Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

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ABSTRACT – We introduce a class of Morrey-type spaces $M_{p,q}^\lambda$, which includes classical Morrey spaces and discuss their properties. We prove a Marcinkiewicz-type interpolation theorem. This theorem is then applied to obtaining the boundedness in the introduced Morrey-type spaces of the Riesz potential and singular integral operator.

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1. Introduction

Let $0 < p \leq \infty$ and $0 \leq \lambda \leq \frac{\mu}{p}$. The Morrey spaces $M_p^\lambda$ were defined as the spaces of all functions $f \in L_p^{\infty}(\mathbb{R}^n)$ such that

$$||f||_{M_p^\lambda} \equiv ||f||_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \sup_{t > 0} t^{-\lambda} ||f||_{L_p(B_t(x))} < \infty,$$

where $B_t(x)$ is the open ball of radius $t > 0$ with center at the point $x \in \mathbb{R}^n$.

If $\lambda = 0$, then $M_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, while if $\lambda = \frac{\mu}{p}$, then $M_p^{\mu/p}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

If $\lambda < 0$ or $\lambda > \frac{\mu}{p}$, then $M_p^\lambda = \Theta$, where $\Theta$ is the set of all functions that are equivalent to zero on $\mathbb{R}^n$. See [1].

Classical Morrey spaces were introduced by Morrey in 1938 and arose in connection with some problems of the theory of partial differential equations and theory of variations [1]. There is a number of books and survey papers on Morrey and Morrey-type spaces and classical operators of real analysis in Morrey-type spaces, for example, [1]-[14].

This paper is devoted to the interpolation properties of Morrey-type spaces. Some results for classical Morrey spaces were obtained in Stampacchia [15], Campanato and Murthy [16], Peetre [6]. In particular in [6] it is proved that

$$(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta, \infty} \subset M_p^\lambda,$$

where $1 \leq p < \infty$, $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$. In Ruiz and Vega [17], Blasco, Ruiz and Vega [18] it is proved that this inclusion is strict.

In [5] a more detailed investigation of the interpolation problem for Morrey spaces was carried out. In particular, it was proved that the inclusion

$$(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta, \infty} \subset M_p^\lambda,$$

where $1 \leq p_0, p_1 < \infty$, $0 < \lambda_0 < \frac{\mu}{p_0}$, $0 < \lambda_1 < \frac{\mu}{p_1}$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$, $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ holds if and only if $p_0 = p_1$.

The case of local Morrey-type spaces $LM_p^\lambda$ was considered in Burenkov and Nursultanov [19]. Let $0 < p, q \leq \infty$, $\lambda \geq 0$. A function $f \in LM_p^\lambda$ if $f \in L_p^{\infty}(\mathbb{R}^n)$ and

$$||f||_{LM_p^\lambda} = ||t^{-\lambda}||f||_{L_p(B_1(0))}||_{L_q(0, \infty)} < \infty.$$

In [19] it was proved, in particular, that local Morrey-type spaces $LM_p^\lambda$ form an interpolation scale when $p$ is fixed, i.e.

$$(LM_p^{\lambda_0}, LM_p^{\lambda_1})_{\theta, q} = LM_p^\lambda,$$

where $0 < p, q, q_1, q \leq \infty$, $0 < \theta < 1$ and $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$.

Further generalization of interpolation properties for general local Morrey-type spaces $LM_p^\lambda(G, \mu)$ was discussed in [20], namely, it was proved that

$$(LM_p^{\lambda_0}(G, \mu), LM_p^{\lambda_1}(G, \mu))_{\theta, q} = LM_p^\lambda(G, \mu),$$
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under the same assumptions on the numerical parameters.

In this paper we introduce a generalized class of Morrey-type spaces $M^{\lambda}_{p,q}(\Omega)$, which coincide with the classical Morrey space $M^\lambda_p$ in the case of $q = \infty$ and $\Omega = \mathbb{R}^n$. According to the above results and also [21] it follows that the classical interpolation theorems for this scale of the spaces do not take place. Nevertheless, in contrast to the classical interpolation theorems we prove some analog of interpolation theorem. Compared with the classical interpolation theorems the condition of theorem is formulated in terms of local Morrey-type spaces $LM^{\lambda}_{p,q,x}$ and the statement in terms of generalized Morrey-type spaces $M^{\lambda}_{p,q}(\Omega)$. At the same time we are saying that this theorem is an interpolation theorem of Marcinkiewicz-type, because it allows us to obtain from the weak estimates for linear and quasi-linear operators in terms of local Morrey-type spaces the stronger estimates in terms of generalized Morrey-type spaces.

Let $f \in L_1^{\text{loc}}$. In this paper we also study estimates for the norm of Riesz operator, which is defined as follows

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\gamma}}, \quad 0 < \gamma < n.$$

Let us consider the well-known Hardy-Littlewood-Sobolev inequality. Let $1 < p < q < \infty$, $\gamma = \frac{n}{p} - \frac{n}{q}$, then operator $I_\gamma$ is bounded from $L_p$ to $L_q$.

The boundedness of $I_\gamma$ in classical Morrey spaces and their generalizations was investigated by Adams [9], Chiarenza and Frasca [10], Peetre [6], Nakai [11], Sawano and Tanaka [12], and others. Further results of the boundedness of Riesz potential for local Morrey-type spaces were obtained in Burenkov, Gogatishvili, Guliyev and Mustafayev [13]. In this paper we continue the study of the boundedness of $I_\gamma$ in the case of Morrey-type spaces. In particular, by applying the Marcinkiewicz-type interpolation theorem we obtain estimate for the norm of Riesz operator in the Morrey-type spaces.

Also we study the boundedness of singular operators. The classical results for Calderon-Zygmund operators of different classes state that if $1 < p < \infty$, then this operator is bounded from $L_p$ to $L_p$ (see, for example, [22], [23]). J. Peetre [7] studied the boundedness of singular operator in Morrey spaces. Further results of the boundedness of Calderon-Zygmund operator for local Morrey-type spaces were obtained in Burenkov, Guliyev, Serbetci and Tararykova [14], and others.

Given functions $F$ and $G$, in this paper $F \lesssim G$ means that $F \leq CG$, where $C$ is a positive number depending only on numerical parameters that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$. 

