Primary group rings

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Abstract – Let $R$ be an associative ring with identity and let $J(R)$ denote the Jacobson radical of $R$. We say that $R$ is primary if $R/J(R)$ is simple Artinian and $J(R)$ is nilpotent. In this paper we obtain necessary and sufficient conditions for the group ring $RG$, where $G$ is a nontrivial abelian group, to be primary.

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1. Introduction

Throughout this paper all rings are associative with identity. For a ring $R$, let $J(R)$ denote its Jacobson radical. We say that $R$ is primary if $R/J(R)$ is simple Artinian and $J(R)$ is nilpotent. The ring $R$ is said to be semiprimary if $R/J(R)$ is Artinian and $J(R)$ is nilpotent. A primary ring is clearly semiprimary. The aim of this paper is to obtain necessary and sufficient conditions for the group ring $RG$, where $G$ is a nontrivial abelian group, to be primary. Our main result is the following:

Theorem 1.1. Let $R$ be a ring and let $G \neq \{1\}$ be an abelian group. Then $RG$ is primary if and only if $R$ is primary with char $R/J(R) = p$ for some prime $p$ and $G$ is a finite $p$-group.

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We shall prove Theorem 1.1 in Section 2. In Section 3 we consider conditions for a group algebra to be primary. As a consequence, we obtain an example of a clean ring which is not primary.

2. Proof of Theorem 1.1

We first obtain some sufficient conditions for a group ring to be primary.

**Proposition 2.1.** Let $R$ be a ring and let $G \neq \{1\}$ be a group. If $R$ is primary with $\text{char } R/J(R) = p$ for some prime $p$ and $G$ is a finite $p$-group, then $RG$ is primary.

In order to prove Proposition 2.1, we shall need the aid of the following results:

**Theorem 2.2.** (Tan [4, Theorem, p. 261]) Let $R$ be a ring and let $G$ be a group. Then $RG$ is semiprimary if and only if $R$ is semiprimary and $G$ is finite.

**Proposition 2.3.** Let $R$ be a ring and let $G \neq \{1\}$ be a group. If $G$ is a locally finite $p$-group for some prime $p$, $J(R) = \{0\}$ and $p = 0$ in $R$, then $J(RG) = \Delta$, the augmentation ideal of $RG$.

**Proof.** See [2, Proposition 16(iv), p. 683].

**Proposition 2.4.** Let $R$ be a ring and let $G$ be a group. If $R$ is Artinian or $G$ is locally finite, then $J(RG) \subseteq J(R)$.

**Proof.** See [2, Proposition 9, p. 665].

We are now ready for the proof of Proposition 2.1.

**Proof.** Since $R$ is primary (hence, semiprimary) and $G$ is finite, it follows by Theorem 2.2 that $RG$ is semiprimary. Thus, we only need to show that $RG/J(RG)$ is simple.

Let $\bar{R} = R/J(R)$. Then $J(\bar{R}) = \{0\}$ and $p = 0$ in $\bar{R}$. It follows by Proposition 2.3 that $J(\bar{RG}) = \bar{\Delta}$, the augmentation ideal of $\bar{RG}$. Since $G$ is locally finite, we have by Proposition 2.4 that $J(R)G \subseteq J(RG)$. Then

$$RG/J(RG) \cong (RG/J(R)G) / (J(RG)/J(R)G)$$
$$\cong (RG/J(R)G) / J(RG/J(R)G)$$
$$\cong \bar{RG}/\bar{J}(\bar{RG}) \cong \bar{R}.$$

Since $\bar{R}$ is simple, so is $RG/J(RG)$. This completes the proof of Proposition 2.1.
In what follows we show that the converse of Proposition 2.1 is true when $G$ is abelian. We first prove the following:

**Proposition 2.5.** Let $R$ be a ring and let $G$ be a group. If $RG$ is primary, then $R$ is primary and $G$ is finite.

We will make use of the following lemma to prove Proposition 2.5.

**Lemma 2.6.** Let $R$ be a ring such that $R/J(R)$ is simple. If $S$ is a homomorphic image of $R$, then $S/J(S)$ is also simple.

