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Marcinkiewicz-type interpolation theorem for Morrey-type spaces and its corollaries

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Abstract – We introduce a class of Morrey-type spaces $M^{\lambda}_{p,q,\Omega}$, which includes the classical Morrey spaces and discuss their properties. We prove a Marcinkiewicz-type interpolation theorem. This theorem is then applied to obtaining the boundedness in the introduced Morrey-type spaces of the Riesz potential and singular operators.

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1. Introduction

Let $0 < p \leq \infty$ and $0 \leq \lambda \leq \frac{n}{p}$. The Morrey spaces M_p^{λ} are the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$||f||_{M_p^{\lambda}} \equiv ||f||_{M_p^{\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{t>0} t^{-\lambda} \|f\|_{L_p(B_t(x))} < \infty,$$

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where $B_t(x)$ is the open ball of radius $t > 0$ with center at the point $x \in \mathbb{R}^n$ (see [19]). If $\lambda = 0$, then $M_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = \frac{n}{p}$, then $M_p^{n/p}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then $M_p^\lambda = \Theta$, where Θ is the set of all functions that are equivalent to zero on $\mathbb{R}^n$.

These spaces were introduced by Morrey [19] in 1938 and arose in connection with some problems of the theory of partial differential equations and theory of variations. There is a number of books and survey papers on the Morrey and Morrey-type spaces and classical operators of real analysis in the Morrey-type spaces, see, for example, [4], [5], [17], [20], [25], [26], [2].

This paper is devoted to the interpolation properties of the Morrey-type spaces. Some results for the classical Morrey spaces were obtained in Stampacchia [29], Campanato and Murthy [15], Peetre [24]. In particular in [24] it is proved that

$$(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta,\infty} \subset M_p^\lambda \,,$$

where $1 \le p < \infty$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$. In Ruiz and Vega [27], Blasco, Ruiz and Vega [3] it is proved that this inclusion is strict.

In [18] a more detailed investigation of the interpolation problem for the Morrey spaces was carried out. In particular, it was proved that the inclusion

$$(M_{p_0}^{\lambda_0}, M_{p_1}^{\lambda_1})_{\theta,\infty} \subset M_p^\lambda \,,$$

where $1 \le p_0, p_1 < \infty$, $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, holds if and only if $p_0 = p_1$.

The case of the local Morrey-type spaces $LM_{p,q}^\lambda$ was considered in [6], [7], [14]. Let $0 < p, q \le \infty$, $\lambda > 0$ if $q < \infty, \lambda \ge 0$ if $q = \infty$. A function $f \in LM_{p,q}^\lambda$ if $f \in L_p^{loc}(\mathbb{R}^n)$ and

$$||f||_{LM_{p,q}^\lambda} = \left(\int_0^\infty \left(t^{-\lambda} \|f\|_{L_p(B_t(0))}\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

In [14] it was proved, in particular, that the local Morrey-type spaces $LM_{p,q}^\lambda$ form an interpolation scale when p is fixed, i.e.

$$(LM_{p,q_0}^{\lambda_0}, LM_{p,q_1}^{\lambda_1})_{\theta,q} = LM_{p,q}^\lambda \,,$$

where $0 < p, q_0, q_1, q \le \infty$, $0 < \theta < 1$, $\lambda_0 > 0$ if $q_0 < \infty$, $\lambda_0 \ge 0$ if $q_0 = \infty$, $\lambda_1 > 0$ if $q_1 < \infty$, $\lambda_1 \ge 0$ if $q_1 = \infty$, $\lambda_0 \ne \lambda_1$ and $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

Further generalizations of interpolation properties for the general local Morrey-type spaces $LM_{p,q}^\lambda(G,\mu)$ were discussed in [6], namely, it was proved that

$$(LM_{p,q_0}^{\lambda_0}(G,\mu), LM_{p,q_1}^{\lambda_1}(G,\mu))_{\theta,q} = LM_{p,q}^\lambda(G,\mu) \,,$$

under the same assumptions on the numerical parameters.

In this paper, we introduce the generalized Morrey-type spaces $M^{\lambda}_{p,q,\Omega}$, which coincide with the classical Morrey spaces M^{λ}_{p} in the case $q = \infty$ and $\Omega = \mathbb{R}^n$. According to the above results it follows that the classical interpolation theorems for this scale of the spaces do not take place. Nevertheless, we prove a certain analogue of an interpolation theorem. Compared with the classical interpolation theorems the assumptions of the theorem are formulated in terms of the local Morrey-type spaces $LM^{\lambda}_{p,q,x}$ and the statement in terms of the generalized Morrey-type spaces $M^{\lambda}_{p,q,\Omega}$. We say that this theorem is an interpolation theorem of Marcinkiewicz-type, because it allows us to obtain from in a certain sense "weak estimates" for quasi-additive operators in terms of the local Morrey-type spaces "strong estimates" in terms of the generalized Morrey-type spaces.

Let $f \in L^{loc}_1(\mathbb{R}^n)$. In this paper we also study estimates for the norm of the Riesz operator

$$I_\gamma f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\gamma}}, \quad 0 < \gamma < n.$$

Let us recall the well-known Hardy-Littlewood-Sobolev inequality. Let

$$1 < p < q < \infty, \text{ and } \gamma = n\Big(\frac{1}{p} - \frac{1}{q}\Big),$$

then the operator I_γ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

The boundedness of I_γ in the classical Morrey spaces and their generalizations was investigated by Peetre [24], Adams [1], Nakai [20], Sawano and Tanaka [28], and others. Further results of the boundedness of the Riesz potential for the local Morrey-type spaces were obtained in [11, 12, 8].

In this paper, we continue the study of the boundedness of I_γ in the case of the Morrey-type spaces. In particular, by applying the Marcinkiewicz-type interpolation theorem we obtain an estimate for the norm of the Riesz operator in the Morrey-type spaces.

Also we study the boundedness for a certain class of singular operators. The classical results for Calderon-Zygmund operators of different classes state that if $1 < p < \infty$, then this operator is bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ (see, for example, [21], [30]). Peetre [25] studied the boundedness of singular operator in the Morrey spaces. Further results on the boundedness of Calderon-Zygmund operators for the local Morrey-type spaces were obtained in Burenkov, Guliyev, Serbetci and Tararykova [13], and others.

Given functions F and G, in this paper $F \lesssim G$ means that $F \le CG$, where C is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$.

2. Spaces $M^{\lambda}_{p,q,\Omega}$

Let $\Omega \subset \mathbb{R}^n$, $0 < p \le \infty$, $0 < q \le \infty$ and $0 \le \lambda \le \frac{n}{p}$. We consider the generalized Morrey-type spaces $M^{\lambda}_{p,q,\Omega}$ that are defined for $q < \infty$ as the spaces of all functions

$f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$||f||_{M^\lambda_{p,q,\Omega}} = \left(\int\limits_0^\infty \left(t^{-\lambda} \sup_{x\in\Omega} \|f\|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

and for $q = \infty$,

$$||f||_{M^\lambda_{p,\infty,\Omega}} = \sup_{x\in\Omega} \sup_{t>0} t^{-\lambda} \|f\|_{L_p(B_t(x))}.$$

Note that the introduced spaces coincide with the classical Morrey spaces in the case $q = \infty$ and $\Omega = \mathbb{R}^n$, i.e.

$$M^\lambda_{p,\infty,\mathbb{R}^n} = M^\lambda_p.$$

However, these spaces differ from the global Morrey-type spaces $GM^\lambda_{p,q,\Omega}$, which are defined as the spaces of all functions f Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{GM^\lambda_{p,q,\Omega}} = \sup_{x\in\Omega} \left(\int\limits_0^\infty \left(t^{-\lambda} \|f\|_{L_p(B_t(x))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

if $q < \infty$ and usual modification if $q = \infty$. For $\Omega = \mathbb{R}^n$ they were introduced by Burenkov and Guliyev [9, 10].