Let $f \in L_1^{\text{loc}}$. In this paper we also study estimates for the norm of Riesz operator, which is defined as follows

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\gamma}}, \quad 0 < \gamma < n.$$
2. Spaces $M_{p,q}^\lambda (\Omega)$

Let $\Omega \subset \mathbb{R}^n$, $0 < p < \infty$, $0 < q \leq \infty$ and $0 \leq \lambda \leq \frac{n}{p'}$. We consider generalized Morrey-type spaces $M_{p,q}^\lambda (\Omega)$ that are defined for $q < \infty$ as the spaces of all functions $f \in L^{loc}_{p} (\mathbb{R}^n)$ such that

$$
\|f\|_{M_{p,q}^\lambda (\Omega)} = \left( \int_0^\infty \left( t^{-\lambda} \sup_{x \in \Omega} \|f\|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},
$$

and for $q = \infty$,

$$
\|f\|_{M_{p,\infty}^\lambda (\Omega)} = \sup_{x \in \Omega} \sup_{t > 0} t^{-\lambda \cdot \frac{1}{p'}} \|f\|_{L_p(B_t(x))}.
$$

Note that if $p = \infty$, then $M_{\infty,q}^\lambda (\Omega) = \Theta$. Also note that the introduced spaces coincide with the classical Morrey spaces in the case of $q = \infty$ and $\Omega = \mathbb{R}^n$, i.e.

$$
M_{p,\infty}^\lambda (\mathbb{R}^n) = M_p^\lambda.
$$

If $\Omega = \{x\}$ – one-point set, then

$$
M_{p,q}^\lambda (\Omega) = L M_{p,q,x}^\lambda,
$$

where $LM_{p,q,x}^\lambda$ – local Morrey-type spaces [24].

However, these spaces are distinct from the global Morrey-type spaces $G M_{p,q}^\lambda$, which were introduced by V.I. Burenkov, V.S. Guliyev and H.V. Guliyev [24] and defined as the space of all functions $f$ Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norm

$$
\|f\|_{GM_{p,q}^\lambda} = \sup_{x \in \mathbb{R}^n} \left( \int_0^\infty \left( t^{-\lambda} \|f\|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
$$

if $q < \infty$ and usual modification if $q = \infty$.

Note also that the generalized Morrey-type spaces are closed to “net” spaces $N_{p,q}(M)$ [25]. We begin with the following lemma, which describes the properties of spaces $M_{p,q}^\lambda (\Omega)$.

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$.

1. If $0 < p_0 < p_1 < \infty$ and $\lambda_0 = \lambda_1 + n (\frac{1}{p_1} - \frac{1}{p_0})$, then the space $M_{p_0,q}^{\lambda_1} (\Omega)$ is continuously embedded into $M_{p_1,q}^{\lambda_0} (\Omega)$.

2. If $0 < q_0 < q_1 \leq \infty$, then the space $M_{p,q_0}^{\lambda} (\Omega)$ is continuously embedded into $M_{p,q_1}^{\lambda} (\Omega)$. 
Let $f \in M_{p_0}^{\lambda}(\Omega)$. Applying Hölder’s inequality and noting that $|B_t(x)| \asymp t^n$, we get

$$
\|f\|_{M_{p_0}^{\lambda}(\Omega)} = \left( \int_{0}^{\infty} \left( t^{-\lambda_0} \sup_{x \in \Omega} \|f\|_{L_{p_0}(B_t(x))} \right) ^{q_0} \frac{dt}{t} \right)^{\frac{1}{q}}
$$

$$
\leq \left( \int_{0}^{\infty} \left( t^{-\lambda_0} \sup_{x \in \Omega} \|f\|_{L_{p_1}(B_t(x))} |B_t(x)|^{\frac{1}{p_0} - \frac{1}{p_1}} \right) ^{q_0} \frac{dt}{t} \right)^{\frac{1}{q}}
$$

$$
\leq \left( \int_{0}^{\infty} \left( t^{-\lambda_0 - n\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} \sup_{x \in \Omega} \|f\|_{L_{p_1}(B_t(x))} \right) ^{q_0} \frac{dt}{t} \right)^{\frac{1}{q}} \|f\|_{M_{p_0}^{\lambda}(\Omega)},
$$

which means inclusion $M_{p_0,q}^{\lambda}(\Omega) \hookrightarrow M_{p_0}^{\lambda}(\Omega)$.

(ii). First let $q_1 = \infty$ and $f \in M_{p_0}^{\lambda}(\Omega)$. Let us proof that $M_{p_0,q_1}^{\lambda}(\Omega) \hookrightarrow M_{p_0}^{\lambda}(\Omega)$.

$$
\|f\|_{M_{p_0}^{\lambda}(\Omega)} = \sup_{t>0} t^{-\lambda} \sup_{x \in \Omega} \|f\|_{L_p(B_t(x))}
$$

$$
= (\lambda q_0) \sup_{t>0} \left( \int_{t}^{\infty} t^{-\lambda q_0} \frac{dt}{\tau} \right) ^{\frac{1}{q}} \sup_{x \in \Omega} \|f\|_{L_p(B_t(x))}.
$$

Note that $B_t(x) \subset B_\tau(x)$ if $t \leq \tau$, then we have

$$
\|f\|_{M_{p,\infty}^{\lambda}(\Omega)} \leq (\lambda q_0) \sup_{t>0} \left( \int_{t}^{\infty} t^{-\lambda q_0} \frac{dt}{\tau} \right) ^{\frac{1}{q}} \|f\|_{M_{p_0}^{\lambda}(\Omega)}.
$$

If $q_1 < \infty$, then

$$
\|f\|_{M_{p,q_1}^{\lambda}(\Omega)} = \left( \int_{t}^{\infty} \left( \tau^{-\lambda} \sup_{x \in \Omega} \|f\|_{L_p(B_\tau(x))} \right) ^{q_1} \frac{dt}{\tau} \right) ^{\frac{1}{q_1}}
$$

$$
\leq \left( \|f\|_{M_{p,q_1}^{\lambda}(\Omega)} \right) ^{1 - \frac{q_1}{q_1}} \|f\|_{M_{p_0}^{\lambda}(\Omega)} ^{\frac{q_0}{q_1}} \lesssim \|f\|_{M_{p_0}^{\lambda}(\Omega)}.
$$

This completes the proof of the second part of the lemma.

