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Primary group rings

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ABSTRACT – Let R be an associative ring with identity and let $J(R)$ denote the Jacobson radical of R. We say that R is primary if $R/J(R)$ is simple Artinian and $J(R)$ is nilpotent. In this paper we obtain necessary and sufficient conditions for the group ring RG, where G is a nontrivial abelian group, to be primary.

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1. Introduction

Throughout this paper all rings are associative with identity. For a ring R, let $J(R)$ denote its Jacobson radical. We say that R is primary if $R/J(R)$ is simple Artinian and $J(R)$ is nilpotent. The ring R is said to be semiprimary if $R/J(R)$ is Artinian and $J(R)$ is nilpotent. A primary ring is clearly semiprimary. The aim of this paper is to obtain necessary and sufficient conditions for the group ring RG, where G is a nontrivial abelian group, to be primary. Our main result is the following:

THEOREM 1.1. *Let R be a ring and let $G \neq \{1\}$ be an abelian group. Then RG is primary if and only if R is primary with char $R/J(R) = p$ for some prime p and G is a finite p-group.*

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We shall prove Theorem 1.1 in Section 2. In Section 3 we consider conditions for a group algebra to be primary. As a consequence, we obtain an example of a clean ring which is not primary.

2. Proof of Theorem 1.1

We first obtain some sufficient conditions for a group ring to be primary.

PROPOSITION 2.1. *Let R be a ring and let $G \neq \{1\}$ be a group. If R is primary with char $R/J(R) = p$ for some prime p and G is a finite p-group, then RG is primary.*

In order to prove Proposition 2.1, we shall need the aid of the following results:

THEOREM 2.2. *(Tan* [4, Theorem, p. 261]*) Let R be a ring and let G be a group. Then RG is semiprimary if and only if R is semiprimary and G is finite.*

PROPOSITION 2.3. *Let R be a ring and let $G \neq \{1\}$ be a group. If G is a locally finite p-group for some prime p, $J(R) = \{0\}$ and $p = 0$ in R, then $J(RG) = \Delta$, the augmentation ideal of RG.*

PROOF. See [2, Proposition 16(iv), p. 683]. □

PROPOSITION 2.4. *Let R be a ring and let G be a group. If R is Artinian or G is locally finite, then $J(R)G \subseteq J(RG)$.*

PROOF. See [2, Proposition 9, p. 665]. □

We are now ready for the proof of Proposition 2.1.

PROOF. Since R is primary (hence, semiprimary) and G is finite, it follows by Theorem 2.2 that RG is semiprimary. Thus, we only need to show that $RG/J(RG)$ is simple.

Let $\bar{R} = R/J(R)$. Then $J(\bar{R}) = \{0\}$ and $p = 0$ in $\bar{R}$. It follows by Proposition 2.3 that $J(\bar{R}G) = \bar{\Delta}$, the augmentation ideal of $\bar{R}G$. Since G is locally finite, we have by Proposition 2.4 that $J(R)G \subseteq J(RG)$. Then

$$\begin{aligned} RG/J(RG) &\cong (RG/J(R)G)\,/\,(J(RG)/J(R)G) \\ &= (RG/J(R)G)\,/\,J\,(RG/J(R)G) \\ &\cong \bar{R}G/J(\bar{R}G) = \bar{R}G/\bar{\Delta} \cong \bar{R}. \end{aligned}$$

Since $\bar{R}$ is simple, so is $RG/J(RG)$. This completes the proof of Proposition 2.1. □

In what follows we show that the converse of Proposition 2.1 is true when G is abelian. We first prove the following:

PROPOSITION 2.5. *Let R be a ring and let G be a group. If RG is primary, then R is primary and G is finite.*

We will make use of the following lemma to prove Proposition 2.5.