Clearly

$$M^\lambda_{p,q,\Omega} \subset GM^\lambda_{p,q,\Omega}$$

and

$$\|f\|_{GM^\lambda_{p,q,\Omega}} \le \|f\|_{M^\lambda_{p,q,\Omega}}.$$

If $\Omega = \{x\}$ is a singleton, then

$$M^\lambda_{p,q,\Omega} = GM^\lambda_{p,q,\Omega} \equiv LM^\lambda_{p,q,x},$$

where $LM^\lambda_{p,q,x}$ are the local Morrey-type spaces [9, 10].

Note also that the generalized Morrey-type spaces are close to the net spaces $N_{p,q}(M)$ introduced by Nursultanov [22, 23].

We begin with the following lemma, which describes the properties of the spaces $M^\lambda_{p,q,\Omega}$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$.*
(i) *If $0 < p_0 < p_1 < \infty$ and $\lambda_0 = \lambda_1 + n\left(\frac{1}{p_1} - \frac{1}{p_0}\right)$, then the space $M^{\lambda_1}_{p_1,q,\Omega}$ is continuously embedded in $M^{\lambda_0}_{p_0,q,\Omega}$.*
(ii) *If $0 < q_0 < q_1 \le \infty$, then the space $M^\lambda_{p,q_0,\Omega}$ is continuously embedded in $M^\lambda_{p,q_1,\Omega}$.*

PROOF. (i). Let $f \in M^{\lambda_1}_{p_1,q,\Omega}$. By applying Hölder's inequality and noting that $|B_t(x)| \asymp t^n$, we get

$$\begin{aligned}
\|f\|_{M^{\lambda_0}_{p_0,q,\Omega}} &= \left(\int_0^\infty \left(t^{-\lambda_0} \sup_{x\in\Omega} \|f\|_{L_{p_0}(B_t(x))}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\le \left(\int_0^\infty \left(t^{-\lambda_0} \sup_{x\in\Omega} \|f\|_{L_{p_1}(B_t(x))} |B_t(x)|^{\frac{1}{p_0}-\frac{1}{p_1}}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \left(\int_0^\infty \left(t^{-\left(\lambda_0 - n\left(\frac{1}{p_1}-\frac{1}{p_0}\right)\right)} \sup_{x\in\Omega} \|f\|_{L_{p_1}(B_t(x))}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|f\|_{M^{\lambda_1}_{p_1,q,\Omega}},
\end{aligned}$$

which implies continuous embedding $M^{\lambda_1}_{p_1,q,\Omega} \hookrightarrow M^{\lambda_0}_{p_0,q,\Omega}$.

(ii). First let $q_1 = \infty$ and $f \in M^{\lambda}_{p,q_0,\Omega}$. Let us prove that $M^{\lambda}_{p,q_0,\Omega} \hookrightarrow M^{\lambda}_{p,\infty,\Omega}$, taking into account that

$$\begin{aligned}
\|f\|_{M^{\lambda}_{p,\infty,\Omega}} &= \sup_{t>0} t^{-\lambda} \sup_{x\in\Omega} \|f\|_{L_p(B_t(x))} \\
&= (\lambda q_0)^{\frac{1}{q_0}} \sup_{t>0} \left(\int_t^\infty \tau^{-\lambda q_0} \frac{d\tau}{\tau}\right)^{\frac{1}{q_0}} \sup_{x\in\Omega} \|f\|_{L_p(B_t(x))}.
\end{aligned}$$

Since $B_t(x) \subset B_\tau(x)$ if $t \le \tau$, we have

$$\begin{aligned}
\|f\|_{M^{\lambda}_{p,\infty,\Omega}} &\le (\lambda q_0)^{\frac{1}{q_0}} \sup_{t>0} \left(\int_t^\infty \left(\tau^{-\lambda} \sup_{x\in\Omega} \|f\|_{L_p(B_\tau(x))}\right)^{q_0} \frac{d\tau}{\tau}\right)^{\frac{1}{q_0}} \\
&= (\lambda q_0)^{\frac{1}{q_0}} \|f\|_{M^{\lambda}_{p,q_0,\Omega}}.
\end{aligned}$$

If $q_1 < \infty$, then

$$\begin{aligned}
\|f\|_{M^{\lambda}_{p,q_1,\Omega}} &= \left(\int_t^\infty \left(\tau^{-\lambda} \sup_{x\in\Omega} \|f\|_{L_p(B_\tau(x))}\right)^{q_1-q_0} \left(\tau^{-\lambda} \sup_{x\in\Omega} \|f\|_{L_p(B_\tau(x))}\right)^{q_0} \frac{d\tau}{\tau}\right)^{\frac{1}{q_1}} \\
&\le (\|f\|_{M^{\lambda}_{p,\infty,\Omega}})^{1-\frac{q_0}{q_1}} (\|f\|_{M^{\lambda}_{p,q_0,\Omega}})^{\frac{q_0}{q_1}} \lesssim \|f\|_{M^{\lambda}_{p,q_0,\Omega}}.
\end{aligned}$$

This completes the proof of the second part of the lemma. □

LEMMA 2.2. *Let* $0 < p, q \le \infty,\ 0 < \lambda < \infty,$ *then for any* $\Omega \subset \mathbb{R}^n$

$$2^{-\lambda} (\ln 2)^{\frac{1}{q}} \left(\sum_{m\in\mathbb{Z}} \left(2^{-\lambda m} \sup_{x\in\Omega} \|f\|_{L_p(B_{2^m}(x))}\right)^q\right)^{1/q} \le \|f\|_{M^{\lambda}_{p,q,\Omega}}$$

$$\le 2^{\lambda}(\ln 2)^{\frac{1}{q}}\left(\sum_{m\in\mathbb{Z}}\left(2^{-\lambda m}\sup_{x\in\Omega}\|f\|_{L_p(B_{2^m}(x))}\right)^q\right)^{1/q}.$$

Proof. Let $t\in(0,\infty)$, then there exists $m\in\mathbb{Z}$, such that $2^m\le t<2^{m+1}$.

Therefore,

$$\left(\int_0^\infty\left(t^{-\lambda}\sup_{x\in\Omega}\|f\|_{L_p(B_t(x))}\right)^q\frac{dt}{t}\right)^{1/q} = \left(\sum_{m\in\mathbb{Z}}\int_{2^m}^{2^{m+1}}\left(t^{-\lambda}\sup_{x\in\Omega}\|f\|_{L_p(B_t(x))}\right)^q\frac{dt}{t}\right)^{1/q}.$$

Thus,

$$\left(\sum_{m\in\mathbb{Z}}\int_{2^m}^{2^{m+1}}\left(t^{-\lambda}\sup_{x\in\Omega}\|f\|_{L_p(B_t(x))}\right)^q\frac{dt}{t}\right)^{1/q} \le 2^{\lambda}(\ln 2)^{\frac{1}{q}}\left(\sum_{m\in\mathbb{Z}}\left(2^{-\lambda(m+1)}\sup_{x\in\Omega}\|f\|_{L_p(B_{2^{m+1}}(x))}\right)^q\right)^{1/q}.$$

On the other hand,

$$\left(\sum_{m\in\mathbb{Z}}\int_{2^m}^{2^{m+1}}\left(t^{-\lambda}\sup_{x\in\Omega}\|f\|_{L_p(B_t(x))}\right)^q\frac{dt}{t}\right)^{1/q} \ge 2^{-\lambda}(\ln 2)^{\frac{1}{q}}\left(\sum_{m\in\mathbb{Z}}\left(2^{-\lambda m}\sup_{x\in\Omega}\|f\|_{L_p(B_{2^m}(x))}\right)^q\right)^{1/q},$$

and we obtain the required equivalence. □

3. Interpolation theorem

Theorem 3.1. *Let* $\Omega\subset\mathbb{R}^n$, $0<\alpha_0,\alpha_1,\beta_0,\beta_1<\infty$, $\alpha_0\ne\alpha_1$, $\beta_0\ne\beta_1$, $0<p,q\le\infty$, $0<\sigma\le\tau\le\infty$, $0<\theta<1$ *and*

$$\alpha=(1-\theta)\alpha_0+\theta\alpha_1,\ \beta=(1-\theta)\beta_0+\theta\beta_1.$$