**Lemma 2.2.** Let $0 < p, q \leq \infty$, $0 < \lambda < \infty$, then for any $\Omega \subset \mathbb{R}^n$

$$
2^{-\lambda} (\ln 2)^{\frac{1}{q}} \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda m} \sup_{x \in \Omega} \|f\|_{L_p(B_2^m(x))} \right) ^{q_0} \right) ^{\frac{1}{q}} \lesssim \|f\|_{M_{p,q}^{\lambda}(\Omega)}
$$
\[ \leq 2^\lambda (\ln 2)^{\frac{1}{q}} \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda m} \sup_{x \in \Omega} \| f \|_{L_p(B_{2^m}(x))} \right)^q \right)^{1/q}. \]

**Proof.** Let \( t \in (0, \infty) \), then there exists \( m \in \mathbb{Z} \), such that \( 2^m \leq t < 2^{m+1} \).

Therefore,
\[
\left( \int_0^\infty \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q} = \left( \sum_{m \in \mathbb{Z}} \int_{2^m}^{2^{m+1}} \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q}.
\]

Thus,
\[
\left( \sum_{m \in \mathbb{Z}} \int_{2^m}^{2^{m+1}} \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q} \leq \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda m} \sup_{x \in \Omega} \| f \|_{L_p(B_{2^m+1}(x))} \right)^q \right)^{1/q}.
\]

On the other hand,
\[
\left( \sum_{m \in \mathbb{Z}} \int_{2^m}^{2^{m+1}} \left( t^{-\lambda} \sup_{x \in \Omega} \| f \|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{1/q} \geq \left( \sum_{m \in \mathbb{Z}} \left( 2^{-\lambda m} \sup_{x \in \Omega} \| f \|_{L_p(B_{2^m}(x))} \right)^q \right)^{1/q},
\]

and we obtain the required equivalence.

\[ \Box \]

### 3. Interpolation theorem

An operator \( T \) is called quasi-additive if for some \( A > 0 \) and for almost all \( y \in \mathbb{R}^n \) the following inequality
\[ |T(f + g)(y)| \leq A(|(Tf)(y)| + |(Tg)(y)|) \]
holds for all functions \( f, g \in LM_{q,\sigma,x}^{\beta_0} + LM_{q,\sigma,x}^{\beta_1} \).

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^n \), \( 0 < \alpha_0, \alpha_1, \beta_0, \beta_1 < \infty, \alpha_0 \neq \alpha_1, \beta_0 \neq \beta_1, 0 < p, q \leq \infty, 0 < \sigma \leq \tau \leq \infty, 0 < \theta < 1 \) and
\[ \alpha = (1 - \theta)\alpha_0 + \theta\alpha_1, \beta = (1 - \theta)\beta_0 + \theta\beta_1. \]

Let \( T \) be a quasi-additive operator given on \( LM_{q,\sigma,x}^{\beta_0} + LM_{q,\sigma,x}^{\beta_1}, x \in \Omega \). Suppose that for some \( M_1, M_2 > 0 \) the following inequalities hold
\[ (3.1) \quad \| Tf \|_{LM_{p,\sigma,x}^{\beta_i}} \leq M_i \| f \|_{LM_{q,\sigma,x}^{\beta_i}}, \quad x \in \Omega, f \in LM_{q,\sigma,x}^{\beta_i}, i = 0, 1, \]
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then

\[ \|Tf\|_{M^\alpha_{p,s}(\Omega)} \leq cAM_0^{1-\theta} M_{\alpha} \|f\|_{M^\alpha_{q,s}(\Omega)} \]

for all functions \( f \in M^\alpha_{q,s}(\Omega) \) and \( c > 0 \) depends only on \( \alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta \).

**Proof.** 1. Let \( f \in M^\alpha_{q,s}(\Omega) \). For every \( x \in \Omega, s > 0 \), we define the functions

\[ f_{0,s} = f \chi_{B_s(x)}, \quad f_{1,s} = f - f_{0,s}, \]

where \( \chi_{B_s(x)} \) denotes the characteristic function of a ball \( B_s(x) \). Then \( f = f_{0,s} + f_{1,s} \) and

\[ \|Tf\|_{L^p(B_s(x))} = \|T(f_{0,s} + f_{1,s})\|_{L^p(B_s(x))} \]

\[ \leq A\|Tf_{0,s}\|_{L^p(B_s(x))} + \|Tf_{1,s}\|_{L^p(B_s(x))} \]

By inequality (3.1) we have

\[ \|Tf_{0,s}\|_{L^p(B_s(x))} = t^{\alpha_0} \|f_{0,s}\|_{L^p(B_s(x))} \]

\[ \leq t^{\alpha_0} \sup_{r > 0} r^{-\alpha_0} \|Tf_{0,s}\|_{L^p(B_s(x))} = t^{\alpha_0} \|Tf_{0,s}\|_{LM^\alpha_{r,\infty,s}} \]

\[ \leq M_0 t^{\alpha_0} \|f_{0,s}\|_{LM^\alpha_{r,\infty,s}} = M_0 t^{\alpha_0} \left( \int_0^s \left( r^{-\beta_0} \|f_{0,s}\|_{L^q(B_s(x))} \right) \frac{dr}{r} \right)^{\frac{1}{\theta}} \]

\[ = M_0 t^{\alpha_0} \left( \int_0^s \left( r^{-\beta_0} \|f_{0,s}\|_{L^q(B_s(x))} \right) \frac{dr}{r} + \int_s^\infty \left( r^{-\beta_0} \|f_{0,s}\|_{L^q(B_s(x))} \right) \frac{dr}{r} \right)^{\frac{1}{\theta}} \]

\[ \leq 2^{1-\frac{1}{\theta}} M_0 t^{\alpha_0} \left( \int_0^s \left( r^{-\beta_0} \|f_{0,s}\|_{L^q(B_s(x))} \right) \frac{dr}{r} \right)^{\frac{1}{\theta}} \]

\[ + \left( \int_s^\infty \left( r^{-\beta_0} \|f_{0,s}\|_{L^q(B_s(x))} \right) \frac{dr}{r} \right)^{\frac{1}{\theta}}. \]

