3(F) = ⟨[x, y]Γ_3(F)⟩$ is cyclic of exponent dividing $p^r$ if $p \neq 2$ and $2^{r-1}$ otherwise (we have

$$1 = (xy)^{2^r} ≡ x^{2^r}y^{2^r}[y, x]^{(x^2)} (mod Γ_3(F)),$$

so $[y, x]^{2^{r-1}} ≡ 1 (mod Γ_3(F))$. Then $|F'/Γ_3(F)| ≤ p^r$ if $p \neq 2$ or $≤ 2^{r-1}$ otherwise. Finally, we have $|Γ_3(F)| ≤ p^2$, because there are only two basic commutators of weight 3. This implies $|F| ≤ p^{3r+2}$ if $p \neq 2$, $|F| ≤ 2^{3r+1}$ otherwise.

We now construct a group $F_0$, belonging to the variety $W$, which has order either $p^{3r+2}$ if $p \geq 3$, or $2^{3r+1}$. So it will be $F_0 ≃ Fr_2(W)$.

Let $N = [A]⟨x⟩$ be the semidirect product of the abelian group $A = ⟨u⟩ × ⟨v_1⟩ × ⟨v_2⟩$,

with the cyclic group $⟨x⟩$ of order $p^r$; where $o(v_1) = o(v_2) = p$ and either $o(u) = p^r$ if $p \geq 3$, or $o(u) = 2^{r-1}$ otherwise.

The action of $x$ on $A$ is given by:

$$u^x = uv_1, \; v_1^x = v_1, \; v_2^x = v_2.$$

Then we consider the group $F_0 = [N]⟨y⟩$, where $y$ is a cyclic group of order $p^r$ and the action of $y$ on $N$ is given by:

$$x^y = xu, \; u^y = uv_2, \; v_1^y = v_1, \; v_2^y = v_2.$$
We can immediately verify that
\[ u = [x, y], \quad v_1 = [u, x] = [x, y, x], \quad v_2 = [u, y] = [x, y, y]. \]

So \( F_0 = \langle x, y \rangle \). Moreover
\[
F'_0 = A, \quad F_0' = \langle xF'_0 \rangle \times (yF'_0) \cong \mathbb{Z}_p \times \mathbb{Z}_{p'^r}, \quad F'_0/\Gamma_3(F_0) = \langle u\Gamma_3(F_0) \rangle,
\]

\[
\Gamma_3(F_0) = \langle v_1 \rangle \times (v_2) \cong \mathbb{Z}_p \times \mathbb{Z}_p, \quad \Gamma_3(F_0) \leq Z(F_0).
\]

We observe that, if \( p \geq 3 \), then \( \langle u\Gamma_3(F_0) \rangle \cong \mathbb{Z}_{p'^r} \), while if \( p = 2 \), then \( \langle u\Gamma_3(F_0) \rangle \cong \mathbb{Z}_{2^{r-1}} \). By the above conditions we deduce that \( F_0 \) is nilpotent of class three with \( |F_0| = p^{3r+2} \) if \( p \geq 3 \), while, if \( p = 2 \), then \( |F_0| = 2^{3r+1} \).

It remains to show that the exponent of \( F_0 \) is \( p^r \) for all \( p \).

First of all we prove that the exponent of \( N \) is \( p^r \) for all \( p \). (We note that for \( p \geq 3 \) we have \( \text{exp}(N) = p^r \), and for \( p \geq 5 \) we have \( \text{exp}(F_0) = p^r \) by the regularity of these groups).

Let \( w \in N \) where \( w = ax^k \) with \( a \in A \). Since \( N \) is of class two we have
\[
w^n = (ax^k)^n = a^n x^{kn} [x^k, a]^{(2)}.
\]

Since \([x^k, a]^{(2)} \in \Gamma_3(F_0)\) which has exponent \( p \) and \( r \geq 2 \), we have that
\[
[x^k, a]^{(2)} = 1. \quad \text{So } (ax^k)^{p^r} = 1.
\]

If \( w \) is an element of \( N \) we set
\[
[w, y^h] = a_1 \in A, \quad [a_1, w] = c_1 \in \Gamma_3(F_0), \quad [a_1, y^h] = c_2 \in \Gamma_3(F_0).
\]

For \( n \geq 2 \) it is easy to prove by induction the following results
\[
[w, y^{kn}] = a_1^n c_2^{(2)}
\]

and
\[
(wy^h)^n = w^n y^{hn} a_1^{(2)} c_1^{-1} c_2^{-2(2)} c_2^{-2(2)}.
\]

Since \( N \) is of exponent \( p^r \) and \( a \) has order \( 2^{r-1} \) for \( p = 2 \), we have by (2) that the exponent of \( F_0 \) is \( p^r \) for all \( p \).

From now on we identify \( F_0 \) with \( F \).

b) By the structure of \( F \) we can write an element \( z \in F \) in the form
\[
z = u^k v_1^{m} x^i y^j.
\]

We have \( z \in Z(F) \) if and only if \([z, x] = [z, y] = 1\). So
\[
1 = [z, y] = [u^k x^i y^j, y] = [u^k, y][u^k, x^i][x^i, y] = [u, y^k][x, y]^{(2)} v_1^{(2)} v_1^{k+i} = u^i v_1^{(2)} v_2^{k+i}.
\]
Similarly we have

\[(4) \quad 1 = [z, x] = u^{-j}v_1^k v_2^{-j/2}.\]

Therefore, for \(p \geq 3\) we have \(i \equiv j \equiv 0 \pmod{p'}\) and \(k \equiv 0 \pmod{p}\). It follows \(z = u^{p}v_1^k v_2^{m}\) with \(k = pk_1\). This implies

\[Z(F) = \langle v_1 \rangle \times \langle v_2 \rangle \simeq \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p.\]

