Berezin transform and Stratonovich-Weyl correspondence for the multi-dimensional Jacobi group

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ABSTRACT – We study the Berezin transform and the Stratonovich-Weyl correspondence associated with a holomorphic representation of the multi-dimensional Jacobi group.


KEYWORDS. Berezin quantization; Berezin transform; quasi-Hermitian Lie group; unitary representation; holomorphic representation; reproducing kernel Hilbert space; multi-dimensional Jacobi group; Stratonovich-Weyl correspondence.

1. Introduction

This paper is part of a program to study Berezin transforms and Stratonovich-Weyl correspondences associated with holomorphic representations. The notion of Stratonovich-Weyl correspondence was introduced in [31] in order to extend the usual Weyl correspondence between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$ (see [1], [21]) to the general setting of a Lie group acting on a homogeneous space. Stratonovich-Weyl correspondences were systematically studied by J.M. Gracia-Bondía, J.C. Várilly and various co-workers, see in particular [23], [20], [18] and [22]. The following definition is taken from [22].

DEFINITION 1.1. Let $G$ be a Lie group and $\pi$ a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Let $M$ be a homogeneous $G$-space and $\mu$ a (suitably normalized) $G$-invariant measure on $M$. Then a Stratonovich-Weyl correspondence for the triple $(G, \pi, M)$ is an isomorphism $W$ from a vector space of operators on $\mathcal{H}$ to a vector space of (generalized) functions on $M$ satisfying the following properties:

1. $W$ maps the identity operator of $\mathcal{H}$ to the constant function 1;
(2) Reality: the function $W(A^*)$ is the complex-conjugate of $W(A)$;

(3) Covariance: we have $W(\pi(g) A \pi(g)^{-1})(x) = W(A)(g^{-1} \cdot x)$;

(4) Unitarity: we have

$$\int_M W(A)(x)W(B)(x) \, d\mu(x) = \text{Tr}(AB).$$

In this context, $M$ is generally a coadjoint orbit of $G$ which is associated with $\pi$ by the Kirillov-Kostant method of orbits [25]. For instance, consider the case when $G$ is the $(2n+1)$-dimensional Heisenberg group $H_n$. Each non-degenerate coadjoint orbit $M$ of $G$ is then diffeomorphic to $\mathbb{R}^{2n}$ and is associated with a Schrödinger representation $\pi$ of $H_n$ on $L^2(\mathbb{R}^n)$. In this case, the classical Weyl correspondence gives a Stratonovich-Weyl correspondence for the triple $(H_n, \pi, M)$ [21], [22].

In the case when $G$ is a quasi-Hermitian Lie group and $\pi$ is a unitary representation of $G$ (on a Hilbert space $\mathcal{H}$) which is holomorphically induced from a unitary character of a compactly embedded subgroup $K$ of $G$, we can apply an idea of [20] and we obtain a Stratonovich-Weyl correspondence by modifying suitably the Berezin correspondence $S$ [14] (see also [2] and [3]).

More precisely, recall that $S$ is an isomorphism from the Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}$ (endowed with the Hilbert-Schmidt norm) onto a space of square integrable functions on a homogeneous complex domain [32]. The map $S$ satisfies (1), (2) and (3) of Definition 1.1 but not (4). A Stratonovich-Weyl correspondence $W$ is then obtained by taking the isometric part in the polar decomposition of $S$, that is, $W := (SS^*)^{-1/2}S$. Let us mention that $B := SS^*$ is then the so-called Berezin transform which have been studied by many authors, see in particular [19], [27], [28], [32], [33].

In [14], we considered the case when the Lie algebra $\mathfrak{g}$ of $G$ is reductive. In this case, we proved that $B$ can be extended to a class of functions which contains $S(d\pi(X_1X_2\cdots X_p))$ for $X_1, X_2, \ldots, X_p \in \mathfrak{g}$ and that the restrictions to each simple ideal of $\mathfrak{g}$ of the mappings $X \to S(d\pi(X))$ and $X \to W(d\pi(X))$ are proportional (see also [12] and [13]).

The case when $\mathfrak{g}$ is not reductive is more delicate. In [16] we investigated the case of the diamond group and, in [17], we studied $B$ and $W$ in the case of the Jacobi group.

The aim of the present paper is to generalize the results of [17] to the case of the multi-dimensional Jacobi group, which is technically more complicated. The multi-dimensional Jacobi group plays a central role in different areas of Mathematics and Physics and its holomorphic unitary representations were studied intensively, see [26], [9], [10], [4], [6]. In particular, the metaplectic factorization should be used to reduce the study of the highest weight representations of a quasi-Hermitian Lie group to that of some generalized multi-dimensional Jacobi group [26]. Then the study of the case of the multi-dimensional Jacobi group can be considered as a first step towards the general case.
In this paper, we begin by some generalities on the multi-dimensional Jacobi group (Section 2) and its holomorphic representations (Section 3). Then we introduce the Berezin correspondence $S$, the Berezin transform $B$ and the Stratonovich-Weyl correspondence $W$ (Section 4). In Section 5, we show that, under some technical assumptions, the Berezin transform of $S(d\pi(X_1X_2\cdots X_p))$ is well-defined for each $X_1, X_2, \ldots, X_p \in g$. In Section 6, we identify a class of functions which is stable under $B$ and contains $S(d\pi(X))$ for each $X \in g$. We also give an expression of $W(d\pi(X))$ in terms of some integrals of Hua’s type (see [24]).

2. The multi-dimensional Jacobi group

The material of this section and of the following section is essentially taken from [21], Chapter 4, [26], Chapters VII and XII and [15].

Consider the symplectic form $\omega$ on $\mathbb{C}^n \times \mathbb{C}^n$ defined by

$$\omega((z, w), (z', w')) = \frac{i}{2} \sum_{k=1}^{2n} (z_kw'_k - z'_kw_k).$$

for $z, w, z', w' \in \mathbb{C}^n$. The $(2n + 1)$-dimensional real Heisenberg group is

$$H := \{((z, \bar{z}), c) : z \in \mathbb{C}^n, c \in \mathbb{R}\}$$

endowed with the multiplication

$$((z, \bar{z}), c) \cdot ((z', \bar{z}'), c') = ((z + z', \bar{z} + \bar{z}''), c + c' + \frac{1}{i} \omega((z, \bar{z}), (z', \bar{z}'))).$$

Then the complexification $H^c$ of $H$ is

$$H^c := \{((z, w), c) : z, w \in \mathbb{C}^n, c \in \mathbb{C}\}$$

and the multiplication of $H^c$ is obtained by replacing $(z, \bar{z})$ by $(z, w)$ and $(z', \bar{z}')$ by $(z', w')$ in the preceding equality. We denote by $\mathfrak{h}$ and $\mathfrak{h}^c$ the Lie algebras of $H$ and $H^c$.

Now consider the group $S := Sp(n, \mathbb{C}) \cap SU(n, n) \simeq Sp(n, \mathbb{R})$ [26], p. 501, [21], p. 175. Then $S$ consists of all matrices

$$h = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P, Q \in M_n(\mathbb{C}), \quad PP^* = QQ^* = I_n, \quad PQ^t = QP^t$$

and $S^c = Sp(n, \mathbb{C})$.

The group $S$ acts on $H$ by

$$h \cdot ((z, \bar{z}), c) = (h(z, \bar{z}), c) = (Pz + Q\bar{z}, \bar{Q}z + \bar{P}\bar{z}, c)$$

where the elements of $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^n$ are considered as column vectors. Then we can form the semi-direct product $G := H \rtimes S$ called the multi-dimensional Jacobi
The elements of $G$ can be written as $((z, \bar{z}), c, h)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $h \in S$. The multiplication of $G$ is thus given by

$$(z, \bar{z}), c, h) \cdot ((z', \bar{z}'), c', h') = ((z, \bar{z}) + h(z', \bar{z}'), c + c' + \frac{1}{2}\omega((z, \bar{z}), h(z', \bar{z}')), hh').$$

The complexification $G^c$ of $G$ is then the semi-direct product $G^c = H^c \rtimes Sp(n, \mathbb{C})$ whose elements can be written as $((z, w), c, h)$ where $z, w \in \mathbb{C}^n$, $c \in \mathbb{C}$, $h \in Sp(n, \mathbb{C})$ and the multiplication of $G^c$ is obtained by replacing $z$ by $z'$ with $w$ in the preceding formula.

