Split objects with respect to a fully invariant short exact sequence in abelian categories

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Dedicated to the memory of Professor John Clark

Abstract – We introduce and investigate (dual) relative split objects with respect to a fully invariant short exact sequence in abelian categories. We compare them with (dual) relative Rickart objects, and we study their behaviour with respect to coproducts and classes all of whose objects are (dual) relative split. We also introduce and study (dual) strongly relative split objects. Applications are given to Grothendieck categories, module and comodule categories.

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Introduction

Right Rickart rings have the root in the work of Rickart [32] on certain Banach algebras. In an arbitrary ring-theoretic setting they were considered by Maeda [27], who defined them as rings in which the right annihilator of any element is generated by an idempotent. Independently, Hattori [20] defined right p.p. rings as rings in which every principal right ideal is projective. Later on, it turned out that the two concepts are actually equivalent, and their theory was further developed by many mathematicians. Some classical examples of right Rickart rings include right semihereditary rings, Baer rings and von Neumann regular rings.

A natural development of the theory of Rickart rings was towards module theory. Thus, Lee, Rizvi and Roman introduced and extensively studied Rickart modules and dual Rickart modules in a series of papers [24, 25, 26], which extended the theory of Baer modules [33] and dual Baer modules [23]. A further generalization was considered by Crivei and Kör [5], who investigated relative Rickart objects and dual relative Rickart objects in abelian categories (also, see [6, 7, 8]). If $M$ and $N$ are objects of an abelian category, then $N$ is called $M$-Rickart if for every morphism $f : M \to N$, $\ker(f)$ is a section, while $N$ is called dual $M$-Rickart if for every morphism $f : M \to N$, $\coker(f)$ is a retraction. Their motivation was to set a unified theory of relative Rickart objects with versatile applications, which allows one to deduce naturally properties of dual relative Rickart objects (by the duality principle), relative regular objects (which are relative Rickart and dual relative Rickart) in the sense of Dăscălescu, Năstăsescu, Tudorache and Dăuş [12, 13] as well as relative Baer objects (as particular relative Rickart objects) and dual relative Baer objects (by the duality principle) [5, 6, 7, 8]. In recent years, Rickart modules and dual Rickart modules were generalized to (dual) $t$-Rickart modules by Asgari and Haghnay [1], (dual) $T$-Rickart modules by Ebrahimi Atani, Khoramdel and Dolati Pish Hesari [15, 17] and (dual) $F$-inverse split modules by Ungor, Halicioglu.
and Harmanci [37, 38], the latter developing a theory using arbitrary fully invariant submodules.

In the present paper we study (dual) relative split objects with respect to a fully invariant short exact sequence in abelian categories. Note that any preradical \( r \) of an abelian category gives rise to a fully invariant short exact sequence \( 0 \rightarrow r(M) \rightarrow M \rightarrow M/r(M) \rightarrow 0 \) for every object \( M \). If \( M \) is an object and \( 0 \rightarrow F \xrightarrow{i} N \xrightarrow{d} C \rightarrow 0 \) is a fully invariant short exact sequence, then \( N \) is \( M \)-\( F \)-split if and only if for every morphism \( g : M \rightarrow N \), \( \ker(dg) \) is a section, while \( N \) is dual \( M \)-\( F \)-split if and only if for every morphism \( g : N \rightarrow M \), \( \text{coker}(gi) \) is a retraction. For \( M = N \), one has the notion of (dual) self-\( F \)-split object. We also introduce a strong version of (dual) relative split object, which generalizes (dual) strong Rickartness in the sense of [7, 8, 16, 41]. Relative split objects generalize all the above Rickart-type modules as well as extending modules. Note that \( N \) is \( M \)-Rickart if and only if \( N \) is \( M \)-0-split, while \( N \) is dual \( M \)-Rickart if and only if \( N \) is dual \( M \)-\( M \)-split. Moreover, \( M \) is (dual) self-\( F \)-split if and only if \( M \cong F \oplus C \) and \( C \) is self-Rickart (\( F \) is dual self-Rickart). When \( F = r(M) \) for some (pre)radical \( r \) of \( A \), this shows that self-\( F \)-split and dual self-\( F \)-split objects are some particular objects for which their (pre)torsion part splits off (see the splitting problem discussed by Chase [3], Goodearl [19], Kaplansky [21], Năstăsescu and Torrecillas [29] or Teply [36]). Applications are given to Grothendieck categories, module and comodule categories.

Our results will usually have two dual parts, out of which we only prove the first one, while the second one follows by the duality principle in abelian categories. For all technical or auxiliary results in abelian categories we have only stated and proved one of the two dual statements. Next we briefly mention some of the main results of the paper.

