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PURE-MINIMAL CHAIN COMPLEXES

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Abstract. We introduce a notion of *pure-minimality* for chain complexes of modules and show that it coincides with (homotopic) minimality in standard settings, while being a more useful notion for complexes of flat modules. As applications, we characterize von Neumann regular rings and left perfect rings.

Introduction

Given a chain complex, it is natural to ask whether it has a "smallest" subcomplex of the same homotopy type, as such a subcomplex would carry all pertinent information of the ambient complex without homotopic redundancy. Initiated by Eilenberg and Zilber [10] in the context of simplicial complexes, this perspective has come to play a significant role in the homological study of rings and modules.

Let R be an associative unital ring. A chain complex M of R-modules is called *minimal* if every homotopy equivalence $M \to M$ is an isomorphism; see Avramov and Martsinkovsky [4] and Roig [15]. Many homological invariants of modules, such as their injective and projective dimension, can be read off from minimal resolutions, provided that they exist. Minimal injective and minimal flat resolutions exist for every R-module; indeed, any resolution constructed from injective envelopes or from flat covers is a minimal chain complex; see e.g. Thompson [17]. Over a perfect ring every module has a minimal projective resolution, and over a semi-perfect ring every finitely generated module has a minimal projective resolution.

While minimal flat resolutions exist, they do not quite behave as one might hope. For example, let p be a prime and consider the local ring $\mathbb{Z}_{(p)}$ with $p\mathbb{Z}_{(p)}$-adic completion $\mathbb{Z}^{\wedge}_{(p)}$. It is elementary to verify that $F = 0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}^{\wedge}_{(p)} \to 0$ is a minimal chain complex of $\mathbb{Z}_{(p)}$-modules; see Example (3.9). Evidently, it is a flat resolution of the module $\mathbb{Z}^{\wedge}_{(p)}/\mathbb{Z}_{(p)}$; however, $\mathbb{Z}^{\wedge}_{(p)}/\mathbb{Z}_{(p)}$ is a flat $\mathbb{Z}_{(p)}$-module and, as such, a minimal flat resolution of itself. This non-uniqueness of minimal flat resolutions is, perhaps, unsurprising as flat resolutions do not come with comparison maps the way projective and injective resolutions do. The difference between the flat resolutions F and $\mathbb{Z}^{\wedge}_{(p)}/\mathbb{Z}_{(p)}$ is the pure subcomplex $P = 0 \to \mathbb{Z}_{(p)} \xrightarrow{=} \mathbb{Z}_{(p)} \to 0$; indeed, there is a pure exact sequence $0 \to P \to F \to \mathbb{Z}^{\wedge}_{(p)}/\mathbb{Z}_{(p)} \to 0$ of chain complexes. This points to a notion of minimality that forbids the existence such subcomplexes.

In this paper, a chain complex is called *pure-minimal* if the zero complex is the only pure-acyclic pure subcomplex. This definition is inspired by the example recounted above and by the fact that in a minimal chain complex the zero complex

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is the only contractible direct summand. In settings where minimality is well-understood, such as for chain complexes of projective modules over a perfect ring, we show that pure-minimality coincides with minimality (Theorem (3.13)).

Our central construction (Theorem (5.1)) shows that given a chain complex M of R-modules, there exists a pure-minimal chain complex that is isomorphic to M in the derived category over R. Moreover, in settings where minimal chain complexes are known to exist, the construction recognizes them (Corollary (5.3)).

As an application of our construction, we show (Theorem (5.6)) that for every chain complex M of R-modules there exists a pure-minimal semi-flat[1] complex F that is isomorphic to M in the derived category over R, and that the flat dimension of M can be read off from F. In fact, pure-minimality is also an appropriate notion of minimality for degreewise finitely generated semi-projective complexes over a noetherian ring (Theorem (5.4)). As further applications, we characterize von Neumann regular rings (Corollary (5.2)) and left perfect rings (Theorem (5.7)).

The paper is organized as follows. In Section 1 we study pure-acyclic chain complexes and give a characterization of von Neumann regular rings in terms of pure homological algebra (Theorem (1.11)); in Section 2 we continue with a few technical results on pure quasi-isomorphisms. In Section 3 we define pure-minimality and compare it with other notions of minimality. We focus separately on minimality of acyclic chain complexes in Section 4. The main results advertised above are found in Section 5. In the appendix we establish sufficient conditions for acyclicity of chain complexes.

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Throughout, R is an associative algebra over a commutative unital ring $\Bbbk$ which, if no other choice is more appealing, can be $\mathbb{Z}$. The term *R-module* refers to a left R-module, while a right R-module is considered to be a (left) module over the opposite ring R°.

A chain complex of R-modules is for short called an *R-complex*. The category of R-complexes is denoted $\mathcal{C}(R)$. For a homologically indexed R-complex M, write ∂^M for the differential and define the subcomplexes $\mathrm{Z}(M)$ and $\mathrm{B}(M)$ with zero differentials by specifying their modules: $\mathrm{Z}_i(M) = \mathrm{Ker}(\partial_i^M)$ and $\mathrm{B}_i(M) = \mathrm{Image}(\partial_{i+1}^M)$. Further, set $\mathrm{C}(M) = M/\mathrm{B}(M)$ and $\mathrm{H}(M) = \mathrm{Z}(M)/\mathrm{B}(M)$. A complex M is said to be *acyclic* if the sequence $0 \to \mathrm{Z}_i(M) \to M_i \to \mathrm{Z}_{i-1}(M) \to 0$ is exact for every $i \in \mathbb{Z}$; that is, $\mathrm{H}(M)$ is the zero complex. A complex M is called *contractible* if the identity 1^M is null-homotopic; a contractible complex is acyclic.

Homology is a functor on $\mathcal{C}(R)$. A morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ is called a *quasi-isomorphism* if $\mathrm{H}(\alpha)$ is an isomorphism. Prominent quasi-isomorphisms are homotopy equivalences; they are morphisms that have an inverse up to homotopy.

For R-complexes L and M, the total Hom complex is denoted $\mathrm{Hom}_R(L, M)$. For an R°-complex N and an R-complex M, the total tensor product complex is written $N \otimes_R M$.

1. Pure-acyclic complexes

In this first section we recall fundamentals of pure homological algebra, with a focus on pure-acyclicity, and give a characterization of von Neumann regular rings.

[1] In the literature, e.g. in [2] by Avramov and Foxby, such complexes are also called dg-flat.

(1.1) **Resolutions of complexes.** An R-complex F is called *semi-flat* if it consists of flat R-modules and the functor $- \otimes_R F$ preserves acyclicity. A complex F of flat modules with $F_i = 0$ for $i \ll 0$ is semi-flat, this follows for example from Corollary (A.2). Similarly, a complex I (a complex P) is called *semi-injective* (*semi-projective*) if it consists of injective modules (projective modules) and the functor $\mathrm{Hom}_R(-, I)$ preserves acyclicity ($\mathrm{Hom}_R(P, -)$ preserves acyclicity). Every semi-projective complex is semi-flat. A complex P of projective modules with $P_i = 0$ for $i \ll 0$ is semi-projective, and a complex I of injective modules with $I_i = 0$ for $i \gg 0$ is semi-injective, this follows from Propositions (A.1) and (A.3).

Every R-complex M has a semi-projective resolution and a semi-injective resolution; that is, there are quasi-isomorphisms

$$P \xrightarrow{\simeq} M \xrightarrow{\simeq} I$$

with P semi-projective and I semi-injective; see [2]. For a module, a classical projective (injective) resolution is a semi-projective (-injective) resolution.

(1.2) **Purity in the category of modules.** An exact sequence of R-modules $0 \to L \to M \to N \to 0$ is called *pure* if the induced sequence of $\Bbbk$-modules

$$0 \longrightarrow \mathrm{Hom}_R(A, L) \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow \mathrm{Hom}_R(A, N) \longrightarrow 0$$

is exact for every finitely presented R-module A. Equivalently, the sequence of $\Bbbk$-modules $0 \to B \otimes_R L \to B \otimes_R M \to B \otimes_R N \to 0$ is exact for every R°-module B.