For \( 0 < r \leq s \) and \( y \in B_r(x) \) we have that \( f_{0,s}(y) = f(y) \chi_{B_s(x)}(y) = f(y) \), therefore \( \|f_{0,s}\|_{L^q(B_s(x))} = \|f\|_{L^q(B_s(x))} \). For \( r > s \) and \( y \notin B_r(x) \) we get that \( f_{0,s}(y) = 0 \), therefore \( \|f_{0,s}\|_{L^q(B_s(x))} = \|f\|_{L^q(B_s(x))} \). Hence,

\[ \left( \int_0^s \left( r^{-\beta_0} \|f_{0,s}\|_{L^q(B_s(x))} \right) \frac{dr}{r} \right)^{\frac{1}{\theta}} = \left( \int_0^s \left( r^{-\beta_0} \|f\|_{L^q(B_s(x))} \right) \frac{dr}{r} \right)^{\frac{1}{\theta}} \]
Thus, for all \( s > 0 \) and \( r \in R^n \), we get that

\[
\| T f \|_{L_p(B_r(x))} = t^{\alpha_1} t^{-\alpha_1} \left[ \int_0^1 \left( \int_s^r \| f \|_{L_p(B_r)} \| f \|_{L_p(B_s)} \right)^{\sigma \frac{dr}{r}} \right] \]

Similarly, according to inequality (3.1), since \( f_1, s(y) = 0 \) if \( y \in B_s(x) \) and \( |f_1, s(y)| \leq |f(y)| \) if \( y \in R^n \), we get that

\[
\| T f \|_{L_p(B_r(x))} = t^{\alpha_1} t^{-\alpha_1} \left[ \int_0^1 \left( \int_s^r \| f \|_{L_p(B_r)} \| f \|_{L_p(B_s)} \right)^{\sigma \frac{dr}{r}} \right] \]

So, for all \( t > 0 \) and \( s > 0 \) we obtain

\[
\| T f \|_{L_p(B_r(x))} \leq c_2 A \left( M_0 t^{\alpha_0} \left[ \int_0^1 \left( \int_s^r \| f \|_{L_p(B_r)} \| f \|_{L_p(B_s)} \right)^{\sigma \frac{dr}{r}} \right] \right) \]

where \( c_2 > 0 \) depends only on \( p, \beta_0, \beta_1 \) and \( \sigma \).
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2. Suppose that $\beta_0 < \beta_1, \alpha_0 < \alpha_1$ and set $s = ct^\gamma$, where $\gamma = \frac{\alpha_1 - \alpha_0}{\beta_1 - \beta_0}$, and $c > 0$ will be taken further. Then

$$\|Tf\|_{M^\alpha_{p, r}(\Omega)} = \left( \int_0^\infty \left( t^{-\alpha} \sup_{x \in \Omega} \|Tf\|_{L^p(B_r(x))} \right)^r \frac{dr}{t} \right)^{\frac{1}{r}} \leq 3^{\frac{1}{r} + 1} c_2 A(M_0 I_1 + M_0 I_2 + M_1 I_3),$$

where

$$I_1 = \left( \int_0^\infty \left( t^{\alpha_0 - \alpha} \int_0^{ct^\gamma} \left( r^{-\beta_0} \sup_{x \in \Omega} \|f\|_{L^q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dt}{t} \right)^{\frac{1}{r}},$$

$$I_2 = (ct^\gamma)^{\beta_1 - \beta_0} \left( \int_0^\infty \left( t^{\alpha_0 - \alpha} \int_0^{ct^\gamma} \left( r^{-\beta_1} \sup_{x \in \Omega} \|f\|_{L^q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dt}{t} \right)^{\frac{1}{r}},$$

and

$$I_3 = \left( \int_0^\infty \left( t^{\alpha_0 - \alpha} \int_0^{ct^\gamma} \left( r^{-\beta_0} \sup_{x \in \Omega} \|f\|_{L^q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dt}{t} \right)^{\frac{1}{r}}.$$

Making the change of variable $ct^\gamma = u$, we obtain

$$I_1 = \gamma^{-\frac{\gamma}{\sigma}} c^{\theta_1 - \beta_0} J_1, \quad I_2 = \gamma^{-\frac{\gamma}{\sigma}} c^{\theta_1 - \beta_0} J_2, \quad I_3 = \gamma^{-\frac{\gamma}{\sigma}} c^{-(1 - \theta)(\beta_1 - \beta_0)} J_3,$$

where

$$J_1 = \left( \int_0^\infty \left( y^{\alpha_0 - \alpha} \left( \int_y^{\gamma} \left( r^{-\beta_0} \sup_{x \in \Omega} \|f\|_{L^q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dr}{r} \right)^{\frac{1}{r}} \frac{dy}{y} \right)^{\frac{1}{r}},$$

$$J_2 = \left( \int_0^\infty \left( y^{1 - \theta(\beta_1 - \beta_0)} \left( \int_y^{\gamma} \left( r^{-\beta_0} \sup_{x \in \Omega} \|f\|_{L^q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dy}{y} \right)^{\frac{1}{r}},$$

and

$$J_3 = \left( \int_0^\infty \left( y^{(1 - \theta)(\beta_1 - \beta_0)} \left( \int_y^{\gamma} \left( r^{-\beta_0} \sup_{x \in \Omega} \|f\|_{L^q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dy}{y} \right)^{\frac{1}{r}}.$$

Note that $\gamma^{-\frac{\gamma}{\sigma}} \leq \max\{1, \gamma^{-\frac{\gamma}{\sigma}}\}$.