We denote by $s$, $s^c$, $g$ and $g^c$ the Lie algebras of $S$, $S^c$, $G$ and $G^c$. The Lie brackets of $g^c$ are given by

$$[((z, w), c), ((z', w'), c', A')] = (A(z', w') - A'(z, w), \omega((z, w), (z', w')), [A, A']).$$

Let $\theta$ denote conjugation over the real form $g$ of $G^c$. For $X \in g^c$, we set $X^* = -\theta(X)$. We can easily verify that if $X = ((z, w), c, (\begin{smallmatrix} A & B \\ C & -A \end{smallmatrix})) \in g^c$ then we have

$$X^* = (((-\bar{w}, -z), -c, (\begin{smallmatrix} A' & -B \\ -C & -A' \end{smallmatrix})).$$

Also, we denote by $g \rightarrow g^*$ the involutive anti-automorphism of $G^c$ which is obtained by exponentiating $X \rightarrow X^*$ to $G^c$.

Let $K$ be the subgroup of $G$ consisting of all elements $((0, 0), c, (\begin{smallmatrix} P & 0 \\ 0 & p \end{smallmatrix}))$ where $c \in \mathbb{R}$ and $P \in U(n)$. Then the Lie algebra $\mathfrak{k}$ of $K$ is a maximal compactly embedded subalgebra of $g$ and the subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ consisting of all elements of the form $((0, 0), c, A)$ where $A$ is diagonal is a compactly embedded Cartan subalgebra of $g$ [26], p. 250. Following [26], p. 532, we set

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} : y \in \mathbb{C}^n, Y \in M_n(\mathbb{C}), Y^t = Y \right\}$$

and

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} : v \in \mathbb{C}^n, V \in M_n(\mathbb{C}), V^t = V \right\}.$$

Then we have the decomposition $g^c = \mathfrak{p}^+ \oplus \mathfrak{t}^c \oplus \mathfrak{p}^-.$

Henceforth we denote by $a(y, Y)$ the element $((y, 0), 0, (\begin{smallmatrix} 0 & Y \\ Y & 0 \end{smallmatrix}))$ of $\mathfrak{p}^+$. Also, we denote by $p_\mathfrak{p}^+$, $p_\mathfrak{t}^c$ and $p_\mathfrak{p}^-$ the projections of $g^c$ onto $\mathfrak{p}^+$, $\mathfrak{t}^c$ and $\mathfrak{p}^-$ associated with the above direct decomposition.

Let $P^+$ and $P^-$ be the analytic subgroups of $G^c$ with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Then we have

$$P^+ = \left\{ \begin{pmatrix} I_n & Y \\ 0 & I_n \end{pmatrix} : y \in \mathbb{C}^n, Y \in M_n(\mathbb{C}), Y^t = Y \right\}$$

and

$$P^- = \left\{ \begin{pmatrix} I_n & 0 \\ V & I_n \end{pmatrix} : v \in \mathbb{C}^n, V \in M_n(\mathbb{C}), V^t = V \right\}.$$

In particular, we see that $G$ is a group of the Harish-Chandra type [26], p. 507 (see also [30]), that is, the following properties are satisfied:
(1) \( g^c = p^+ \oplus \pi^c \oplus p^- \) is a direct sum of vector spaces, \((p^+)^* = p^-\) and \([\pi^c, \pi^c] \subset p^-;\)

(2) The multiplication map \( P^+K^cP^- \to G^c, (z, k, y) \to zky \) is a biholomorphic
diffeomorphism onto its open image;

(3) \( G \subset P^+K^cP^- \) and \( G \cap K^cP^- = K.\)

We can introduce an action (defined almost everywhere) of \( g \) as follows. For \( y, Y \in D \) we define the element \( g \cdot Z \) of \( p^+ \) by \( g \cdot Z := \log(g \exp Z). \) From the above formula for the action of \( g \) as follows. We set

\[
\begin{align*}
\text{if and only if } \det(D) \neq 0 \text{ and, in this case, we have } y = z_0 - BD^{-1}w_0, & \\
& \text{we denote by } z := \log(g \exp Z). \text{ From the above formula for the action of } \, g \text{ as follows. We set } \, g \cdot Z := \log(g \exp Z). \text{ From the above formula for the action of } \, g \text{ as follows. We set } \, g \cdot Z := \log(g \exp Z). \text{ From the above formula for the action of } \, g \text{ as follows. We set } \, g \cdot Z := \log(g \exp Z).
\end{align*}
\]

This implies that

\[
D := G \cdot 0 = \{a(y, Y) \in p^+ : I_n - YY^* > 0\} \cong \mathbb{C}^n \times \mathcal{B}.
\]

where \( \mathcal{B} := \{Y \in M_n(\mathbb{C}) : Y^* = Y, \, I_n - YY^* > 0\}. \)

Now we introduce a useful section \( Z \to g_Z \) for the action of \( G \) on \( D. \) Let \( Z = a(y, Y) \in D. \) Define \( g_z := ((z_0, z), 0, (P, Q)) \in G \) as follows. We set

\[
\begin{align*}
z_0 &= (I_n - YY^*)^{-1}(y + Y\bar{y}), & P = (I_n - YY^*)^{-1/2}, & Q = (I_n - YY^*)^{-1/2}Y.
\end{align*}
\]

Then one has \( g_z \cdot 0 = Z. \)

From the above formula for the action of \( G \) on \( D, \) we can deduce the \( G \)-invariant measure \( \mu \) on \( D. \) Let \( \mu_L \) denote the Lebesgue measure on \( D \simeq \mathbb{C}^n \times \mathcal{B}. \) Thus, we easily obtain that \( d\mu(Z) = \det(1 - YY^*)^{-(n+2)} d\mu_L(y, Y), \) see for instance [5]. This result can be also deduced from the general formula for the invariant measure, see [26], p. 538.

In the rest of the paper, we fix the normalization of the Lebesgue measure as follows. For \( y \in \mathbb{C}^n, \) write \( y = (a_1 + ib_1, a_2 + ib_2, \ldots, a_n + ib_n) \) with \( a_j, b_j \in \mathbb{R} \) for \( j = 1, 2, \ldots, n. \) Then we take the measure Lebesgue on \( \mathbb{C}^n \) to be \( dy := da_1db_1da_2db_2 \cdots da_ndb_n. \) Similarly, writing \( y \in \mathcal{B} \) as \( y = (y_{kl}), \) we denote by \( dY \) the Lebesgue measure on \( \mathcal{B} \) defined by \( dY := \prod_{kl} dy_{kl}. \) Thus we set \( d\mu_L(y, Y) := dy_Y. \)
Now we aim to compute the adjoint and coadjoint actions of $G^c$. First, we compute the adjoint action of $G^c$ as follows. Let $g = (v_0, c_0, h_0) \in G^c$ where $v_0 \in \mathbb{C}^{2n}$, $c_0 \in \mathbb{C}$ and $h_0 \in S^c = Sp(n, \mathbb{C})$ and $X = (w, c, U) \in \mathfrak{g}^c$ where $w \in \mathbb{C}^{2n}$, $c \in \mathbb{C}$ and $U \in \mathfrak{s}^c$. We set $\exp(tX) = (w(t), c(t), \exp(tU))$. Then, since the derivatives of $w(t)$ and $c(t)$ at $t = 0$ are $w$ and $c$, we find that

$$\text{Ad}(g)X = \frac{d}{dt}(g \exp(tX)g^{-1})|_{t=0}$$

$$= (h_0w - (\text{Ad}(h_0)U)v_0, c + \omega(v_0, h_0w) - \frac{1}{2}\omega(v_0, (\text{Ad}(h_0)U)v_0), (\text{Ad}(h_0)U)v_0).$$

On the other hand, let us denote by $\xi = (u, d, \varphi)$, where $u \in \mathbb{C}^{2n}$, $d \in \mathbb{C}$ and $\varphi \in (\mathfrak{g}^c)^*$, the element of $(\mathfrak{g}^c)^*$ defined by

$$\langle \xi, (w, c, U) \rangle = \omega(u, w) + dc + \langle \varphi, U \rangle.$$ 

Moreover, for $u, v \in \mathbb{C}^{2n}$, we denote by $v \times u$ the element of $(\mathfrak{s}^c)^*$ defined by

$$\langle v \times u, U \rangle := \omega(u, Uv)$$

for $U \in \mathfrak{s}^c$.