**Proof.** Let $f : R \to S$ be a ring epimorphism and let $\pi : R/J(R) \to S/J(S)$ be the mapping induced by $f$. That is,

$$\pi(r + J(R)) = f(r) + J(S), \quad r \in R.$$  

It is straightforward to verify that $\pi$ is a well-defined ring epimorphism. Then since $R/J(R)$ is simple, so is $S/J(S)$. \qed

We now prove Proposition 2.5.

**Proof.** Since $RG$ is primary (hence, semiprimary), it follows readily by Theorem 2.2 that $R$ is semiprimary and $G$ is finite. It remains to show that $R/J(R)$ is simple. But this follows readily by Lemma 2.6 since $R$ is a homomorphic image of $RG$ and $RG/J(RG)$ is simple. We thus have that $R$ is primary, as required. \qed

If $G \neq \{1\}$ is an abelian group, Proposition 2.5 can be made more precise as follows:

**Proposition 2.7.** Let $R$ be a ring and let $G \neq \{1\}$ be an abelian group. If $RG$ is primary, then $R$ is primary with $\text{char} R/J(R) = p$ for some prime $p$ and $G$ is a finite $p$-group.

We first give some preliminaries of the proof of Proposition 2.7. Let $R$ be a ring and let $G$ be a group. Let $\delta : RG \to R$ be the norm epimorphism, that is, for any $\alpha = \sum_{g \in G} r_g g \in RG$, $\delta(\alpha) = \sum_{g \in G} r_g$. Let $\psi : R \to R/J(R)$ naturally and let $\phi = \psi \delta : RG \to R/J(R)$. Note that $\text{Ker} \phi = \{\alpha \in RG \mid \phi(\alpha) = J(R)\}$. Since $\phi$ is onto, we have that $\phi(J(RG)) \subseteq \{J(R)\}$. Therefore, $J(RG) \subseteq \text{Ker} \phi$.

**Lemma 2.8.** Let $R$ be a ring such that $R/J(R)$ is simple Artinian and let $G \neq \{1\}$ be a torsion abelian group. For any $x \in RG$ such that $\phi(x) \neq J(R)$, assume that there exist $a, b \in RG$ such that $axb = 1$. Then $\text{char} R/J(R) = p$ for some prime $p$ and $G$ is a $p$-group.
PROOF. Let \( g \in G \), \( g \neq 1 \) and let \( n \) be the order of \( g \). Suppose that \( \text{char } R/J(R) = 0 \). Then \( \phi(\sum_{i=0}^{n-1} g^i) = \psi(\sum_{i=0}^{n-1} g^i) = \psi(n1) = n1 + J(R) \neq J(R) \). By the hypothesis, we have \( a, b \in RG \) such that \( a \left( \sum_{i=0}^{n-1} g^i \right) b = 1 \). Therefore, \( 1 - g = \left( a \left( \sum_{i=0}^{n-1} g^i \right) b \right) (1 - g) = a \left( \sum_{i=0}^{n-1} g^i \right) (1 - g) b = a(0)b = 0 \). This gives us \( g = 1 \); a contradiction. Hence, \( \text{char } R/J(R) \neq 0 \). Now since \( R/J(R) \) is simple Artinian (hence, completely reducible), so \( R/J(R) \) is isomorphic to a ring of square matrices over some division ring. Since \( \text{char } R/J(R) \neq 0 \), we must then have that \( \text{char } R/J(R) = p \) for some prime \( p \).

Next we show that \( G \) is a \( p \)-group. Write \( n = p^ak \), where \( p \) and \( k \) are relatively prime, and assume that \( k \geq 1 \). Since \( \phi \left( \sum_{i=0}^{k-1} g^{ip^a} \right) = \psi \left( \sum_{i=0}^{k-1} g^{ip^a} \right) = \psi(k1) = k1 + J(R) \neq J(R) \), it follows from the hypothesis that there exist \( u, v \in RG \) such that \( u \left( \sum_{i=0}^{k-1} g^{ip^a} \right) v = 1 \). Therefore,

\[
1 - g^{p^n} = \left( u \left( \sum_{i=0}^{k-1} g^{ip^a} \right) v \right) (1 - g^{p^n}) = u \left( \sum_{i=0}^{k-1} g^{ip^a} \right) (1 - g^{p^n}) v = u(0)v = 0
\]

which gives us \( g^{p^n} = 1 \); a contradiction. Thus, \( k = 1 \). Then since \( g \) is an arbitrary element of \( G \), it follows that \( G \) is a \( p \)-group.