If \(p = 2\) we must have \(i \equiv j \equiv 0 \pmod{2^{r-1}}\) and \(k \equiv 0 \pmod{2}\). So we have \(z = u^{2k_1}v_1^{2r}v_2^{2^{r-1}i}y^{2^{r-1}j1}\) where \(k = 2k_1\), \(i = 2^{r-1}i_1\), \(j = 2^{r-1}j_1\). Then, if \(r \geq 3\) we get

\[Z(F) = \langle u^2 \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \times \langle x^{2^{r-1}} \rangle \times \langle y^{2^{r-1}} \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.\]

If \(p = 2\) and \(r = 2\), we have \(u^2 = 1\), so \(z = v_1^i v_2^j x^{2i} y^{2j} w^{j-1}\) where \(i = 2i_1\), \(j = 2j_1\). But the condition \(j_1 \equiv 0 \pmod{2}\) implies \(i_1(2i_1 - 1) \equiv 0 \pmod{2}\). So \(i_1 \equiv 0 \pmod{2}\). Similarly we have \(j_1 \equiv 0 \pmod{2}\). Therefore \(z = v_1^i v_2^j\) and

\[Z(F) = \langle v_1 \rangle \times \langle v_2 \rangle = \Gamma_3(F) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.\]

c) Observe that for all \(a, b, c \in F\) we have \([a^p, b, c] = [a, b, c]p = 1\). So \([F^p, F] \leq Z(F)\).

d) We have \([\Phi(F), F] = [F', F^p, F]\). Since \([F', F] = \Gamma_3(F) \leq Z(F)\) and \([F^p, F] \leq Z(F)\) by Part c), it follows that \([\Phi(F), F] \leq Z(F)\). So by Lemma 2.4 every proper subgroup of \(F\) is nilpotent of class two.

q.e.d.

**Theorem 3.3.** Let \(p\) be a prime and \(r \geq 2\).

a) Let \(G\) be a 2-generator group in \(\text{Min}(N_2)\) with \(\text{exp} G = p^r\). Then

\[|G| \leq \begin{cases} p^{3r} & \text{if } p \geq 3 \\ 2^{4r-1} & \text{if } p = 2, r \geq 3 \\ 2^{2r-1} & \text{if } p = 2, r = 2 \end{cases}\]

b) For each one of the above three cases, there is a group of exponent \(p^r\) in \(\text{Min}(N_2)\) whose order attains the upper bound.

**Proof.** a) Every 2-generator group \(G \in \text{Min}(N_2)\) of exponent \(p^r\) is a quotient \(F/H\) of \(F\) where \(H \cap Z(F)\) does not contain \(\Gamma_3(F)\) because \(G \cong F/H\) is of class three. As \(Z(G)\) is cyclic by Lemma 2.1, also \(Z(F)/(H \cap Z(F))\) must be cyclic. Then \(H \cap Z(F)\) is abelian of rank \(\geq 2\) if \(p \neq 2\); of rank \(\geq 4\) if \(p = 2\) and \(r \geq 3\); of rank 1 if \(p = 2\) and \(r = 2\). Thus if \(p \geq 3\) we have \(|H| \geq p^2\) and \(|G| \leq p^{3r}\); if \(p = 2\) and \(r \geq 3\) we have \(|H| \geq 2^4\) and \(|G| \leq 2^{2(r-1)}\). Finally we observe that, if \(p = 2\) and \(r = 2\), no quotient of \(F\) by a proper subgroup of \(Z(F)\) is in \(\text{Min}(N_2)\). In fact, there are only three proper subgroups of \(Z(F)\), namely \(H_1 = \langle v_1 \rangle, H_2 = \langle v_2 \rangle, H_3 = \langle v_1 v_2 \rangle\). We see that in each quotient
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For $F/H_i$ ($i = 1, 2, 3$) there are couples of independent elements of $Z(F/H_i)$: for example, $x^2H_1, v_2H_1$ in $Z(F/H_1)$; $y^2H_2, v_1H_2$ in $Z(F/H_2)$ and $(xy)^2H_3, v_1v_2H_3$ in $Z(F/H_3)$. So no $F/H_i$ belongs to $Min(N_2)$ and therefore $|G| \leq 2^{3r-1}$.

b) For the first two cases of a) we consider respectively the subgroups of $Z(F)$:

\[ R_1 = \langle v_2, u^{p^{r-1}}v_1 \rangle \]  \quad if \quad p \geq 3, \ r \geq 2, \ \text{and} \ \\\\\\\\\\ \\\\\\\\\( R_2 = \langle v_2, v_1u^{p^{r-2}}, v_1x^{p^{2r-1}}, v_1y^{2^{r-1}} \rangle \]  \quad if \quad p = 2, \ r \geq 4, \ \text{and} \ \\\\\\\\\\ \\\\\\\\( R_3 = \langle v_2, u^2, x^k, y^l \rangle \]  \quad if \quad p = 2, \ r = 3.

We want to show that $G_t = F/R_t \in Min(N_2)$ ($t = 1, 2, 3$). First, since $R_t$ does not contain $\Gamma_3(F)$ it follows that $G_t$ is of class three. Moreover, as every proper subgroup of $F$ is of class two, the same holds for $G_t$. By definition of $G_t$, we also have $|\Gamma_3(G_t)| = p$. Therefore, by Lemma 2.1, it is sufficient to show that $Z(G_t)$ is cyclic.