Let $\xi = (u, d, \varphi) \in (\mathfrak{g}^c)^*$ and $g = (v_0, c_0, h_0) \in G^c$. Then, by using the relation $\langle \text{Ad}^*(g)\xi, X \rangle = \langle \xi, \text{Ad}(g^{-1})X \rangle$ for $X \in \mathfrak{g}^c$, we obtain

$$\text{Ad}^*(g)\xi = (h_0u - dv_0, d, \text{Ad}^*(h_0)\varphi + v_0 \times (h_0u - \frac{d}{2}v_0)).$$

By restriction, we also get the formula for the coadjoint action of $G$. The following lemma will be needed later.

**Lemma 2.1.** [15] The elements $\xi_0$ of $\mathfrak{g}^*$ fixed by $K$ are the elements of the form $(0, d, \varphi_\lambda)$ where $d, \lambda \in \mathbb{R}$ and $\varphi_\lambda \in (\mathfrak{s}^c)^*$ is defined by $\langle \varphi_\lambda, (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \rangle = i\lambda \text{Tr}(A)$.

### 3. Holomorphic representations

The holomorphic representations of the multi-dimensional Jacobi group were studied by many authors, see in particular [26], [9], [10], [4], [5] and [6]. We follow here the general presentation of [26], Chapter XII (see also [14]).

Let $\chi$ be a unitary character of $K$. The extension of $\chi$ to $K^c$ is also denoted by $\chi$. We set $K_\chi(Z, W) := \chi(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$ and $J_\chi(g, Z) := \chi(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. We consider the Hilbert space $\mathcal{H}_\chi$ of all holomorphic functions $f$ on $\mathcal{D}$ such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2 K_\chi(Z, Z)^{-1} c_\chi d\mu(Z) < +\infty$$

where the constant $c_\chi$ is defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} K_\chi(Z, Z)^{-1} d\mu(Z).$$
We shall see that, under some hypothesis on $\chi$, $c\chi$ is well-defined and $\mathcal{H}_\chi \neq (0)$. In that case, $\mathcal{H}_\chi$ contains the polynomials [26], p. 546. Moreover, the formula
\[
\pi_\chi(g)f(Z) = J_\chi(g^{-1}, Z)f(g^{-1} \cdot Z)
\]
defines a unitary representation of $G$ on $\mathcal{H}_\chi$ which is a highest weight representation [26], p. 540.

The space $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, if we set $e_Z(W) := K_\chi(W, Z)$ then we have we have the reproducing property
\[
f(Z) = \langle f, e_Z \rangle_{\mathcal{H}_\chi}
\]
for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [26], p. 540. Here $(\cdot, \cdot)_{\mathcal{H}_\chi}$ denotes the inner product on $\mathcal{H}_\chi$.

Here we fix $\chi$ as follows. Let $\gamma \in \mathbb{R}$ and $m \in \mathbb{Z}$. Then, for each $k = ((0, 0), c, (\begin{smallmatrix} P & 0 \\
0 & P \end{smallmatrix})) \in K$, we set $\chi(k) := e^{i\gamma c}(\det P)^m$.

We need the following lemma.

**Lemma 3.1.** [24] Let $\lambda \in \mathbb{R}$. The integral
\[
J_n(\lambda) := \int_B \det(I_n - Y\bar{Y})^m dY
\]
is convergent if $\lambda > -1$ and in this case we have
\[
J_n(\lambda) = \pi^{n(n+1)/2} \frac{\Gamma(2\lambda + 3)\Gamma(2\lambda + 5) \cdots \Gamma(2\lambda + 2n - 1)}{(\lambda + 1)(\lambda + 2) \cdots (\lambda + n)\Gamma(2\lambda + n + 2)\Gamma(2\lambda + n + 3) \cdots \Gamma(2\lambda + 2n)}.
\]

Then we have the following result.

**Proposition 3.2.**
1. Let $Z = a(Y, Y) \in \mathcal{D}$ and $W = a(v, V) \in \mathcal{D}$. We set
\[
E(y, v, Y, V) := 2y(I_n - \bar{Y}V)^{-1}\bar{v} + y(I_n - \bar{Y}V)^{-1}\bar{v} + \bar{v}Y(I_n - \bar{Y}V)^{-1}\bar{v}.
\]

Then we have
\[
K_\chi(Z, W) = \det(I_n - YV)^m \exp\left(\frac{\gamma}{4} E(y, v, Y, V)\right).
\]

2. We have $\mathcal{H}_\chi \neq (0)$ if and only if $\gamma > 0$ and $m + n + 1/2 < 0$. In this case, we also have $c^{-1} = (2\pi)^n \gamma^{-n} J_n(-m - n - 1/2)$.

3. For each $g = ((z_0, \bar{z}_0), c_0, (\begin{smallmatrix} Q & P \\
0 & P \end{smallmatrix})) \in G$ and each $Z = a(y, Y) \in \mathcal{D}$, we have
\[
J(g, Z) = e^{i\gamma c_0} \det(QY + P)^{-m} \exp\left(\frac{\gamma}{4} \left(\bar{z}_0 z_0 + 2\bar{z}_0 y_0^t P + y_0^t P^t \bar{Q} y - (\bar{z}_0 + \bar{Q} y)^t (PY + Q)(QY + P)^{-1}(z_0 + Q y)\right)\right)
\]
Proof. We can verify (1) and (3) by computations based on the formula for \( \kappa \) given in Section 2. To prove (2), recall that, by [26], Theorem XII.5.6, we have \( \mathcal{H}_\chi \neq \{0\} \) if and only if

\[
I_\chi := \int_{D} K_\chi(Z,Z)^{-1} d\mu(Z) < \infty.
\]

Then we have to study the convergence of \( I_\chi \). By taking into account the expression of \( \mu \) given in Section 2, we get

\[
I_\chi = \int_{\mathcal{C}_n \times B} \operatorname{Det}(I_n - YY^t)^{-m-n-2} d\mu_L(y,Y)
\]

and, by making the change of variables \( y \rightarrow (I_n - YY^t)^{1/2}y \) whose Jacobian is \( \operatorname{Det}(I_n - YY^t) \), we find that

\[
I_\chi = \int_{\mathcal{C}_n \times B} \operatorname{Det}(I_n - YY^t)^{-m-n-3/2} dY
\]

for \( \gamma > 0 \). The result then follows from Lemma 3.1.

Note that we can deduce from (3) of Proposition 3.2 an explicit but rather complicated expression for \( \pi_\chi(g) \). Now we consider the derived representation \( d\pi_\chi \).

Here we use the following notation. If \( L \) is a Lie group and \( X \) is an element of the Lie algebra of \( L \) then we denote by \( X^+ \) the right invariant vector field on \( L \) generated by \( X \), that is, \( X^+(h) = \frac{d}{dt}(\exp(tX))h|_{t=0} \) for \( h \in L \).

By differentiating the multiplication map from \( P^+ \times K^c \times P^- \) onto \( P^+ K^c P^- \), we can easily prove the following result.

Lemma 3.3. Let \( X \in g^c \) and \( g = zky \) where \( z \in P^+, k \in K^c \) and \( y \in P^- \). We have

1. \( d\sigma(X^+(g)) = (\operatorname{Ad}(z)p_{p^+}(\operatorname{Ad}(z^{-1})X))^+(z) \).
2. \( d\sigma_k(X^+(g)) = (p_{p^+}(\operatorname{Ad}(z^{-1})X))^+(k) \).
3. \( d\eta_k(X^+(g)) = (\operatorname{Ad}(k^{-1})p_{p^-}(\operatorname{Ad}(z^{-1})X))^+(y) \).

From this, we easily deduce the following proposition (see also [26], p. 515).