We are now ready for the proof of Proposition 2.7.

PROOF. By Proposition 2.5 it follows readily that \( R \) is primary and \( G \) is finite. Thus, it remains to show that \( \text{char } R/J(R) = p \) for some prime \( p \) and \( G \) is a \( p \)-group.

We first note that \( \text{Ker } \phi = J(RG) \). Indeed, we have seen that \( J(RG) \subseteq \text{Ker } \phi \). Hence, \( \text{Ker } \phi/J(RG) \) is an ideal of \( RG/J(RG) \). But since \( RG/J(RG) \) is simple and \( \text{Ker } \phi \neq RG \), it follows that \( \text{Ker } \phi = J(RG) \).

Now let \( z \in RG \) such that \( \phi(z) \neq J(R) \). Then \( z \notin \text{Ker } \phi = J(RG) \) and hence, \( (RGzRG + J(RG))/J(RG) \) is a nonzero ideal of \( RG/J(RG) \). Since \( RG/J(RG) \) is simple, it follows that \( (RGzRG + J(RG))/J(RG) = RG/J(RG) \). Therefore, \( 1 - uzv \in J(RG) \) for some \( u, v \in RG \). We then have that \( uzv = 1 - (1 - uzv) \) is a unit of \( RG \). Hence, the hypothesis in Lemma 2.8 is satisfied. It then follows by Lemma 2.8 that \( \text{char } R/J(R) = p \) for some prime \( p \) and \( G \) is a \( p \)-group. This completes the proof of Proposition 2.7.

Finally, by combining Propositions 2.1 and 2.7, we obtain the proof of Theorem 1.1.

3. Some related results

In the case of group algebras, we obtain the following:
Theorem 3.1. Let $K$ be a field of characteristic $p > 0$ and let $G \neq \{1\}$ be a group. Then $KG$ is primary if and only if $G$ is a finite $p$-group.

We shall need the aid of the following lemma to prove Theorem 3.1.

Lemma 3.2. Let $R$ be a ring and let $G \neq \{1\}$ be a group. If $J(RG) = \Delta$, the augmentation ideal of $RG$, then $G$ is a $p$-group for some prime $p$, $J(R) = \{0\}$ and $p = 0$ in $R$.

Proof. See [2, Proposition 16(iii), p. 683].

We now prove Theorem 3.1.

Proof. Suppose that $KG$ is primary. Then by Proposition 2.5 we have that $G$ is a finite group. Note that $\Delta$, the augmentation ideal of $KG$, is a maximal ideal of $KG$ since $KG/\Delta \cong K$. Therefore, $J(KG) \subseteq \Delta$ and hence, $\Delta/J(KG)$ is an ideal of $KG/J(KG)$. But since $KG/J(KG)$ is simple, it follows that $J(KG) = \Delta$. We then have by Lemma 3.2 that $G$ is a $p$-group.

Conversely, if $G$ is a finite $p$-group, it follows readily by Proposition 2.1 that $KG$ is primary.

We conclude this paper with the following remarks.

(1) If $K$ is a field with char $K = 0$ and $G \neq \{1\}$ is an abelian group, then $G$ being finite is not sufficient for $KG$ to be primary. Indeed, since char $K = 0$ and $G$ is abelian, Amitsur (see [1, Theorem 3, p. 252]) has shown that $J(KG) = \{0\}$. Therefore, $KG/J(KG) \cong KG$ is not a simple ring and therefore, $KG$ is not primary.

(2) A ring is said to be clean if every element in the ring can be written as the sum of a unit and an idempotent in the ring. It is known that primary rings are semiperfect and semiperfect rings are clean; hence, primary rings are clean. If $K$ is a field with char $K = 0$ and $G \neq \{1\}$ is an abelian group, then $G$ being finite is sufficient for the group algebra $KG$ to be clean (by [3, Corollary 2.10, p. 406]). Thus $KG$ is an example of a clean ring which is not primary.

References


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