Let us consider a typical element $zR_t \in G_t$ with $z = u^kv_1^ly^kx^iy^j \in F$. Then $zR_t \in Z(F/R_t)$ if and only if $[z, y] \in R_t$ and $[z, x] \in R_t$. By (3) and (4), this holds if and only if

\[ u^i v_1^j, v_2^k \in R_t \]

and

\[ u^{-j} v_1^i v_2^{(-1)} \in R_t \]

For $p \geq 3$ this happens if and only if there are $\alpha, \beta \in \mathbb{Z}$ such that

\[ u^i v_1^j = (u^{p^{r-1}}v_1)\alpha \]

(5)

\[ u^{-j} v_1^{(i)} = (u^{p^{r-1}}v_1)^\beta \]

(6)

By equation (5) we obtain that $i \equiv \alpha p^{r-1} \pmod{p^r}$ and $\frac{i(\alpha-1)}{2} \equiv \alpha \pmod{p}$. So

\[ i(1 - \frac{i-1}{2}p^{r-1}) \equiv 0 \pmod{p^r} \]

(7)

which gives $i \equiv 0 \pmod{p^r}$.

By Equation (6) we get $-j \equiv \beta p^{r-1} \pmod{p^r}$ and $k \equiv \beta \pmod{p}$. So

\[ j \equiv -p^{r-1}k \pmod{p^r}. \]

(8)

Therefore, we have that $zR_1 \in Z(F/R_1)$ if and only if

\[ z = u^k v_1^{l} y^{-(p^{r-1}k)} = (uy^{-(p^{r-1})})^k v_1^{l}. \]

We observe that

\[ (uy^{-(p^{r-1})})^{p^{r-1}} = u^{-p^{r-1}} y^{p^{2r-2}} = u^{-p^{r-1}}. \]
Let $u^v w^{-1} v_1 \in R$, we have $v_1 R_1 = u^{-p^r -1} R_1 = (uy^{-p^{r-1}})^{-1} R_1$. Then $zR_1 = (uy^{-p^{r-1}})^k R_1$. Thus $Z(F/R_1) = (uy^{-p^{r-1}})^k R_1$ is cyclic.

If $p = 2$ and $r \geq 3$ an analogous calculation yields

\begin{equation}
(9) \quad u^v w^{-1} = (u^{2r-2} v_1)^3
\end{equation}

and

\begin{equation}
(10) \quad u^v w^{-1} = (u^{2r-2} v_1)^3
\end{equation}

By (9) and (10) we obtain

$$i(1 - (i - 1)2^{r-3}) \equiv 0 \pmod{2^{r-1}}$$

and

$$j \equiv -2^{r-2} k \pmod{2^{r-1}}.$$

So if $r \geq 4$, we obtain $i \equiv 0 \pmod{2^{r-1}}$; while if $r = 3$ we have $i \equiv 0 \pmod{2}$.

In the case $p = 2, r \geq 4$ it follows that $zR_2 \in Z(F/R_2)$ if and only if $z = (uy^{-p^{r-1}})^k v_1 x^{2r-1} w$ with $i = 2^{r-1} i_1$. Since $(uy^{-p^{r-1}})^{-2} = u^{-2r-2}$, we have

$$u^{-2r-2} R_2 = v_1 R_2 = x^{2r-1} R_2 = y^{2r-1} R_2.$$  Thus $zR_2 = (uy^{-p^{r-1}})^k x^{2r-2} (i + i_1)$ and $Z(F/R_2) = (uy^{-2^{r-2}}) R_2$ is cyclic.

In the case $p = 2, r = 3$ we have $zR_3 \in Z(F/R_3)$ if and only if $z = (uy^{-2})^k v_1$. Since $(uy^{-2})^{-2} = u^{-2} y^4 = u^{-2}$ and $u^{-2} R_3 = v_1 R_3$, we have $zR_3 \equiv (uy^{-2})^k R_3$. Thus, $Z(F/R_3) = (uy^{-2}) R_3$ is cyclic.

Finally, in the case $p = 2$ and $r = 2$, we consider the normal (non central) subgroup

$$R_4 = \langle x, y^2 \rangle.$$

Then $zR_4 \in Z(F/R_4)$ if and only if $z = u^k v_1 x^i y^j$ with $k \equiv 0 \pmod{2}$, $j \equiv 0 \pmod{2}$, $i \equiv 0 \pmod{2}$ and $\frac{i+i_1}{2} \equiv 0 \pmod{2}$. The last two conditions implies $i \equiv 0 \pmod{4}$. Then $zR_4 = v_1 R_4$ and thus $Z(F/R_4) = (v_1) R_4$ is cyclic. q.e.d.

**Theorem 3.4.** Let $p$ be a prime and $r \geq 2$. If $p \geq 3$ or $p = 2$ and either $r = 3$ or $r = 2$, then all 2-generator groups in $Min(N_2)$ of exponent $p^r$ and maximal order are isomorphic.

**Proof.** Using the same notation as in the proof of Theorem 3.3, let $F/H \in Min(N_2)$ be of exponent $p^r$ ($p \geq 3$) and maximal order $|F/H| = p^{3r}$. By the proof of Theorem 3.3 it follows that $H \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. We will show that there exists an automorphism $\varphi$ of $F$ with $\varphi(H) = R_1$ and so $F/H \simeq F/R_1$. Since $F/H$ is of nilpotency class three, we have that $\Gamma_3(F) \not\subseteq H$. As $Z(F)$ is of rank three and
$H \cong \mathbb{Z}_p \times \mathbb{Z}_p$, we get $|H \cap \Gamma_3(F)| = p$. We construct the automorphism $\varphi$ in two steps. First we give an automorphism which maps $H \cap \Gamma_3(F)$ onto the subgroup $\langle v_2 \rangle$ of $R_1$.