An R-module F is flat if and only if every exact sequence $0 \to L \to M \to F \to 0$ is pure. For a flat R-module F, an exact sequence $0 \to L \to F \to N \to 0$ is pure if and only if L and N are flat. See for example Lam [14, sec. 4J] for details.

On the other hand, an R-module E is fp-injective if and only if every exact sequence $0 \to E \to M \to N \to 0$ is pure. For an fp-injective R-module E, an exact sequence $0 \to L \to E \to N \to 0$ is pure if and only if L is fp-injective.

(1.3) **Remark.** In view of (1.2) the following conditions are equivalent.

(i) Every R-module is flat.
(ii) Every short exact sequence of R-modules is pure.
(iii) Every R-module is fp-injective.

The rings that satisfy these conditions are precisely the von Neumann regular rings—also called absolutely flat rings.

(1.4) **Definition.** An exact sequence of R-complexes $0 \to L \to M \to N \to 0$ is called *degreewise pure* (*degreewise split*) if the sequence $0 \to L_i \to M_i \to N_i \to 0$ of R-modules is pure (split) for every $i \in \mathbb{Z}$. A subcomplex $L \subseteq M$, and a quotient complex M/L, are called degreewise pure (degreewise split) if the canonical exact sequence $0 \to L \to M \to M/L \to 0$ is degreewise pure (degreewise split).

(1.5) **Lemma.** *Let M be an R-complex. Under each of the conditions* (a)–(d) *below, every exact sequence of R-complexes $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ that is degreewise pure is degreewise split.*

(a) *R is left noetherian; M is a complex of injective R-modules.*
(b) *R is left perfect; M is a complex of projective R-modules.*
(c) *R is semi-perfect; M is a complex of finitely generated projective R-modules.*
(d) *R is left noetherian; M is a complex of finitely generated projective R-modules.*

Proof. Consider a degreewise pure exact sequence of R-complexes

$$(*) \qquad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0\,.$$

(a): It follows from (1.2) that L is a complex of fp-injective modules. As R is left noetherian, fp-injective R-modules are injective. Thus, $(*)$ is degreewise an exact sequence of injective modules, in particular it is degreewise split exact.

(b): It follows from (1.2) that N is a complex of flat modules. As R is left perfect, flat R-modules are projective. Thus, $(*)$ is degreewise an exact sequence of projective modules, in particular it is degreewise split exact.

(c) & (d): The complex N is degreewise finitely generated and as in (b) a complex of flat modules. Over a semi-perfect or left noetherian ring, finitely generated flat modules are projective, so as in (b) the sequence $(*)$ is degreewise split exact. □

(1.6) **Definition.** An R-complex M is called *pure-acyclic* if it is acyclic and the exact sequence $0 \to \mathrm{Z}_i(M) \to M_i \to \mathrm{Z}_{i-1}(M) \to 0$ is pure for every $i \in \mathbb{Z}$.

(1.7) **Example.** Every acyclic semi-flat complex is pure-acyclic, see [8, thm. 7.3]. On the other hand, the $\mathbb{Z}/4\mathbb{Z}$-complex known as the *Dold complex*,

$$\cdots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \cdots\,,$$

is an acyclic complex of flat modules which is not pure acyclic. Indeed, the cycle submodules $2\mathbb{Z}/4\mathbb{Z}$ are torsion and hence not flat, cf. (1.2).

Recall that an R-module P is *pure-projective* if the sequence

$$0 \longrightarrow \mathrm{Hom}_R(P,L) \longrightarrow \mathrm{Hom}_R(P,M) \longrightarrow \mathrm{Hom}_R(P,N) \longrightarrow 0$$

is exact for every pure exact sequence $0 \to L \to M \to N \to 0$ of R-modules. Pure-injective modules are defined dually.

(1.8) **Remark.** An R-complex M is by definition pure-acyclic if (and only if) $\mathrm{Hom}_R(A,M)$ is acyclic for every finitely presented R-module A. Emmanouil [11, thm. 3.6] shows that M is pure-acyclic (if and) only if $\mathrm{Hom}_R(A,M)$ is acyclic for every complex A of pure-projective R-modules. Dually, M is pure-acyclic if and only if $\mathrm{Hom}_R(M,E)$ is acyclic for every pure-injective R-module E, equivalently, every complex E of pure-injective R-modules. This was proved by Šťovíček [16, thm. 5.4]; see also Bazzoni, Cortés Izurdiaga, and Estrada [5, rmk. 4.7].

From the proof of [5, cor. 2.6] one can extract:

(1.9) **Fact.** *Every pure-acyclic complex of pure-projective modules is contractible.*

For complexes of projective modules this follows from an earlier result of Benson and Goodearl [6, thm. 2.5]; see [8, prop. 7.6]. From the proof of [5, cor. 4.5] one can extract the dual result:

(1.10) **Fact.** *Every pure-acyclic complex of pure-injective modules is contractible.*

To close the section we apply these two facts to characterize von Neumann regular rings in pure homological terms.

(1.11) **Theorem.** *The following conditions are equivalent.*

(i) *R is von Neumann regular.*

(ii) *Every acyclic R-complex is pure-acyclic.*

(*iii*) *Every R-complex is semi-flat.*
(*iv*) *Every complex of pure-projective R-modules is semi-projective.*
(*v*) *Every complex of pure-injective R-modules is semi-injective.*

Proof. The (bi)implications $(i) \Leftrightarrow (ii)$ and $(iii) \Rightarrow (i)$ are clear from Remark (1.3).

$(i) \Rightarrow (iii)$: Let M be an R-complex and A be an acyclic R°-complex. As every R-module is flat, it follows from Corollary (A.2) that $A \otimes_R M$ is acyclic, whence M is semi-flat.

$(ii) \Rightarrow (v)$: Let J be a complex of pure-injective R-modules. By assumption every short exact sequence of R-modules is pure, so J is a complex of injective modules. Let $\iota\colon J \xrightarrow{\simeq} I$ be a semi-injective resolution. The complex $\operatorname{Cone}\iota$ is an acyclic, hence pure-acyclic, complex of injective R-modules, so by (1.10) it is contractible. For an acyclic R-complex A, there is a triangle

$$\operatorname{Hom}_R(A,J) \longrightarrow \operatorname{Hom}_R(A,I) \longrightarrow \operatorname{Hom}_R(A,\operatorname{Cone}\iota) \longrightarrow$$

in the derived category of $\Bbbk$-complexes. The middle complex is acyclic as I is semi-injective, and the right-hand complexes is even contractible; it follows that $\operatorname{Hom}_R(A,J)$ is acyclic, whence J is semi-injective.

$(v) \Rightarrow (i)$: A ring is von Neumann regular if and only if the opposite ring is so; it is thus sufficient to show that every R°-module is flat. The character dual $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ of an R°-module M is a pure-injective R-module. It follows from the assumption that $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ is injective, whence M is flat.

$(iii) \Rightarrow (iv)$: Let P be a complex of pure-projective R-modules. By assumption each module P_i is flat and hence a pure quotient of a free R-module L_i, cf. (1.2). By pure-projectivity of P_i, the homomorphism $\operatorname{Hom}_R(P_i,L_i) \to \operatorname{Hom}_R(P_i,P_i)$ is surjective, whence P_i is a summand of L_i. Thus, P is a complex of projective R-modules and semi-flat. Now proceed in parallel with the proof of $(ii) \Rightarrow (v)$ above, but invoke (1.9) instead of (1.10). Alternately see [8, thm. 7.8].

