Marcinkiewicz-type interpolation theorem
for Morrey-type spaces and its corollaries

BURENKOV V.I. – CHIGAMBAYEVA D.K. – NURSULTANOV E.D. *

Abstract – We introduce a class of Morrey-type spaces $M^\lambda_{p,q;\Omega}$, which includes the classical Morrey spaces and discuss their properties. We prove a Marcinkiewicz-type interpolation theorem. This theorem is then applied to obtaining the boundedness in the introduced Morrey-type spaces of the Riesz potential and singular operators.

Mathematics Subject Classification (2010). 42B35, 47B06, 47B38, 46B70, 47G10.

Keywords. Morrey spaces, Morrey-type spaces, interpolation theorem, Riesz potential, singular operators.

1. Introduction

Let $0 < p \leq \infty$ and $0 \leq \lambda \leq \frac{n}{p}$. The Morrey spaces $M^\lambda_p$ are the spaces of all functions $f \in L_{loc}^p(\mathbb{R}^n)$ such that

$$
||f||_{M^\lambda_p} \equiv ||f||_{M^\lambda_p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \sup_{t > 0} t^{-\lambda} \|f\|_{L_p(B_t(x))} < \infty,
$$

*BThe publication was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.0008).

Burenkov V.I., RUDN University, S.M. Nikol’skii Mathematical Institute, 6 Mikiulko-Maklay St, Moscow, 117198, and Cardiff School of Mathematics, Cardiff University, Cardiff CF24 4AG, UK
E-mail: burenkov@cardiff.ac.uk

Chigambayeva D.K., L.N. Gumilyov Eurasian National University, 2 Satpayev St, Astana, Kazakhstan, 010008, and RUDN University, S.M. Nikol’skii Mathematical Institute, 6 Mikiulko-Maklay St, Moscow, 117198
E-mail: d.darbaeva@yandex.kz

Nursultanov E.D., Lomonosov Moscow State University (Kazakhstan branch), 11 Kazhimukan St, Astana, Kazakhstan, 010010, and RUDN University, S.M. Nikol’skii Mathematical Institute, 6 Mikiulko-Maklay St, Moscow, 117198
E-mail: er-nurs@yandex.ru
where $B_t(x)$ is the open ball of radius $t > 0$ with center at the point $x \in \mathbb{R}^n$ (see [19]). If $\lambda = 0$, then $M^0_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = \frac{p}{p-1}$, then $M^{p/p}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then $M^\lambda_p = \Theta$, where $\Theta$ is the set of all functions that are equivalent to zero on $\mathbb{R}^n$.

These spaces were introduced by Morrey [19] in 1938 and arose in connection with some problems of the theory of partial differential equations and theory of variations. There is a number of books and survey papers on the Morrey and Morrey-type spaces and classical operators of real analysis in the Morrey-type spaces, see, for example, [4], [5], [17], [20], [25], [26], [2].

This paper is devoted to the interpolation properties of the Morrey-type spaces. Some results for the classical Morrey spaces were obtained in Stampacchia [29], Campanato and Murthy [15], Peetre [24]. In particular in [24] it is proved that

$$(M^\lambda_0, M^\lambda_1)_{\theta, \infty} \subset M^\lambda_p,$$

where $1 \leq p < \infty$, $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$. In Ruiz and Vega [27], Blasco, Ruiz and Vega [3] it is proved that this inclusion is strict.

In [18] a more detailed investigation of the interpolation problem for the Morrey spaces was carried out. In particular, it was proved that the inclusion

$$(M^\lambda_{p_0}, M^\lambda_{p_1})_{\theta, \infty} \subset M^\lambda_p,$$

where $1 \leq p_0, p_1 < \infty$, $0 < \lambda_0 < \frac{p}{p_0}$, $0 < \lambda_1 < \frac{p}{p_1}$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1}$, $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$, holds if and only if $p_0 = p_1$.

The case of the local Morrey-type spaces $L^{\lambda}_{p,q}$ was considered in [6], [7], [14].

Let $0 < p, q \leq \infty$, $\lambda > 0$ if $q < \infty$, $\lambda \geq 0$ if $q = \infty$. A function $f \in L^{\lambda}_{p,q}$ if $f \in L^\infty_{loc}(\mathbb{R}^n)$ and

$$||f||_{L^{\lambda}_{p,q}} = \left( \int_0^\infty \left( t^{-\lambda} ||f||_{L_p(B_t(0))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$ 

In [14] it was proved, in particular, that the local Morrey-type spaces $L^{\lambda}_{p,q}$ form an interpolation scale when $p$ is fixed, i.e.

$$(L^{\lambda}_{p,q_0}, L^{\lambda}_{p,q_1})_{\theta, q} = L^{\lambda}_{p,q},$$

where $0 < p, q_0, q_1 \leq \infty$, $0 < \theta < 1$, $\lambda_0 > 0$ if $q_0 < \infty$, $\lambda_0 \geq 0$ if $q_0 = \infty$, $\lambda_1 > 0$ if $q_1 < \infty$, $\lambda_1 \geq 0$ if $q_1 = \infty$, $\lambda_0 \neq \lambda_1$ and $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$.

Further generalizations of interpolation properties for the general local Morrey-type spaces $L^{\lambda}_{p,q}(G, \mu)$ were discussed in [6], namely, it was proved that

$$(L^{\lambda}_{p,q_0}(G, \mu), L^{\lambda}_{p,q_1}(G, \mu))_{\theta, q} = L^{\lambda}_{p,q}(G, \mu),$$

under the same assumptions on the numerical parameters.
In this paper, we introduce the generalized Morrey-type spaces $M_{p,q}^\lambda$, which coincide with the classical Morrey spaces $M_p^\lambda$ in the case $q = \infty$ and $\Omega = \mathbb{R}^n$. According to the above results it follows that the classical interpolation theorems for this scale of the spaces do not take place. Nevertheless, we prove a certain analogue of an interpolation theorem. Compared with the classical interpolation theorems the assumptions of the theorem are formulated in terms of the local Morrey-type spaces $L_{p,q}^\lambda$ and the statement in terms of the generalized Morrey-type spaces $M_{p,q}^\lambda$. We say that this theorem is an interpolation theorem of Marcinkiewicz-type, because it allows us to obtain from in a certain sense “weak estimates” for quasi-additive operators in terms of the local Morrey-type spaces “strong estimates” in terms of the generalized Morrey-type spaces.

Let $f \in L_{1}^{\infty}(\mathbb{R}^n)$. In this paper we also study estimates for the norm of the Riesz operator

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\gamma}}, \quad 0 < \gamma < n.$$ 

Let us recall the well-known Hardy-Littlewood-Sobolev inequality. Let

$$1 < p < q < \infty, \quad \text{and} \quad \gamma = n\left(\frac{1}{p} - \frac{1}{q}\right),$$

then the operator $I_\gamma$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

The boundedness of $I_\gamma$ in the classical Morrey spaces and their generalizations was investigated by Peetre [24], Adams [1], Nakai [20], Sawano and Tanaka [28], and others. Further results of the boundedness of the Riesz potential for the local Morrey-type spaces were obtained in [11, 12, 8].