If $H \cap \Gamma_3(F) = \langle v_1 \rangle$, we consider the automorphism $\alpha$ of $F$ with $\alpha(x) = y$ and $\alpha(y) = x$. In this case we have $\alpha([x, y, x]) = [y, x, y] = [x, y, y]^{-1}$, that is $\alpha(v_1) = v_2^{-1} \in R_1$.

If $H \cap \Gamma_3(F) = \langle v_2v_1^h \rangle$ for some $h \in \mathbb{Z}$, we consider the automorphism $\beta$ of $F$ with $\beta(x) = x$ and $\beta(y) = x^{-h}y$. Then we have $\beta(v_1) = [x, x^{-h}y, x] = v_1$ and $\beta(v_2) = [x, x^{-h}y, x^{-h}y] = [x, y][x, y, x]^{-h} = v_2v_1^{-h}$. So we have $\beta(v_2v_1^h) = v_2v_1^{-h}v_2^{-1} = v_2 \in R_1$.

In both cases we have now found an automorphism of $F$ which maps $H$ onto a subgroup $H^*$ of $Z(F)$ with

$$H^* \cap \Gamma_3(F) = \langle v_2 \rangle.$$ 

Therefore we may assume that $H^* = \langle v_2, v_1^nu_1up^{-1} \rangle$ with $m, n \in \mathbb{Z}$ and $n \neq 0 (\mod p)$. Since $n \neq 0 (\mod p)$, we have

$$H^* = \langle v_2, v_1^hu_1p^{-1} \rangle$$

with $h \equiv mn^{-1} (\mod p)$. First let $h \not\equiv 0 (\mod p)$. We consider the automorphism $\gamma$ of $F$ such that $\gamma(x) = x^h$ and $\gamma(y) = y$. We have $\gamma(v_2v_1^h) = v_2^h \in H^*$ and

$$\gamma([x, y, x][x, y]^{p^{-1}}) = [x, y, x]^h[x, y]^{p^{-1}} = ([x, y, x]^h[x, y]^{p^{-1}})^h \in H^*.$$ 

So $\gamma(v_1u_1p^{-1}) = (v_1^hu_1p^{-1})^h$ and $R_1^h = H^*$.

Finally let $h \equiv 0 (\mod p)$. So $H^* = \langle v_2, u_1p^{-1} \rangle$. Since $[x^{p^{-1}}, y] = u_1^{p^{-1}} \in H^*$, we have that $x^{p^{-1}}H^* \subset Z(F/H^*)$. Similarly $y^{p^{-1}}H^* \subset Z(F/H^*)$. But the images of $x^{p^{-1}}$ and $y^{p^{-1}}$ under the canonical epimorphism of $F/H^*$ onto $F/F' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ are independent, and so the center of $F/H^*$ is not cyclic. This case does not occur.

Let $F/H \in Min(N_2)$ be of exponent $2^3$ and maximal order $2^6$. Then $|H| = 2^4$ and $H$ must contain exactly one of the three subgroups $\langle v_1 \rangle$, $\langle v_2 \rangle$, $\langle v_1v_2 \rangle$ of $\Gamma_3(F)$. The automorphism $\alpha$ of $F$, defined by $\alpha(x) = y$ and $\alpha(y) = x^{-1}y^{-1}$, is of order 3 and acts transitively on the non-identity elements of $\Gamma_3(F)$. So without loss of generality we may assume $H \cap \Gamma_3(F) = \langle v_2 \rangle$ and $v_1 \not\in H$. Now consider the intersection of $H$ with the subgroup $E = \langle v_1, v_2, u_2 \rangle = \Omega_1(F')$.

Since $E/H \cong E \cap H \leq Z(F/H)$ which is cyclic, we get $|E \cap H| = 2^2$. The subgroups of $E$ of order $2^2$ that contain $v_2$ but not $v_1$ are precisely $L_1 = \langle v_2, u_2 \rangle$ and $L_2 = \langle v_2, u_2^2v_1 \rangle$. If $L_2 \leq H$, then $v_1L_2, x^2L_2, uy^2L_2 \in Z(F/L_2)$. So $Z(F/H)$ is not cyclic, because $Z(F/H) \cong Z((F/L_2)/(H/L_2))$ contains $Z(F/L_2)/(H/L_2)$ and $x^2L_2, uy^2L_2 \not\in H/L_2$ since $H \leq Z(F)$. Therefore $L_1 \leq H$ and $H/L_1$ is a subgroup of rank 2 of $Z(F/L_1)$ that does not contain $v_1L_1$. Since $|Z(F)/L_1| = 2^3$, we get the following four subgroups:

$$H_1 = \langle v_2, u_2, v_1x^4, v_1y^4 \rangle, \quad H_2 = \langle v_2, u_2^2, x^4, v_1y^4 \rangle, \quad H_3 = \langle v_2, u_2, v_1x^4, y^4 \rangle, \quad H_4 = \langle v_2, u_2^2, x^4, y^4 \rangle.$$
By a simple calculation, using the relations (3) and (4), we see that $F/H_1$ and $F/H_2$ have cyclic center, while the centers of the two remaining quotients are not cyclic. Finally, the theorem for the case $p = 2$ and $r = 3$ is proved by the automorphism $\beta$ defined by $\beta(x) = xy, \beta(y) = y$ that fixes $v_2$ and $u^2$ and maps $H_1$ onto $H_2$.