Let T *be a quasi-additive operator* [1] *given on* $LM^{\beta_0}_{q,\sigma,x}+LM^{\beta_1}_{q,\sigma,x}$, $x\in\Omega$. *Suppose that for some* $M_1,M_2>0$ *the following inequalities hold*

$$\|Tf\|_{LM^{\alpha_i}_{p,\infty,x}}\le M_i\|f\|_{LM^{\beta_i}_{q,\sigma,x}},\quad x\in\Omega,\ f\in LM^{\beta_i}_{q,\sigma,x},\ i=0,1, \tag{3.1}$$

then

$$\|Tf\|_{M^{\alpha}_{p,\tau,\Omega}}\le cAM_0^{1-\theta}M_1^{\theta}\|f\|_{M^{\beta}_{q,\tau}(\Omega)} \tag{3.2}$$

for all functions $f\in M^{\beta}_{q,\tau,\Omega}$, *where* $c>0$ *depends only on* $\alpha_0,\alpha_1,\beta_0,\beta_1,p,q,\sigma,\theta$.

[1] that is for some $A>0$ and for almost all $y\in\mathbb{R}^n$ $|T(f+g)(y)|\le A(|(Tf)(y)|+|(Tg)(y)|)$ for all functions $f,g\in LM^{\beta_0}_{q,\sigma,x}+LM^{\beta_1}_{q,\sigma,x}$

PROOF. Step 1. Let $f \in M^{\beta}_{q,\tau,\Omega}$. For every $x \in \Omega$, $s > 0$, we define the functions

$$f_{0,s} = f\chi_{B_s(x)}, \quad f_{1,s} = f - f_{0,s},$$

where $\chi_{B_s(x)}$ denotes the characteristic function of a ball $B_s(x)$. Then $f = f_{0,s} + f_{1,s}$ and

$$\begin{aligned}\|Tf\|_{L_p(B_t(x))} &= \|T(f_{0,s} + f_{1,s})\|_{L_p(B_t(x))} \\ &\le A\| \, |Tf_{0,s}| + |Tf_{1,s}| \, \|_{L_p(B_t(x))} \\ &\le 2^{(\frac{1}{p}-1)_+} A(\|Tf_{0,s}\|_{L_p(B_t(x))} + \|Tf_{1,s}\|_{L_p(B_t(x))}).\end{aligned}$$

By inequality (3.1) we have

$$\begin{aligned}&\|Tf_{0,s}\|_{L_p(B_t(x))} = t^{\alpha_0} t^{-\alpha_0} \|Tf_{0,s}\|_{L_p(B_t(x))} \\ &\le t^{\alpha_0} \sup_{r>0} r^{-\alpha_0} \|Tf_{0,s}\|_{L_p(B_r(x))} = t^{\alpha_0} \|Tf_{0,s}\|_{LM^{\alpha_0}_{p,\infty,x}} \\ &\le M_0 t^{\alpha_0} \|f_{0,s}\|_{LM^{\beta_0}_{q,\sigma,x}} = M_0 t^{\alpha_0} \left(\int_0^\infty \left(r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \\ &= M_0 t^{\alpha_0} \left(\int_0^s \left(r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))} \right)^\sigma \frac{dr}{r} + \int_s^\infty \left(r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \\ &\le 2^{(\frac{1}{\sigma}-1)_+} M_0 t^{\alpha_0} \left[\left(\int_0^s \left(r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \right. \\ &\left. + \left(\int_s^\infty \left(r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} \right].\end{aligned}$$

For $0 < r \le s$ and $y \in B_r(x)$ we have that $f_{0,s}(y) = f(y)\chi_{B_s(x)}(y) = f(y)$, therefore $\|f_{0,s}\|_{L_q(B_r(x))} = \|f\|_{L_q(B_r(x))}$. For $r > s$ and $y \notin B_r(x)$ we get that $f_{0,s}(y) = 0$, therefore $\|f_{0,s}\|_{L_q(B_r(x))} = \|f\|_{L_q(B_s(x))}$. Hence,

$$\left(\int_0^s \left(r^{-\beta_0} \|f_{0,s}\|_{L_q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}} = \left(\int_0^s \left(r^{-\beta_0} \|f\|_{L_q(B_r(x))} \right)^\sigma \frac{dr}{r} \right)^{\frac{1}{\sigma}}$$

and

$$\left(\int\limits_s^\infty \big(r^{-\beta_0}\|f_{0,s}\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}} = \|f\|_{L_q(B_s(x))}\left(\int\limits_s^\infty r^{-\beta_0\sigma}\frac{dr}{r}\right)^{\frac{1}{\sigma}}$$
$$= (\beta_0\sigma)^{-\frac{1}{\sigma}}\|f\|_{L_q(B_s(x))}s^{-\beta_0}$$
$$= (\beta_0\sigma)^{-\frac{1}{\sigma}}\|f\|_{L_q(B_s(x))}s^{-\beta_0}(\beta_1\sigma)^{\frac{1}{\sigma}}s^{\beta_1}\left(\int\limits_s^\infty r^{-\beta_1\sigma}\frac{dr}{r}\right)^{\frac{1}{\sigma}}$$
$$= \big(\frac{\beta_1}{\beta_0}\big)^{\frac{1}{\sigma}}s^{\beta_1-\beta_0}\left(\int\limits_s^\infty \big(r^{-\beta_1}\|f\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}.$$

Thus, for all $\sigma > 0$
$\|Tf_{0,s}\|_{L_p(B_t(x))}$

$$\le c_1 M_0 t^{\alpha_0}\left[\left(\int\limits_0^s \big(r^{-\beta_0}\|f\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}} + s^{\beta_1-\beta_0}\left(\int\limits_s^\infty \big(r^{-\beta_1}\|f\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\right].$$

Similarly, according to inequality (3.1), since $f_{1,s}(y) = 0$ if $y \in B_s(x)$ and $|f_{1,s}(y)| \le |f(y)|$ if $y \in \mathbb{R}^n$, we get that

$$\|Tf_{1,s}\|_{L_p(B_t(x))} = t^{\alpha_1}t^{-\alpha_1}\|Tf_{1,s}\|_{L_p(B_t(x))}$$
$$\le M_1 t^{\alpha_1}\left(\int\limits_0^s \big(r^{-\beta_1}\|f_{1,s}\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r} + \int\limits_s^\infty \big(r^{-\beta_1}\|f_{1,s}\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}$$
$$\le M_1 t^{\alpha_1}\left(\int\limits_s^\infty \big(r^{-\beta_1}\|f\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}.$$

So, for all $t > 0$ and $s > 0$ we obtain

$$\|Tf\|_{L_p(B_t(x))} \le c_2 A\Bigg(M_0 t^{\alpha_0}\Bigg[\left(\int\limits_0^s \big(r^{-\beta_0}\|f\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}$$
$$+ s^{\beta_1-\beta_0}\left(\int\limits_s^\infty \big(r^{-\beta_1}\|f\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\Bigg]$$
$$+ M_1 t^{\alpha_1}\left(\int\limits_s^\infty \big(r^{-\beta_1}\|f\|_{L_q(B_r(x))}\big)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\Bigg),$$

where $c_2 > 0$ depends only on p, β_0, β_1 and σ.