In this paper, we continue the study of the boundedness of $I_\gamma$ in the case of the Morrey-type spaces. In particular, by applying the Marcinkiewicz-type interpolation theorem we obtain an estimate for the norm of the Riesz operator in the Morrey-type spaces.

Also we study the boundedness for a certain class of singular operators. The classical results for Calderon-Zygmund operators of different classes state that if $1 < p < \infty$, then this operator is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see, for example, [21], [30]). Peetre [25] studied the boundedness of singular operator in the Morrey spaces. Further results on the boundedness of Calderon-Zygmund operators for the local Morrey-type spaces were obtained in Burenkov, Guliyev, Serbetci and Tararykova [13], and others.

Given functions $F$ and $G$, in this paper $F \lesssim G$ means that $F \leq CG$, where $C$ is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$.

2. Spaces $M_{p,q}^\lambda$

Let $\Omega \subset \mathbb{R}^n$, $0 < p \leq \infty$, $0 < q \leq \infty$ and $0 \leq \lambda \leq \frac{n}{p}$. We consider the generalized Morrey-type spaces $M_{p,q}^\lambda$ that are defined for $q < \infty$ as the spaces of all functions
Let \( f \in L^\text{loc}_p(\mathbb{R}^n) \) such that
\[
\|f\|_{M^\lambda_{p,q},\Omega} = \left( \int_0^\infty \left( t^{-\lambda} \sup_{x \in \Omega} \|f\|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},
\]
and for \( q = \infty \),
\[
\|f\|_{M^\lambda_{p,\infty},\Omega} = \sup_{x \in \Omega} \sup_{t > 0} t^{-\lambda} \|f\|_{L_p(B_t(x))}.
\]

Note that the introduced spaces coincide with the classical Morrey spaces in the case \( q = \infty \) and \( \Omega = \mathbb{R}^n \), i.e.
\[
M^\lambda_{p,\infty,\mathbb{R}^n} = M^\lambda_p.
\]

However, these spaces differ from the global Morrey-type spaces \( GM^\lambda_{p,q,\Omega} \), which are defined as the spaces of all functions \( f \) Lebesgue measurable on \( \mathbb{R}^n \) with finite quasi-norm
\[
\|f\|_{GM^\lambda_{p,q,\Omega}} = \sup_{x \in \Omega} \left( \int_0^\infty \left( t^{-\lambda} \|f\|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
if \( q < \infty \) and usual modification if \( q = \infty \). For \( \Omega = \mathbb{R}^n \) they were introduced by Burenkov and Guliyev [9, 10].

Clearly
\[
M^\lambda_{p,q,\Omega} \subset GM^\lambda_{p,q,\Omega}
\]
and
\[
\|f\|_{GM^\lambda_{p,q,\Omega}} \leq \|f\|_{M^\lambda_{p,q,\Omega}}.
\]

If \( \Omega = \{x\} \) is a singleton, then
\[
M^\lambda_{p,q,\Omega} = GM^\lambda_{p,q,\Omega} = LM^\lambda_{p,q,x},
\]
where \( LM^\lambda_{p,q,x} \) are the local Morrey-type spaces [9, 10].

Note also that the generalized Morrey-type spaces are close to the net spaces \( N_{p,q}(M) \) introduced by Nursultanov [22, 23].

We begin with the following lemma, which describes the properties of the spaces \( M^\lambda_{p,q,\Omega} \).

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^n \).

(i) If \( 0 < p_0 < p_1 < \infty \) and \( \lambda_0 = \lambda_1 + n(\frac{1}{p_1} - \frac{1}{p_0}) \), then the space \( M^\lambda_{p_1,q,\Omega} \) is continuously embedded in \( M^\lambda_{p_0,q,\Omega} \).

(ii) If \( 0 < q_0 < q_1 \leq \infty \), then the space \( M^\lambda_{p,q_0,\Omega} \) is continuously embedded in \( M^\lambda_{p,q_1,\Omega} \).
Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

Proof. (i). Let \( f \in M_{p;\alpha}^{\lambda} \). By applying Hölder’s inequality and noting that \( |B_t(x)| \approx t^n \), we get

\[
\| f \|_{M_{p;\alpha}^{\lambda}} = \left( \int_0^\infty \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right) \frac{dt}{t} \right)^\frac{1}{\lambda} \\
\leq \left( \int_0^\infty \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \| B_t(x) \|_p^{\frac{1}{p} - \frac{1}{q}} \right) \frac{dt}{t} \right)^\frac{1}{\lambda} \\
\leq C \left( \int_0^\infty \left( t^{-\lambda} \left( \frac{n}{p} \right)^{\frac{1}{q}} \right) \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right) \frac{dt}{t} \right)^\frac{1}{\lambda} = \| f \|_{M_{p;\alpha}^{\lambda}},
\]

which implies continuous embedding \( M_{p;\alpha}^{\lambda} \hookrightarrow M_{p;\alpha}^{\lambda} \).

(ii). First let \( q_1 = \infty \) and \( f \in M_{p;\alpha}^{\lambda} \). Let us prove that \( M_{p;\alpha}^{\lambda} \hookrightarrow M_{p;\alpha}^{\lambda} \), taking into account that

\[
\| f \|_{M_{p;\alpha}^{\lambda}} = \sup_{t > 0} t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \\
= (\lambda q_0) \frac{1}{\alpha} \sup_{t > 0} \left( \int \tau^{-\lambda} q_0 \frac{d\tau}{\tau} \right) \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))}.
\]

Since \( B_t(x) \subset B_r(x) \) if \( t \leq \tau \), we have

\[
\| f \|_{M_{p;\alpha}^{\lambda}} \leq (\lambda q_0) \frac{1}{\alpha} \sup_{t > 0} \left( \int \tau^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_\tau(x))} \right) \frac{q_0 d\tau}{\tau} \frac{1}{\alpha} \\
= (\lambda q_0) \frac{1}{\alpha} \| f \|_{M_{p;\alpha}^{\lambda}}.
\]

If \( q_1 < \infty \), then

\[
\| f \|_{M_{p;\alpha}^{\lambda}} = \left( \int t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^{q_1} \left( \tau^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_\tau(x))} \right)^{q_0} \frac{d\tau}{\tau} \frac{1}{\alpha} \\
\leq \left( \| f \|_{M_{p;\alpha}^{\lambda}} \right)^{q_1} \left( \| f \|_{M_{p;\alpha}^{\lambda}} \right)^{q_0} \frac{d\tau}{\tau} \frac{1}{\alpha} \| f \|_{M_{p;\alpha}^{\lambda}}.
\]

This completes the proof of the second part of the lemma.