$(iv) \Rightarrow (ii)$: A finitely presented module is pure-projective, so by assumption finitely presented R-modules are projective. Now it follows from Remark (1.8) that every acyclic R-complex is pure-acyclic. □

2. Pure quasi-isomorphisms

We continue with a discussion of morphisms with pure-acyclic mapping cones. Recall that a morphism α of complexes is a homotopy equivalence if and only if its mapping cone, $\operatorname{Cone}\alpha$, is contractible, while α is a quasi-isomorphism if and only if $\operatorname{Cone}\alpha$ is acyclic. Pure quasi-isomorphisms are an intermediate type of morphisms.

(2.1) **Definition.** A morphism of R-complexes is called a *pure quasi-isomorphism* if its mapping cone is a pure-acyclic complex.

(2.2) **Remark.** A morphism α of R-complexes is a pure quasi-isomorphism if and only if $\operatorname{Hom}_R(A,\alpha)$ is a quasi-isomorphism for every finitely presented R-module A, equivalently for every complex A of pure-projective R-modules. This follows in view of Remark (1.8) from the isomorphism $\operatorname{Hom}_R(A,\operatorname{Cone}\alpha) \cong \operatorname{Cone}\operatorname{Hom}_R(A,\alpha)$.

(2.3) **Example.** A contractible complex is pure-acyclic, so every homotopy equivalence is a pure quasi-isomorphism. Further, every quasi-isomorphism of semi-flat complexes is a pure quasi-isomorphism by [8, cor. 7.4] and Remark (2.2).

(2.4) **Purity in the category of complexes.** For R-complexes L and M, the hom-set is denoted $\hom_{\mathcal{C}(R)}(L,M)$; it relates to the total Hom complex through the equality $\hom_{\mathcal{C}(R)}(L,M) = \mathrm{Z}_0(\mathrm{Hom}_R(L,M))$ of graded $\Bbbk$-modules.

An exact sequence of R-complexes $0 \to L \to M \to N \to 0$ is called *pure* if the sequence of $\Bbbk$-complexes

$$0 \longrightarrow \hom_{\mathcal{C}(R)}(A,L) \longrightarrow \hom_{\mathcal{C}(R)}(A,M) \longrightarrow \hom_{\mathcal{C}(R)}(A,N) \longrightarrow 0$$

is exact for every bounded complex A of finitely presented R-modules. It follows from [12, thm. 5.1.3] and [8, thm. 4.5] that every pure exact sequence of R-complexes is degreewise pure.

A subcomplex $L \subseteq M$, and a quotient complex M/L, are called pure if the canonical exact sequence $0 \to L \to M \to M/L \to 0$ is pure.

The next results help to recognize pure quasi-isomorphisms (homotopy equivalences) and pure-acyclic pure subcomplexes (contractible split subcomplexes). In the literature, contractible complexes are at times called split; we emphasize that by a (degreewise) split subcomplex we mean a (degreewise) direct summand.

(2.5) **Proposition.** *Let* $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ *be a degreewise pure exact sequence in* $\mathcal{C}(R)$. *The following assertions hold.*

(a) *The complex* L *is pure-acyclic if and only if* β *is a pure quasi-isomorphism.*

(b) *The complex* N *is pure-acyclic if and only if* α *is a pure quasi-isomorphism.*

Moreover, if L *or* N *is pure-acyclic, then the sequence is pure in* $\mathcal{C}(R)$.

In particular, a pure-acyclic subcomplex is a pure subcomplex if and only if it is a degreewise pure subcomplex.

Proof. For every finitely presented R-module A, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(A,L) \xrightarrow{\mathrm{Hom}_R(A,\alpha)} \mathrm{Hom}_R(A,M) \xrightarrow{\mathrm{Hom}_R(A,\beta)} \mathrm{Hom}_R(A,N) \longrightarrow 0$$

is exact, as the given exact sequence is degreewise pure. The complex $\mathrm{Hom}_R(A,L)$ is acyclic if and only if $\mathrm{Hom}_R(A,\beta)$ is a quasi-isomorphism, and $\mathrm{Hom}_R(A,N)$ is acyclic if and only if $\mathrm{Hom}_R(A,\alpha)$ is a quasi-isomorphism. Now (a) and (b) follow from Remark (2.2).

Finally, given a bounded complex A of finitely presented R-modules, we must verify exactness of the sequence

$$0 \longrightarrow \hom_{\mathcal{C}(R)}(A,L) \longrightarrow \hom_{\mathcal{C}(R)}(A,M) \longrightarrow \hom_{\mathcal{C}(R)}(A,N) \longrightarrow 0\,.$$

As $\hom_{\mathcal{C}(R)}(A,-)$ is left exact and one has $\hom_{\mathcal{C}(R)}(A,-) = \mathrm{Z}_0(\mathrm{Hom}_R(A,-))$, this amounts to showing that the map

$$\mathrm{Z}_0(\mathrm{Hom}_R(A,M)) \xrightarrow{\mathrm{Z}_0(\mathrm{Hom}_R(A,\beta))} \mathrm{Z}_0(\mathrm{Hom}_R(A,N))$$

is surjective. First notice that the morphism $\mathrm{Hom}_R(A,\beta)$ is surjective as the given exact sequence is degreewise pure. If L is pure-acyclic, then it follows from (a) and Remark (2.2) that $\mathrm{Hom}_R(A,\beta)$ is a quasi-isomorphism and, therefore, surjective on cycles. If N is pure-acyclic, then $\mathrm{Hom}_R(A,N)$ is acyclic by Remark (1.8). As every surjective chain map is surjective on boundaries, it follows that $\mathrm{Hom}_R(A,\beta)$ is surjective on cycles. $\square$

(2.6) **Proposition.** *Let* $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ *be a degreewise split exact sequence in* $\mathcal{C}(R)$. *The following assertions hold.*

(a) *The complex L is contractible if and only if β is a homotopy equivalence.*

(b) *The complex N is contractible if and only if α is a homotopy equivalence.*

Moreover, if L or N is contractible, then the sequence splits in $\mathcal{C}(R)$.

In particular, a contractible subcomplex is a split subcomplex if and only if it is a degreewise split subcomplex.

Proof. Part (a) follows from [4, lem. 1.6], and part (b) has an analogous proof. □

In the sequel we use following two-of-three property of pure quasi-isomorphisms.

(2.7) **Lemma.** *Let $\alpha\colon L \to M$ and $\beta\colon M \to N$ be morphisms of R-complexes. If any two of α, β, and $\beta\alpha$ are pure quasi-isomorphisms, then so is the third.*

Proof. For every finitely presented R-module A one has

$$\mathrm{H}(\mathrm{Hom}_R(A,\beta))\,\mathrm{H}(\mathrm{Hom}_R(A,\alpha)) \;=\; \mathrm{H}(\mathrm{Hom}_R(A,\beta\alpha))\,,$$

which shows that if any two of the morphisms $\mathrm{H}(\mathrm{Hom}_R(A,\alpha))$, $\mathrm{H}(\mathrm{Hom}_R(A,\beta))$, and $\mathrm{H}(\mathrm{Hom}_R(A,\beta\alpha))$ are isomorphisms, then so is the third. Now the statement follows from Remark (2.2). □

It is an elementary observation that an acyclic semi-projective (-injective) complex is contractible and, hence, a quasi-isomorphism of semi-projective (-injective) complexes is a homotopy equivalence. In the same vein one has the following immediate consequences of (1.9) and (1.10).

(2.8) **Corollary.** *A pure quasi-isomorphism of complexes of pure-projective modules is a homotopy equivalence.* □

(2.9) **Corollary.** *A pure quasi-isomorphism of complexes of pure-injective modules is a homotopy equivalence.* □

3. Flavors of minimality

We introduce the notion of *pure-minimality* and explore how it compares to notions found in the literature. This section paves the way for our main results in Section 5.