Step 2. Suppose that $\beta_0 < \beta_1, \alpha_0 < \alpha_1$ and set $s = ct^\gamma$, where $\gamma = \frac{\alpha_1-\alpha_0}{\beta_1-\beta_0}$, and $c > 0$ will be taken further. Then

$$\|Tf\|_{M^{\alpha}_{p,\tau,\Omega}} = \left(\int_0^\infty \big(t^{-\alpha}\sup_{x\in\Omega}\|Tf\|_{L_p(B_t(x))}\big)^\tau \frac{dt}{t}\right)^{\frac{1}{\tau}}$$
$$\leq 3^{(\frac{1}{\tau}-1)_+} c_2 A(M_0 I_1 + M_0 I_2 + M_1 I_3),$$

where

$$I_1 = \left(\int_0^\infty \left(t^{\alpha_0-\alpha}\int_0^{ct^\gamma}\left(r^{-\beta_0}\sup_{x\in\Omega}\|f\|_{L_q(B_r(x))}\right)^\sigma \frac{dr}{r}\right)^{\frac{\tau}{\sigma}}\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$I_2 = (ct^\gamma)^{\beta_1-\beta_0}\left(\int_0^\infty \left(t^{\alpha_0-\alpha}\int_{ct^\gamma}^\infty\left(r^{-\beta_1}\sup_{x\in\Omega}\|f\|_{L_q(B_r(x))}\right)^\sigma \frac{dr}{r}\right)^{\frac{\tau}{\sigma}}\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

and

$$I_3 = \left(\int_0^\infty \left(t^{\alpha_1-\alpha}\int_{ct^\gamma}^\infty\left(r^{-\beta_1}\sup_{x\in\Omega}\|f\|_{L_q(B_r(x))}\right)^\sigma \frac{dr}{r}\right)^{\frac{\tau}{\sigma}}\frac{dt}{t}\right)^{\frac{1}{\tau}}.$$

Making the change of variable $ct^\gamma = u$, we obtain

$$I_1 = \gamma^{-\frac{1}{\tau}}c^{\theta(\beta_1-\beta_0)}J_1,\ I_2 = \gamma^{-\frac{1}{\tau}}c^{\theta(\beta_1-\beta_0)}J_2,\ I_3 = \gamma^{-\frac{1}{\sigma}}c^{-(1-\theta)(\beta_1-\beta_0)}J_3,$$

where

$$J_1 = \left(\int_0^\infty \left(y^{-\theta(\beta_1-\beta_0)}\left(\int_0^y\left(r^{-\beta_0}\sup_{x\in\Omega}\|f\|_{L_q(B_r(x))}\right)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$J_2 = \left(\int_0^\infty \left(y^{(1-\theta)(\beta_1-\beta_0)}\left(\int_y^\infty\left(r^{-\beta_0}\sup_{x\in\Omega}\|f\|_{L_q(B_r(x))}\right)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

and

$$J_3 = \left(\int_0^\infty \left(y^{(1-\theta)(\beta_1-\beta_0)}\left(\int_y^\infty\left(r^{-\beta_1}\sup_{x\in\Omega}\|f\|_{L_q(B_r(x))}\right)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}}.$$

Note that $\gamma^{-\frac{1}{\tau}} \leq \max\{1, \gamma^{-\frac{1}{\sigma}}\}$.

Step 3. For the integral estimates J_1, J_2, J_3 apply the following variants of Hardy inequalities: if $\mu > 0, -\infty < \nu < \infty$ and $0 < \sigma \leq \tau \leq \infty$, then

$$\left(\int_0^\infty \left(y^{-\mu}\left(\int_0^y \left(r^{-\nu}|g(r)|\right)^\sigma\frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}} \leq (\mu\sigma)^{-\frac{1}{\sigma}}\left(\int_0^\infty\left(y^{-\mu-\nu}|g(y)|\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}$$

and

$$\left(\int\limits_0^\infty \left(y^\mu \left(\int\limits_y^\infty \left(r^{-\nu}|g(r)|\right)^\sigma \frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau \frac{dy}{y}\right)^{\frac{1}{\tau}} \leq (\mu\sigma)^{-\frac{1}{\sigma}} \left(\int\limits_0^\infty \left(y^{\mu-\nu}|g(y)|\right)^\tau \frac{dy}{y}\right)^{\frac{1}{\tau}}.$$

According to these inequalities we have

$$\begin{aligned} J_1 &\leq (\theta(\beta_1-\beta_0)\sigma)^{-\frac{1}{\sigma}} \left(\int\limits_0^\infty \left(y^{-\beta} \sup_{x\in\Omega} \|f\|_{L_q(B_y(x))}\right)^\tau \frac{dy}{y}\right)^{\frac{1}{\tau}} \\ &= (\theta(\beta_1-\beta_0)\sigma)^{-\frac{1}{\sigma}} \|f\|_{M^\beta_{q,\tau,\Omega}}, \\ J_2 &\leq (\theta(\beta_1-\beta_0)\sigma)^{-\frac{1}{\sigma}} \|f\|_{M^\beta_{q,\tau,\Omega}}, \end{aligned}$$

and

$$J_3 \leq ((1-\theta)(\beta_1-\beta_0)\sigma)^{-\frac{1}{\sigma}} \|f\|_{M^\beta_{q,\tau,\Omega}}.$$

Hence,

$$\|Tf\|_{M^\alpha_{p,\tau,\Omega}} \leq c_3(M_0 c^{-\theta(\beta_1-\beta_0)} + M_1 c^{-(1-\theta)(\beta_1-\beta_0)})\|f\|_{M^\beta_{q,\tau,\Omega}},$$

where $c_3 > 0$ depends only on $\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma$ and θ.

Let now $c = \left(\frac{M_1}{M_0}\right)^{\frac{1}{\beta_1-\beta_0}}$, then

$$\|Tf\|_{M^\alpha_{p,\tau,\Omega}} \leq 2c_3 M_0^{1-\theta} M_1^\theta \|f\|_{M^\beta_{q,\tau,\Omega}}.$$

Step 4. If $\beta_1 < \beta_0, \alpha_0 < \alpha_1$ or $\beta_0 < \beta_1, \alpha_1 < \alpha_0$ or $\beta_1 < \beta_0, \alpha_1 < \alpha_0$, then the assumptions of Part 2 and 3 are similar, changing only the choice of parameters γ and c. In the first case $\gamma = \frac{\alpha_1-\alpha_0}{\beta_0-\beta_1}$, $c = \left(\frac{M_1}{M_0}\right)^{\frac{1}{\beta_0-\beta_1}}$, in the second $\gamma = \frac{\alpha_0-\alpha_1}{\beta_1-\beta_0}$, $c = \left(\frac{M_1}{M_0}\right)^{\frac{1}{\beta_0-\beta_1}}$, in the third $\gamma = \frac{\alpha_0-\alpha_1}{\beta_0-\beta_1}$, $c = \left(\frac{M_1}{M_0}\right)^{\frac{1}{\beta_0-\beta_1}}$, and the proof of the theorem is completed.

□

Corollary 3.1. *Let* $0 < \alpha_0, \alpha_1, \beta_0, \beta_1 < \infty$, $\alpha_0 \neq \alpha_1$, $\beta_0 \neq \beta_1$, $0 < p, q \leq \infty$, $0 < \sigma \leq \tau \leq \infty$, $0 < \theta < 1$ *and*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1,\ \beta = (1-\theta)\beta_0 + \theta\beta_1.$$

Then there exists $c > 0$, *depending only on* $\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta$, *such that if* $x \in \mathbb{R}^n$, T *is a quasi-additive operator on* $LM^{\beta_0}_{q,\sigma,x} + LM^{\beta_1}_{q,\sigma,x}$, *and for some* $M_1, M_2 > 0$ *the following inequalities hold*

$$\|Tf\|_{LM^{\alpha_i}_{p,\infty,x}} \leq M_i \|f\|_{LM^{\beta_i}_{q,\sigma,x}}$$

for all functions $f \in LM^{\beta_i}_{q,\sigma,x}$, $i = 0, 1$, *then*

$$\|Tf\|_{LM^\alpha_{p,\tau,x}} \leq cAM_0^{1-\theta} M_1^\theta \|f\|_{LM^\beta_{q,\tau,x}}$$

for all functions $f \in LM^\beta_{q,\tau,x}$.