Lemma 2.2. Let \( 0 < p, q \leq \infty, 0 < \lambda < \infty \), then for any \( \Omega \subset \mathbb{R}^n \)

\[
2^{-\lambda} (\ln 2)^{\frac{1}{2}} \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda m} \sup_{x \in \Omega} \| f \|_{L_p(B_{2^m}(x))} \right)^q \right)^{1/q} \leq \| f \|_{M_{p;\alpha}^{\lambda}}.
\]
\[
\leq 2^\lambda (\ln 2)^{\frac{1}{2}} \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda m} \sup_{x \in \Omega} \| f \|_{L_p(B_{2^m}(x))} \right)^q \right)^{1/q}.
\]

**Proof.** Let \( t \in (0, \infty) \), then there exists \( m \in \mathbb{Z} \), such that \( 2^m \leq t < 2^{m+1} \).

Therefore,

\[
\left( \int_0^\infty \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q} = \left( \sum_{m \in \mathbb{Z}} \int_{2^m}^{2^{m+1}} \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q}.
\]

Thus,

\[
\left( \sum_{m \in \mathbb{Z}} \int_{2^m}^{2^{m+1}} \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q} \leq 2^\lambda (\ln 2)^{\frac{1}{2}} \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda (m+1)} \sup_{x \in \Omega} \| f \|_{L_p(B_{2^m+1}(x))} \right)^q \right)^{1/q}.
\]

On the other hand,

\[
\left( \sum_{m \in \mathbb{Z}} \int_{2^m}^{2^{m+1}} \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q} \geq 2^{-\lambda} (\ln 2)^{\frac{1}{2}} \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda m} \sup_{x \in \Omega} \| f \|_{L_p(B_{2^m}(x))} \right)^q \right)^{1/q},
\]

and we obtain the required equivalence. \( \square \)

**3. Interpolation theorem**

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^n \), \( 0 < \alpha_0, \alpha_1, \beta_0, \beta_1 < \infty \), \( \alpha_0 \neq \alpha_1, \beta_0 \neq \beta_1 \), \( 0 < p, q \leq \infty \), \( 0 < \sigma \leq \tau \leq \infty \), \( 0 < \theta < 1 \) and

\[
\alpha = (1 - \theta)\alpha_0 + \theta \alpha_1, \quad \beta = (1 - \theta)\beta_0 + \theta \beta_1.
\]

Let \( T \) be a quasi-additive operator \(^1\) given on \( LM^{\beta_0}_{q,\sigma, x} + LM^{\beta_1}_{q,\sigma, x} \), \( x \in \Omega \). Suppose that for some \( M_1, M_2 > 0 \) the following inequalities hold

\[
(3.1) \quad \| T f \|_{LM^{\beta_0}_{p,\tau, x}} \leq M_1 \| f \|_{LM^{\beta_0}_{q,\sigma, x}}, \quad x \in \Omega, \quad f \in LM^{\beta_0}_{q,\sigma, x}, \quad i = 0, 1,
\]

then

\[
(3.2) \quad \| T f \|_{LM^{\beta}_{p,\tau, x}} \leq c M_1^{-\theta} M_1^{\beta} \| f \|_{LM^{\beta}_{p,\tau, \Omega}}
\]

for all functions \( f \in LM^{\beta}_{q,\tau, \Omega} \), where \( c > 0 \) depends only on \( \alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta \).

\(^1\)that is for some \( A > 0 \) and for almost all \( y \in \mathbb{R}^n \) \( |T(f + g)(y)| \leq A(|(Tf)(y)| + |(Tg)(y)|) \) for all functions \( f, g \in LM^{\beta_0}_{q,\sigma, x} + LM^{\beta_1}_{q,\sigma, x} \).
Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

**Proof.** Step 1. Let \( f \in M^q_{q,r,\Omega} \). For every \( x \in \Omega, s > 0 \), we define the functions

\[
\begin{align*}
    f_{0,s} &= f \chi_{B_s(x)}, \\
    f_{1,s} &= f - f_{0,s},
\end{align*}
\]

where \( \chi_{B_s(x)} \) denotes the characteristic function of a ball \( B_s(x) \). Then \( f = f_{0,s} + f_{1,s} \) and

\[
\|Tf\|_{L_p(B_r(x))} = \|T(f_{0,s} + f_{1,s})\|_{L_p(B_r(x))} \\
\leq A \|Tf_{0,s}\| + \|Tf_{1,s}\|_{L_p(B_r(x))} \\
\leq 2^{1 - \frac{1}{q}} A (\|Tf_{0,s}\|_{L_p(B_r(x))} + \|Tf_{1,s}\|_{L_p(B_r(x))}).
\]

By inequality (3.1) we have

\[
\begin{align*}
    \|Tf_{0,s}\|_{L_p(B_r(x))} &= t^\alpha t^{-\alpha} \|Tf_{0,s}\|_{L_p(B_r(x))} \\
    &\leq t^\alpha \sup_{r > 0} \|Tf_{0,s}\|_{L_p(B_r(x))} = t^\alpha \|Tf_{0,s}\|_{LM^\alpha_p,\infty, x} \\
    &\leq M_0 t^\alpha (\int_0^s (r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))}) \frac{\sigma}{r} \, dr)^\frac{1}{2} \\
    &= M_0 t^\alpha \left( \int_0^s (r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))}) \frac{\sigma}{r} \, dr + \int_s^\infty (r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))}) \frac{\sigma}{r} \, dr \right)^\frac{1}{2} \\
    &\leq 2^{1 - \frac{1}{q}} M_0 t^\alpha \left[ \left( \int_0^s (r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))}) \frac{\sigma}{r} \, dr \right)^\frac{1}{2} + \left( \int_s^\infty (r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))}) \frac{\sigma}{r} \, dr \right)^\frac{1}{2} \right].
\end{align*}
\]

For \( 0 < r \leq s \) and \( y \in B_r(x) \) we have that \( f_{0,s}(y) = f(y) \chi_{B_s(x)}(y) = f(y) \), therefore \( \|f_{0,s}\|_{L_q(B_r(x))} = \|f\|_{L_q(B_r(x))} \). For \( r > s \) and \( y \notin B_r(x) \) we get that \( f_{0,s}(y) = 0 \), therefore \( \|f_{0,s}\|_{L_q(B_r(x))} = \|f\|_{L_q(B_r(x))} \). Hence,

\[
\left( \int_0^s (r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))}) \frac{\sigma}{r} \, dr \right)^\frac{1}{2} = \left( \int_0^s (r^{-\beta_0} \|f\|_{L_q(B_r(x))}) \frac{\sigma}{r} \, dr \right)^\frac{1}{2}.
\]
Thus, for all $\sigma > 0$

$$\|Tf_{0,s}\|_{Lq(B_s(x))} \leq c_1 M_0 t^\alpha 0 \leq c_1 M_0 t^\alpha \left[ \left( \int_s^\infty (r^{-\beta_0} \|f\|_{Lq(B_r(x))}) \sigma \frac{dr}{r} \right)^\frac{1}{\beta_0} + \int_s^\infty (r^{-\beta_1} \|f\|_{Lq(B_r(x))}) \sigma \frac{dr}{r} \right]^\frac{1}{\beta_1}.$$

Similarly, according to inequality (3.1), since $f_1,s(y) = 0$ if $y \in B_s(x)$ and $|f_1,s(y)| \leq |f(y)|$ if $y \in \mathbb{R}^n$, we get that

$$\|Tf_{1,s}\|_{Lp(B_t(x))} = t^\alpha 1 t^{-\alpha_1} \|Tf_{1,s}\|_{Lp(B_t(x))} \leq M_1 t^\alpha 1 \left( \int_s^\infty (r^{-\beta_1} \|f_1,s\|_{Lq(B_r(x))}) \sigma \frac{dr}{r} \right)^\frac{1}{\beta_1}.$$