Proposition 3.4. For \( X \in g^c \), \( f \in \mathcal{H}_\chi \) and \( Z \in D \), we have

\[
d\pi_\chi(X)f(Z) = d\chi(p_{p^+}(e^{-\operatorname{ad}Z}X))f(Z) - (df)_Z \left( p_{p^+}(e^{-\operatorname{ad}Z}X) \right)
\]

In particular, we have
(1) If $X \in \mathfrak{p}^+$ then $d\pi_\chi(X)f(Z) = -(df)_Z(X)$.
(2) If $X \in \mathfrak{p}^+$ then $d\pi_\chi(X)f(Z) = d\chi(X)f(Z) + (df)_Z([Z,X])$.
(3) If $X \in \mathfrak{p}^-$ then

\[ d\pi_\chi(X)f(Z) = (d\chi \circ p_\pi) \left( -[Z,X] + \frac{1}{2}[Z,[Z,X]] \right) f(Z) 
- (df_Z \circ p_\pi) \left( -[Z,X] + \frac{1}{2}[Z,[Z,X]] \right). \]

Now we need to introduce some notation. As usual, we write $Z \in \mathcal{D}$ as $Z = a(y,Y)$ where $y = (y_j)_{1 \leq j \leq n} \in \mathbb{C}^n$ and $Y = (y_{kl})_{1 \leq k,l \leq n} \in \mathcal{B}$. Define

\[ \mathcal{I} := \{1,2,\ldots,n\} \cup \{(k,l) : 1 \leq k,l \leq n\} \]

and consider $i \in \mathcal{I}$. Then we define $\partial_i$ as follows. If $i \in \{1,2,\ldots,n\}$ then $\partial_i$ is the partial derivative with respect to $y_i$ and if $i = (k,l)$ then $\partial_i$ is the partial derivative with respect to $y_{kl}$. Moreover, we say that $P(Z)$ is a polynomial of degree $\leq q$ if $P(a(y,Y))$ is a polynomial of degree $\leq q$ in the variables $y_j$ and $y_{kl}$.

From the preceding proposition we deduce the following result.

**Proposition 3.5.** For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$ the operator $d\pi_\chi(X_1X_2\cdots X_q)$ is a sum of terms of the form $P(Z)\partial_{i_1}\partial_{i_2}\cdots\partial_{i_r}$ where $r \leq q$, $i_1, i_2, \ldots, i_r \in \mathcal{I}$ and $P(Z)$ is a polynomial of degree $\leq 2q$.

**Proof.** By Proposition 3.4 we see that, for each $X \in \mathfrak{g}^c$, $d\pi_\chi(X)$ is of the form $P^0(Z) + \sum_i P^i(Z)\partial_i$ where $P^0(Z)$, $P^i(Z)$ are polynomials of degree $\leq 2$. The result then follows by induction on $q$. \qed

4. Generalities on the Stratonovich-Weyl correspondence

In this section, we review some general facts about the Berezin correspondence, the Berezin transform and the Stratonovich-Weyl correspondence.

First at all, recall that the Berezin correspondence on $\mathcal{D}$ is defined as follows. Consider an operator (not necessarily bounded) $A$ on $\mathcal{H}_\chi$ whose domain contains $e_Z$ for each $Z \in \mathcal{D}$. Then the Berezin symbol of $A$ is the function $S_\chi(A)$ defined on $\mathcal{D}$ by

\[ S_\chi(A)(Z) := \langle Ae_Z, e_Z \rangle_\chi. \]

We can verify that each operator is determined by its Berezin symbol and that if an operator $A$ has adjoint $A^*$ then we have $S_\chi(A^*) = S_\chi(A)^*$ [7], [8]. Moreover, for each operator $A$ on $\mathcal{H}_\chi$ whose domain contains the coherent states $e_Z$ for each $Z \in \mathcal{D}$ and each $g \in G$, the domain of $\pi_\chi(g^{-1})A\pi_\chi(g)$ also contains $e_Z$ for each $Z \in \mathcal{D}$ and we have

\[ S_\chi(\pi_\chi(g^{-1})A\pi_\chi(g))(Z) = S_\chi(A)(g \cdot Z), \]
that is, $S_\chi$ is $G$-equivariant, see [14]. We have also the following result.

**Proposition 4.1.** [14]

1. For $g \in G$ and $Z \in \mathcal{D}$, we have
   \[ S_\chi(\pi_\chi(g))(Z) = \chi(\kappa(\exp Z^* g^{-1} \exp Z)^{-1} \kappa(\exp Z^* \exp Z)). \]

2. For $X \in \mathfrak{g}^e$ and $Z \in \mathcal{D}$, we have
   \[ S_\chi(d\pi_\chi(X))(Z) = d\chi(p_{\mathfrak{g}^e}(\operatorname{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*) X)). \]

Let $\xi$ be the linear form on $\mathfrak{g}^e$ defined by $\xi = -idy$ on $\mathfrak{p}^e$ and $\xi = 0$ on $\mathfrak{p}^\pm$. Then we have $\xi(\mathfrak{g}) \subset \mathbb{R}$ and the restriction $\xi_\chi$ of $\xi$ to $\mathfrak{g}$ is an element of $\mathfrak{g}^*$. In the notation of Section 2 we have $\xi_\chi = (0, \gamma, -m\varphi_0)$ where $\varphi_0 \in \mathfrak{s}^*$ is defined by $\langle \varphi_0, ( P Q ) \rangle = i \operatorname{Tr}(P)$.

We denote by $\mathcal{O}(\xi_\chi)$ the orbit of $\xi_\chi$ in $\mathfrak{g}^*$ for the coadjoint action of $G$. This orbit is said to be associated with $\pi_\chi$ by the Kostant-Kirillov method of orbits, see [25], [14]. Moreover, we have the following result.

**Proposition 4.2.** [14]

1. For each $Z \in \mathcal{D}$, let $\Psi_\chi(Z) := \operatorname{Ad}^*(\exp(-Z^*) \zeta(\exp Z^* \exp Z)) \xi_\chi$. Then, for each $X \in \mathfrak{g}^e$ and each $Z \in \mathcal{D}$, we have
   \[ S(d\pi_\chi(X))(Z) = i(\Psi_\chi(Z), X). \]

2. For each $g \in G$ and each $Z \in \mathcal{D}$, we have $\Psi_\chi(g \cdot Z) = \operatorname{Ad}^*(g) \Psi_\chi(Z)$.

3. The map $\Psi_\chi$ is a diffeomorphism from $\mathcal{D}$ onto $\mathcal{O}(\xi_\chi)$.

In order to make the expression of $\Psi_\chi$ more explicit, we introduce the following notation. For $\varphi \in \mathfrak{s}^*$, let $\alpha(\varphi)$ the unique element of $\mathfrak{s}$ such that $\langle \varphi, X \rangle = \operatorname{Tr}(\alpha(\varphi)X)$ for each $X \in \mathfrak{s}$. In particular, one has $\alpha(\varphi_0) = \frac{1}{2}(\begin{smallmatrix} 1 & 0 \\ -i & -1 \end{smallmatrix})$. Moreover, for $u = (x, \hat{x}) \in \mathbb{C}^{2n}$ and $u = (y, \hat{y}) \in \mathbb{C}^{2n}$ we have
\[ \theta(v \times u) = \frac{1}{2} \begin{pmatrix} -iyx^t & iyx^t \\ -iyx^t & iyx^t \end{pmatrix}. \]

Note also that $\theta$ intertwines $\operatorname{Ad}^*$ and $\operatorname{Ad}$. Then we have the following result.