We start by recalling that an R-complex M is *minimal* if every homotopy equivalence $M \to M$ is an isomorphism or, equivalently, every morphism $M \to M$ that is homotopic to the identity 1^M is an isomorphism. In our context, this definition is best known from [4]. It is also an instance of Roig's [15] notion of S-left or S-right minimality: the one where S in [15, def. 1.1] is the class of homotopy equivalences.

Every complex has a minimal semi-injective resolution; see [3] or [13, prop. B.2]. Minimal semi-projective resolutions are more tricky: If R is left perfect—such that flat R-modules are projective—then every R-complex has a minimal semi-projective resolution. If R is semi-perfect—such that finitely generated flat R-modules are projective—then every R-complex M with $\mathrm{H}(M)$ degreewise finitely generated and $\mathrm{H}_i(M) = 0$ for $i \ll 0$ has a minimal semi-projective resolution, see [3]. For the case of resolutions of modules over a perfect ring, one can refer to Eilenberg [9].

(3.1) **Example.** The $\mathbb{Z}$-complex $F = 0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Q} \longrightarrow 0$, where ι is the natural embedding, is minimal. Indeed, the morphisms $\pm 1^F$ are the only homotopy equivalences $F \to F$, because there no nonzero homomorphisms $\mathbb{Q} \to \mathbb{Z}$.

The following weaker condition already detects minimality of complexes of injective or projective modules; see Proposition (3.4)[2] below.

(3.2) **Definition.** An R-complex M is called *split-minimal* if the zero complex is the only contractible split subcomplex of M.

(3.3) **Example.** Let $p \geqslant 5$ be a prime and denote by $\mathbb{Z}_p$ the integers localized at the powers of p. The $\mathbb{Z}$-complex $F = 0 \longrightarrow \mathbb{Z}_p \xrightarrow{2} \mathbb{Z}_p \longrightarrow 0$ is split-minimal but not minimal. Indeed, every $\mathbb{Z}$-submodule of $\mathbb{Z}_p$ is cyclic, so any two non-zero submodules have a nonzero intersection. Hence $\mathbb{Z}_p$ has no non-trivial direct summands, and the zero complex is the only acyclic split subcomplex of F. However, the morphism 3^F is homotopic to 1^F (the homotopy is given by the identity on $\mathbb{Z}_p$) but not an isomorphism.

To frame the next result we point out that the complex F in Example (3.3) is a complex of flat modules.

(3.4) **Proposition.** *Let M be an R-complex. If M is minimal, then it is split-minimal; the converse holds under each of the following conditions:*

(a) *M is a complex injective R-modules.*

(b) *R is left perfect; M is a complex of projective R-modules.*

(c) *R is semi-perfect; M is a complex of finitely generated projective R-modules.*

Proof. A minimal R-complex has by [4, prop. 1.7] no nonzero contractible split subcomplexes, hence it is split-minimal.

(a): Let M be a split-minimal complex of injective R-modules; it has by [3] or [13, prop. B.2] a decomposition $M = M' \oplus M''$, where M' is minimal and M'' is contractible. Thus M'' is a contractible split subcomplex of M, and so $M'' = 0$.

(b) & (c): Suppose that R is left perfect and M is a split-minimal complex of projective R-modules, or that R is semi-perfect and M is a split-minimal complex of finitely generated projective R-modules. In either case, M has a decomposition $M = M' \oplus M''$, where M' is minimal and M'' is contractible; see [3]. As above, it follows that $M'' = 0$, and so M is minimal. $\square$

In search of a useful notion of minimality for complexes of flat modules, we turn to purity to formulate an analogue of split-minimality. Given a semi-flat complex F with flat dimension n, the cokernels $\mathrm{C}_i(F)$ are flat for $i \geqslant n$, and it follows that $P = \cdots \to F_{n+1} \to \mathrm{B}_n(F) \to 0$ is a pure-acyclic degreewise pure subcomplex of F.

At the least, a "minimal" semi-flat complex ought to vanish beyond the flat dimension, but a minimal semi-flat complex need not meet that requirement; see Example (3.9). Inspired by this example, we introduce a notion of minimality that forbids nonzero pure acyclic pure (equivalently, degreewise pure) subcomplexes.

(3.5) **Definition.** An R-complex M is called *pure-minimal* if the zero complex is the only pure-acyclic pure subcomplex of M.

[2] The gist of this result is that minimality and split-minimality are equivalent notions for complexes of injective modules and complexes of projective modules. For the former this is true as stated, and for the latter it is true over rings where minimal complexes of projective modules are known to exist, in the strong sense that every complex of projectives decomposes as a direct sum of a minimal complex and a contractible one. The proof references the unpublished [3], and the result serves to encapsulate this reference in the sense that one can henceforth focus on split-minimality of complexes of projective modules.

(3.6) **Remark.** Every pure-minimal complex is split-minimal. Indeed every contractible complex is pure-acyclic and every split subcomplex is a pure subcomplex.

(3.7) **Example.** The minimal $\mathbb{Z}$-complex F from Example (3.1) is pure-minimal. To see this, recall that a $\mathbb{Z}$-module is flat if and only if it is torsion-free; this means that 0 and $\mathbb{Z}$ are the only pure submodules of $\mathbb{Z}$; cf. (1.2). The only candidate for a nonzero pure-acyclic pure subcomplex of F is, therefore, $0 \longrightarrow \mathbb{Z} \xrightarrow{=} \mathbb{Z} \longrightarrow 0$, but $\mathbb{Z}$ is not a pure submodule of $\mathbb{Q}$, as $\mathbb{Q}/\mathbb{Z}$ is torsion.

(3.8) **Example.** The $\mathbb{Z}$-complex F in Example (3.3) is pure-minimal but not minimal. Indeed, as every $\mathbb{Z}$-submodule of $\mathbb{Z}_p$ is cyclic, it follows that every non-trivial quotient is torsion. As in Example (3.7) it then follows that the flat $\mathbb{Z}$-module $\mathbb{Z}_p$ has no non-trivial pure submodules. Thus, as F is not acyclic, the zero complex is the only acyclic pure subcomplex of F.

(3.9) **Example.** Let p be a prime and $\mathbb{Z}^{\wedge}_{(p)}$ denote the p-adic completion of $\mathbb{Z}_{(p)}$. The $\mathbb{Z}_{(p)}$-complex $F = 0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}^{\wedge}_{(p)} \to 0$ is minimal but not pure-minimal. To see this recall, for example from [1, lem. 3.3], that one has $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(\mathbb{Z}^{\wedge}_{(p)}, \mathbb{Z}_{(p)}) = 0$. It follows that every homotopy equivalence $F \to F$ is an isomorphism. Thus, F and $\mathbb{Z}^{\wedge}_{(p)}/\mathbb{Z}_{(p)}$ are both minimal semi-flat resolutions of $\mathbb{Z}^{\wedge}_{(p)}/\mathbb{Z}_{(p)}$; see [14, thm. (4.74)]. However, F is not pure-minimal as the subcomplex $0 \to \mathbb{Z}_{(p)} \xrightarrow{=} \mathbb{Z}_{(p)} \to 0$ is degreewise pure and hence pure per Proposition (2.5).

(3.10) **Remark.** If one takes the class S in [15, def. 1.1] to be that of pure quasi-isomorphisms, then an S-right minimal complex in the sense of Roig is pure-minimal. Indeed, let M be an R-complex and P a pure-acyclic pure subcomplex of M; by Proposition (2.5) the canonical map $\pi \colon M \to M/P$ is a pure quasi-isomorphism. Thus, if M is S-right minimal, then π has a left inverse, which implies that π is injective and hence $P = 0$. The converse fails: with F as in Example (3.3), the map 3^F is a pure quasi-isomorphism, see Example (2.3), with no left inverse.

The next two corollaries paraphrase parts of Propositions (2.5) and (2.6).