So, for all $t > 0$ and $s > 0$ we obtain

$$\|Tf\|_{Lp(B_t(x))} \leq c_2 A \left( M_0 t^\alpha 0 \left[ \left( \int_s^\infty (r^{-\beta_0} \|f\|_{Lq(B_r(x))}) \sigma \frac{dr}{r} \right)^\frac{1}{\beta_0} + \int_s^\infty (r^{-\beta_1} \|f\|_{Lq(B_r(x))}) \sigma \frac{dr}{r} \right]^\frac{1}{\beta_1} + M_1 t^\alpha 1 \left( \int_s^\infty (r^{-\beta_1} \|f\|_{Lq(B_r(x))}) \sigma \frac{dr}{r} \right)^\frac{1}{\beta_1},$$

where $c_2 > 0$ depends only on $p, \beta_0, \beta_1$ and $\sigma$. 
Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

Step 2. Suppose that $\beta_0 < \beta_1, \alpha_0 < \alpha_1$ and set $s = ct^\gamma$, where $\gamma = \frac{\alpha_1 - \alpha_0}{\beta_1 - \beta_0}$, and $c > 0$ will be taken further. Then

$$\|Tf\|_{M^{\alpha_1,\alpha_0}_{p_1,p_2},\Omega} = \left( \int_0^\infty \left( t^{-\alpha} \sup_{x \in \Omega} \|Tf\|_{L^p(B_t(x))} \right)^\sigma \frac{dt}{t} \right)^{\frac{1}{\sigma}} \leq 3(\frac{1}{2} - 1)^2 c_2 A (M_0 I_1 + M_0 I_2 + M_1 I_3),$$

where

$$I_1 = \left( \int_0^\infty \left( t^{\alpha - \alpha} \int_0^{ct^\gamma} \left( \sup_{x \in \Omega} \|f\|_{L^p(B_t(x))} \right)^\sigma \frac{dt}{t} \right)^{\frac{1}{\sigma}} \right),$$

$$I_2 = (ct^\gamma)^{p_1 - \beta_0} \left( \int_0^\infty \left( t^{\alpha - \alpha} \int_0^{ct^\gamma} \left( \sup_{x \in \Omega} \|f\|_{L^p(B_t(x))} \right)^\sigma \frac{dt}{t} \right)^{\frac{1}{\sigma}} \right),$$

and

$$I_3 = \left( \int_0^\infty \left( t^{\alpha - \alpha} \int_0^{ct^\gamma} \left( \sup_{x \in \Omega} \|f\|_{L^p(B_t(x))} \right)^\sigma \frac{dt}{t} \right)^{\frac{1}{\sigma}} \right).$$

Making the change of variable $ct^\gamma = u$, we obtain

$$I_1 = \gamma^{-\frac{1}{\sigma}} c^\theta (\beta_1 - \beta_0) J_1, I_2 = \gamma^{-\frac{1}{\sigma}} c^\theta (\beta_1 - \beta_0) J_2, I_3 = \gamma^{-\frac{1}{\sigma}} c^{-1} (\beta_1 - \beta_0) J_3,$$

where

$$J_1 = \left( \int_0^\infty \left( y^{-\theta (\beta_1 - \beta_0)} \left( \int_0^y \left( \sup_{x \in \Omega} \|f\|_{L^p(B_t(x))} \right)^\sigma \frac{dt}{t} \right) \right)^{\frac{1}{\sigma}} \right),$$

$$J_2 = \left( \int_0^\infty \left( y^{1-\theta (\beta_1 - \beta_0)} \left( \int_0^y \left( \sup_{x \in \Omega} \|f\|_{L^p(B_t(x))} \right)^\sigma \frac{dt}{t} \right) \right)^{\frac{1}{\sigma}} \right),$$

and

$$J_3 = \left( \int_0^\infty \left( y^{1-\theta (\beta_1 - \beta_0)} \left( \int_0^y \left( \sup_{x \in \Omega} \|f\|_{L^p(B_t(x))} \right)^\sigma \frac{dt}{t} \right) \right)^{\frac{1}{\sigma}} \right).$$

Note that $\gamma^{-\frac{1}{\sigma}} \leq \max\{1, \gamma^{-\frac{1}{\sigma}}\}$.

Step 3. For the integral estimates $J_1, J_2, J_3$ apply the following variants of Hardy inequalities: if $\mu > 0$, $-\infty < \nu < \infty$ and $0 < \sigma \leq \tau \leq \infty$, then

$$\left( \int_0^\infty \left( \int_0^y \left( g(r) \right)^\nu \left( \frac{dr}{r} \right) \right)^\frac{1}{\nu} \frac{dy}{y} \right)^\frac{1}{\tau} \leq (\mu^\sigma)^{-\frac{1}{\tau}} \left( \int_0^\infty \left( g(y) \right)^\nu \left( \frac{dy}{y} \right) \right)^\frac{1}{\tau}. $$
and
\[
\left( \int_0^\infty \left( \int_0^y (r^{-\nu} |g(r)|)^{\sigma} \frac{dr}{r} \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}} \leq (\mu \sigma)^{-\frac{1}{\tau}} \left( \int_0^\infty (y^{\mu-\nu} |g(y)|) \tau \frac{dy}{y} \right)^{\frac{1}{\tau}}.
\]

According to these inequalities we have
\[
J_1 \leq (\theta(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\tau}} \left( \int_1^\infty (y^{\beta_1 - \beta_0}) \frac{dy}{y} \right)^{\frac{1}{\tau}} = (\theta(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\tau}} \|f\|_{M_{q,.1}^0},
\]
\[
J_2 \leq (\theta(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\tau}} \|f\|_{M_{q,.1}^0},
\]
and
\[
J_3 \leq ((1 - \theta)(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\tau}} \|f\|_{M_{q,.1}}.
\]

Hence,
\[
\|Tf\|_{M_{p,.1}^0} \leq c_3 (M_0 e^{-\theta(\beta_1 - \beta_0)} + M_1 e^{(1-\theta)(\beta_1 - \beta_0)}) \|f\|_{M_{q,.1}^0},
\]
where \(c_3 > 0\) depends only on \(\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma\) and \(\theta\).