**Proposition 4.3.** [15] The map $\psi_\chi : \mathcal{D} \to \mathcal{O}(\xi_\chi)$ is given by
\[ \psi_\chi(a(y, \hat{y})) = (-d(y_1, \bar{y}_1), \gamma, \varphi(y, \hat{y})) \]
where $y_1 = (I_n - YY)^{-1}(y + Y\hat{y})$ and
\[ \varphi(y, \hat{y}) := -m \operatorname{Ad}^* \begin{pmatrix} (I_n - YY)^{-1/2} & (I_n - YY)^{-1/2}Y \\ (I_n - YY)^{-1/2}Y & (I_n - YY)^{-1/2} \end{pmatrix} \varphi_0 - \frac{\gamma}{2} (y_1, \bar{y}_1) \times (y_1, \bar{y}_1). \]
Moreover, we have
\[
\alpha(\varphi(y, Y)) = -\frac{\gamma}{4} \begin{pmatrix} -iy_{1}y_{2}^{2} & iy_{1}y_{4}^{2} \\ -iy_{1}y_{2}^{2} & iy_{1}y_{4}^{2} \end{pmatrix} \frac{m}{2} \begin{pmatrix} A(Y) & B(Y) \\ -B(Y) & A(Y) \end{pmatrix},
\]
where
\[
A(Y) := (I + Y\bar{Y})(I - Y\bar{Y})^{-1/2}(I - \bar{Y}Y)^{-1/2};
\]
\[
B(Y) := -2Y(I - \bar{Y}Y)^{-1/2}(I - YY)^{-1/2}.
\]

Now we recall briefly the construction of the Stratonovich-Weyl correspondence [20], [13], [14]. Denote by \( L_{2}(\mathcal{H}_{\chi}) \) the space of all Hilbert-Schmidt operators on \( \mathcal{H}_{\chi} \) and by \( \mu_{\chi} \) the \( G \)-invariant measure on \( \mathcal{D} \) defined by \( d\mu_{\chi}(Z) = c_{\chi}d\mu(Z) \). Then the map \( S_{\chi} \) is a bounded operator from \( L_{2}(\mathcal{H}_{\chi}) \) into \( L^{2}(\mathcal{D}, \mu_{\chi}) \) which is one-to-one and has dense range [29], [32]. Moreover, the Berezin transform is the operator on \( L^{2}(\mathcal{D}, \mu_{\chi}) \) defined by \( B_{\chi} := S_{\chi}\hat{S}_{\chi}^{*} \). We can easily verify that we have the following integral formula for \( B_{\chi} \)
\begin{equation}
B_{\chi}F(Z) = \int_{\mathcal{D}} F(W) \frac{|\langle e_{z}, e_{w} \rangle|_{\chi}^{2}}{\langle e_{z}, e_{z} \rangle_{\chi}\langle e_{w}, e_{w} \rangle_{\chi}} d\mu_{\chi}(W)
\end{equation}
(see [7], [32], [33] for instance).

Let \( \rho \) be the left-regular representation of \( G \) on \( L^{2}(\mathcal{D}, \mu_{\chi}) \). As a consequence of the equivariance property for \( S_{\chi} \), we see that \( B_{\chi} \) commute with \( \rho(g) \) for each \( g \in G \).

Consider the polar decomposition of \( S_{\chi} : S_{\chi} = (S_{\chi}\hat{S}_{\chi}^{*})^{1/2}W_{\chi} = B_{\chi}^{1/2}W_{\chi} \) where \( W_{\chi} := B_{\chi}^{-1/2}S_{\chi} \) is a unitary operator from \( L_{2}(\mathcal{H}_{\chi}) \) onto \( L^{2}(\mathcal{D}, \mu_{\chi}) \). Note that, by (2) of Proposition 4.2, the measure \( \mu_{0} := (\Psi_{\chi}^{-1})^{*}(\mu_{\chi}) \) is a \( G \)-invariant measure on \( \mathcal{O}(\xi) \). The following proposition is then immediate.

**Proposition 4.4.** 1) The map \( W_{\chi} : L_{2}(\mathcal{H}_{\chi}) \rightarrow L^{2}(\mathcal{D}, \mu_{\chi}) \) is a Stratonovich-Weyl correspondence for the triple \((G, \pi_{\chi}, \mathcal{D})\), that is, we have
\begin{enumerate}
\item \( W_{\chi}(A^{*}) = W_{\chi}(A) \);
\item \( W_{\chi}(\pi_{\chi}(g)A\pi_{\chi}(g)^{-1})(Z) = W_{\chi}(A)(g^{-1}: Z) \);
\item \( W_{\chi} \) is unitary.
\end{enumerate}
2) Similarly, the map \( W_{\chi} : L_{2}(\mathcal{H}_{\chi}) \rightarrow L^{2}(\mathcal{O}(\xi), \mu_{0}) \) defined by \( W_{\chi}(A) = W_{\chi}(A)\circ\Psi_{\chi}^{-1} \) is a Stratonovich-Weyl correspondence for the triple \((G, \pi_{\chi}, \mathcal{O}(\xi))\).

Note that we have relaxed here (1) of Definition 1.1 which is not adapted to the present setting since \( I \) is not Hilbert-Schmidt. However, this requirement should be hold in some generalizable sense, see for instance [22].
5. Extension of the Berezin transform

The aim of this section is to extend the Berezin transform to a class of functions which contains $S_\chi(d\pi_\chi(X))$ for each $X \in \mathfrak{g}^+$, in order to define and study $W_\chi(d\pi_\chi(X))$. This question was already investigated in [14] in the case of a reductive Lie group and in [17] in the case of the Jacobi group.

For $Z, W \in \mathcal{D}$, we set $l_Z(W) := \log \exp Z^* \exp W \in \mathfrak{p}^-$. 

**Lemma 5.1.**  
(1) For each $Z, W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have
\[
\frac{d}{dt} e_Z(W + tV)|_{t=0} = -e_Z(W)(d\chi \circ p_T) \left( [l_Z(W), V] + \frac{1}{2} [l_Z(W), [l_Z(W), V]] \right).
\]

(2) For each $Z, W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have
\[
\frac{d}{dt} l_Z(W + tV)|_{t=0} = p_T \left( [l_Z(W), V] + \frac{1}{2} [l_Z(W), [l_Z(W), V]] \right).
\]

(3) For each $i_1, i_2, \ldots, i_q \in I$ and $Z \in \mathcal{D}$, the function $(\partial_{i_1} \partial_{i_2} \cdots \partial_{i_q} e_Z)(W)$ is of the form $e_Z(W)Q(l_Z(W))$ where $Q$ is a polynomial on $\mathfrak{p}^-$ of degree $\leq 2q$.

(4) For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^+$, the function $S_\chi(d\pi_\chi(X_1 X_2 \cdots X_q))(Z)$ is a sum of terms of the form $P(Z)Q(l_Z(Z))$ where $P$ and $Q$ are polynomials of degree $\leq 2q$.

**Proof.** The proof of this lemma is similar to those of Lemma 6.2 of [14] and Lemma 5.2 of [17]. Note that the proof of (1) is essentially based on Lemma 3.3, that (3) is a consequence of (1) and (2) and, finally, that (4) follows from (3) and Proposition 3.5.

We can then establish the main result of this section.

**Proposition 5.2.** If $q < \frac{1}{2}(-m - 2n)$ then for each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^+$, the Berezin transform of $S_\chi(d\pi_\chi(X_1 X_2 \cdots X_q))$ is well-defined.

**Proof.** First, we fix $Z \in \mathcal{D}$ and we make the change of variables $W \rightarrow g_Z \cdot W$ in Equation 4.1. Then we obtain
\[
(B_\chi F)(Z) = \int_{\mathcal{D}} F(g_Z \cdot W)(e_W, e_W)^{-1} d\mu_\chi(W).
\]

We take $F = S_\chi(d\pi_\chi(X_1 X_2 \cdots X_q))$ and we set $Y_k := \text{Ad}(g_Z^{-1})X_k$ for $k = 1, 2, \ldots, q$. Then, by $G$-invariance of $S_\chi$, we have
\[
F(g_Z \cdot W) = S_\chi(d\pi_\chi(Y_1 Y_2 \cdots Y_q))(W)
\]
for each $W \in \mathcal{D}$. Recall that, by the preceding lemma, the function
\[
S_\chi(d\pi_\chi(Y_1 Y_2 \cdots Y_q))(W)
\]
is a sum of terms of the form $P(W)Q(l_W(W))$ where $P$ and $Q$ are polynomials of degree $\leq 2q$. Then we have to prove that, for each $q < \frac{1}{4}(-m - 2n)$ and each polynomials $P$ and $Q$ of degree $\leq 2q$, the integral

$$I := \int_D P(W)Q(l_W(W))\,d\mu(W)$$

is convergent.