(3.11) **Corollary.** *Let M be an R-complex. The next conditions are equivalent.*

(i) *M is pure-minimal.*

(ii) *The zero complex is the only pure-acyclic degreewise pure subcomplex of M.*

(iii) *In a degreewise pure exact sequence $0 \longrightarrow L \longrightarrow M \xrightarrow{\beta} N \longrightarrow 0$ the morphism β is a pure quasi-isomorphism if and only if it is an isomorphism.* □

(3.12) **Corollary.** *Let M be an R-complex. The next conditions are equivalent.*

(i) *M is split-minimal.*

(ii) *The zero complex is the only contractible degreewise split subcomplex of M.*

(iii) *In a degreewise split exact sequence $0 \longrightarrow L \longrightarrow M \xrightarrow{\beta} N \longrightarrow 0$ the morphism β is a homotopy equivalence if and only if it is an isomorphism.* □

We now show that split-minimality and pure-minimality coincide in standard settings while we already saw in Example (3.9) that a (split-)minimal complex of flat modules need not be pure-minimal.

(3.13) **Theorem.** *Let M be an R-complex. Under each of the conditions* (a)–(d) *below, every pure-acyclic pure subcomplex of M is contractible and a split subcomplex; in particular, the complex M is split-minimal if and only if it is pure-minimal.*

(a) *R is left noetherian; M is a complex of injective R-modules.*
(b) *R is left perfect; M is a complex of projective R-modules.*
(c) *R is semi-perfect; M is a complex of finitely generated projective R-modules.*
(d) *R is left noetherian; M is a complex of finitely generated projective R-modules.*

Proof. Let P be a pure-acyclic pure subcomplex of M and consider the pure exact sequence

$$(*) \qquad 0 \longrightarrow P \longrightarrow M \to M/P \longrightarrow 0\,.$$

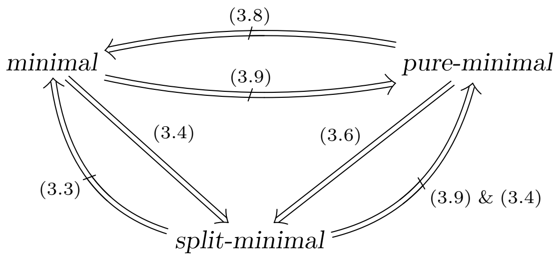
Under any one of the assumptions (a)–(d) the sequence is degreewise split exact by Lemma (1.5). Applied to the exact sequences $0 \to \mathrm{Z}_i(P) \to P_i \to \mathrm{Z}_{i-1}(P) \to 0$ the same lemma shows that P is contractible. Now it follows from Proposition (2.6) that the sequence $(*)$ splits.

By Remark (3.6) every pure-minimal complex is split-minimal, and the argument above shows that the converse holds under each of the conditions (a)–(d). $\square$

(3.14) **Corollary.** *Let R and M be as in* (3.13)(a), (b), *or* (c). *The R-complex M is pure-minimal if and only if it is split-minimal if and only if it is minimal.*

Proof. Combine Proposition (3.4) and Theorem (3.13). $\square$

(3.15) The diagram below summarizes the (non-)implications among the notions of minimality considered in this section. We stress that while the three notions agree under each of the assumptions (a), (b), or (c) in Theorem (3.13), the examples that lie behind the non-implications deal with semi-flat complexes over PIDs.



4. Minimality of acyclic complexes

The zero complex is minimal as can be; this short section complements the preceding one by spelling out what the flavors of minimality mean for acyclic complexes.

We start by noticing that the Dold complex from Example (1.7) is an acyclic complex of projective and injective modules which is both minimal and pure-minimal, as $\mathbb{Z}/4\mathbb{Z}$ has no non-trivial pure submodule. Thus, nonzero minimal and pure-minimal acyclic complexes exist over quasi-Frobenius rings.

(4.1) **Proposition.** *The zero complex is the only contractible split-minimal complex. In particular, the zero complex is the only*

- *acyclic split-minimal semi-injective complex,*
- *acyclic split-minimal semi-projective complex,*

- *pure-acyclic split-minimal complex of pure-injective modules,*
- *pure-acyclic split-minimal complex of pure-projective modules.*

Proof. The first assertion is immediate from Definition (3.2) and the remaining follow from [2, 1.3.P and 1.3.I], (1.9), and (1.10). □

(4.2) **Example.** Assume that R is semi-simple; that is, every acyclic R-complex is contractible. It follows from Proposition (4.1) that the zero complex is the only acyclic split-minimal R-complex. Even more, an R-complex M is split-minimal if and only if the zero complex is the only acyclic subcomplex of M. Indeed, the "if" is trivial and the "only if" follows from Proposition (2.6).

(4.3) **Proposition.** *The zero complex is the only pure-acyclic pure-minimal complex; in particular it is the only acyclic pure-minimal semi-flat complex.*

Proof. The first assertion is immediate from Definition (3.5) and the second follows from [8, thm. 7.3]. □

(4.4) **Example.** Assume that R is von Neumann regular; that is, every acyclic R-complex is pure-acyclic; see Theorem (1.11). It follows from Proposition (4.3) that the zero complex is the only acyclic pure-minimal R-complex. (In fact, this property characterizes von Neumann regular rings; see Corollary (5.2).) Even more, an R-complex M is pure-minimal if and only if the zero complex is the only acyclic subcomplex of M. Here "only if" follows from Proposition (2.5) and "if" is clear.

For work in the derived category of chain complexes—computation of derived functors for example—the emphasis is on distinguished complexes of injective (projective or flat) modules, namely the semi-injective (-projective or -flat) complexes. The next result shows that the noetherian hypothesis in Theorem (3.13)(a), so to speak, does not impact work in the derived category.

(4.5) **Proposition.** *Let I be a semi-injective R-complex. The following conditions are equivalent:*

(i) *I is minimal.*
(ii) *I is split-minimal.*
(iii) *I is pure-minimal.*
(iv) *The zero complex is the only acyclic subcomplex of I.*

Proof. In view of Proposition (3.4)(a) and Remark (3.6) it suffices to show that (i) implies (iv). Assume that I is minimal and let A be an acyclic subcomplex. As I is semi-injective, the quasi-isomorphism $\pi\colon I \to I/A$ has a left inverse up to homotopy. That is, there is a morphism $\gamma\colon I/A \to I$ such that $\gamma\pi$ is homotopic to 1^I. As I is minimal, it follows that $\gamma\pi$ is an isomorphism and, therefore, $A = 0$. □

Pure-minimal semi-flat complexes have a similar characterization.[3]

(4.6) **Proposition.** *Let F be a semi-flat R-complex. The following conditions are equivalent:*

(i) *F is pure-minimal.*

[3] When they exist, minimal semi-projective complexes are characterized by having only the trivial acyclic quotient complex.

(*ii*) *The zero complex is the only acyclic pure subcomplex of F.*

Proof. It is clear that (*ii*) implies (*i*). For the converse, let P be an acyclic pure subcomplex of F. It follows from [8, prop. 6.2] that P is semi-flat, hence P is pure acyclic by [8, thm. 7.3] and, therefore, $P = 0$. $\square$

5. Pure-minimal replacements

Every chain complex M has a minimal, hence pure-minimal, semi-injective resolution $M \xrightarrow{\simeq} I$; see Proposition (4.5). In particular, M and I are isomorphic in the derived category. We proceed to show that every complex is isomorphic, in the derived category, to a pure-minimal semi-flat complex.

The gist of the next theorem, which is our central construction, is that every complex has a pure-acyclic pure subcomplex, such that the associated quotient is pure-minimal.

(5.1) **Theorem.** *Let M be an R-complex. There is a pure exact sequence in $\mathcal{C}(R)$*

$$0 \longrightarrow P \longrightarrow M \longrightarrow M/P \longrightarrow 0$$

with P pure-acyclic and M/P pure-minimal. Consequently, the map $M \longrightarrow M/P$ is a pure quasi-isomorphism.