3. For the integral estimates $J_1, J_2, J_3$ apply the following variants of Hardy inequalities: if $\mu > 0, -\infty < \nu < \infty$ and $0 < \sigma \leq \tau \leq \infty$, then

$$\left( \int_0^\infty \left( \int_0^y \left( r^{-\nu} |g(r)| \right)^{\sigma} \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dy}{y} \right)^{\frac{1}{\tau}} \leq (\mu \sigma)^{-\frac{\tau}{\sigma}} \left( \int_0^\infty \left( \int_0^y |g(y)| \right)^{\frac{\nu}{y}} \frac{dy}{y} \right)^{\frac{1}{r}}.$$
and
\[
\left( \int_0^\infty \left( \int_y^\infty r^{-\nu} |g(r)| \right)^\sigma \frac{dr}{r} \right)^{\frac{\nu}{\sigma}} \frac{dy}{y} \right)^{\frac{1}{\nu}} \leq (\mu\sigma)^{-\frac{1}{\nu}} \left( \int_0^\infty \left( y^{-\nu} |g(y)| \right)^\sigma \frac{dy}{y} \right)^{\frac{1}{\sigma}}.
\]

According to these inequalities we have
\[
J_1 \leq (\theta(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\sigma}} \left( \int_0^\infty (y^{-\beta_1} \sup_{x \in \Omega} |f|_{L_\gamma(B_y(x))})^{\frac{\sigma}{\nu}} \frac{dy}{y} \right)^{\frac{1}{\sigma}} = (\theta(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\sigma}} \|f\|_{L^{\sigma,\nu}_0(\Omega)},
\]
\[
J_2 \leq (\theta(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\sigma}} \|f\|_{L^{\sigma,\nu}_0(\Omega)},
\]
and
\[
J_3 \leq ((1 - \theta)(\beta_1 - \beta_0)\sigma)^{-\frac{1}{\sigma}} \|f\|_{L^{\sigma,\nu}_0(\Omega)}.
\]

Hence,
\[
\|Tf\|_{L^{\sigma,\nu}_0(\Omega)} \leq c_3(\mathbf{M}_0 c^{-\theta(\beta_1 - \beta_0)} + \mathbf{M}_1 c^{-1(1-\theta)(\beta_1 - \beta_0)}) \|f\|_{L^{\sigma,\nu}_0(\Omega)},
\]
where \(c_3 > 0\) depends only on \(\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma\) and \(\theta\).

Let now \(c = \left(\frac{\mathbf{M}_1}{\mathbf{M}_0}\right)^{\frac{1}{\sigma-\nu}}\), then
\[
\|Tf\|_{L^{\sigma,\nu}_0(\Omega)} \leq 2c_3 \mathbf{M}_0^{1-\theta} \mathbf{M}_1^\theta \|f\|_{L^{\sigma,\nu}_0(\Omega)}.
\]

4. If \(\beta_1 < \beta_0, \alpha_0 < \alpha_1\) or \(\beta_0 < \beta_1, \alpha_1 < \alpha_0\) or \(\beta_1 < \beta_0, \alpha_1 < \alpha_0\), then the assumptions of Part 2 and 3 are similar, changing only the choice of parameters \(\gamma\) and \(c\). In the first case \(\gamma = \frac{\alpha_1 - \alpha_0}{\beta_1 - \beta_0}, \ c = \left(\frac{\mathbf{M}_1}{\mathbf{M}_0}\right)^{\frac{1}{\sigma-\nu}}, \) in the second \(\gamma = \frac{\alpha_0 - \alpha_1}{\beta_1 - \beta_0}, \ c = \left(\frac{\mathbf{M}_1}{\mathbf{M}_0}\right)^{\frac{1}{\sigma-\nu}}, \) in the third \(\gamma = \frac{\alpha_0 - \alpha_1}{\beta_0 - \beta_1}, \ c = \left(\frac{\mathbf{M}_1}{\mathbf{M}_0}\right)^{\frac{1}{\sigma-\nu}}.\)

\(\blacksquare\)

Corollary 3.1. Let \(0 < \alpha_0, \alpha_1, \beta_0, \beta_1 < \infty, \alpha_0 \neq \alpha_1, \beta_0 \neq \beta_1, 0 < p, q \leq \infty, \ 0 < \sigma \leq \tau \leq \infty, \ 0 < \theta < 1\) and
\[
\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \ \beta = (1-\theta)\beta_0 + \theta\beta_1.
\]

Then there exists \(c > 0\), depending only on \(\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta\), such that if \(x \in \mathbb{R}^n, T\) is a quasi-additive operator on \(L^{\beta_0}_{p,q,x} + L^{\beta_1}_{p,q,x}\), and for some \(M_1, M_2 > 0\) the following inequalities hold
\[
\|Tf\|_{L^{\beta_0}_{p,q,x}} \leq M_1 \|f\|_{L^{\beta_0}_{p,q,x}}
\]
for all \(f \in L^{\beta_0}_{p,q,x}, i = 0, 1, \) then
\[
\|Tf\|_{L^{\beta_i}_{p,q,x}} \leq cAM_0^{1-\theta} M_1^\theta \|f\|_{L^{\beta_i}_{p,q,x}}
\]
for all \(f \in L^{\beta_i}_{p,q,x}\).
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Remark 3.1. If $T$ is a linear operator, then the statement of Corollary 3.1 follows using the standard arguments of interpolation theory from the equality

$$ (LM_{p,r_0,x}^0,LM_{p,r_1,x}^0)_{\theta,r} = LM_{p,r,x}^\lambda, $$

where $0 < p, r_0, r_1 \leq \infty, 0 < \lambda_0, \lambda_1 < \infty, \lambda_0 \neq \lambda_1, 0 < \theta < 1, \lambda = (1-\theta)\lambda_0 + \theta\lambda_1,$ and respectively by equivalence of quasi-norm, which were proved in [19] under the additional assumptions $\lambda_0, \lambda_1 \leq \frac{n}{p}$ and in general case in [20]. In the case of $T$ is quasi-linear operator, then the statement does not follow from (3.2).

4. Riesz potentials in Morrey-type spaces

Lemma 4.1. Let $1 < p, q, r < \infty, \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$, $g \in L_{r,\infty}(D), f \in L_p(B-D)$, where $B, D \subset \mathbb{R}^n$ are measurable sets, then

$$ \left\| \int_D f(\cdot - y)g(y)dy \right\|_{L_q(B)} \leq c \|f\|_{L_p(B-D)} \|g\|_{L_{r,\infty}(D)}, $$

where $c > 0$ depends only on the parameters $p, q$ and $r$.