LEMMA 2.6. *Let R be a ring such that $R/J(R)$ is simple. If S is a homomorphic image of R, then $S/J(S)$ is also simple.*

PROOF. Let $f : R \to S$ be a ring epimorphism and let $\pi : R/J(R) \to S/J(S)$ be the mapping induced by f. That is,

$$\pi(r + J(R)) = f(r) + J(S), \quad r \in R.$$

It is straightforward to verify that π is a well-defined ring epimorphism. Then since $R/J(R)$ is simple, so is $S/J(S)$. □

We now prove Proposition 2.5.

PROOF. Since RG is primary (hence, semiprimary), it follows readily by Theorem 2.2 that R is semiprimary and G is finite. It remains to show that $R/J(R)$ is simple. But this follows readily by Lemma 2.6 since R is a homomorphic image of RG and $RG/J(RG)$ is simple. We thus have that R is primary, as required. □

If $G \neq \{1\}$ is an abelian group, Proposition 2.5 can be made more precise as follows:

PROPOSITION 2.7. *Let R be a ring and let $G \neq \{1\}$ be an abelian group. If RG is primary, then R is primary with char $R/J(R) = p$ for some prime p and G is a finite p-group.*

We first give some preliminaries of the proof of Proposition 2.7. Let R be a ring and let G be a group. Let $\delta : RG \to R$ be the norm epimorphism, that is, for any $\alpha = \sum_{g \in G} r_g g \in RG$, $\delta(\alpha) = \sum_{g \in G} r_g$. Let $\psi : R \to R/J(R)$ naturally and let $\phi = \psi\delta : RG \to R/J(R)$. Note that Ker $\phi = \{\alpha \in RG \mid \phi(\alpha) = J(R)\}$. Since ϕ is onto, we have that $\phi(J(RG)) \subseteq \{J(R)\}$. Therefore, $J(RG) \subseteq$ Ker ϕ.

LEMMA 2.8. *Let R be a ring such that $R/J(R)$ is simple Artinian and let $G \neq \{1\}$ be a torsion abelian group. For any $x \in RG$ such that $\phi(x) \neq J(R)$, assume that there exist $a, b \in RG$ such that $axb = 1$. Then char $R/J(R) = p$ for some prime p and G is a p-group.*

Proof. Let $g \in G, g \neq 1$ and let n be the order of g. Suppose that char $R/J(R) = 0$. Then $\phi(\sum_{i=0}^{n-1} g^i) = \psi\delta\left(\sum_{i=0}^{n-1} g^i\right) = \psi(n1) = n1 + J(R) \neq J(R)$. By the hypothesis, we have $a, b \in RG$ such that $a\left(\sum_{i=0}^{n-1} g^i\right)b = 1$. Therefore, $1 - g = \left(a\left(\sum_{i=0}^{n-1} g^i\right)b\right)(1-g) = a\left(\sum_{i=0}^{n-1} g^i\right)(1-g)b = a(0)b = 0$. This gives us $g = 1$; a contradiction. Hence, char $R/J(R) \neq 0$. Now since $R/J(R)$ is simple Artinian (hence, completely reducible), so $R/J(R)$ is isomorphic to a ring of square matrices over some division ring. Since char $R/J(R) \neq 0$, we must then have that char $R/J(R) = p$ for some prime p.

Next we show that G is a p-group. Write $n = p^u k$, where p and k are relatively prime, and assume that $k > 1$. Since $\phi\left(\sum_{i=0}^{k-1} g^{ip^u}\right) = \psi\delta\left(\sum_{i=0}^{k-1} g^{ip^u}\right) = \psi(k1) = k1 + J(R) \neq J(R)$, it follows from the hypothesis that there exist $u, v \in RG$ such that $u\left(\sum_{i=0}^{k-1} g^{ip^u}\right)v = 1$. Therefore,

$$\begin{aligned} 1 - g^{p^u} &= \left(u\left(\sum_{i=0}^{k-1} g^{ip^u}\right)v\right)(1 - g^{p^u}) = u\left(\sum_{i=0}^{k-1} g^{ip^u}\right)(1 - g^{p^u})v \\ &= u(0)v = 0 \end{aligned}$$

which gives us $g^{p^u} = 1$; a contradiction. Thus, $k = 1$. Then since g is an arbitrary element of G, it follows that G is a p-group. □

We are now ready for the proof of Proposition 2.7.