Let now \(c = (\frac{M_1}{M_0})^{\frac{1}{\alpha_0 - \alpha_1}}\), then
\[
\|Tf\|_{M_{p,.1}^0} \leq 2c_3 M_{1}^{1-\theta} M_{q,.1}^0 \|f\|_{M_{q,.1}^0}.
\]

Step 4. If \(\beta_1 < \beta_0, \alpha_0 < \alpha_1\) or \(\beta_0 < \beta_1, \alpha_1 < \alpha_0\) or \(\beta_1 < \beta_0, \alpha_1 < \alpha_0\), then the assumptions of Part 2 and 3 are similar, changing only the choice of parameters \(\gamma\) and \(c\). In the first case \(\gamma = \frac{q_0 - q_1}{q_0 - q_1}, c = (\frac{M_1}{M_0})^{\frac{1}{\alpha_0 - \alpha_1}},\) in the second \(\gamma = \frac{q_0 - q_1}{q_0 - q_1}, c = (\frac{M_1}{M_0})^{\frac{1}{\alpha_0 - \alpha_1}},\) and in the third \(\gamma = \frac{q_0 - q_1}{q_0 - q_1}, c = (\frac{M_1}{M_0})^{\frac{1}{\alpha_0 - \alpha_1}},\) and the proof of the theorem is completed.

\[\square\]

**Corollary 3.1.** Let \(0 < \alpha_0, \alpha_1, \beta_0, \beta_1 < \infty, \alpha_0 \neq \alpha_1, \beta_0 \neq \beta_1, 0 < p, q \leq \infty, 0 < \sigma \leq \tau \leq \infty, 0 < \theta < 1\) and
\[
\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1, \beta = (1 - \theta)\beta_0 + \theta\beta_1.
\]
Then there exists \(c > 0\), depending only on \(\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta\), such that if \(x \in \mathbb{R}^n, T\) is a quasi-additive operator on \(LM_{q,.1}^0\) and \(LM_{q,.1}^0\), and for some \(M_1, M_2 > 0\) the following inequalities hold
\[
\|Tf\|_{LM_{p,.1}^0} \leq M_1 \|f\|_{LM_{q,.1}^0}
\]
for all functions \(f \in LM_{q,.1}^0, i = 0, 1,\) then
\[
\|Tf\|_{LM_{p,.1}^0} \leq cAM_1^{1-\theta} M_{q,.1}^0 \|f\|_{LM_{q,.1}^0}
\]
for all functions \(f \in LM_{q,.1}^0\).
Proof. It suffices to take \( \Omega = \{ x \} \) in Theorem 3.1. \( \square \)

Remark 3.1. If \( T \) is a linear operator, then the statement of Corollary 3.1 follows using the standard arguments of interpolation theory from the equality

\[
(\text{LM}_{p;r_0,x}^{\lambda_0}, \text{LM}_{p;r_1,x}^{\lambda_1})_{\theta;r} = \text{LM}_{p,r,x}^{\lambda},
\]

where \( 0 < p, r_0, r_1, \tau \leq \infty, 0 < \lambda_0, \lambda_1 < \infty, \lambda_0 \neq \lambda_1, 0 < \theta < 1, \lambda = (1-\theta)\lambda_0 + \theta\lambda_1 \), and respectively by equivalence of quasi-norm, which was proved in [14] under the additional assumptions \( \lambda_0, \lambda_1 \leq \frac{n}{p} \) and in general case in [6].

4. Riesz potentials in Morrey-type spaces

Let, for \( 0 < p, \theta \leq \infty \), \( L_{p,\theta}(\mathbb{R}^n) \) be the Lorentz space of functions defined on \( \mathbb{R}^n \), and, for a measurable set \( D \subset \mathbb{R}^n \), \( L_{p,\theta}(D) \) be the set of all functions \( f \) measurable on \( D \) for which extensions of \( f \) by 0 outside \( D \) belong to \( L_{p,\theta}(\mathbb{R}^n) \).

Lemma 4.1. Let \( 1 < p, q, r < \infty \), \( \frac{1}{r} + \frac{1}{\tau} = 1 + \frac{1}{q} \), \( f \in L_{r,\infty}(B - D) \), \( g \in L_{p}(D) \), where \( B, D \subset \mathbb{R}^n \) are measurable sets, then

\[
\left\| \int_D f(\cdot - y)g(y)dy \right\|_{L_q(B)} \leq c\|f\|_{L_{r,\infty}(B - D)}\|g\|_{L_p(D)},
\]

where \( c > 0 \) depends only on the parameters \( n, p, q \) and \( r \).

Proof. Let \( f \in L_{r,\infty}(B - D) \), \( g \in L_{p}(D) \). Set

\[
\tilde{g}(y) = \begin{cases} 
  g(y), & y \in D, \\
  0, & y \notin D
\end{cases}
\]

and

\[
\tilde{f}(y) = \begin{cases} 
  f(y), & y \in B - D, \\
  0, & y \notin B - D.
\end{cases}
\]

Using O’Neil’s inequality, we have

\[
\left\| \int_D f(\cdot - y)g(y)dy \right\|_{L_q(B)} = \left\| \int_D \tilde{f}(\cdot - y)\tilde{g}(y)dy \right\|_{L_q(B)} = \left\| \int_{\mathbb{R}^n} \tilde{f}(\cdot - y)\tilde{g}(y)dy \right\|_{L_q(B)}
\]

\[
\leq \left\| \int_{\mathbb{R}^n} \tilde{f}(\cdot - y)\tilde{g}(y)dy \right\|_{L_q(\mathbb{R}^n)} \leq c\|\tilde{f}\|_{L_{r,\infty}(\mathbb{R}^n)}\|\tilde{g}\|_{L_p(\mathbb{R}^n)}
\]

\[
= c\|f\|_{L_{r,\infty}(B - D)}\|g\|_{L_p(D)},
\]

which completes the proof of the lemma. \( \square \)
Let \( z \in \mathbb{R}^n \), \( 1 < p < q < \infty \), \( 0 < \nu \leq \lambda < \frac{n}{q} \), \( 1 + \frac{1}{q} = \frac{1}{p} + \frac{\lambda - \nu}{\nu} \) and \( \frac{1}{s} = \frac{1}{r} - \frac{\nu}{\nu} \). If \( f \in LM_{p,1}^{\nu} \), \( k \in L_{r,\infty}(\mathbb{R}^n) \) and
\[
(4.1) \quad M = \sup_{m \in \mathbb{Z}} 2^{\nu m} \| k \|_{L_s,\infty(D_m)} < \infty,
\]
where \( D_m = B_{2m+1}(0) \setminus B_{2m}(0) \). Then the convolution
\[
(k \ast f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy
\]
exists for almost all \( x \in \mathbb{R}^n \) and the following estimate
\[
\| k \ast f \|_{LM_{q,\infty,\infty}^{\lambda}} \leq c(\| k \|_{L_{r,\infty}(\mathbb{R}^n)} + M)\| f \|_{LM_{p,1}^{\nu}},
\]
holds, where \( c > 0 \) depends only on the parameters \( n, p, q, r, \nu, \lambda \).