Proof. Let $f \in L_p(B-D), g \in L_{r,\infty}(D)$. Set

$$ \tilde{g}(y) = \begin{cases} g(y), & y \in D, \\ 0, & y \notin D \end{cases} $$

and

$$ \tilde{f}(y) = \begin{cases} f(y), & y \in B-D, \\ 0, & y \notin B-D \end{cases} $$

Using O’Neil’s inequality, we have

$$ \left\| \int_D f(\cdot - y)g(y)dy \right\|_{L_q(B)} = \left\| \int_D \tilde{f}(\cdot - y)\tilde{g}(y)dy \right\|_{L_q(B)} = \left\| \int_{\mathbb{R}^n} \tilde{f}(\cdot - y)\tilde{g}(y)dy \right\|_{L_q(\mathbb{R}^n)} \leq c \|\tilde{f}\|_{L_p(\mathbb{R}^n)} \|\tilde{g}\|_{L_{r,\infty}(\mathbb{R}^n)} = c \|f\|_{L_p(B-D)} \|g\|_{L_{r,\infty}(D)}, $$

which completes the proof of the lemma.

Theorem 4.1. Let $z \in \mathbb{R}^n, 1 < p < q < \infty, 0 < \nu \leq \lambda < \frac{q}{r}, 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{\lambda-\nu}{n}$ and $\frac{1}{s} = \frac{1}{r} - \frac{\nu}{n}$. If $k \in L_{r,\infty}(\mathbb{R}^n)$ and

$$ \sup_{m \in \mathbb{Z}} 2^{\nu m} \|k\|_{L_{s,\infty}(D_m)} \leq M, $$

(4.1)
where $D_m = B_{2^{m+1}}(0) \setminus B_{2^m}(0)$. Then convolution operator

$$(k * f)(x) = \int \kappa(x-y)f(y)dy$$

is bounded from $LM^r_{p,1,z}$ to $LM^r_{q,\infty,z}$. Moreover, the following estimate holds

$$\|k * f\|_{LM^r_{q,\infty,z}} \leq c(\|k\|_{L_{r,\infty}(\mathbb{R}^n)} + M\|f\|_{LM^r_{p,1,z}}).$$

**Proof.** Let $z \in \mathbb{R}^n$, $l \in Z$ and $D_m = B_{2^{m+1}}(0) \setminus B_{2^m}(0)$, $m \in Z$. Applying Minkowski’s inequality, we get

$$\|k * f\|_{L_q(B_l(z))} = \left\| \int k(y)f(x-y)dy \right\|_{L_q(B_l(z))}$$

$$\leq \left\| \int k(y)f(x-y)dy \right\|_{L_q(B_l(z))} + \sum_{m=1}^{\infty} \left\| \int k(y)f(x-y)dy \right\|_{L_q(B_l(z))}$$

$$= I_1 + I_2.$$

Let us estimate each part. Applying Lemma 4.1 with $1/r_1 = 1 + 1/q - 1/p$ we obtain

$$I_1 \leq \|k\|_{L_{r_1,\infty}(B_{l+1}(0))}\|f\|_{L_p(B_{l+1}(z))}$$

$$\lesssim 2^{l1}\|k\|_{L_{r_1,\infty}(B_{l+1}(0))}\|f\|_{LM^r_{p,\infty,z}}.$$
According Lemma 2.1 we have

\[ 2^{-\lambda l} \| k \ast f \|_{L_q(B_{2l}(z))} \]

\[ \lesssim \left( 2^{(\nu-\lambda)l} \| k \|_{L^\infty(B_{2l}(0))} \| f \|_{LM_p^\nu(\mathbb{R}^n)} + \sup_{m \in \mathbb{Z}} 2^{\nu m} \| k \|_{L^\infty(D_m)} \| f \|_{LM_p^\nu(\mathbb{R}^n)}^2 \right). \]

Since \( \nu \leq \lambda \), then

\[ 2^{(\nu-\lambda)l} \| k \|_{L^\infty(B_{2l}(0))} \lesssim 2^{(\nu-\lambda)l} \sup_{0<s<2^n} s^{\lambda/n} k^*(s) \]

\[ = 2^{(\nu-\lambda)l} \sup_{0<s<2^n} s^{\lambda/n} s^{1/r} - s^{\lambda/n} k^*(s) \lesssim \sup_{s>0} s^{1/r} - s^{\lambda/n} k^*(s). \]

Therefore,

\[ 2^{(\nu-\lambda)l} \| k \|_{L^\infty(B_{2l}(0))} \lesssim \| k \|_{L^\infty(\mathbb{R}^n)}, \]

where \( \frac{1}{p} = \frac{1}{r_l} - \frac{\lambda-n}{n} \).

Given an arbitrariness of \( l \), by Lemma 2.2 we get

\[ \| k \ast f \|_{LM_p^\nu(\mathbb{R}^n)} \lesssim \sup_{t \in \mathbb{Z}} 2^{-\lambda l} \| k \ast f \|_{L_p(B_{2l}(z))} \lesssim (\| k \|_{L^\infty(\mathbb{R}^n)} + M) \| f \|_{LM_p^\nu(\mathbb{R}^n)}, \]

which completes the proof of the theorem.

Note that the condition (4.1) in Theorem 4.1 is essential, i.e. the following statement does not hold

(4.2) \[ \| T \|_{LM_p^\nu(\mathbb{R}^n) \rightarrow LM_p^\nu(\mathbb{R}^n)} \leq c \| k \|_{L^\infty(\mathbb{R}^n)} = c \sup_{t>0} t^{\frac{1}{r}} k^*(t). \]

**Example 4.1.** Let \( z = 0 \), \( 1 < p \leq q < \infty \), \( 0 < \nu \leq \lambda < \frac{n}{q} \), \( 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{\lambda-n}{n} \), \( \alpha > \frac{1}{r^q} \). Define the functions

\[ f(x) = \begin{cases} 1, & |x| \in [l^\alpha, l^\alpha + 1], \ l \in \mathbb{Z}_+ \\ 0, & \text{otherwise} \end{cases} \]

and

\[ k(x) = \begin{cases} (l+1)^{-\frac{1}{r}}, & |x| \in [l^\alpha, l^\alpha + 1], \ l \in \mathbb{Z}_+ \\ 0, & \text{otherwise}. \end{cases} \]

We show that the function \( f \) belongs to the space \( LM_p^\nu_{(p,1,0)} \). Indeed,

\[ \| f \|_{LM_p^\nu_{(p,1,0)}} = \int_0^\infty \frac{1}{t^{p+1}} \left( \int_{-t}^{t} |f(y)|^p \, dy \right)^{1/p} \, dt. \]
Furthermore,

$$\int_{-t}^{+t} |f(y)|^p dy = 2 \min \left( t, \sum_{t_i \leq t} 1 \right) \leq 2 \min(t, t^{1/\alpha}),$$

hence,

$$\|f\|_{L^{p,1,0}_\infty} < \infty.$$

Since $k^*(t) \leq t^{-1/r}$, therefore $k \in L_{r,\infty}(\mathbb{R})$, but also for $x \in [-1/2, 1/2]$ we have

$$(f * k)(x) = \int_{-\infty}^{+\infty} k(x - y)f(y)dy \geq \frac{1}{2} \sum_{l=1}^{\infty} (l + 1)^{-1/r} = \infty,$$

and the inequality of type (4.2) is impossible.