First, we note that if $W = a(y, Y)$ then

$$l_W(W) = \begin{pmatrix} 0, -(I_n - YY)^{-1}(\bar{y} + \bar{Y}y), 0, -(I_n - YY)^{-1}\bar{Y} \end{pmatrix}.$$  

Thus we have

$$I = c_x \int_D P(y, Y)Q(-(I_n - YY)^{-1}(\bar{y} + \bar{Y}y), -(I_n - YY)^{-1}\bar{Y})$$

$$\times \exp \left( -\frac{\gamma}{4}(2y^t(I_n - YY)^{-1}\bar{y} + y^t(I_n - YY)^{-1}\bar{Y}y + \bar{y}^t(I_n - YY)^{-1}\bar{Y} + \bar{y}^t(I_n - YY)^{-1}\bar{Y}) \right)$$

$$\times \det(I_n - YY)^{-m-n-2}d\mu_L(y, Y).$$

As in the proof of Proposition 3.2, we make the change of variables $y \to (I_n - YY)^{1/2}y$ and we find that

$$I = c_x \int_D P((I_n - YY)^{1/2}y, Y)Q(-(I_n - YY)^{-1/2}(\bar{y} + \bar{Y}y), -(I_n - YY)^{-1}\bar{Y})$$

$$\times \exp \left( -\frac{\gamma}{4}(2y^t\bar{y} + y^t\bar{Y}y + \bar{y}^t\bar{Y}\bar{y}) \right) \det(I_n - YY)^{-m-n-1}d\mu_L(y, Y).$$

Now we make the following remarks.

1. Since $P$ is a polynomial of degree $\leq 2q$ and $B$ is bounded, there exists a constant $C_0 > 0$ such that

$$|P((I_n - YY)^{1/2}y, Y)| \leq C_0 \sum_{r \leq 2q} |y|^r$$

for each $(y, Y) \in \mathbb{C}^n \times B$.

2. By using the classical formula for the inverse of a matrix, for each $Y \in B$ we have

$$(I_n - YY)^{-1} = \det(I_n - YY)^{-1}C(I_n - YY)^t$$

where $C(A)$ denotes the cofactor matrix of a matrix $A$. From this we deduce that there exists a constant $C_0' > 0$ such that

$$|Q(-(I_n - YY)^{-1}(I_n - YY)^{1/2}(\bar{y} + \bar{Y}y), -(I_n - YY)^{-1}\bar{Y})|$$

$$\leq C_0' \det(I_n - YY)^{-2q} \sum_{r \leq 2q} |y|^r$$
for each \((y, Y) \in \mathbb{C}^n \times \mathcal{B}\).

(3) For each \((y, Y) \in \mathbb{C}^n \times \mathcal{B}\), we have
\[2y^t \bar{y} + y^t \bar{Y} y + \bar{y}^t Y \bar{y} = 2(y^t y + \text{Re}(y^t \bar{Y} y)) \geq 2(1 - \|Y\|)\|y\|^2.\]

Here \(\|\cdot\|\) denotes the operator norm corresponding to the Hermitian norm on \(\mathbb{C}^n\).

By using these remarks, we can reduce the study of the convergence of \(I\) to that of the integral
\[I' := \int_{\mathcal{D}} \text{Det}(I_n - YY^*)^{-2q-m-n-1}|y|^{4q}e^{-\frac{\gamma}{2} |y|^2} (1 - \|Y\|) d\mu_L(y, Y).\]

We set
\[I(Y) := \int_{\mathbb{C}^n} |y|^{4q}e^{-\frac{\gamma}{2} |y|^2} (1 - \|Y\|) dy\]
and, passing to spherical coordinates, we see that there exists some constants \(C, C' > 0\) such that, for each \(Y \in \mathcal{B}\), we have
\[I(Y) = C \int_0^{+\infty} x^{4q+2m-2n-1}e^{-\frac{\gamma}{2} (1 - \|Y\|)x^2} dx = C' (1 - \|Y\|)^{-2q-n}.\]

Then we have to study the integral
\[I'' := \int_{\mathcal{B}} \text{Det}(I_n - Y Y^*)^{-2q-m-n-1}(1 - \|Y\|)^{-2q-n} dY.\]

Now denote by \(\lambda_s(Y Y^*)\) the maximum of the eigenvalues of \(Y Y^*\) and recall that \(\|Y\|^2 = \lambda_s(Y Y^*)\). Then we have
\[\text{Det}(I_n - Y Y^*) \leq 1 - \lambda_s(Y Y^*) = 1 - \|Y\|^2 \leq 2(1 - \|Y\|)\]
for each \(Y \in \mathcal{B}\). Thus we obtain
\[\text{Det}(I_n - Y Y^*)^{-2q-m-n-1}(1 - \|Y\|)^{-2q-n} \leq 2^{2q+n} \text{Det}(I_n - Y Y^*)^{-4q-m-2n-1}\]
for each \(Y \in \mathcal{B}\). But by Lemma 3.1, we see that \(J_n(-4q - m - 2n - 1)\) hence \(I''\) converges if \(q < \frac{1}{2}(-m - 2n)\). This ends the proof.

\[\Box\]

6. Stratonovich-Weyl symbols of derived representation operators

Here we assume that \(-m > 2n+4\). Then, by Proposition 5.2, \(B_\chi(S_\chi(d\pi_\chi(X)))\) is well-defined for each \(X \in \mathfrak{g}^c\). We aim to define also \(W_\chi(d\pi_\chi(X))\) for \(X \in \mathfrak{g}^c\). To this goal, we first introduce a space of functions on \(\mathcal{D}\) which is stable under \(B_\chi\) and contains \(S_\chi(d\pi_\chi(X))\) for each \(X \in \mathfrak{g}^c\).
Recall that, by Proposition 4.2 we have $S(\pi(X))(Z) = i\xi(\text{Ad}(gZ^{-1})X)$ for each $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$. This leads us to introduce the vector space $S$ generated by the functions $Z \to \phi_0(\text{Ad}(gZ^{-1})X)$ where $X \in \mathfrak{g}^c$ and $\phi_0$ is an element of $(\mathfrak{g}^c)^*$ which is $\text{Ad}^*(K)$-invariant. Such elements $\phi_0$ were determined in [15], see Lemma 2.1 above. The following proposition is analogous to Proposition 6.2 of [17].

**Proposition 6.1.** Let $\phi : \mathcal{D} \times \mathfrak{g}^c \to \mathbb{C}$ be a function such that

(i) For each $Z \in \mathcal{D}$, the map $X \to \phi(Z, X)$ is linear;

(ii) For each $X \in \mathfrak{g}^c$, $g \in G$ and $Z \in \mathcal{D}$, we have $\phi(gZ, X) = \phi(Z, \text{Ad}(g^{-1})X)$.

Then

(1) The element $\phi_0$ of $(\mathfrak{g}^c)^*$ defined by $\phi_0(X) := \phi(0, X)$ is fixed by $K$;

(2) For each $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$, we have

$$\phi(Z, X) = \phi_0(\text{Ad}(gZ^{-1})X).$$

We also have

$$\phi(Z, X) = \phi_0 \left( \text{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*) \right) X
= \left( \phi_0 \circ p_{\mathbb{R}} \right) \left( \text{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X \right).$$

(3) For each $X \in \mathfrak{g}^c$, the function $\psi : \mathcal{D} \times \mathfrak{g}^c \to \mathbb{C}$ given by $\psi(\cdot, X) = B_\chi(\phi(\cdot, X))$ is well-defined and satisfies (i) and (ii).

(4) The vector space $S$ is generated by all the functions $Z \to \phi(Z, X)$ for $\phi$ as above and $X \in \mathfrak{g}^c$. Moreover, $S$ is stable under $B_\chi$.

**Proof.** (1) By (ii), for each $k \in K$ and $X \in \mathfrak{g}^c$, we have

$$(\text{Ad}^*(k)\phi_0)(X) = \phi_0(\text{Ad}(k^{-1})X) = \phi(0, \text{Ad}(k^{-1})X) = \phi(k0, X) = \phi(0, X) = \phi_0(X).$$

Then $\phi_0$ is fixed by $K$.