**Proof.** Let \( z \in \mathbb{R}^n \) and \( l \in \mathbb{Z} \). By applying Minkowski’s inequality, we get
\[
\| k \ast f \|_{L_q(B_{2l}(z))} = \left\| \int_{B_{2l}(0)} k(y)f(x-y)dy + \sum_{k=1}^{\infty} \int_{D_{l+k}} k(y)f(x-y)dy \right\|_{L_q(B_{2l}(z))}
\]
\[
\leq \left\| \int_{B_{2l}(0)} k(y)f(x-y)dy \right\|_{L_q(B_{2l}(z))} + \sum_{m=1}^{\infty} \left\| \int_{D_m} k(y)f(x-y)dy \right\|_{L_q(B_{2l}(z))}
\]
\[
= I_1 + I_2.
\]

Let us estimate each summand separately. By applying Lemma 4.1 with \( 1/r_1 = 1 + 1/q - 1/p \) we obtain
\[
I_1 \leq \| k \|_{L_{\infty,\infty}(B_{2l}(0))}\| f \|_{L_p(B_{2l+1}(z))}
\]
\[
\leq 2^{\lambda l} \| k \|_{L_{\infty,\infty}(B_{2l}(0))}\| f \|_{LM_{p,1}^{\nu},\infty}.
\]

For the second estimate we use Hölder’s inequality, Lemma 4.1, Lemma 2.2, and we get
\[
I_2 \leq 2^{\lambda l} \sum_{m=1}^{\infty} \left\| \int_{D_m} k(y)f(x-y)dy \right\|_{L_q(B_{2l}(z))}
\]
\[
\leq 2^{\lambda l} \sum_{m=1}^{\infty} \| k \|_{L_{s,\infty}(D_m)}\| f \|_{L_p(B_{2l}(z) - D_m)}
\]
\[
\leq 2^{\lambda l} \sum_{m=1}^{\infty} 2^{\nu(m+2)} \| k \|_{L_{s,\infty}(D_m)}(2^{-\nu(m+2)}\| f \|_{L_p(B_{2m+2}(z))})
\]
\[
< 2^{\lambda l} \sup_{m \geq l} 2^{\nu(m+2)} \| k \|_{L_{s,\infty}(D_m)} \sum_{m=2}^{\infty} (2^{-\nu(m+2)}\| f \|_{L_p(B_{2m+2}(z))})
\]
\[
\leq 2^{\lambda l} \sup_{m \geq l} 2^{\nu(m+2)} \| k \|_{L_{s,\infty}(D_m)}\| f \|_{LM_{p,1}^{\nu},\infty}.
\]
Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

where \(1/q_1 = 1/q - \lambda/n\), \(1/s = 1 + 1/q - \lambda/n - 1/p = 1/r - \nu/n\).

According to Lemma 2.1 we have

\[
2^{-\lambda t} \|k \ast f\|_{L_q(B_r(z))} \lesssim \left(2^{(\nu-\lambda)t} \|k\|_{L_{r,\infty}(B_r(0))} \|f\|_{LM_{p,\infty,z}} + \sup_{m \in \mathbb{Z}} 2^{\nu m} \|k\|_{L_{s,\infty}(D_m)} \|f\|_{LM_{p,\infty,z}}\right)
\]

\[
\lesssim \left(2^{(\nu-\lambda)t} \|k\|_{L_{r,\infty}(B_r(0))} + \sup_{m \in \mathbb{Z}} 2^{\nu m} \|k\|_{L_{s,\infty}(D_m)}\right) \|f\|_{LM_{p,\infty,z}}.
\]

Since \(\nu \leq \lambda\), we get

\[
2^{(\nu-\lambda)t} \|k\|_{L_{r,\infty}(B_r(0))} \lesssim 2^{(\nu-\lambda)t} \sup_{0 < s \leq 2^l} s^{\frac{1}{r} - \frac{\lambda}{n}} k^* (s)
\]

\[
= 2^{(\nu-\lambda)t} \sup_{0 < s \leq 2^l} s^{\frac{1}{r} - \frac{\lambda}{n}} - s^{\frac{1}{r} - \frac{\nu}{n}} k^* (s) \lesssim \sup_{s > 0} s^{\frac{1}{r} - \frac{\nu}{n}} k^* (s).
\]

Therefore,

\[
2^{(\nu-\lambda)t} \|k\|_{L_{r,\infty}(B_r(0))} \lesssim \|k\|_{L_{r,\infty}(\mathbb{R}^n)},
\]

where \(\frac{1}{r} = \frac{1}{r_1} - \frac{\lambda}{n}\).

Since \(l \in \mathbb{Z}\) is arbitrary, by Lemma 2.2 we get

\[
\|k \ast f\|_{LM_{p,\infty,z}} \gtrsim \sup_{l \in \mathbb{Z}} 2^{-\lambda t} \|k \ast f\|_{L_p(B_r(z))} \lesssim \left(\|k\|_{L_{r,\infty}(\mathbb{R}^n)} + M\right) \|f\|_{LM_{p,\infty,z}},
\]

which completes the proof of the theorem. \(\square\)

Note that assumption (4.1) in Theorem 4.1 is essential, i.e. the direct analogue of the O’Neil inequality

\[
(4.2) \quad \|k \ast f\|_{LM_{p,\infty,z}} \leq c\|k\|_{L_{r,\infty}} \|f\|_{LM_{p,\infty,z}}
\]

does not hold for any \(c > 0\) depending only on the parameters \(n, p, q, r, \nu, \lambda\).

**Example 4.1.** Let \(z = 0\), \(1 < p \leq q < \infty\), \(0 < \nu \leq \lambda < \frac{n}{q}\), \(1 + \frac{1}{q} = \frac{1}{p} + \alpha + \frac{\lambda - \nu}{n}\), \(\alpha > \frac{1}{r}p\). Define the functions

\[
f(x) = \begin{cases} 1, & |x| \in [l^\alpha, l^\alpha + 1], \quad l \in \mathbb{Z}_+ \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
k(x) = \begin{cases} (l + 1)^{-\frac{1}{r}}, & |x| \in [l^\alpha, l^\alpha + 1], \quad l \in \mathbb{Z}_+ \\ 0, & \text{otherwise} \end{cases}
\]

We show that the function \(f\) belongs to the space \(LM_{q,1,0}\). Indeed,

\[
\|f\|_{LM_{q,1,0}} = \int_0^\infty \frac{1}{t^{\nu+1}} \left(\int_{-t}^t |f(y)|^q \, dy\right)^{1/p} \, dt.
\]
Furthermore,
\[ \int_{-t}^{t} |f(y)|^p \, dy = 2 \min\left(t, \sum_{l \leq t} 1\right) \leq 2 \min(t, t^{1/\alpha}), \]
hence,
\[ \|f\|_{LM^{p,1,0}} < \infty. \]

Since \( k^*(t) \leq t^{-1/r} \), it follows that \( k \in L_{r,\infty}(\mathbb{R}) \), but also for \( x \in [-1/2, 1/2] \) we have
\[ (k * f)(x) = \int_{-\infty}^{+\infty} k(x-y)f(y) \, dy \geq \frac{1}{2} \sum_{l=1}^{\infty} (l+1)^{-1/r} = \infty, \]
and an inequality of type (4.2) is impossible.

**Theorem 4.2.** Let \( \Omega \subset \mathbb{R}^n \), \( 1 < p < q < \infty \), \( 0 < \tau \leq \infty \), \( 0 < \nu \leq \lambda < \frac{n}{q} \) and \( \gamma = n\left(\frac{1}{p} - \frac{1}{q}\right) + \lambda - \nu \),
\[ I_{\gamma} f(x) = \int_{\mathbb{R}^n} f(x-y) \frac{dy}{|y|^n-\gamma}. \]

If \( f \in M^{\nu}_{p,\tau,\Omega} \), then \( I_{\gamma} f \in M^{\lambda}_{q,\tau,\Omega} \), and the following inequality holds
\[ (4.3) \quad \| I_{\gamma} f \|_{M^{\lambda}_{q,\tau,\Omega}} \leq c \| f \|_{M^{\nu}_{p,\tau,\Omega}}, \]
where \( c > 0 \) depends only on \( n, p, q, \tau, \nu, \lambda \).