Proof. It suffices to take $\Omega = \{x\}$ in Theorem 3.1. □

Remark 3.1. *If T is a linear operator, then the statement of Corollary 3.1 follows using the standard arguments of interpolation theory from the equality*

$$(LM^{\lambda_0}_{p,r_0,x}, LM^{\lambda_1}_{p,r_1,x})_{\theta,\tau} = LM^{\lambda}_{p,\tau,x}, \tag{3.3}$$

where $0 < p, r_0, r_1, \tau \le \infty, 0 < \lambda_0, \lambda_1 < \infty, \lambda_0 \ne \lambda_1, 0 < \theta < 1,\ \lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, and respectively by equivalence of quasi-norm, which was proved in [14] *under the additional assumptions $\lambda_0, \lambda_1 \le \frac{n}{p}$ and in general case in* [6].

4. Riesz potentials in Morrey-type spaces

Let, for $0 < p, \theta \le \infty,\ L_{p,\theta}(\mathbb{R}^n)$ be the Lorentz space of functions defined on $\mathbb{R}^n$, and, for a measurable set $D \subset \mathbb{R}^n,\ L_{p,\theta}(D)$ be the set of all functions f measurable on D for which extensions of f by 0 outside D belong to $L_{p,\theta}(\mathbb{R}^n)$.

Lemma 4.1. *Let $1 < p, q, r < \infty,\ \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q},\ f \in L_{r,\infty}(B - D),\ g \in L_p(D)$, where $B, D \subset \mathbb{R}^n$ are measurable sets, then*

$$\left\| \int_D f(\cdot - y) g(y) dy \right\|_{L_q(B)} \le c \|f\|_{L_{r,\infty}(B-D)} ||g||_{L_p(D)},$$

where $c > 0$ depends only on the parameters n, p, q and r.

Proof. Let $f \in L_{r,\infty}(B - D),\ g \in L_p(D)$. Set

$$\tilde{g}(y) = \begin{cases} g(y), & y \in D, \\ 0, & y \notin D \end{cases}$$

and

$$\tilde{f}(y) = \begin{cases} f(y), & y \in B - D, \\ 0, & y \notin B - D. \end{cases}$$

Using O'Neil's inequality, we have

$$\begin{aligned} \left\| \int_D f(\cdot - y) g(y) dy \right\|_{L_q(B)} &= \left\| \int_D \tilde{f}(\cdot - y) g(y) dy \right\|_{L_q(B)} = \left\| \int_{\mathbb{R}^n} \tilde{f}(\cdot - y) \tilde{g}(y) dy \right\|_{L_q(B)} \\ &\le \left\| \int_{\mathbb{R}^n} \tilde{f}(\cdot - y) \tilde{g}(y) dy \right\|_{L_q(\mathbb{R}^n)} \le c \|\tilde{f}\|_{L_{r,\infty}(\mathbb{R}^n)} ||\tilde{g}||_{L_p(\mathbb{R}^n)} \\ &= c \|f\|_{L_{r,\infty}(B-D)} ||g||_{L_p(D)}, \end{aligned}$$

which completes the proof of the lemma.

□

Theorem 4.1. *Let* $z \in \mathbb{R}^n$, $1 < p < q < \infty$, $0 < \nu \le \lambda < \frac{n}{q}$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{\lambda-\nu}{n}$ *and* $\frac{1}{s} = \frac{1}{r} - \frac{\nu}{n}$. *If* $f \in LM^{\nu}_{p,1,z}$, $k \in L_{r,\infty}(\mathbb{R}^n)$ *and*

$$M = \sup_{m\in\mathbb{Z}} 2^{\nu m}\|k\|_{L_{s,\infty}(D_m)} < \infty, \tag{4.1}$$

where $D_m = B_{2^{m+1}}(0) \setminus B_{2^m}(0)$. *Then the convolution*

$$(k * f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy$$

exists for almost all $x \in \mathbb{R}^n$ *and the following estimate*

$$||k * f||_{LM^{\lambda}_{q,\infty,z}} \le c(||k||_{L_{r,\infty}(\mathbb{R}^n)} + M)||f||_{LM^{\nu}_{p,1,z}}$$

holds, where $c > 0$ *depends only on the parameters* n, p, q, r, ν, λ.

Proof. Let $z \in \mathbb{R}^n$ and $l \in \mathbb{Z}$. By applying Minkowski's inequality, we get

$$\|k * f\|_{L_q(B_{2^l}(z))} = \left\| \int_{B_{2^l}(0)} k(y)f(x-y)dy + \sum_{k=1}^{\infty} \int_{D_{l+k}} k(y)f(x-y)dy \right\|_{L_q(B_{2^l}(z))}$$

$$\le \left\| \int_{B_{2^l}(0)} k(y)f(x-y)dy \right\|_{L_q(B_{2^l}(z))} + \sum_{m=l}^{\infty} \left\| \int_{D_m} k(y)f(x-y)dy \right\|_{L_q(B_{2^l}(z))}$$

$$= I_1 + I_2.$$

Let us estimate each summand separately. By applying Lemma 4.1 with $1/r_1 = 1 + 1/q - 1/p$ we obtain

$$\begin{aligned} I_1 &\le \|k\|_{L_{r_1,\infty}(B_{2^l}(0))}\|f\|_{L_p(B_{2^{l+1}}(z))} \\ &\lesssim 2^{\nu l}\|k\|_{L_{r_1,\infty}(B_{2^l}(0))}\|f\|_{LM^{\nu}_{p,\infty,z}}. \end{aligned}$$

For the second estimate we use Hölder's inequality, Lemma 4.1, Lemma 2.2, and we get

$$\begin{aligned} I_2 &\lesssim 2^{\lambda l} \sum_{m=l}^{\infty} \left\| \int_{D_m} k(y)f(x-y)dy \right\|_{L_{q_1}(B_{2^l}(z))} \\ &\le 2^{\lambda l} \sum_{m=l}^{\infty} \|k\|_{L_{s,\infty}(D_m)}\|f\|_{L_p(B_{2^l}(z)-D_m)} \\ &\le 2^{\lambda l} \sum_{m=l}^{\infty} 2^{\nu(m+2)}\|k\|_{L_{s,\infty}(D_m)}(2^{-\nu(m+2)}\|f\|_{L_p(B_{2^{m+2}}(z))}) \\ &< 2^{\lambda l} \sup_{m\ge l} 2^{\nu(m+2)}\|k\|_{L_{s,\infty}(D_m)} \sum_{m\in\mathbb{Z}} (2^{-\nu(m+2)}\|f\|_{L_p(B_{2^{m+2}}(z))}) \\ &\asymp 2^{\lambda l} \sup_{m\ge l} 2^{\nu(m+2)}\|k\|_{L_{s,\infty}(D_m)} \|f\|_{LM^{\nu}_{p,1,z}}, \end{aligned}$$

where $1/q_1 = 1/q - \lambda/n,\ 1/s = 1 + 1/q - \lambda/n - 1/p = 1/r - \nu/n.$

According to Lemma 2.1 we have

$$\begin{aligned}
&2^{-\lambda l}\|k * f\|_{L_q(B_{2^l}(z))}\\
&\lesssim \left(2^{(\nu-\lambda)l}\|k\|_{L_{r_1,\infty}(B_{2^l}(0))}\|f\|_{LM^\nu_{p,\infty,z}} + \sup_{m\in\mathbb{Z}} 2^{\nu m}\|k\|_{L_{s,\infty}(D_m)}\|f\|_{LM^\nu_{p,1,z}}\right)\\
&\lesssim \left(2^{(\nu-\lambda)l}\|k\|_{L_{r_1,\infty}(B_{2^l}(0))} + \sup_{m\in\mathbb{Z}} 2^{\nu m}\|k\|_{L_{s,\infty}(D_m)}\right)\|f\|_{LM^\nu_{p,1,z}}.
\end{aligned}$$

Since $\nu \le \lambda$, we get

$$\begin{aligned}
2^{(\nu-\lambda)l}||k||_{L_{r_1,\infty}(B_{2^l}(0))} &\asymp 2^{(\nu-\lambda)l}\sup_{0<s<2^{ln}} s^{\frac{1}{r_1}}k^*(s)\\
&= 2^{(\nu-\lambda)l}\sup_{0<s<2^{ln}} s^{\frac{\lambda-\nu}{n}}s^{\frac{1}{r_1}-\frac{\lambda-\nu}{n}}k^*(s) \lesssim \sup_{s>0} s^{\frac{1}{r_1}-\frac{\lambda-\nu}{n}}k^*(s).
\end{aligned}$$

Therefore,

$$2^{(\nu-\lambda)l}||k||_{L_{r_1,\infty}(B_{2^l}(0))} \lesssim ||k||_{L_{r,\infty}(\mathbb{R}^n)},$$

where $\frac{1}{r} = \frac{1}{r_1} - \frac{\lambda-\nu}{n}$.