(2) The first assertion follows from (ii). To prove the second assertion, recall that by [15], there exists $k_Z \in K$ such that $g_{kZ} = \exp(-Z^*\zeta(\exp Z^* \exp Z)kZ^{-1}$. Then we have

$$\phi(Z, X) = \phi_0 \left( \text{Ad}(kZ\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X \right)
= \phi_0 \left( \text{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X \right)$$

and, noting that $\phi_0|_{\mathbb{R}} = 0$ by Lemma 2.1, we can conclude that

$$\phi(Z, X) = \left( \phi_0 \circ p_{\mathbb{R}} \right) \left( \text{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X \right).$$

(3) By using the same arguments as in the proof of Proposition 5.2, we can verify that, for each $X \in \mathfrak{g}^c$, the Berezin transform of $\phi(\cdot, X)$ is well-defined. The second assertion follows from the fact that $B_\chi$ commutes to the $\rho(g), g \in G$.

(4) This follows from the preceding statements. □
Now we need the following lemmas.

**Lemma 6.2.** For each \( Y \in \mathcal{B} \), we have

\[
I_1(Y) := \int_{\mathbb{C}^n} y^i \bar{y} \exp \left( -\frac{\gamma}{4} (2y^i \bar{y} + y^i \bar{Y} y + \bar{y}^i Y \bar{y}) \right) \, dy
\]

\[
= \frac{2}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \det(I_n - Y \bar{Y})^{-1/2} \text{Tr} \left( (I_n - Y \bar{Y})^{-1} \right)
\]

\[
I_2(Y) := \int_{\mathbb{C}^n} y^i \bar{Y} y \exp \left( -\frac{\gamma}{4} (2y^i \bar{y} + y^i \bar{Y} y + \bar{y}^i Y \bar{y}) \right) \, dy
\]

\[
= \frac{2}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \det(I_n - Y \bar{Y})^{-1/2} \left( n - \text{Tr} \left( (I_n - Y \bar{Y})^{-1} \right) \right)
\]

**Proof.** For \( s \in [0, 1] \) and \( Y \in \mathcal{B} \), let us introduce

\[
J_s(Y) = \int_{\mathbb{C}^n} \exp \left( -\frac{\gamma}{4} (2y^i \bar{y} + sy^i \bar{Y} y + s\bar{y}^i Y \bar{y}) \right) \, dy.
\]

By [21], p. 258, we have

\[
J_s(Y) = \left( \frac{2\pi}{\gamma} \right)^n \det(I_n - s^2 Y \bar{Y})^{-1/2}
\]

Then, by computing the derivative of \( J_s(Y) \) at \( s = 1 \), we get

\[
\int_{\mathbb{C}^n} (y^i \bar{Y} y + \bar{y}^i Y \bar{y}) \exp \left( -\frac{\gamma}{4} (2y^i \bar{y} + y^i \bar{Y} y + \bar{y}^i Y \bar{y}) \right) \, dy
\]

\[
= -\frac{4}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \det(I_n - Y \bar{Y})^{-1/2} \text{Tr} \left( (I_n - Y \bar{Y})^{-1} Y \bar{Y} \right).
\]

Thus we have

\[
(6.1) \quad I_2(Y) + \overline{I_2(Y)} = -\frac{4}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \det(I_n - Y \bar{Y})^{-1/2} \text{Tr} \left( (I_n - Y \bar{Y})^{-1} \right).
\]

On the other hand, by integrating by parts, we get

\[
J_1(Y) = -\int_{\mathbb{C}^n} y_k \frac{\partial}{\partial k} \left( \exp \left( -\frac{\gamma}{4} (2y^i \bar{y} + y^i \bar{Y} y + \bar{y}^i Y \bar{y}) \right) \right) \, dy
\]

\[
= \frac{\gamma}{4} \int_{\mathbb{C}^n} y_k (2\bar{y} + 2e_k Y \bar{Y} y) \exp \left( -\frac{\gamma}{4} (2y^i \bar{y} + y^i \bar{Y} y + \bar{y}^i Y \bar{y}) \right) \, dy
\]

for each \( k = 1, 2, \ldots, k \). By summing up over \( k \), we obtain

\[
(6.2) \quad nJ_1(Y) = \frac{\gamma}{2} (I_1(Y) + I_2(Y)).
\]
This last equation implies that $I_2(Y)$ is real since $J_1(Y)$ and $I_1(Y)$ are real. Consequently, Equation 6.1 gives the desired value for $I_2(Y)$ hence Equation 6.2 provides the desired value for $I_1(Y)$.

The following lemma gives a useful expression for $K_\chi(Z, Z)$ which will be used in the proof of Proposition 6.4.

**Lemma 6.3.** For each $Z = a(y, Y)$, let $z_0 := (I_n - Y\bar{Y})^{-1}(y + Y\bar{y})$. Then we have
\[ K_\chi(Z, Z) = \exp \left( \frac{7}{4} (2z_0^T\bar{z}_0 - z_0^T\bar{z}_0 - \bar{z}_0^T Yz_0) \right) \text{Det}(I_n - Y\bar{Y})^m. \]

**Proof.** The result follows from Proposition 3.2 by a routine computation. Alternatively, by [14], Lemma 4.1, we have
\[ \left\langle e_Z, e_Z \right\rangle_\chi = \left\langle e_{gz_0}, e_{gz_0} \right\rangle_\chi = \left( J(gz, 0) \pi(gz) e_0, J(gz, 0) \pi(gz) e_0 \right)_\chi = |J(gz, 0)|^2 = |\chi(s(gz))|^2 \]
and, by taking into account the expressions of $\chi$ and $g_Z$, we then recover the desired formula for $K_\chi(Z, Z)$.

Let us introduce the following integral of Hua’s type:
\[ K_n(\lambda) := \int_{\mathcal{B}} \text{Tr}((I_n - Y\bar{Y})^{-1}) \text{Det}(I_n - Y\bar{Y})^\lambda dY. \]

Since the maximum of the eigenvalues of $(I_n - Y\bar{Y})^{-1}$ is $(1 - \lambda_s(Y\bar{Y}))^{-1}$, we have
\[ \text{Tr}((I_n - Y\bar{Y})^{-1}) \leq n(1 - \lambda_s(Y\bar{Y}))^{-1} \leq n \text{Det}(I_n - Y\bar{Y})^{-1} \]
and then we see that $K_n(\lambda)$ converges for $\lambda > 2$ since $J_n(\lambda)$ converges for $\lambda > 1$, see Lemma 3.1.

Also, we denote by $\phi^1$ and $\phi^2$ the elements of $\mathcal{S}$ defined by $\phi^0_0 = (0, 1, 0)$ and $\phi^2_0 = (0, 0, \varphi_0)$. We are now in position to establish the following proposition.

**Proposition 6.4.** Let
\[ \mu_n := \frac{c_\chi}{n\gamma} \left( \frac{2\pi}{\gamma} \right)^n K_n(-m - n - 3/2); \quad \nu_n := -1 + \frac{2c_\chi}{n} \left( \frac{2\pi}{\gamma} \right)^n K_n(-m - n - 3/2). \]

Let $\phi \in \mathcal{S}$ defined by $\phi_0 = (0, d, \lambda \varphi_0)$ with $d, \lambda \in \mathbb{C}$. Let $\psi \in \mathcal{S}$ such that $\psi(\cdot, X) = B_\chi(\phi(\cdot, X))$ for each $X \in \mathfrak{g}^c$. Then we have $\psi_0 = (0, d, d\mu + \lambda\nu_n)$.

**Proof.** We will use the formula
\[ \psi_0(X) = \int_{\mathcal{D}} \phi_0(\text{Ad}(g^{-1}_Z X)) K_\chi(Z, Z)^{-1} c_\chi d\mu(Z) \]
in order to compute the Berezin transforms $\psi^1(\cdot, X)$ and $\psi^2(\cdot, X)$ of $\phi^1(\cdot, X)$ and $\phi^2(\cdot, X)$.