Let $F/H \in \text{Min}(N_2)$ be of exponent 4 and maximal order $2^5$. Then $|H| = 4$ and $F/H$ is nilpotent of class 3 with cyclic center (see Lemma 2.1). Since $\Gamma_3(F) = \langle v_1, v_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we must have $|H \cap \Gamma_3(F)| = 2$. As in the previous case, without loss we may assume $H \cap \Gamma_3(F) = \langle v_2 \rangle$. Let $L = \langle v_2 \rangle$. It is easy to see that $Z(F/L) = \langle y^2 L \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now $H/L \leq F/L$ and $|H/L| = 2$. If $v_1 L \in H/L$, then $\Gamma_3(F) = \langle v_1, v_2 \rangle \leq L$ and so $F/L$ would be of class two, a contradiction. Hence $v_1 \notin H/L$, and hence either $H = \langle v_2, y^2 \rangle$ or $H = \langle v_2, v_1 y^2 \rangle$. But the automorphism $\gamma$ of $F$, defined by $\gamma(x) = x$ and $\gamma(y) = x^2 y$, centralizes $\Gamma_3(F)$ and maps $y^2$ to $v_1 y^2$. Therefore all the quotients $F/H \in \text{Min}(N_2)$ of order $2^5$ are isomorphic.

**Remark 3.1.** In the case $p = 2$ and $r \geq 4$, there are non-isomorphic groups in $\text{Min}(N_2)$ of exponent $2^r$ and maximal order $2^{3(r-1)}$. In fact, the two quotients $F/R_2$ and $F/R_2'$, where $R_2 = \langle v_2, v_1 u^{2^{r-2}}, v_1 x^{2^{r-1}}, v_1 y^{2^{r-1}} \rangle$ and $R_2' = \langle v_2, v_1 u^{2^{r-2}}, x^{2^{r-1}}, y^{2^{r-1}} \rangle$, have cyclic center but one can check that the power $2^{r-1}$ of an element $g = u^k v_1^{r_1} x^i y^j$ in $F$ is
\[ g^{2^{r-1}} = (u^k x^i y^j)^{2^{r-1}} = (x^{2^{r-1}})^i (y^{2^{r-1}})^j (u^{-2^{r-2}(2^{r-1})})^{ij}; \]
so we have
\[ g^{2^{r-1}} R_2 = v_1^{r_1+j} R_2 \]
and
\[ g^{2^{r-1}} R_2' = v_1^{ij} R_2'. \]
It follows that the number of the elements of order $2^r$ is different in the two quotients and $F/R_2, F/R_2'$ are not isomorphic.

**Remark 3.2.** The referee suggested to investigate the existence of groups in $\text{Min}(N_2)$ of exponent $p^r$ and order $p^k$ for all $k$ with $r + 2 \leq k < 3r$. He gave an example of minimal order $p^{r+2}$. Namely the group:
\[ G_1 = \langle x, y, x^{p^r} = 1 = y^{p^r} = u^{p^r}, [x, y] = [u, x] = x^{p^{r-1}}, [u, y] = 1 \rangle. \]
We have $G_1 = F/L_1$ where $L_1 = \langle v_2, u^{p^r}, x^{p^{r-1}} v_1^{-1}, y^{p^r} \rangle$. An other example of minimal order non-isomorphic to the previous one is given by
\[ G_2 = \langle x, y, x^{p^r} = 1 = y^{p^r} = u^{p^r}, [x, y] = [u, x] = x^{p^{r-1}}, [u, y] = 1 \rangle; \]
in fact, $G_2$ has an abelian maximal subgroup $(\overline{u}, \overline{y})$, while $G_1$ has no abelian maximal subgroup. This is the quotient of $F$ by the subgroup:

$L_2 = \langle v_2, u^p, x^p, y^{p^{r-1}} v_1^{-1} \rangle$.

Other examples of order $p^{r+\frac{r+1}{2}}$, with $r = 2h + 1$, are given by splitting metacyclic groups:

$M_h = \langle x, y, y^{2h+1} = 1 = x^{2h+1}, [y, x] = y^{p^h} \rangle$.

These are the quotients of $F$ by the subgroups:

$N_h = \langle v_2, u y^p, x^{p^{h+1}}, v_1 y^{2h+1} \rangle$.

The problem of the existence of groups in $\text{Min}(N_2)$ of order other than of the maximal one seems of non easy solution. We have to construct quotients $F/L$ of $F$ with cyclic center. Considering the automorphisms $\alpha$ and $\beta$ used in the proof of the Theorem 3.4, we can assume, W.L.O.G., that $L \geq H^* = \langle v_2, u^{p^{r-1}} \rangle$. We prove that the orders of such quotients cannot be greater than $p^{2r+1}$. Since $Z(F/H^*) \cong \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p$, $L$ has to contain a subgroup isomorphic to $\mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p$. In fact $F/L \cong (F/H^*)/(L/H^*)$ and $Z(F/L) \geq (Z(F/H^*)/(L/H^*))/(L/H^*)$; since $Z(F/H^*) = \langle u^p H^*, y_1 H^*, x^{p^{r-1}} H^*, y^{p^{r-1}} H^* \rangle$ and $(Z(F/H^*)/(L/H^*/H^*)$ has to be cyclic, it follows that $L/H^*$ has to contain a complement of $\langle v_1 H^* \rangle$ in $Z(F/H^*)$. Thus $[L] \geq p^{r+1}$ and $|F/L| \leq p^{2r+1}$.

4. $\text{Min}(N_2)$-groups with three generators

It follows from Lemma 2.3 that a group $G \in \text{Min}(N_2)$, with three generators and exponent $3^r$ ($r \geq 1$), belongs to the variety $V$ of all 2-Engel groups of exponent $3^r$. So $G$ is a quotient of $F_{3}(V)$.