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^n$, $1 < p < q < \infty$, $0 < \tau \leq \infty$, $0 < \nu \leq \lambda < \frac{n}{q}$ and $\gamma = \frac{n}{p} - \frac{n}{q} + \lambda - \nu > 0$,

$$I_\gamma f(x) = \int_{\mathbb{R}^n} f(x - y) \frac{dy}{|y|^{n-\gamma}}.$$  

If $f \in M^{\nu}_{p,\tau}(\Omega)$, then convolution operator $I_\gamma f \in M^{\lambda\nu}_{q,\tau}(\Omega)$, and the following inequality holds

$$(4.3) \quad \|I_\gamma f\|_{M^{\lambda\nu}_{q,\tau}(\Omega)} \leq c\|f\|_{M^{\nu}_{p,\tau}(\Omega)}.$$ 

**Proof.** Let $k(x) = 1/|x|^{n-\gamma}$. Check the conditions of Theorem 4.1

$$\|k\|_{L_{r,\infty}(\mathbb{R}^n)} = \sup_{t > 0} t^{1/r} k^*(t) \leq \sup_{t > 0} t^{1/r} \frac{1}{t^{1-\tau}} = 1,$$

$$\sup_{m \in \mathbb{Z}} 2^{\nu m} \|k\|_{L_{r,\infty}(D_m)} \leq \sup_{m \in \mathbb{Z}} 2^{\nu m} \sup_{0 < t \leq 2^m} t^{1-\frac{\nu}{p}} \frac{1}{(2^m + t)^{1-\frac{\nu}{p}}} \leq 1.$$  

Then applying Theorem 4.1 we obtain the weak inequality

$$\|I_\gamma f\|_{L^{\lambda\nu}_{q,\infty}(\Omega)} \leq c\|f\|_{L^{\nu}_{p,1\ldots,1}, z \in \Omega}.$$  

Next, since $0 < \nu < \frac{n}{p}$, $0 < \lambda < \frac{n}{q}$, then there exist $\nu_0, \nu_1$ and $\lambda_0, \lambda_1$, such that

$$0 < \nu_0 < \nu < \nu_1 < \frac{n}{p}, \quad 0 < \lambda_1 < \lambda < \lambda_0 < \frac{n}{q}.$$  

Let $\nu_0 - \lambda_0 = \nu_1 - \lambda_1 = \nu - \lambda$. Then the weak inequalities hold

$$\|I_\gamma f\|_{L^{\lambda\nu}_{q,\infty}(\Omega)} \leq c\|f\|_{L^{\nu}_{p,1\ldots,1}, i = 0, 1, z \in \Omega}.$$
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Hence, by interpolation Theorem 3.1 the strong inequality holds

$$
\|I_\gamma f\|_{M^\lambda_{q',\nu}(\Omega)} \leq c \|f\|_{M^\lambda_{p',\nu}(\Omega)},
$$

where $0 < \tau \leq \infty$, $\nu = (1 - \theta)\nu_0 + \theta\nu_1$, $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ and $0 < \theta < 1$,

$$
1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{(1 - \theta)(\lambda_0 - \nu_0) + \theta(\lambda_1 - \nu_1)}{n} = 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{\lambda - \nu}{n},
$$

which completes the proof of the theorem.

Note that a similar description of the boundedness of $I_\gamma f$ in classical Morrey spaces with restriction $\lambda q = \nu p$ was given by Adams [9], Chiarenza and Frasca [10], Sawano and Tanaka [12] and Burenkov in survey paper [3], who used the different methods of the proof. In our case instead of this condition we have condition $\nu \leq \lambda$, which is necessary in Theorem 4.2. Indeed, from (4.3), in particular, it follows that

$$
\|I_\gamma f\|_{LM^\lambda_{q',\nu,0}(\Omega)} \leq c \|f\|_{LM^\lambda_{p',\nu,0}(\Omega)},
$$

that is equivalent to inequality

(4.4) $$
\|I_\gamma f\|_{L_q(|x|^{-\nu})} \leq c \|f\|_{L_p(|x|^{-\nu})},
$$

and in this case the condition $\nu \leq \lambda$ is necessary. Indeed, let $N \in \mathbb{N}$, \(z_N = (N, 0, ..., 0)\),

$$
f(x) = \chi_{B_2(z_N)}(x)|x|^{\nu},
$$

$$
I_\gamma f(y) = \int_{\mathbb{R}^n} \frac{f(x - y)}{|x|^\gamma} dx.
$$

Then by (4.4) we have

$$
|B_2|^{1/p} c \geq \|I_\gamma f\|_{L_q(|x|^{-\nu})}
$$

$$
\geq \left( \int_{B_1(z_N)} \left| \frac{1}{|y|^\lambda} \int_{B_2(z_N) + y} \frac{|x-y|^\nu}{|x|^{\gamma}} dx \right|^q dy \right)^{1/q}.
$$

Since $B_2(z_N) + y \supset B_1(0)$ for any $y \in B_1(z_N)$ we get for $N > 2$
5. Singular operators in Morrey-type spaces

**Definition 5.1.** Let \( 1 \leq a < b \leq \infty \). We call a linear operator \( T \) an \((a; b)\)-singular if

1. the operator \( T \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) for \( a < p < b \);
2. there exists a measurable function \( K \) on \( \mathbb{R}^n \) such that for all \( x, y \in \mathbb{R}^n \),

\[
|K(x, y)| \leq \frac{C}{|x - y|^n}
\]

and for any finite, locally integrable function \( f \) with compact support the operator \( T \) is defined by

\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy \text{ a.e. on } \mathbb{R}^n \setminus \text{supp } f.
\]

**Theorem 5.1.** Let \( \Omega \subset \mathbb{R}^n \), \( 1 \leq a < b \leq \infty \) and operator \( T \) is an \((a, b)\)-singular. Then for \( p \in (a, b) \), \( 0 < \nu < \frac{a}{p} \), \( 0 < \tau \leq \infty \) operator \( T \) is bounded from \( M^p_{\nu, \tau}(\Omega) \) to \( M^p_{\nu, \tau}(\Omega) \).