Proof. By Proposition 2.5 it follows readily that R is primary and G is finite. Thus, it remains to show that char $R/J(R) = p$ for some prime p and G is a p-group.

We first note that Ker $\phi = J(RG)$. Indeed, we have seen that $J(RG) \subseteq$ Ker ϕ. Hence, Ker $\phi/J(RG)$ is an ideal of $RG/J(RG)$. But since $RG/J(RG)$ is simple and Ker $\phi \neq RG$, it follows that Ker $\phi = J(RG)$.

Now let $z \in RG$ such that $\phi(z) \neq J(R)$. Then $z \notin$ Ker $\phi = J(RG)$ and hence, $(RGzRG + J(RG))/J(RG)$ is a nonzero ideal of $RG/J(RG)$. Since $RG/J(RG)$ is simple, it follows that $(RGzRG + J(RG))/J(RG) = RG/J(RG)$. Therefore, $1 - uzv \in J(RG)$ for some $u, v \in RG$. We then have that $uzv = 1 - (1 - uzv)$ is a unit of RG. Hence, the hypothesis in Lemma 2.8 is satisfied. It then follows by Lemma 2.8 that char $R/J(R) = p$ for some prime p and G is a p-group. This completes the proof of Proposition 2.7. □

Finally, by combining Propositions 2.1 and 2.7, we obtain the proof of Theorem 1.1.

3. Some related results

In the case of group algebras, we obtain the following:

Theorem 3.1. *Let K be a field of characteristic $p > 0$ and let $G \neq \{1\}$ be a group. Then KG is primary if and only if G is a finite p-group.*

We shall need the aid of the following lemma to prove Theorem 3.1.

Lemma 3.2. *Let R be a ring and let $G \neq \{1\}$ be a group. If $J(RG) = \Delta$, the augmentation ideal of RG, then G is a p-group for some prime p, $J(R) = \{0\}$ and $p = 0$ in R.*

Proof. See [2, Proposition 16(iii), p. 683]. □

We now prove Theorem 3.1.

Proof. Suppose that KG is primary. Then by Proposition 2.5 we have that G is a finite group. Note that Δ, the augmentation ideal of KG, is a maximal ideal of KG since $KG/\Delta \cong K$. Therefore, $J(KG) \subseteq \Delta$ and hence, $\Delta/J(KG)$ is an ideal of $KG/J(KG)$. But since $KG/J(KG)$ is simple, it follows that $J(KG) = \Delta$. We then have by Lemma 3.2 that G is a p-group.

Conversely, if G is a finite p-group, it follows readily by Proposition 2.1 that KG is primary. □

We conclude this paper with the following remarks.

(1) If K is a field with char $K = 0$ and $G \neq \{1\}$ is an abelian group, then G being finite is not sufficient for KG to be primary. Indeed, since char $K = 0$ and G is abelian, Amitsur (see [1, Theorem 3, p. 252]) has shown that $J(KG) = \{0\}$. Therefore, $KG/J(KG) \cong KG$ is Artinian and $J(KG) = \{0\}$ is nilpotent. However, we note that the augmentation ideal Δ of KG is a nontrivial ideal of KG. Thus, $KG/J(KG) \cong KG$ is not a simple ring and therefore, KG is not primary.

(2) A ring is said to be clean if every element in the ring can be written as the sum of a unit and an idempotent in the ring. It is known that primary rings are semiperfect and semiperfect rings are clean; hence, primary rings are clean. If K is a field with char $K = 0$ and $G \neq \{1\}$ is an abelian group, then G being finite is sufficient for the group algebra KG to be clean (by [3, Corollary 2.10, p. 406]). Thus KG is an example of a clean ring which is not primary.

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