Since $l \in \mathbb{Z}$ is arbitrary, by Lemma 2.2 we get

$$\|k * f\|_{LM^\lambda_{p,\infty,z}} \asymp \sup_{l\in\mathbb{Z}} 2^{-\lambda l}\|k * f\|_{L_p(B_{2^l}(z))} \lesssim \left(\|k\|_{L_{r,\infty}(\mathbb{R}^n)} + M\right)\|f\|_{LM^\nu_{p,1,z}},$$

which completes the proof of the theorem. □

Note that assumption (4.1) in Theorem 4.1 is essential, i.e. the direct analogue of the O'Neil inequality

$$\|k * f\|_{LM^\lambda_{q,\infty,z}} \le c\|k\|_{L_{r,\infty}}\|f\|_{LM^\nu_{p,1,z}} \tag{4.2}$$

does not hold for any $c > 0$ depending only on the parameters n, p, q, r, ν, λ.

Example 4.1. *Let* $z = 0,\ 1 < p \le q < \infty,\ 0 < \nu \le \lambda < \frac{n}{q},\ 1+\frac{1}{q} = \frac{1}{p}+\frac{1}{r}+\frac{\lambda-\nu}{n}$, $\alpha > \frac{1}{\nu p}$. *Define the functions*

$$f(x) = \begin{cases} 1, & |x| \in [l^\alpha, l^\alpha + 1], \quad l \in \mathbb{Z}_+\\ 0, & \text{otherwise}\end{cases}$$

and

$$k(x) = \begin{cases} (l+1)^{-\frac{1}{r}}, & |x| \in [l^\alpha, l^\alpha + 1], \quad l \in \mathbb{Z}_+\\ 0, & \text{otherwise.}\end{cases}$$

We show that the function f belongs to the space $LM^\nu_{p,1,0}$. Indeed,

$$\|f\|_{LM^\nu_{p,1,0}} = \int_0^\infty \frac{1}{t^{\nu+1}}\left(\int_{-t}^{t} |f(y)|^p dy\right)^{1/p} dt.$$

Furthermore,

$$\int_{-t}^{t} |f(y)|^p dy = 2\min\Big(t, \sum_{l^\alpha \le t} 1\Big) \le 2\min(t, t^{1/\alpha}),$$

hence,

$$\|f\|_{LM^{\nu}_{p,1,0}} < \infty.$$

Since $k^*(t) \le t^{-1/r}$*, it follows that* $k \in L_{r,\infty}(\mathbb{R})$*, but also for* $x \in [-1/2, 1/2]$ *we have*

$$(k*f)(x) = \int_{-\infty}^{+\infty} k(x-y)f(y)dy \ge \frac{1}{2}\sum_{l=1}^{\infty}(l+1)^{-1/r} = \infty,$$

and an inequality of type (4.2) *is impossible.*

Theorem 4.2. *Let* $\Omega \subset \mathbb{R}^n$, $1 < p < q < \infty$, $0 < \tau \le \infty$, $0 < \nu \le \lambda < \frac{n}{q}$ *and* $\gamma = n\big(\frac{1}{p} - \frac{1}{q}\big) + \lambda - \nu$,

$$I_\gamma f(x) = \int_{\mathbb{R}^n} f(x-y)\frac{dy}{|y|^{n-\gamma}}.$$

If $f \in M^{\nu}_{p,\tau,\Omega}$*, then* $I_\gamma f \in M^{\lambda}_{q,\tau,\Omega}$*, and the following inequality holds*

$$\|I_\gamma f\|_{M^{\lambda}_{q,\tau,\Omega}} \le c\|f\|_{M^{\nu}_{p,\tau,\Omega}}, \tag{4.3}$$

where $c > 0$ *depends only on* $n, p, q, \tau, \nu, \lambda$.

Proof. Let $k(x) = 1/|x|^{n-\gamma}$. Check the conditions of Theorem 4.1. By the assumptions of Theorems 4.1 and 4.2 $\frac{1}{r} + \frac{\gamma}{n} - 1 = 0$. Moreover, $k^*(t) \asymp t^{\frac{\gamma}{n}-1}$ on $(0,\infty)$, $\big(k\chi_{D_m}\big)^*(t) = 0$ if $|t| \ge |D_m|$ and $\big(k\chi_{D_m}\big)^*(t) \asymp 2^{m(\gamma-n)}$ on $(0, |D_m|)$ (because $k(x) \asymp 2^m$ on D_m). Hence, it follows that

$$\|k\|_{L_{r,\infty}(\mathbb{R}^n)} = \sup_{t>0} t^{1/r}k^*(t) \asymp \sup_{t>0} t^{\frac{1}{r}+\frac{\gamma}{n}-1} = 1,$$

$$\begin{aligned}\sup_{m\in\mathbb{Z}} 2^{\nu m}\|k\|_{L_{s,\infty}(D_m)} &= \sup_{m\in\mathbb{Z}} 2^{\nu m} \sup_{0<t\le|D_m|} t^{\frac{1}{s}}\big(k\chi_{D_m}\big)^*(t) \\ &\asymp \sup_{m\in\mathbb{Z}} 2^{\nu m}\big(\sup_{0<t\le 2^{mn}} t^{\frac{1}{n}-\frac{\nu}{n}}\big)2^{m(\gamma-n)} = 1.\end{aligned}$$

Then by applying Theorem 4.1 we obtain the weak inequality

$$\|I_\gamma f\|_{LM^{\lambda}_{q,\infty,z}} \le c\|f\|_{LM^{\nu}_{p,1,z}},\ z \in \Omega.$$

Next, since $0<\nu<\frac{n}{p},\ 0<\lambda<\frac{n}{q}$, then there exist ν_0,ν_1 and λ_0,λ_1, such that

$$0<\nu_1<\nu<\nu_0<\frac{n}{p},\quad 0<\lambda_1<\lambda<\lambda_0<\frac{n}{q}.$$

Let $\nu_0-\lambda_0=\nu_1-\lambda_1=\nu-\lambda$. Then the weak inequalities hold

$$\|I_\gamma f\|_{LM^{\lambda_i}_{q,\infty,z}}\le c\,\|f\|_{LM^{\nu_i}_{p,1,z}}\,,\ i=0,1,\ z\in\Omega.$$

Hence, by interpolation Theorem 3.1 the strong inequality holds

$$\|I_\gamma f\|_{M^{\lambda}_{q,\tau,\Omega}}\le c\|f\|_{M^{\nu}_{p,\tau,\Omega}},$$

where $0<\tau\le\infty,\ \nu=(1-\theta)\nu_0+\theta\nu_1,\ \lambda=(1-\theta)\lambda_0+\theta\lambda_1$ and $0<\theta<1$,

$$\begin{aligned}1+\frac{1}{q}&=\frac{1}{p}+\frac{1}{r}+\frac{(1-\theta)(\lambda_0-\nu_0)+\theta(\lambda_1-\nu_1)}{n}\\&=1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}+\frac{\lambda-\nu}{n},\end{aligned}$$

which completes the proof of the theorem. □

If $\tau=\infty$ and $\Omega=\mathbb{R}^n$, then inequality (4.3) takes the form

$$\|I_\gamma f\|_{M^{\lambda}_{q}}\le c\|f\|_{M^{\nu}_{p}}.$$

Note that the assumption $\nu\le\lambda$ of Theorem 4.2 in this case is not required, and this inequality holds for any $1<p<q<\infty, 0<\nu<\frac{n}{p}, 0<\lambda<\frac{n}{q}, \gamma=n(\frac{1}{p}-\frac{1}{q})+\lambda-\nu$. For $\lambda=\nu$ this was proved in [24], under the assumption $q\lambda=p\nu$ in [1] (in this case $\nu>\lambda$), and, without any additional assumptions on ν and λ, in [11, 12].