Proof. Consider the set of all pure-acyclic pure subcomplexes of M, ordered by containment. Let Λ be a chain in this set, and let U be the union $\operatorname{colim}_{A\in\Lambda} A$. It is standard that U is a subcomplex of M and we proceed to show that it is a pure subcomplex and pure-acyclic.

First we verify that U is a pure subcomplex. Let F be a bounded complex of finitely presented R-modules. For every $A \in \Lambda$ there is an exact sequence

$$0 \longrightarrow \hom_{\mathcal{C}(R)}(F, A) \longrightarrow \hom_{\mathcal{C}(R)}(F, M) \longrightarrow \hom_{\mathcal{C}(R)}(F, M/A) \longrightarrow 0\,.$$

Recall, e.g. from [8, thm. 4.5], that as Λ is filtered there is a natural isomorphism,

$$\operatorname*{colim}_{A\in\Lambda} \hom_{\mathcal{C}(R)}(F, -) \xrightarrow{\cong} \hom_{\mathcal{C}(R)}(F, \operatorname*{colim}_{A\in\Lambda}(-))\,.$$

Using this, along with the fact that filtered colimits are exact, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname*{colim}_{A\in\Lambda} \hom_{\mathcal{C}(R)}(F, A) \arrow[r] \arrow[d, "\cong"] & \operatorname*{colim}_{A\in\Lambda} \hom_{\mathcal{C}(R)}(F, M) \arrow[r] \arrow[d, "\cong"] & \operatorname*{colim}_{A\in\Lambda} \hom_{\mathcal{C}(R)}(F, M/A) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \hom_{\mathcal{C}(R)}(F, \operatorname*{colim}_{A\in\Lambda} A) \arrow[r] & \hom_{\mathcal{C}(R)}(F, \operatorname*{colim}_{A\in\Lambda} M) \arrow[r] & \hom_{\mathcal{C}(R)}(F, \operatorname*{colim}_{A\in\Lambda} M/A) &
\end{tikzcd}$$

It follows from a simple diagram chase that

$$\hom_{\mathcal{C}(R)}(F, \operatorname*{colim}_{A\in\Lambda} M) \longrightarrow \hom_{\mathcal{C}(R)}(F, \operatorname*{colim}_{A\in\Lambda} M/A) \longrightarrow 0$$

is exact. In view of the canonical isomorphisms

$$M/\operatorname*{colim}_{A\in\Lambda} A \;\cong\; \operatorname*{colim}_{A\in\Lambda} M/\operatorname*{colim}_{A\in\Lambda} A \;\cong\; \operatorname*{colim}_{A\in\Lambda} M/A$$

it thus follows that $U = \operatorname{colim}_{A\in\Lambda} A$ is a pure subcomplex of M.

Next we argue that U is pure-acyclic. Let F be a finitely presented R-module; we have to show that $\mathrm{Hom}_R(F,U)$ is acyclic. Since the functor $\mathrm{Hom}_R(F,-)$ preserves filtered colimits, one has

$$\mathrm{Hom}_R(F,U) \;=\; \mathrm{Hom}_R(F,\operatorname*{colim}_{A\in\Lambda} A) \;\cong\; \operatorname*{colim}_{A\in\Lambda}\mathrm{Hom}_R(F,A)\,.$$

As each complex A is pure-acyclic, the complexes $\mathrm{Hom}_R(F,A)$ are acyclic. Finally, $\mathrm{colim}_{A\in\Lambda}(-)$ preserves acyclicity and so $\mathrm{Hom}_R(F,U)$ is acyclic.

By Zorn's lemma, there exists a maximal pure-acyclic pure subcomplex P of M. To show that M/P is pure-minimal, let $P'/P \subseteq M/P$ be a pure-acyclic pure subcomplex. Consider the commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P \arrow[r, equal] \arrow[d] & P \arrow[d] & & \\
0 \arrow[r] & P' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/P' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P'/P \arrow[r] \arrow[d] & M/P \arrow[r] \arrow[d] & M/P' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The bottom row and the middle column are pure sequences in $\mathcal{C}(R)$ by the assumptions and by what we have shown above. Hence, for every bounded complex F of finitely presented R-modules, application of $\mathrm{hom}_{\mathcal{C}(R)}(F,-)$ yields another commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathrm{hom}_{\mathcal{C}(R)}(F,P) \arrow[r, equal] \arrow[d] & \mathrm{hom}_{\mathcal{C}(R)}(F,P) \arrow[d] & & \\
0 \arrow[r] & \mathrm{hom}_{\mathcal{C}(R)}(F,P') \arrow[r] \arrow[d] & \mathrm{hom}_{\mathcal{C}(R)}(F,M) \arrow[r] \arrow[d] & \mathrm{hom}_{\mathcal{C}(R)}(F,M/P') \arrow[d, equal] & \\
0 \arrow[r] & \mathrm{hom}_{\mathcal{C}(R)}(F,P'/P) \arrow[r] & \mathrm{hom}_{\mathcal{C}(R)}(F,M/P) \arrow[r] \arrow[d] & \mathrm{hom}_{\mathcal{C}(R)}(F,M/P') \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

A diagram chase shows that the morphism $\mathrm{hom}_{\mathcal{C}(R)}(F,M) \longrightarrow \mathrm{hom}_{\mathcal{C}(R)}(F,M/P')$ is surjective, whence P' is a pure subcomplex of M. To see that P' is pure-acyclic it is now by Proposition (2.5) sufficient to show that the canonical map $M \to M/P'$ is a pure quasi-isomorphism. This map is the composite of canonical maps

$$M \longrightarrow M/P \longrightarrow M/P'\,,$$

both of which are pure quasi-isomorphisms, again by Proposition (2.5). Now it follows from Lemma (2.7) that $M \to M/P'$ is a pure quasi-isomorphism. As P is a maximal pure acyclic pure subcomplex of M, one gets $P'/P = 0$, and it follows that M/P is pure-minimal.

An application of Proposition (2.5) now shows that the map $M \to M/P$ is a pure quasi-isomorphism. $\square$

(5.2) **Corollary.** *The following conditions are equivalent.*

(i) *R is von Neumann regular.*

(ii) *The zero complex is the only acyclic pure-minimal R-complex.*

(iii) *An R-complex M is pure-minimal if and only if the zero complex is the only acyclic subcomplex of M.*

Proof. It was noted in Example (4.4) that (iii) follows from (i) by way of Proposition (2.5), and (iii) clearly implies (ii). To see that (ii) implies (i), let M be an acyclic R-complex. By Theorem (5.1) there is a pure-acyclic subcomplex P of M such that the quotient M/P is pure-minimal and acyclic. By assumption, $M/P = 0$ so $M = P$ is pure-acyclic. It now follows from Theorem (1.11) that R is von Neumann regular. $\square$

In classical settings, such as in Theorem (3.13)(a,b,c), a complex decomposes as a direct sum of a minimal complex and a contractible one. These decompositions are recovered by Theorem (5.1)—together with Corollary (3.14)—which also yields a similar decomposition for complexes of finitely generated projective modules over noetherian rings.

(5.3) **Corollary.** *Let R and M be as in* (3.13)(a), (b), (c), *or* (d). *The exact sequence $0 \to P \to M \to M/P \to 0$ from* (5.1) *is split in $\mathcal{C}(R)$ and yields a decomposition $M \cong P \oplus (M/P)$, where P is contractible and M/P is pure-minimal.*

Proof. Immediate from Theorem (3.13). $\square$

(5.4) **Theorem.** *Let R be left noetherian and M be an R-complex with $\mathrm{H}(M)$ degreewise finitely generated and $\mathrm{H}_i(M) = 0$ for $i \ll 0$. There is a semi-projective resolution $L \xrightarrow{\simeq} M$ with L pure-minimal and degreewise finitely generated. Furthermore, for every such resolution $L \xrightarrow{\simeq} M$ one has*

$$\operatorname{proj.dim}_R M \;=\; \sup\{i \mid L_i \neq 0\}\,.$$

Proof. Notice first that if M is acyclic, then one can take $L = 0$. Assume that M is not acyclic, and let $L' \xrightarrow{\simeq} M$ be a semi-projective resolution with L' degreewise finitely generated; see [2, rmk. 1.7]. By Corollary (5.3) the complex L' has a pure-minimal summand L. The complex L is semi-projective, degreewise finitely generated, and isomorphic to M in the derived category. By [2, 1.4.P] there is a quasi-isomorphism $L \to M$.