**Proposition 4.1.** Let $F = F_{3}(V)$ be the relatively free group with free generators $x, y, z$ in the variety $V$. Then:

- $a)$ $|\Gamma_3(F)| = 3$ and $|F| = 3^{3r+1}$.
- $b)$ $Z(F) \cong \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_3$.
- $c)$ Every proper subgroup of $F$ is nilpotent of class two.
- $d)$ $F$ belongs to $\text{Min}(N_2)$ if and only if $r = 1$.
- $e)$ Let $F/H$ be a quotient of $F$ of class three. Then $F/H \in \text{Min}(N_2)$ if and only if $Z(F/H)$ is cyclic.

**Proof.** a) Note that $F/F'$ is a 3-generated group of exponent $3^r$, so $|F/F'| \leq 3^{3r}$. Similarly, we have $|F'/\Gamma_3(F)| \leq 3^{3r}$. Now we show that $|\Gamma_3(F)| = 3$. In fact, $\Gamma_3(F)$ is generated by the basic commutators of weight three and, as $F$ is 2-Engel, they are all equal to 1, except at most $[y, x, z]$ and $[z, x, y]$ (see, for example [2, p. 54]). Moreover, in a 2-Engel group $G$, for all $x_1, x_2, x_3 \in G$ the following conditions hold:
[1, x_3, x_2] = [x_1, x_2, x_3]^{-1}

ii) [x_1^{-1}, x_2] = [x_1, x_2^{-1}] = [x_1, x_2]^{-1}

(see (2) and (3) in the proof of Satz 6.5 in [1, p. 288]).

So we get

\[ [z, x, y] = [[x, z], y] = \text{by ii) } \]
\[ = [x, y, z] = [[y, x], z] = \text{by ii) } \]
\[ = [y, x, z]^{-1}. \]

Hence \( \Gamma_3(F) = \langle [x, y, z] \rangle \) is cyclic of order 3 (see [4, p. 358]) and \(|F| \leq 3^{3r+1} \).

We now construct a group \( F_0 \), belonging to the variety \( \mathbf{V} \), which has order \( 3^{3r+1} \). Then it follows that \( F_0 \cong F \) and \(|F| = 3^{3r+1} \).

Let \( A \) be the abelian group of exponent \( 3^r \) defined by

\[ A = \langle z \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \times \langle v_3 \rangle \cong \mathbb{Z}_{3^r} \times \mathbb{Z}_{3^r} \times \mathbb{Z}_{3^r} \times \mathbb{Z}_3 \]

and let \( Q \) be the group of exponent \( 3^r \) and of nilpotency class 2 defined by

\[ Q = \langle x, y | x^{3^r} = y^{3^r} = 1, u = [x, y], u^{3^r} = 1, [u, x] = [u, y] = 1 \rangle. \]

Let \( F_0 = [A]Q \) be the semidirect product of \( A \) and \( Q \) with the action of \( Q \) on \( A \) defined by

\[ z^v = zv_2^{-1}, \quad v_1^v = v_1v_3, \quad v_2^v = v_2, \quad v_3^v = v_3, \]
\[ z^w = zw_1, \quad v_1^w = v_1, \quad v_2^w = v_2v_3, \quad v_3^w = v_3. \]

Since

\[ u = [x, y], \quad v_1 = [z, y], \quad v_2 = [x, z], \quad v_3 = [v_1, x] = [v_2, y], \]

we obtain that \( F_0 = \langle x, y, z \rangle \) and we have \(|F_0| = |A|Q| = 3^{3r+1}3^{3r} = 3^{6r+1} \).

Also we have

\[ [z, u] = v_3, \quad [u, v_1] = [u, v_2] = [u, v_3] = 1. \]

So \( F_0' = \langle u, v_1, v_2, v_3 \rangle \) and \( \Gamma_3(F_0) = \langle v_3 \rangle \) is of order 3. Therefore \( F_0 \) is nilpotent of class 3.

To prove a) we only need to show that the group \( F_0 \) we have constructed belongs to the variety \( \mathbf{V} \). In other words, we have to show that \( F_0 \) is a 2-Engel group of exponent \( 3^r \). Since the right 2-Engel elements form a subgroup of a group (see [3]), it is sufficient to check that the generators \( x, y, z \) of \( F_0 \) are right 2-Engel elements. In fact, by the definition of \( F_0 \), it is easy to see that the basic commutators of weight three on the generators, are the following:

\[ [x, y, y] = [x, y, x] = [z, x, x] = [z, y, z] = [z, y, y] = [z, x, z] = 1 \]
\[ [x, y, z] = v_3^{-1}, \quad [z, y, x] = v_3. \]
On finite $p$-groups minimally of class greater than two

We observe that $v_3 \in Z(F_0)$ by (11) and (12). Then it follows that $F_0$ is nilpotent of class 3 and $\Gamma_3(F_0) = \langle v_3 \rangle$ is of order 3.

Moreover, since $A$ is abelian, the relations (11), (12) yield:

\begin{align*}
[x^a, z^a] &= v_2^{a_1}, [v_1^b, x^a] = v_3^{b_1}, [z^{a_1}, y^{b_1}] = v_1^{a_1 b_1} \\
[v_2^b, y^{b_1}] &= c_3^{b_1}, [u^{c_1}, z^{b_1}] = v_3^{-a_1 b_1}
\end{align*}

where $a, b, c, a_1, b_1, \alpha, \beta, \gamma$ belong to $Z$. Using the above relations, we can directly check that for all $g \in F_0$ we have

\[ [x, g, g] = [y, g, g] = [z, g, g] = 1. \]

Write $g = vw$ with $v \in A$ and $w \in Q$. Since $Q$ is of class 2 and $A$ is abelian, we have $[x, w, w] = [x, v, v] = 1$. So

\[ [x, g, g] = [x, v, w][x, w, v]. \]

Letting $w = y_i^s$, where $s \in (x, u)$, and $v = z^j \tilde{v}$, where $\tilde{v} \in \langle v_1, v_2, v_3 \rangle$, the relations displayed in (11), (12) and (13) yield

\[ [x, v, w] = [x, z^j, w] = [v_3^{j_1}, y^{j_1}] = v_3^{j_1 j} \]

and

\[ [x, w, v] = [x, y_i^s, v] = [u_i^s, z^j] = v_3^{-j_i}. \]

So $[x, g, g] = 1$.