**Proof.** Let \( z \in \Omega \), \( l \in \mathbb{Z} \) and \( D_m(z) = B_{2^{m+1}}(z) \setminus B_2^m(z) \), \( m \in \mathbb{Z} \), \( f \in LM^{p}_{\nu, 1, z} \). Applying Minkowski’s and Hölder’s inequalities we get

\[
\|Tf\|_{L_p(B_{2^l}(z))} \lesssim \|T\left(\sum_{m=l+1}^{\infty} f_{D_m(z)}\right)\|_{L_p(B_{2^l}(z))} \lesssim T\left(\sum_{m=l+1}^{\infty} f_{D_m(z)}\right)\|_{L_q(B_{2^l}(z))} = I_1 + I_2.
\]
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where $1/p = 1/q + \nu/n$.

Let us estimate each part. By definition of the singular operator we have

$$ I_1 \leq \left\| T (f \chi_{B_{2l+1}(z)}) \right\|_{L_p(\mathbb{R}^n)} \leq C \| f \|_{L_p(B_{2l+1}(z))}. $$

Since $B_{2l}(z) \cap (\bigcup_{m=l+1}^{\infty} D_m(z)) = \emptyset$, for operator $T$ and function $f \chi_{D_m(z)}$ if $m \geq l+1$ the representation (5.2) and estimate (5.1) hold for all $x \in B_{2l}(z)$, hence

$$ I_2 \leq 2^{2l} \sum_{m=l+1}^{\infty} \left\| T (f \chi_{D_m(z)}) \right\|_{L_q(B_{2l}(z))} = 2^{2l} \sum_{m=l+1}^{\infty} \left\| \int_{\mathbb{R}^n} K(x,y)f(y)\chi_{D_m(y)}dy \right\|_{L_q(B_{2l}(z))} \lesssim 2^{2l} \sum_{m=l+1}^{\infty} \int_{D_m(z)} \frac{|f(y)|}{|x-y|^n} dy \right\|_{L_q(B_{2l}(z))}. $$

Since $m > l$, $B_{2l}(z) - D_m(z) = B_{2l}(0) + z - (D_m(0) + z) = B_{2l}(0) - D_m(0) \subset D_{m-1}(0) \cup D_m(0) \cup D_{m+1}(0)$, application of Lemma 4.1 yields

$$ I_2 \lesssim 2^{2l} \sum_{m=l+1}^{\infty} \| |x|^{-n} \|_{L_{r,\infty}(B_{2l}(z) - D_m(z))} \| f \|_{L_p(D_m(z))} \leq 2^{2l} \sum_{m=l+1}^{\infty} \| |x|^{-n} \|_{L_{r,\infty}(D_{m-1}(0) \cup D_m(0) \cup D_{m+1}(0))} \| f \|_{L_p(B_{2m+1}(z))}, $$

where $1/r = 1 + 1/q - 1/p = 1 - \nu/n$.

Note that

$$ \| |x|^{-n} \|_{L_{r,\infty}(D_{m-1}(0) \cup D_m(0) \cup D_{m+1}(0))} \leq \sup_{0 < t < 2^{mn}} \frac{t^{1-\nu/n}}{2^{mn} + t} \lesssim 2^{-\nu m}. $$

Therefore,

$$ 2^{-\nu l} \| T f \|_{L_p(B_{2l}(z))} \lesssim \left( 2^{-\nu l} \| f \|_{L_p(B_{2l+1}(z))} + \sum_{m=l+1}^{\infty} 2^{-\nu (m+1)} \| f \|_{L_p(B_{2m+1}(z))} \right) \lesssim \| f \|_{LM_{p,1}^{\nu,z}}. $$

Given an arbitrariness of $l$ for all $z \in \Omega$ we have the weak inequality

$$ \| T f \|_{LM_{p,\infty}^{\nu,z}} \lesssim \| f \|_{LM_{p,1}^{\nu,z}}. $$

Interpolation Theorem 3.1 completes the proof of the theorem. \(\square\)
Note that Theorem 5.1 answers the question what singular operators which are bounded in the spaces $L_p$, will be bounded in the generalized Morrey-type spaces.

Let $T$ be a Calderon-Zygmund operator, i.e. a linear operator taking $C_0^\infty$ into $L_1^\text{loc}$, bounded on $L_2$ and represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \quad \text{a.e. on } \mathbb{R}^n \setminus \text{supp} f$$

for every function $f \in L^\infty(\mathbb{R}^n)$ with compact support. Here $K(x,y)$ is a continuous function away from the diagonal and satisfies the standard estimates: for some $c_1 > 0$ and $0 < \varepsilon \leq 1$

$$|K(x,y)| \leq c|x-y|^{-n},$$

for all $x, y \in \mathbb{R}^n, x \neq y$ and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| 
\leq c_1 \left( \frac{|x-x'|^\varepsilon}{|x-y|} \right) |x-y|^{-n}$$

whenever $2|x-x'| \leq |x-y|$ for some constants $c > 0, \varepsilon \in ]0,1]$. This class of operators was introduced by Coifman and Meyer [26].

**Corollary 5.1.** [3],[12] Let $1 < p < \infty$, $0 \leq \lambda < \frac{n}{p}$. Then Calderon-Zygmund operators $T$ are bounded from $M_p^\lambda$ to $M_p^\lambda$.

**Proof.** It is known that Calderon-Zygmund operator is bounded in the Lebesgue spaces $L_p(\mathbb{R}^n)$ if $p \in (0,\infty)$. Hence, for the interval $(a,b) = (0,\infty)$ the conditions of Theorem 5.1 are satisfied, which completes the proof.

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**References**


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