The condition $\nu\le\lambda$ appears in Theorem 4.2 because arbitrary $0<\tau\le\infty$ and $\Omega\subset\mathbb{R}^n$ are considered. Note that at least for the case $\tau=q$ and $\Omega=\{0\}$ this condition is a necessary condition for the validity of inequality (4.3). Indeed, by (4.3) and Lemma 2.1 (ii) it follows that

$$\|I_\gamma f\|_{LM^{\lambda}_{q,q,0}}\le c_1\|f\|_{LM^{\nu}_{p,p,0}},$$

that is equivalent to the inequality

$$\|I_\gamma f\|_{L_q(|x|^{-q\lambda})}\le c_2\|f\|_{L_p(|x|^{-p\nu})}. \tag{4.4}$$

Indeed, let $N\in\mathbb{N},\ z_N=(N,0,...,0)$,

$$f(x)=\chi_{B_2(z_N)}(x)|x|^{\nu}.$$

Then by (4.4) we have

$$|B_2(0)|^{1/p}c_2 \geq \|I_\gamma f\|_{L_q(|x|^{-q\lambda})}$$
$$\geq \Bigg(\int\limits_{B_1(z_N)} \Bigg| \frac{1}{|x|^\lambda} \int\limits_{x-B_2(z_N)} \frac{|x-y|^\nu}{|y|^{n-\gamma}} dy \Bigg|^q dx \Bigg)^{1/q}.$$

Since $x - B_2(z_N) \supset B_1(0)$ for any $x \in B_1(z_N)$ we get for $N > 2$

$$|B_2(0)|^{1/p}c_2 \geq \Bigg(\int\limits_{B_1(z_N)} \Bigg| \frac{1}{|x|^\lambda} \int\limits_{B_1(0)} \frac{|x-y|^\nu}{|y|^{n-\gamma}} dy \Bigg|^q dx \Bigg)^{1/q}$$
$$\geq \Bigg(\int\limits_{B_1(z_N)} \Bigg| \frac{1}{|x|^\lambda} \int\limits_{B_1(0)} \frac{(|x|-1)^\nu}{|y|^{n-\gamma}} dy \Bigg|^q dx \Bigg)^{1/q}$$
$$\geq \Bigg(\int\limits_{B_1(0)} \frac{dy}{|y|^{n-\gamma}} \Bigg) \Bigg(\int\limits_{B_1(z_N)} \Bigg| \frac{(|x|-1)^\nu}{|x|^\lambda} \Bigg|^q dx \Bigg)^{1/q}$$
$$\geq c_3 \Bigg(\int\limits_{B_1(z_N)} |x|^{(\nu-\lambda)q} dx \Bigg)^{1/q},$$

where $c_1, c_2, c_3 > 0$ are independent of N and this is possible only if $\nu \leq \lambda$.

5. Singular operators in Morrey-type spaces

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$, $1 \leq p < \infty$, $0 < \nu < \frac{n}{p}$, $0 < \tau \leq \infty$. If a subadditive operator T is bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and there exists $C > 0$ such that*

$$|Tf(x)| \leq \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy \tag{5.1}$$

for all locally integrable functions f with compact support and for all $x \in \mathbb{R}^n \setminus$ supp$f$, then T is also bounded from $M^\nu_{p,\tau,\Omega}$ to $M^\nu_{p,\tau,\Omega}$.

Proof. Let $z \in \Omega$, $l \in \mathbb{Z}$ and $D_m(z) = B_{2^{m+1}}(z) \setminus B_{2^m}(z)$, $m \in \mathbb{Z}$, $f \in LM^\nu_{p,1,z}$. By applying Minkowski's and Hölder's inequalities and the subadditivity of T and taking into account that $f = f\chi_{B_{2^{l+1}}(z)} + \sum\limits_{m=l+1}^{\infty} f\chi_{D_m(z)}$ we get

$$\|Tf\|_{L_p(B_{2^{l+1}}(z))} \le \big\|T\big(f\chi_{B_{2^{l+1}}(z)}\big)\big\|_{L_p(B_{2^l}(z))} + \left\|T\Big(\sum_{m=l+1}^{\infty} f\chi_{D_m(z)}\Big)\right\|_{L_p(B_{2^l}(z))}$$

$$\le \big\|T\big(f\chi_{B_{2^{l+1}}(z)}\big)\big\|_{L_p(B_{2^l}(z))} + \left\|\sum_{m=l+1}^{\infty} \big|T\big(f\chi_{D_m(z)}\big)\big|\right\|_{L_p(B_{2^l}(z))}$$

$$\lesssim \big\|T\big(f\chi_{B_{2^{l+1}}(z)}\big)\big\|_{L_p(B_{2^l}(z))} + 2^{\nu l}\left\|\sum_{m=l+1}^{\infty} \big|T\big(f\chi_{D_m(z)}\big)\big|\right\|_{L_q(B_{2^l}(z))} \equiv I_1 + I_2,$$

where q is defined by the equality $\frac{1}{p} = \frac{1}{q} + \frac{\nu}{n}$.

Let us estimate I_1 and I_2. Since T is bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ we get

$$I_1 \le \big\|T\big(f\chi_{B_{2^{l+1}}(z)}\big)\big\|_{L_p(\mathbb{R}^n)} \lesssim \|f\|_{L_p(B_{2^{l+1}}(z))}.$$

Since $B_{2^l}(z) \cap \operatorname{supp} f\chi_{D_m(z)} = \emptyset$ for any $m \ge l+1$ by applying estimate (5.1) with f replaced by $f\chi_{D_m(z)}$ we get

$$I_2 \le 2^{\nu l} \sum_{m=l+1}^{\infty} \big\|T\left(f\chi_{D_m(z)}\right)\big\|_{L_q(B_{2^l}(z))}$$

$$\lesssim 2^{\nu l} \sum_{m=l+1}^{\infty} \left\| \int\limits_{D_m(z)} \frac{|f(y)|}{|x-y|^n} dy \right\|_{L_q(B_{2^l}(z))}.$$

Since for $m \ge l+1$, $B_{2^l}(z) - D_m(z) = B_{2^l}(0) + z - (D_m(0) + z) = B_{2^l}(0) - D_m(0) = D_m(0) - B_{2^l}(0) \subset G_m = D_{m-1}(0) \cup D_m(0) \cup D_{m+1}(0)$, application of Lemma 4.1 yields

$$I_2 \lesssim 2^{\nu l} \sum_{m=l+1}^{\infty} \|\,|x|^{-n}\|_{L_{r,\infty}(B_{2^l}(z)-D_m(z))} \|f\|_{L_p(D_m(z))}$$

$$\le 2^{\nu l} \sum_{m=l+1}^{\infty} \|\,|x|^{-n}\|_{L_{r,\infty}(G_m)} \|f\|_{L_p(B_{2^{m+1}}(z))},$$

where $1/r = 1 + 1/q - 1/p = 1 - \nu/n$.