The proof that $y$ is right 2-Engel is analogous.

For $z$ we observe that, since $A$ is abelian and $[z, Q]$ is contained in $A$, we have

\[ [z, v, v] = [z, v, w] = [z, w, v] = 1. \]

Moreover, letting $w = x^h y^i u^k$, by relations (11), (12) and (13) we have

\[ [z, w] = [z, x^h y^i u^k] = [z, y^i][z, x^h]c, \; c \in Z(F_0). \]

It follows that

\[ [z, w, w] = [v_1^{j_1} v_2^{j_2}, x^h y^i] = v_3^{j_1} v_3^{j_2} = 1. \]

It remains to check that the exponent of $F_0$ is $3^r$. By the Hall-Petrescu identity (see [1, p. 317]) we have

\[ y^{3^r} = (vw)^{3^r} = v^{3^r} w^{3^r} c_1^{(3^r)} c_2^{(3^r)}, \]

where $c_1 \in F_0$ and $c_2 \in \gamma_3(F_0) = \langle v_3 \rangle$. Since $Q, A, F_0$ are of exponent $3^r$ and $|\Gamma_3(F_0)| = 3$, we have $(vw)^{3^r} = 1$.

We can now identify $F$ with $F_0$. 

b) By the relations (11) and (12) we have that $u^3, v_1^3, v_2^3, v_3 \in Z(F)$. Conversely, computing the commutators between an element $g = vw = z^a v_1^b v_2^c x^a y^b u^g$ and the generators $x, y, z$ of $F$, we obtain

\begin{align}
(14) \quad [x, vw] &= [x, u][x, v][x, v, w] = u^\beta v_2^\alpha v_3^{-b} v_2^a y^\beta = v_2^a v_3^\alpha - b u^\beta \\
(15) \quad [y, vw] &= [y, w][y, v][y, v, w] = u^{-\alpha} v_1^{-a} v_3^b v_1^a, x^\alpha = v_1^{-a} v_3^b, c^{-\alpha} u^{-\alpha} \\
(16) \quad [z, vw] &= [z, w] = [z, x^\alpha y^\beta u^g] = [z, u^g] [z, y^\beta x^\alpha] = v_1^\alpha v_2^{-a} v_3^\gamma - \alpha^3
\end{align}

It follows that $g \in Z(F)$ only if $a \equiv \beta \equiv \alpha \equiv 0 \pmod{3'}$ and $b \equiv c \equiv \gamma \equiv 0 \pmod{3}$. So the elements of $Z(G)$ have the following form

$$g = v_1^{b_1} v_2^{b_2} v_3^{b_3} u^{\gamma_1}$$

where $b_1, c_1, \gamma_1 \in \mathbb{Z}_{3^r - 1}$ and $d \in \mathbb{Z}_3$. Thus

$$Z(F) = \langle v_1^3 \rangle \times \langle v_2^3 \rangle \times \langle v_3^3 \rangle \times \langle u^3 \rangle.$$ 

c) It is sufficient to show that every maximal subgroup $M$ of $F$ is of class two. As $F/\Phi(F) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, we have $M = \langle \Phi(F), x_1, x_2 \rangle$ for some $x_1, x_2 \in M$. We want to show that $M' = \langle [x_1, x_2], [x_i, F'] \rangle$ for all $i = 1, 2$ is contained in $Z(M)$. In fact, $F$ is nilpotent of class 3, so $[x_i, F'] \leq Z(F)$. We observe that $Z(F) = \langle v_3, F'^3 \rangle$ is contained in $M$ and then $Z(F) \leq Z(M)$. Therefore $[x_i, F'] \leq Z(M)$. Since the identity $[g_1, g_2] = [g_1, g_2]^{u_n}$ holds in the 2-Engel group $F$, for all $u \in \mathbb{Z}$ and $g_1, g_2 \in F$, we have $[x_i, F'^3] = [x_i, F'^3] \leq F'^3 \leq Z(M)$. In the same way we see that $\Phi(F)' \leq Z(M)$. Finally $[x_1, x_2] \in Z(M)$ because $[x_1, x_2, x_1] = [x_1, x_2, x_2] = 1$ holds in the 2-Engel group $F$.

d) Suppose $r > 1$, then $v_1^3, v_2^3, u^3$ belong to $Z(F)$ (see b). So $Z(F)$ is not cyclic, contradicting Lemma 2.1.

Conversely, let $r = 1$, then $Z(F) = \langle v_3 \rangle$ is cyclic of order three. So, by Lemma 2.1 and c), $F$ belongs to $Min(N_2)$.

e) Let $L = F/H$ be a quotient of $F$ of class precisely three. If $M/H$ is a maximal subgroup of $L$, then $M$ is a maximal subgroup of $F$ and, by c), it is nilpotent of class two. Since $\Gamma_3(L) = \Gamma_3(F)H/H$ is cyclic of order 3, by Lemma 2.1, $L \in Min(N_2)$ if and only if $Z(L)$ is cyclic. q.e.d.