Let $L \xrightarrow{\simeq} M$ be a semi-projective resolution with L pure-minimal and degreewise finitely generated. By [2, thm. 2.4.P] one has $\operatorname{proj.dim}_R M \leqslant \sup\{i \mid L_i \neq 0\}$, and equality holds trivially if $\operatorname{proj.dim}_R M = \infty$. If M has finite projective dimension n, then the complex $L_{\subseteq n} = 0 \to \mathrm{C}_n(L) \to L_{n-1} \to \cdots$ is by [2, thm. 2.4.P] a split subcomplex of L with contractible complement. It follows that the complement is zero, whence one has $L = L_{\subseteq n}$ and $\operatorname{proj.dim}_R M = n = \sup\{i \mid L_i \neq 0\}$. $\square$

(5.5) **Remark.** For any pure-minimal degreewise finitely generated semi-projective complex P, the proof of Theorem (5.4) yields $\operatorname{proj.dim}_R P = \sup\{i \mid P_i \neq 0\}$.

The construction in Theorem (5.1) also applies to yield a complex that detects flat dimension. We recall from [2, exa. 2.9.F] that a complex M need not have a semi-flat resolution that detects its flat dimension, hence we settle for a *semi-flat replacement* of M, i.e. a semi-flat complex isomorphic to M in the derived category.

(5.6) **Theorem.** *For every R-complex M there exists a pure-minimal semi-flat R-complex F isomorphic to M in the derived category. Furthermore, for every such complex F one has*

$$\operatorname{flat\,dim}_R M \;=\; \sup\{i \mid F_i \neq 0\}\,.$$

Proof. If M is acyclic, then one can take $F = 0$, so assume that M is not acyclic. Let $L \xrightarrow{\simeq} M$ be a semi-projective resolution. Theorem (5.1) yields a pure-acyclic pure subcomplex P of L such that the quotient $F = L/P$ is pure-minimal. As L is semi-flat, it follows from [8, prop. 6.2] that F is semi-flat as well. There are now quasi-isomorphisms

$$M \xleftarrow{\;\simeq\;} L \xrightarrow{\;\simeq\;} F\,,$$

so M and F are isomorphic in the derived category.

Let F be a pure-minimal semi-flat R-complex isomorphic to M in the derived category. By [2, thm. 2.4.F] one has $\operatorname{flat\,dim}_R M \leqslant \sup\{i \mid F_i \neq 0\}$, and equality holds trivially if $\operatorname{flat\,dim}_R M = \infty$. If M has finite flat dimension n, then $F_{\subseteq n} = 0 \to \mathrm{C}_n(F) \to F_{n-1} \to \cdots$ is a semi-flat R-complex isomorphic to M in the derived category; see [2, thm. 2.4.F]. Set $K = \operatorname{Ker}(F \to F_{\subseteq n})$ and consider the exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow F_{\subseteq n} \longrightarrow 0\,.$$

It is degreewise pure as $F_{\subseteq n}$ is a complex of flat modules. The morphism $F \longrightarrow F_{\subseteq n}$ is a pure quasi-isomorphism, see Example (2.3), so it follows from Proposition (2.5) that K is a pure-acyclic pure subcomplex of F. Since F is pure-minimal, this means $K = 0$. Hence one has $F = F_{\subseteq n}$ and $\operatorname{flat\,dim}_R M = n = \sup\{i \mid F_i \neq 0\}$. □

Minimal semi-projective resolutions are only known to exist for all R-complexes if R is a left perfect ring. We close this section with a characterization of such rings in terms of existence of pure-minimal semi-projective resolutions.

(5.7) **Theorem.** *The following conditions on R are equivalent.*

(i) *R is left perfect.*

(ii) *Every semi-flat R-complex is semi-projective.*

(iii) *Every R-complex has a pure-minimal semi-projective resolution.*

Proof. Every flat module over a perfect ring is projective, and a semi-flat complex of projective modules is semi-projective; see [8, thm. 7.8]. Thus (i) implies (ii). By Theorem (5.6) every R-complex M has a pure-minimal semi-flat replacement F. Assuming (ii) the complex F is semi-projective, and it follows from [2, 1.4.P] that there is a quasi-isomorphism $F \to M$. Thus (ii) implies (iii). To finish the proof, let F be a flat R-module with pure-minimal semi-projective resolution $\pi\colon P \xrightarrow{\simeq} F$. As $\mathrm{H}(\pi)\colon\ \mathrm{H}(P) \to \mathrm{H}(F) = F$ is an isomorphism, π is surjective, and it follows from (1.2) that $K = \operatorname{Ker}\pi$ is a degreewise pure subcomplex of P. Since P and F are semi-flat complexes, π is a pure quasi-isomorphism, see Example (2.3). From Corollary (3.11) it now follows that π is an isomorphism. Thus every flat R-module is projective, whence R is left perfect. □

Appendix. Sufficient conditions for acyclicity

We collect a few technical results that are useful for establishing acyclicity of Hom and tensor product complexes. The results complement and improve those in [7, sec. 2]; the proofs extend and dualize an argument by Emmanouil [11].

(A.1) **Proposition.** *Let M and N be R-complexes. The complex* $\operatorname{Hom}_R(M,N)$ *is acyclic if the following conditions are satisfied.*

(a) $\operatorname{Hom}_R(M_i,N)$ *is acyclic for every* $i \in \mathbb{Z}$, *and*
(b) $\operatorname{Hom}_R(\mathrm{C}_i(M),N)$ *is acyclic for every* $i \ll 0$.

Proof. Emmanouil's argument for [11, lem. 2.6] can be adapted to apply; see also the argument for the dual result Proposition (A.3) below. $\square$

(A.2) **Corollary.** *Let L be an R°-complex and M be an R-complex. The complex $L \otimes_R M$ is acyclic if the next conditions are satisfied.*

(a) $L \otimes_R M_i$ *is acyclic for every* $i \in \mathbb{Z}$, *and*
(b) $L \otimes_R \mathrm{C}_i(M)$ *is acyclic for every* $i \ll 0$.

Proof. Recall that the complex $L \otimes_R M$ is acyclic if and only if the dual complex $\operatorname{Hom}_{\mathbb{Z}}(L \otimes_R M, \mathbb{Q}/\mathbb{Z})$ is acyclic. The result now follows from Proposition (A.1) by way of Hom–tensor adjunction. $\square$

(A.3) **Proposition.** *Let M and N be R-complexes. The complex* $\operatorname{Hom}_R(M,N)$ *is acyclic if the following conditions are satisfied.*

(a) $\operatorname{Hom}_R(M,N_i)$ *is acyclic for every* $i \in \mathbb{Z}$, *and*
(b) $\operatorname{Hom}_R(M,\mathrm{Z}_i(N))$ *is acyclic for every* $i \gg 0$.

Proof. It is well-known, for example from [7, lem. 2.5], that condition (a) implies that the complex $\operatorname{Hom}_R(M,N_{<m})$ is acyclic for every m. From an application of $\operatorname{Hom}_R(M,-)$ to the degreewise split exact sequence $0 \to N_{<m} \to N \to N_{\geqslant m} \to 0$ it follows that it is sufficient to prove that $\operatorname{Hom}_R(M,N_{\geqslant m})$ is acyclic for some integer m. Thus, without loss of generality assume that $N_i = 0$ holds for $i \ll 0$ and that $\operatorname{Hom}_R(M,\mathrm{Z}_i(N))$ is acyclic for every $i \in \mathbb{Z}$.