**Proof.** Let \( k(x) = 1/|x|^{n-\gamma} \). Check the conditions of Theorem 4.1. By the assumptions of Theorems 4.1 and 4.2 \( \frac{1}{r} + \frac{2}{n} - 1 = 0 \). Moreover, \( k^*(t) \asymp t^{n-1} \) on \((0, \infty)\), \( (k \chi_{D_m})^*(t) = 0 \) if \( |t| \geq |D_m| \) and \( (k \chi_{D_m})^*(t) \asymp 2^m(\gamma-n) \) on \((0, |D_m|)\) (because \( k(x) \asymp 2^m \) on \( D_m \)). Hence, it follows that
\[ \|k\|_{L_{r,\infty}(\mathbb{R}^n)} = \sup_{t>0} t^{1/r} k^*(t) \asymp \sup_{t>0} t^{n+\frac{n}{r}-1} = 1, \]
\[ \sup_{m \in \mathbb{Z}} 2^{\nu m} \|k\|_{L_{r,\infty}(D_m)} \sup_{m \in \mathbb{Z}} 2^{\nu m} \sup_{0 < t \leq |D_m|} t^{\frac{1}{r}} (k \chi_{D_m})^*(t) \]
\[ \asymp \sup_{m \in \mathbb{Z}} 2^{\nu m} \left( \sup_{0 < t \leq 2^m} t^{n-\frac{n}{r}-1} \right) 2^m(\gamma-n) = 1. \]

Then by applying Theorem 4.1 we obtain the weak inequality
\[ \| I_{\gamma} f \|_{LM^{\nu}_{p,\tau,\Omega}} \leq c \| f \|_{LM^{\nu}_{p,\tau,\Omega}}, \quad z \in \Omega. \]
Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

Next, since \( 0 < \nu < \frac{n}{p} \), \( 0 < \lambda < \frac{n}{q} \), then there exist \( \nu_0, \nu_1 \) and \( \lambda_0, \lambda_1 \), such that

\[
0 < \nu_1 < \nu_1 < \nu_0 < \frac{n}{p}, \quad 0 < \lambda_1 < \lambda < \lambda_0 < \frac{n}{q}.
\]

Let \( \nu_0 - \lambda_0 = \nu_1 - \lambda_1 = \nu - \lambda \). Then the weak inequalities hold

\[
\|I_\gamma f\|_{L_{M_i}^{\lambda_0, \infty}} \leq c \|f\|_{L_{M_i}^{\nu_1, z}}, \quad i = 0, 1, z \in \Omega.
\]

Hence, by interpolation Theorem 3.1 the strong inequality holds

\[
\|I_\gamma f\|_{M_i}^{\lambda_0, r, \Omega} \leq c \|f\|_{M_i}^{\nu_1, r, \Omega},
\]

where \( 0 < r \leq \infty \), \( \nu = (1 - \theta)\nu_0 + \theta\nu_1 \), \( \lambda = (1 - \theta)\lambda_0 + \theta\lambda_1 \) and \( 0 < \theta < 1 \),

\[
1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{(1 - \theta)(\lambda_0 - \nu_0) + \theta(\lambda_1 - \nu_1)}{n}
\]

\[
= 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{\lambda - \nu}{n},
\]

which completes the proof of the theorem.

If \( r = \infty \) and \( \Omega = \mathbb{R}^n \), then inequality (4.3) takes the form

\[
\|I_\gamma f\|_{M_i}^{\lambda_0} \leq c \|f\|_{M_i}^{\nu_1}.
\]

Note that the assumption \( \nu \leq \lambda \) of Theorem 4.2 in this case is not required, and this inequality holds for any \( 1 < p < q < \infty \), \( 0 < \nu < \frac{n}{p} \), \( 0 < \lambda < \frac{n}{q} \), \( \gamma = n(\frac{1}{p} - \frac{1}{q}) + \lambda - \nu \). For \( \lambda = \nu \) this was proved in [24], under the assumption \( q\lambda = p\nu \) in [1] (in this case \( \nu > \lambda \)), and, without any additional assumptions on \( \nu \) and \( \lambda \), in [11, 12].

The condition \( \nu \leq \lambda \) appears in Theorem 4.2 because arbitrary \( 0 < r \leq \infty \) and \( \Omega \subset \mathbb{R}^n \) are considered. Note that at least for the case \( r = q \) and \( \Omega = \{0\} \) this condition is a necessary condition for the validity of inequality (4.3). Indeed, by (4.3) and Lemma 2.1 (ii) it follows that

\[
\|I_\gamma f\|_{L_{M_i}^{\lambda_0, \infty}} \leq c_1 \|f\|_{L_{M_i}^{\nu_1, p, \Omega}},
\]

that is equivalent to the inequality

\[
\|I_\gamma f\|_{L_{q,\{x\} = |x|^{-\lambda}}} \leq c_2 \|f\|_{L_{p,(x)} = |x|^{-\nu}}.
\]

Indeed, let \( N \in \mathbb{N}, \quad z_N = (N, 0, \ldots, 0), \)

\[
f(x) = \chi_{B_2(z_N)}(x)|x|^\nu.
\]
Then by (4.4) we have

\[ |B_2(0)|^{1/p} c_2 \geq \|I_{\gamma,f}\|_{L_q(|x|^{-\nu})} \]

\[ \geq \left( \int_{B_1(z_N)} \left| \frac{1}{|x|^\lambda} \int_{x-B_2(z_N)} \frac{|x-y|^{\nu}}{|y|^{n-\gamma}} \, dy \right|^q \, dx \right)^{1/q}. \]

Since \( x - B_2(z_N) \supset B_1(0) \) for any \( x \in B_1(z_N) \) we get for \( N > 2 \)

\[ |B_2(0)|^{1/p} c_2 \geq \left( \int_{B_1(z_N)} \left| \frac{1}{|x|^\lambda} \int_{B_1(0)} \frac{|x-y|^{\nu}}{|y|^{n-\gamma}} \, dy \right|^q \, dx \right)^{1/q} \]

\[ \geq \left( \int_{B_1(0)} \frac{dy}{|y|^{n-\gamma}} \left( \int_{B_1(z_N)} \left| \frac{|x-y|^{\nu}}{|x|^{\lambda}} \right|^q \, dx \right)^{1/q} \right)^{1/q} \]

\[ \geq c_3 \left( \int_{B_1(z_N)} |x|^{(\nu-\lambda)q} \, dx \right)^{1/q}, \]

where \( c_1, c_2, c_3 > 0 \) are independent of \( N \) and this is possible only if \( \nu \leq \lambda \).

5. Singular operators in Morrey-type spaces

**Theorem 5.1.** Let \( \Omega \subset \mathbb{R}^n \), \( 1 \leq p < \infty \), \( 0 < \nu < \frac{n}{p} \), \( 0 < \tau \leq \infty \). If a subadditive operator \( T \) is bounded from \( L_p(\mathbb{R}^n) \) to \( L_p(\mathbb{R}^n) \) and there exists \( C > 0 \) such that

\[ |Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} \, dy \]

for all locally integrable functions \( f \) with compact support and for all \( x \in \mathbb{R}^n \setminus \text{supp} f \), then \( T \) is also bounded from \( M^\nu_{p,\tau,\Omega} \) to \( M^\nu_{p,\tau,\Omega} \).