**Proposition 4.2.**

a) Let $G$ be a 3-generated group in $Min(N_2)$ with $exp(G) = 9$. Then $|G| \leq 3^7$.

b) There are at least two non-isomorphic groups in $Min(N_2)$ of exponent 9 and order $3^7$.
Proof. a) Using the same notation as in the previous theorem, $G$ has to be isomorphic to a quotient $F/H$ of the relatively free group $F$ with $\text{exp}(F) = 3^2$. Since $F/H$ has to be nilpotent of class 3, we have $v_3 \not\in H$. As $Z(F/H)$ must be cyclic and $Z(F)$ is elementary abelian of rank 4, then $H$ must contain a subgroup $K$ of $Z(F)$ which is of rank 3 and $v_3 \not\in H$. Now $Z(F)$ contains 40 subgroups of index 3. Among these, 13 contain $v_3$. So there are 27 subgroups of $Z(F)$ which do not contain $\langle v_3 \rangle$. The subgroup $K_1 = \langle v_1^3, v_2^3, u^3 \rangle = (F')^3$ is characteristic in $F$ and the other 26 form a single orbit under the automorphism $\varphi$ of $F$ defined by

$$x^a = y,$$
$$y^a = z,$$
$$z^a = x^{-1}y.$$

In fact consider the subgroup $K_2 = \langle v_1^3, v_2^3, u^3 \rangle$ of $Z(F)$. We observe that $v_1^3 = v_1^{-1}v_2^{-1}$, $v_2^3 = u$, $u^3 = v_1^{-1}$. A straightforward calculation shows that $F^{v_2} = (v_1^3, v_2^3, u^3) \neq K_2$. As $\varphi$ is an automorphism of order 26, the orbit of $K_2$ has length 26. So we may assume that $H$ contains one of the two subgroups $K_i$, $i = 1, 2$. Consider $F/K_1$. A generic element of $K_1$ can be written in the form

$$v_1^{3l}v_2^{3m}u^{3n} \text{ with } l, m, n \in \{0, 1, 2\}.$$

From the relations (14), (15) and (16) we get that an element $gK_1 = z^a v_1^b v_2^c x^d y^e u^f K_1$ of $F/K_1$ belongs to $Z(F/K_1)$ if and only if

$$v_1^{a-\alpha \beta} v_2^{-\beta} \cdot u^\beta = v_1^{\alpha \beta} v_2^{\beta} u^\beta,$$
$$v_1^{\alpha \beta} v_2^{-\beta} \cdot u^{-\alpha} = v_1^{\alpha \beta} v_2^{\beta} u^{-\alpha},$$
$$v_1^{\alpha \beta} v_2^{-\beta} \cdot y^{-\gamma} = v_1^{\alpha \beta} v_2^{\beta} y^{-\gamma},$$

for some $l, m, n \in \{0, 1, 2\}$; $i = 1, 2, 3$. It follows $a \equiv \beta \equiv b \equiv e \equiv \alpha \equiv \gamma \equiv 0 \pmod{3}$. Let $a = 3a_1$, $\beta = 3\beta_1$, $b = 3b_1$, $e = 3c_1$, $\alpha = 3a_1$ and $\gamma = 3\gamma_1$. Then

$$gK_1 = z^{3a_1} v_1^{3b_1} v_2^{3c_1} x^{3d_1} y^{3e_1} u^{3f_1} K_1 = z^{3a_1} v_1^{3b_1} x^{3d_1} y^{3e_1} K_1.$$

In a similar way we see that $gK_2 \in Z(F/K_2)$ if and only if $\beta \equiv c \equiv 0 \pmod{3}$ and $a \equiv 3b \pmod{9}$ $\alpha \equiv 3\gamma \pmod{9}$. If $\beta = 3\beta_1$, $c = 2c_1$ and $\alpha = 3a_1$, we have that

$$gK_2 = z^{3b_1} v_1^{3b_1} v_2^{3c_1} x^{3d_1} y^{3e_1} u^{3f_1} K_2 = \langle z^{3b_1} v_1^{3b_1} x^{3d_1} u^{3f_1} \rangle g^{3\beta_1} K_2.$$

Therefore $Z(F/K_1)$ and $Z(F/K_2)$ are abelian groups which can be represented as direct product

$$Z(F/K_1) = \langle z^3 K_1 \rangle \times \langle v_3 K_1 \rangle \times \langle x^3 K_1 \rangle \times \langle y^3 K_1 \rangle.$$
In order that a quotient \((F/K_i)/(H/K_i)\), \((i = 1, 2)\) of \(F/K_i\) would be nilpotent of class 3 with cyclic center, we need that \(v_3 K_i \notin Z(F/K_i)\) and that \(H/K_i\) would contains a subgroup of rank 3 of \(Z(F/K_i)\). So the order of a group \(F/H \in Min(N_2)\) is at most \(3^7\).

b) Consider the subgroups

\[ H_1 = \langle v_1^3, v_2^3, u^3, x^3, y^3, z^3 v_1^{-1} \rangle \quad \text{and} \quad H_2 = \langle v_1^3, v_2^3 v_3^{-1}, u^3, x^3 u, y^3, z^3 v_1 \rangle \]

which contain \(K_1\) and \(K_2\), respectively. By the same argument used above to determine the center of \(F/K_i\), one can check easily that the center of \(Z(F/H_i)\) is cyclic. If \(g = z^a v_1^b v_2^c x^d y^e u^f\) is, as before, a generic element of \(F\), we have

\[ g^3 = z^{3\alpha} v_1^{3(b-a\beta)} v_2^{3(c+\alpha a)} x^{3\alpha} y^{3\beta} u^{3(\gamma - a\beta)}. \]

Using this relation we see that the exponent of \(F/H_i\) is 9. Moreover we see that the \(U_1(F/H_1) = \langle v_3 H_1 \rangle\) while \(U_1(F/H_2) = \langle v_1 H_2, v_3 H_2, u H_2 \rangle\) which is not cyclic.

q.e.d.

References


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