Note that

$$\||x|^{-n}\|_{L_{r,\infty}(G_m)} \le \|\,|x|^{-n}\chi_{G_m}(x)\|_{L_{r,\infty}(\mathbb{R}^n)} = \sup_{t>0} t^{\frac{1}{r}} \left(|x|^{-n}\chi_{G_m}(x)\right)^*(t)$$

$$= \sup_{0<t\le|G_m|} t^{\frac{1}{r}} \left(|x|^{-n}\chi_{G_m}(x)\right)^*(t) \le 2^{-n(m-1)} \sup_{0<t\le|G_m|} t^{\frac{1}{r}}$$

$$\lesssim 2^{-m(n-\frac{n}{r})} = 2^{-\nu m}.$$

Hence,

$$I_2 \lesssim 2^{\nu l} \sum_{m=l+1}^{\infty} 2^{-\nu(m+1)} \|f\|_{L_p(B_{2^{m+1}}(z))}.$$

Therefore by Lemma 2.2 with $q = 1$

$$\begin{aligned} &2^{-\nu l}\|Tf\|_{L_p(B_{2^l}(z))} \\ &\lesssim 2^{-\nu l}\|f\|_{L_p(B_{2^{l+1}}(z))} + \sum_{m=l+1}^{\infty} 2^{-\nu(m+1)}\|f\|_{L_p(B_{2^{m+1}}(z))} \\ &\lesssim \|f\|_{LM^{\nu}_{p,1,z}}. \end{aligned}$$

Due to arbitrariness of l by Lemma 2.2 with $q = \infty$ we get that for all $z \in \Omega$

$$\|Tf\|_{LM^{\nu}_{p,\infty,z}} \lesssim \sup_{l\in\mathbb{Z}} 2^{-\nu l}\|Tf\|_{L_p(B_{2^l}(z))} \lesssim \|f\|_{LM^{\nu}_{p,1,z}}.$$

Application of Theorem 3.1 completes the proof of the theorem. □

REMARK 5.1. *Let T be a Calderon-Zygmund operator, i.e. a linear operator taking C_0^{∞} into L_1^{loc}, bounded on L_2 and represented by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad a.e. \quad on\ \mathbb{R}^n \backslash \mathrm{supp} f$$

for every function $f \in L^{\infty}(\mathbb{R}^n)$ with compact support. Here $K(x,y)$ is a continuous function away from the diagonal and satisfies the standard estimates: for some $c_1 > 0$ and $0 < \varepsilon \leq 1$

$$|K(x,y)| \leq c_1|x-y|^{-n},$$

for all $x, y \in \mathbb{R}^n, x \neq y$ and

$$\begin{aligned} &|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \\ &\leq c_1\left(\frac{|x-x'|}{|x-y|}\right)^{\varepsilon}|x-y|^{-n} \end{aligned}$$

whenever $2|x-x'| \leq |x-y|$. This class of operators was introduced by Coifman and Meyer [16].

It is known that a Calderon-Zygmund operator is bounded in the Lebesgue spaces $L_p(\mathbb{R}^n)$ if $p \in (1,\infty)$. Hence, Theorem 5.1 can be applied to all Calderon-Zygmund operators with any $1 < p < \infty$.

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References

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math. Journal, **42** (1975), pp. 765–778.

[2] D.R. Adams, *Morrey spaces*, Birkhäuser Mathematics, Lecture notes in applied and numerical harmonic analysis, (2015), 124 p.

[3] O. Blasco, A. Ruiz, L. Vega, *Non interpolation in Morrey-Campanato and block spaces*, Ann. Scuola Norm. Super. Pisa, **28** (1999), no. 1, pp. 31–40.

[4] V.I. Burenkov, *Recent progress in studing the boundedness of classical operators of real analysis in general Morrey-type spaces I*, Eurasian Mathematical Journal, **3** (2012), no. 3, pp. 11–32.

[5] V.I. Burenkov, *Recent progress in studing the boundedness of classical operators of real analysis in general Morrey-type spaces II*, Eurasian Mathematical Journal, **4** (2013), no. 1, pp. 21–45.

[6] V.I. Burenkov, D.K. Chigambayeva, E.D. Nursultanov, *Description of the interpolation spaces for a pair of local Morrey-type spaces and their generalizations*, Trudy Math. Inst. Steklov, **284** (2014), pp. 97–128.

[7] V.I. Burenkov, D.K. Darbayeva, E.D. Nursultanov, *Description of interpolation spaces for general local Morrey-type spaces*, Eurasian Mathematical Journal, **4** (2013), no. 1, pp. 46-53.

[8] V.I. Burenkov, A. Gogatishvili, V.S. Guliyev, R.Ch. Mustafayev, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal, **35** (2011), no. 1, pp. 67–87.

[9] V.I. Burenkov, H.V. Guliyev, *Nessesary and sufficient conditions for the boundedness of the maximal operator in the local Morrey-type spaces*, Doklady Ross. Akad. Nauk. Matematika, **391** (2003), no. 5, pp. 591–594 (in Russian). English trans. in Russian Acad. Sci. Dokl. Math. (2003).

[10] V.I. Burenkov, H.V. Guliyev, *Nessesary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces*, Studia Mathematica, **163** (2004), no. 2, pp. 157–176.

[11] V.I. Burenkov, H.V. Guliyev, V.S. Guliyev, *Nessesary and sufficient conditions for the boundedness of the Riesz operator in the local Morrey-type spaces*, Doklady Ross. Akad. Nauk. Matematika, **412** (2007), no. 5, pp. 585–589 (in Russian). English transl. in Russian Acad. Sci. Dokl. Math. (2007).

[12] V.I. Burenkov, V.S. Guliyev, *Nessesary and sufficient conditions for the boundedness of the Riesz operator in the local Morrey-type spaces*, Potential Analysis, **30** (2009), no. 3, pp. 211–249.

[13] V.I. Burenkov, V.S. Guliyev, A. Serbetci, T.V. Tararykova, *Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces*, Eurasian Mathematical Journal, **1** (2010), no. 1, pp. 32–53.

[14] V.I. Burenkov, E.D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces*, Trudy Math. Inst. Steklov, **269** (2010), pp. 46–56.

[15] A. Campanato, M.K.V. Murthy, *Una generalizzazione del teorema di Riesz-Thorin*, Ann. Scuola Norm. Super. Pisa, **19** (1965), pp. 87–100.

[16] R. Coifman, Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Société Mathématique de France, Astérisque, **57** (1978), 185 p.

[17] P.G. Lemarié-Rieusset, *The role of Morrey spaces in the study of Navier-Stokes and Euler equations*, Eurasian Mathematical Journal, **3** (2012), no. 3, pp. 62–93.

[18] P.G. Lemarié-Rieusset, *Multipliers and Morrey spaces*, Potential Analysis, **38** (2013), no. 3, pp. 741–752.

[19] C.B. Morrey, *On the solution of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), pp. 126–166.

[20] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr., **166** (1994), pp. 95–103.

[21] F. Nazarov, S. Treil, A. Volberg, *Weak type estimates and Cotlar inequalities for Calderön-Zygmund operators on nonhomogeneous spaces*, Internat. Math. Res. Notices, (1998), pp. 463–487.

[22] E.D. Nursultanov, *Net spaces and the Fourier transform*, Dokl. Akad. Nauk, **361** (1998), no. 5, pp. 597–599.

[23] E.D. Nursultanov, *Net spaces and inequalities of HardyLittlewood type*, Sb. Math., **189** (1998), no. 3, pp. 399–419.

[24] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal., **4** (1969), pp. 71–87.

[25] J. Peetre, *On convolution operators leaving $L^{p,\lambda}$ spaces invariant*, Ann. Mat. Pura ed Appl., **72** (1966), no. 4, pp. 295–304.

[26] M.A. Ragusa, *Operators in Morrey type spaces and applications*, Eurasian Mathematical Journal, **3** (2012), no. 3, pp. 94–109.

[27] A. Ruiz, L. Vega, *Corrigenda to "Unique continuation for Schrödinger operators" and a remark on interpolation on Morrey spaces*, Publ. Mat., Barc., **39** (1995), pp. 405–411.

[28] Y. Sawano, Tanaka H., *Morrey spaces for nondoubling measures*, Acta Mathematica Sinica, **21** (2005), no. 6, pp. 1535–1544.

[29] G. Stampacchia, *$\mathcal{L}_{p,\lambda}$-spaces and interpolation*, Comm. Pure Appl. Math., **17** (1964), pp. 293–306.

[30] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, **43** (1993), 712 p.

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