**Proof.** Let \( z \in \Omega \), \( l \in \mathbb{Z} \) and \( D_m(z) = B_{2^{m+1}}(z) \setminus B_{2^m}(z) \), \( m \in \mathbb{Z} \), \( f \in LM^\nu_{p,1,\Omega} \). By applying Minkowski’s and Hölder’s inequalities and the subadditivity of \( T \) and
Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries 17
taking into account that $f = f \chi_{B_{2^{l+1}}(z)} + \sum_{m=l+1}^{\infty} f \chi_{D_m(z)}$ we get

$$
\|Tf\|_{L_p(B_{2^{l+1}}(z))} \leq \|T(f \chi_{B_{2^{l+1}}(z)})\|_{L_p(B_{2^{l}}(z))} + \|T\left(\sum_{m=l+1}^{\infty} f \chi_{D_m(z)}\right)\|_{L_p(B_{2^{l}}(z))}
$$

$$
\leq \|T(f \chi_{B_{2^{l+1}}(z)})\|_{L_p(B_{2^{l}}(z))} + \left\|\sum_{m=l+1}^{\infty} |T(f \chi_{D_m(z)})|\right\|_{L_p(B_{2^{l}}(z))}
\lesssim \|T(f \chi_{B_{2^{l+1}}(z)})\|_{L_p(B_{2^{l}}(z))} + 2^{ql}\left\|\sum_{m=l+1}^{\infty} |T(f \chi_{D_m(z)})|\right\|_{L_p(B_{2^{l}}(z))} \equiv I_1 + I_2,
$$

where $q$ is defined by the equality $\frac{1}{p} = \frac{1}{q} + \frac{\nu}{n}$.

Let us estimate $I_1$ and $I_2$. Since $T$ is bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ we get

$$
I_1 \leq \|T(f \chi_{B_{2^{l+1}}(z)})\|_{L_p(\mathbb{R}^n)} \lesssim \|f\|_{L_p(B_{2^{l+1}}(z))}.
$$

Since $B_{2^{l}}(z) \cap \text{supp} f \chi_{D_m(z)} = \emptyset$ for any $m \geq l+1$ by applying estimate (5.1) with $f$ replaced by $f \chi_{D_m(z)}$ we get

$$
I_2 \leq 2^{ql}\sum_{m=l+1}^{\infty} \|T(f \chi_{D_m(z)})\|_{L_p(B_{2^{l}}(z))}
$$

$$
\lesssim 2^{ql}\sum_{m=l+1}^{\infty} \left\|\int \frac{|f(y)|}{|x-y|^r} dy\right\|_{L_p(B_{2^{l}}(z))}.
$$

Since for $m \geq l+1$, $B_{2^{l}}(z) - D_m(z) = B_{2^{l}}(0) + z - (D_m(0) + z) = B_{2^{l}}(0) - D_m(0) = D_m(0) - B_{2^{l}}(0) \subset G_m = D_{m-1}(0) \cup D_m(0) \cup D_{m+1}(0)$, application of Lemma 4.1 yields

$$
I_2 \lesssim 2^{ql}\sum_{m=l+1}^{\infty} \|x|^{-n}\|_{L_{r,\infty}(B_{2^{l}}(z) - D_m(z))} \|f\|_{L_p(D_m(z))}
$$

$$
\leq 2^{ql}\sum_{m=l+1}^{\infty} \|x|^{-n}\|_{L_{r,\infty}(G_m)} \|f\|_{L_p(B_{2^{m+1}}(z))},
$$

where $1/r = 1 + 1/q - 1/p = 1 - \nu/n$.

Note that

$$
\|x|^{-n}\|_{L_{r,\infty}(G_m)} \leq \|x|^{-n}\chi_{G_m}(x)\|_{L_{r,\infty}(\mathbb{R}^n)} = \sup_{t>0} t^{\frac{1}{r}} \sup_{0<t \leq |G_m|} |x|^{-n}\chi_{G_m}(x)\|_{L_{r,\infty}(\mathbb{R}^n)}
$$

$$
= \sup_{0<t \leq |G_m|} t^{\frac{1}{r}} \left\|\int_0^t |x|^{-n}\chi_{G_m}(x)\|_{L_{r,\infty}(\mathbb{R}^n)}
$$

$$
\lesssim 2^{-m(\frac{n}{r} - \frac{1}{r})} = 2^{-\nu m}.
$$
Hence,
\[
I_2 \lesssim 2^{\nu l} \sum_{m=l+1}^{\infty} 2^{-\nu(m+1)} \|f\|_{L_p(B_{2m+1}(z))}.
\]
Therefore by Lemma 2.2 with \( q = 1 \)
\[
2^{-\nu l} \|Tf\|_{L_p(B_2(z))} \lesssim 2^{-\nu l} \|f\|_{L_p(B_{2l+1}(z))} + \sum_{m=l+1}^{\infty} 2^{-\nu(m+1)} \|f\|_{L_p(B_{2m+1}(z))}.
\]
Due to arbitrariness of \( l \) by Lemma 2.2 with \( q = \infty \) we get that for all \( z \in \Omega \)
\[
\|Tf\|_{L_{\infty}^{p,\infty}(z)} \lesssim \sup_{l \in \mathbb{Z}} 2^{-\nu l} \|Tf\|_{L_p(B_{2l}(z))} \lesssim \|f\|_{L^p_{\infty,1}(z)}.
\]
Application of Theorem 3.1 completes the proof of the theorem.

**Remark 5.1.** Let \( T \) be a Calderon-Zygmund operator, i.e. a linear operator taking \( C_0^\infty \) into \( L_1^{\text{loc}} \), bounded on \( L^2 \) and represented by
\[
Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad \text{a.e. on } \mathbb{R}^n \setminus \text{supp} f
\]
for every function \( f \in L^\infty(\mathbb{R}^n) \) with compact support. Here \( K(x,y) \) is a continuous function away from the diagonal and satisfies the standard estimates: for some \( c_1 > 0 \) and \( 0 < \varepsilon \leq 1 \)
\[
|K(x,y)| \leq c_1 |x-y|^{-n},
\]
for all \( x, y \in \mathbb{R}^n, x \neq y \) and
\[
|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq c_1 \left( \frac{|x-x'|}{|x-y|} \right)^\varepsilon \frac{|x-y|^{-n}}{|x-y|}
\]
whenever \( 2|x-x'| \leq |x-y| \). This class of operators was introduced by Coifman and Meyer [16].

It is known that a Calderon-Zygmund operator is bounded in the Lebesgue spaces \( L_p(\mathbb{R}^n) \) if \( p \in (1, \infty) \). Hence, Theorem 5.1 can be applied to all Calderon-Zygmund operators with any \( 1 < p < \infty \).

**Acknowledgements.** The work on the paper was started when the authors were staying at the Department of Mathematics of Padova University (Padova, Italy) in Summer 2014. It was supported by the grant of the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.0008).
Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

REFERENCES


Received submission date; revised revision date