A homomorphism $M \to N$ is a cycle in $\operatorname{Hom}_R(M,N)$ if and only if it is a chain map and a boundary if and only if it is null-homotopic. Let $\varphi\colon M \to N$ be a chain map; after shifting and reindexing we may assume that φ has degree zero. The goal is to construct a homotopy from φ to 0, i.e. a family of R-module homomorphisms $\sigma_i\colon M_i \to N_{i+1}$ with $\varphi_i = \partial^N_{i+1}\sigma_i + \sigma_{i-1}\partial^M_i$. Evidently σ_i has to be zero for $i \ll 0$; this provides the basis for an induction argument. Fix n and assume that the desired homomorphisms σ_i have been constructed for $i \leqslant n-2$; assume further that a homomorphism $\tau_{n-1}\colon M_{n-1} \to N_n$ with $\varphi_{n-1} = \partial^N_n\tau_{n-1} + \sigma_{n-2}\partial^M_{n-1}$ has been constructed. The map τ_{n-1} may not have all the properties required of σ_{n-1}, but in the induction step it is modified to yield the desired σ_{n-1}. For $i \ll 0$ one takes $\tau_i = 0$. The next diagram depicts the data from the induction hypothesis.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+1} \arrow[r] \arrow[d, "\varphi_{n+1}"] & M_n \arrow[r] \arrow[d, "\varphi_n"] & M_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] \arrow[dl, "\tau_{n-1}"] & M_{n-2} \arrow[r] \arrow[d, "\varphi_{n-2}"] \arrow[dl, "\sigma_{n-2}"] & M_{n-3} \arrow[r] \arrow[d, "\varphi_{n-3}"] \arrow[dl, "\sigma_{n-3}"] & \cdots \\
\cdots \arrow[r] & N_{n+1} \arrow[r] & N_n \arrow[r] & N_{n-1} \arrow[r] & N_{n-2} \arrow[r] & N_{n-3} \arrow[r] & \cdots
\end{tikzcd}$$

In the induction step we need to construct homomorphisms

(1) $$\tau_n \colon M_n \longrightarrow N_{n+1} \quad \text{and} \quad \upsilon_{n-1} \colon M_{n-1} \longrightarrow N_n$$

such that τ_n and $\sigma_{n-1} = \tau_{n-1} + \upsilon_{n-1}$ satisfy

(2) $$\varphi_n \;=\; \partial^N_{n+1}\tau_n + \sigma_{n-1}\partial^M_n \quad \text{and} \quad \varphi_{n-1} \;=\; \partial^N_n\sigma_{n-1} + \sigma_{n-2}\partial^M_{n-1}\,.$$

In the next computation, the first equality holds as φ is a chain map, and the second follows from the assumption on τ_{n-1},

$$\partial^N_n(\varphi_n - \tau_{n-1}\partial^M_n) \;=\; (\varphi_{n-1} - \partial^N_n\tau_{n-1})\partial^M_n \;=\; \sigma_{n-2}\partial^M_{n-1}\partial^M_n \;=\; 0\,.$$

This shows that $\varphi_n - \tau_{n-1}\partial^M_n$ corestricts to a homomorphism $M_n \to \mathrm{Z}_n(N)$. It is elementary to verify that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+3} \arrow[r] \arrow[d] & M_{n+2} \arrow[r] \arrow[d, "\varphi_{n+1}\partial^M_{n+2}"] & M_{n+1} \arrow[r] \arrow[d, "\varphi_{n+1}"] & M_n \arrow[r] \arrow[d, "\varphi_n - \tau_{n-1}\partial^M_n"] & M_{n-1} \arrow[r] \arrow[d] & \cdots \\
& 0 \arrow[r] & \mathrm{Z}_{n+1}(N) \arrow[r, tail] & N_{n+1} \arrow[r] & \mathrm{Z}_n(N) \arrow[r] & 0 &
\end{tikzcd}$$

is commutative; that is, the vertical maps form a chain map $\varphi' \colon M \to N'$, where N' denotes the bottom row in the diagram. By the assumptions and [7, lem. 2.5] the complex $\mathrm{Hom}_R(M, N')$ is acyclic, so φ' is null-homotopic. In particular, there exist homomorphisms τ_n and υ_{n-1} as in (1) with

$$\varphi_n - \tau_{n-1}\partial^M_n \;=\; \partial^N_{n+1}\tau_n + \upsilon_{n-1}\partial^M_n\,.$$

It is now straightforward to verify that the identities in (2) hold. □

(A.4) **Corollary.** *Let L be an R°-complex and M be an R-complex. The complex $L \otimes_R M$ is acyclic if the next conditions are satisfied.*

(a) *$L_i \otimes_R M$ is acyclic for every $i \in \mathbb{Z}$, and*

(b) *$\mathrm{B}_i(L) \otimes_R M$ is acyclic for every $i \ll 0$.*

Proof. As in the proof of Corollary (A.2) it suffices to show that the complex

$$\mathrm{Hom}_{\mathbb{Z}}(L \otimes_R M, \mathbb{Q}/\mathbb{Z}) \;\cong\; \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}))$$

is acyclic. The cycles of $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z})$ have the form

$$\mathrm{Z}_i(\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z})) \;=\; \mathrm{Hom}_{\mathbb{Z}}(\mathrm{B}_{-i-1}(L), \mathbb{Q}/\mathbb{Z})\,.$$

Now apply Proposition (A.3). □

References

[1] Stephen T. Aldrich, Edgar E. Enochs, and Juan A. Lopez-Ramos, *Derived functors of Hom relative to flat covers*, Math. Nachr. **242** (2002), 17–26. MR1916846

[2] Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155. MR1117631

[3] Luchezar L. Avramov, Hans-Bjørn Foxby, and Stephen Halperin, *Differential graded homological algebra*, preprint (1994–2017).

[4] Luchezar L. Avramov and Alex Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85** (2002), no. 2, 393–440. MR1912056

[5] Silvana Bazzoni, Manuel Cortés Izurdiaga, and Sergio Estrada, *Periodic modules and acyclic complexes*, preprint arXiv:1704.06672 [math.RA].

[6] David J. Benson and Kenneth R. Goodearl, *Periodic flat modules, and flat modules for finite groups*, Pacific J. Math. **196** (2000), no. 1, 45–67. MR1797235

[7] Lars Winther Christensen, Anders Frankild, and Henrik Holm, *On Gorenstein projective, injective and flat dimensions—A functorial description with applications*, J. Algebra **302** (2006), no. 1, 231–279. MR2236602

[8] Lars Winther Christensen and Henrik Holm, *The direct limit closure of perfect complexes*, J. Pure Appl. Algebra **219** (2015), no. 3, 449–463. MR3279365

[9] Samuel Eilenberg, *Homological dimension and syzygies*, Ann. of Math. (2) **64** (1956), 328–336. MR0082489

[10] Samuel Eilenberg and J. A. Zilber, *Semi-simplicial complexes and singular homology*, Ann. of Math. (2) **51** (1950), 499–513. MR0035434

[11] Ioannis Emmanouil, *On pure acyclic complexes*, J. Algebra **465** (2016), 190–213. MR3537821

[12] Juan Ramon García Rozas, *Covers and envelopes in the category of complexes of modules*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 407, Chapman & Hall/CRC, Boca Raton, FL, 1999. MR1693036

[13] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162. MR2157133

[14] Tsit Yuen Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR1653294

[15] Agustí Roig, *Minimal resolutions and other minimal models*, Publ. Mat. **37** (1993), no. 2, 285–303. MR1249232

[16] Jan Šťovíček, *On purity and applications to coderived and singularity categories*, preprint, arXiv:1412.1615v1 [math.CT].

[17] Peder Thompson, *Minimal complexes of cotorsion-flat modules*, Math. Scand., to appear. Preprint arXiv:1702.02985 [math.AC].

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