We write $\psi^1_0 = (0, d_1, \lambda_1 \varphi_0)$ with $d_1, \lambda_1 \in \mathbb{C}$. Let $H_1 := ((0, 0), 1)$. Then we have $\text{Ad}(g^{-1}_Z)H_1 = H_1$ hence $\phi^1_0(\text{Ad}(g^{-1}_Z)H_1) = 1$ for each $Z \in \mathcal{D}$. This gives

$$\psi^1_0(H_1) = \int_{\mathcal{D}} K_\chi(Z, Z)^{-1} c_\chi d\mu(Z) = 1.$$  

On the other hand, we also have $\psi^1_0(H_1) = d_1$. Then we find $d_1 = 1$.

Now, let $H_2 := ((0, 0), 0, (\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}))$. Then, for each $Z \in \mathcal{D}$, we have

$$\text{Ad}(g^{-1}_Z)H_2 = \left( \begin{array}{c} 0 \\ \frac{i}{2} z_0^* \end{array} \right) \left( \begin{array}{c} 1 \\ -(I_n - YY)^{-1} (I_n + YY) - \frac{i}{2} \end{array} \right)$$

where, as usual, $z_0 = (I_n - YY)^{-1}(y + Y\tilde{y})$. Consequently, we have

$$\phi^1_0(\text{Ad}(g^{-1}_Z)H_2) = \frac{i}{2} z_0^* z_0.$$  

Thus, by Lemma 6.3, we get

$$\psi^1_0(H_2) = \frac{ie_\chi}{2} \int_{\mathcal{D}} z_0^* z_0 \exp \left( -\frac{\gamma}{4} (2z_0^* z_0 - z_0^* Y z_0 - \frac{1}{2} z_0^* Y z_0) \right)$$

and we make the change of variables $y = z_0 - Y\tilde{z}_0$ whose Jacobian is $\det(I_n - YY)$. Hence, by using Lemma 6.2, we obtain

$$\psi^1_0(H_2) = \frac{ie_\chi}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n K_n \left( -m - n - \frac{3}{2} \right).$$

On the other hand, it is clear that $\psi^1_0(H_2) = i\lambda_1 n$. Finally, we find that

$$\lambda_1 = \frac{c_\chi}{n^2} \left( \frac{2\pi}{\gamma} \right)^n K_n \left( -m - n - \frac{3}{2} \right) = \mu_n.$$  

Similarly, we write $\psi^2_0 = (0, d_2, \lambda_2 \varphi_0)$. Since we have $\phi^2_0(\text{Ad}(g^{-1}_Z)H_1) = 0$ for each $Z \in \mathcal{D}$, we first obtain $d_2 = \psi^2_0(H_1) = 0$. Moreover, for each $Z = a(y, Y) \in \mathcal{D}$, we also have

$$\phi^2_0(\text{Ad}(g^{-1}_Z)H_2) = i \text{Tr}(I_n - YY)^{-1}(I_n + YY)$$

$$= i \left( -n + 2 \text{Tr} \left( (I_n - YY)^{-1} \right) \right).$$

Then, changing variables $y \rightarrow (I_n - YY)^{1/2} y$, we get

$$\psi^2_0(\text{Ad}(g^{-1}_Z)H_2) = -in + 2ic_\chi \int_{\mathcal{B} \times \mathbb{C}^n} \exp \left( -\frac{\gamma}{4} (2y^* \tilde{y} + y^* \tilde{y} y + \tilde{y}^* Y \tilde{y}) \right)$$

$$\times \text{Tr} \left( (I_n - YY)^{-1} \right) \det(I_n - YY)^{-m-n-1} dY dY.$$
Thus, by using [21], p. 248, we obtain
\[
\psi_0^2(\text{Ad}(g_Z^{-1})H_2) = -in + 2ic_X \left( \frac{2\pi}{\gamma} \right)^n K_n \left( -m - n - \frac{3}{2} \right).
\]

Also, we have \( \psi_0^2(\text{Ad}(g_Z^{-1})H_2) = i\lambda_2 n \). This gives
\[
\lambda_2 = -1 + 2c_X \left( \frac{2\pi}{n} \right)^n K_n \left( -m - n - \frac{3}{2} \right) = \nu_n.
\]
This finishes the proof.

Recall that \( c_x \) can be expressed in terms of the Hua’s integral \( J_n(-m-n-\frac{3}{2}) \) which can be explicitly computed, see Proposition 3.2 and Lemma 3.1. However, it seems difficult to compute \( K_n(-m-n-\frac{3}{2}) \) similarly.

Now we give the matrix of \( B_\chi \) in a suitable basis of \( S \). First, we consider the basis of \( \g^c \) consisting of the elements:

\[
X_i = ((e_i, 0), 0, 0); \quad Y_j = ((0, e_j), 0, 0); \quad F_{ij} = ((0, 0, 0, (0_{E_{ij}})));
\]
\[
G_{ij} = ((0, 0, 0, (0_{E_{ij}})); \quad H_1 = ((0, 0), 1, 0); \quad A_{ij} = ((0, 0, 0, (E_{ij} 0 0 -E_{ij})).
\]

for \( i,j = 1,2,\ldots,n \), \( E_{ij} \) denoting the \( n \times n \) complex matrix whose \( ij \)-th entry is 1 and all of whose other entries are 0.

Note that \( \phi^2(\cdot, X_i) = \phi^2(\cdot, Y_j) = \phi^2(\cdot, H_1) = 0 \). Then, from the preceding proposition, we easily deduce the following result.

**Corollary 6.5.** The functions \( \phi^1(\cdot, X_i), \phi^1(\cdot, Y_j), \phi^1(\cdot, H_1), \phi^1(\cdot, F_{ij}), \phi^1(\cdot, G_{ij}), \phi^1(\cdot, A_{ij}), \phi^2(\cdot, F_{ij}), \phi^2(\cdot, G_{ij}), \phi^2(\cdot, A_{ij}) \) form a basis for \( S \) in which \( B_\chi \) has matrix

\[
\begin{pmatrix}
I_{2n+1} & O & O \\
O & I_{3n^2} & O \\
O & \mu_n I_{3n^2} & \nu_n I_{3n^2}
\end{pmatrix}.
\]

Recall that for each \( X \in \g^c \), we have \( S_\chi(d \pi_\chi(X)) \in S \). Consequently, we see that \( W_\chi(d \pi_\chi(X)) = B_\chi^{-1/2}(S_\chi(d \pi_\chi(X))) \) is well-defined. Moreover, we have the following proposition.

**Proposition 6.6.** For each \( X \in \text{Span}_C\{H_1, X_i, Y_j, 1 \leq i,j \leq n\} \), we have \( W_\chi(d \pi_\chi(X)) = S_\chi(d \pi_\chi(X)) \). For each \( X \in \text{Span}_C\{F_{ij}, G_{ij}, A_{ij}; 1 \leq i,j \leq n\} \), we have

\[
W_\chi(d \pi_\chi(X)) = S_\chi(d \pi_\chi(X)) + i(1 - \nu_n^{-1/2}) \left( \frac{\gamma \mu_n}{1 - \nu_n} + m \right) \phi^2(\cdot, X).
\]
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Proof. For each $X \in \mathfrak{g}^c$ we have

$$S_\chi(d\pi_\chi(X)) = d\chi(\text{Ad}(g_Z^{-1})X) = i\gamma \phi^1(\cdot, X) - im\phi^2(\cdot, X).$$

Now, by using the preceding corollary, we see that the matrix of $B^{-1/2}_\chi$ with respect to the above basis of $S$ is

$$\begin{pmatrix}
I_{2n+1} & O & O \\
O & I_{3n^2} & O \\
O & -\mu_{1+n/2}^{-1/2}I_3 & \nu_n^{-1/2}I_{3n^2}
\end{pmatrix}.$$

This implies that for $X \in \{H_1, X_i, Y_j, 1 \leq i, j \leq n\}$, we have $W_\chi(d\pi_\chi(X)) = S_\chi(d\pi_\chi(X))$ and, for $X \in \{F_{ij}, G_{ij}, A_{ij}, 1 \leq i, j \leq n\}$, we have

$$W_\chi(d\pi_\chi(X)) = i\gamma \left( \phi^1(\cdot, X) - \frac{\mu_n\nu_n^{-1/2}}{1 + \nu_n^{1/2}} \phi^2(\cdot, X) \right) - im\nu_n^{-1/2}\phi^2(\cdot, X).$$

Hence the result follows. \hfill \qed

References


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