In Section 3 we show that an object \( M \) is strongly self-\( F \)-split if and only if \( M \) is self-\( F \)-split and every direct summand of \( M \) which contains \( F \) is fully invariant. Also, if \( \epsilon : M \rightarrow M' \) is an epimorphism, \( m : N' \rightarrow N \) is a monomorphism, the inclusion monomorphism \( u : F \cap N' \rightarrow N' \) is fully invariant and \( N \) is (strongly) \( M \)-\( F \)-split, then \( N' \) is (strongly) \( M'-(F \cap N') \)-split. In particular, the relative splitness property is well behaved with respect to direct summands.

In Section 4 we compare relative self-\( F \)-split and relative self-Rickart objects. We prove that \( M \) is (strongly) self-\( F \)-split if and only if \( M \cong F \oplus C \) and \( C \) is (strongly) self-Rickart. We also show that \( M \) is strongly self-\( F \)-split if and only if \( M \) is self-\( F \)-split and \( \text{End}_A(C) \) is abelian.
In Section 5 we deal with coproducts of relative split objects. If \( M \) and \( N \) are objects, \( N = \bigoplus_{k=1}^{n} N_k \) is a direct sum decomposition, and \( 0 \to F \to N \to C \to 0 \) is a fully invariant short exact sequence, then it is shown that \( N \) is (strongly) \( M \)-\( F \)-split if and only if \( N_k \) is (strongly) \( M \)-\((F \cap N_k)\)-split for every \( k \in \{1, \ldots, n\} \).

In Section 6 we study classes all of whose objects are self-\( F \)-split. We characterize spectral categories, locally finitely generated Grothendieck categories which are \( V \)-categories or regular categories, (semi)hereditary categories with enough projectives and co(semi)hereditary categories with enough injectives in terms of self-\( F \)-splitness or dual self-\( F \)-splitness of objects of certain classes.

Finally, in Section 7 we give some further applications to module and comodule categories. Among them, we show that every (strongly) extending module \( N \) is (strongly) self-\( F \)-split, where \( F \) is the second singular submodule of \( N \).

We note that the transfer via functors between abelian categories of the (dual) relative splitness of objects with respect to a fully invariant short exact sequence is studied in [4].

We shall use the following notation. For every morphism \( f : M \to N \) in an abelian category \( \mathcal{A} \) we denote its kernel, cokernel, coimage and image by \( \ker(f) : \ker(f) \to M \), \( \coker(f) : N \to \coker(f) \), \( \coim(f) : M \to \coim(f) \) and \( \text{im}(f) : \text{im}(f) \to N \) respectively. Since \( \mathcal{A} \) is an abelian category, one has \( \coim(f) \cong \text{im}(f) \). For a short exact sequence \( 0 \to A \to B \to C \to 0 \) in \( \mathcal{A} \), we sometimes write \( C = B/A \). Recall that a morphism \( f : M \to N \) is called a section (retraction) if there exists a morphism \( f' : N \to M \) such that \( f'f = 1_M \) (\( ff' = 1_N \)).

### Fully invariant short exact sequences

In this section we prepare the stage with some technical results concerning fully invariant short exact sequences in abelian categories, which will be needed later on. The module-theoretic concept of fully invariant submodule may be naturally generalized to categories as follows.

**Definition 2.1** ([7, Definition 2.2]). Let \( \mathcal{A} \) be an abelian category.

1. A kernel (monomorphism) \( i : K \to M \) in \( \mathcal{A} \) is called **fully invariant** if for every morphism \( h : M \to M \) in \( \mathcal{A} \), there exists a morphism \( \alpha : K \to K \) such that \( hi = i\alpha \).
(2) A cokernel (epimorphism) \(d : M \to C\) in \(\mathcal{A}\) is called fully coinvariant if for every morphism \(h : M \to M_i\), there exists a morphism \(\beta : C \to C\) such that \(dh = \beta d\).

(3) A short exact sequence \(0 \to A \xrightarrow{i} B \xrightarrow{d} C \to 0\) in \(\mathcal{A}\) is called fully invariant if \(i\) is fully invariant, or equivalently, \(d\) is fully coinvariant [7, Lemma 2.5].

Example 2.2 ([7, Example 2.3]). Consider the category \(\text{Mod}(R)\) of unitary right modules over a ring \(R\) with identity. Then a kernel \(i : K \to M\) is fully invariant if and only if for every endomorphism \(h : M \to M\), \(hi = i\alpha\) for some homomorphism \(\alpha : K \to K\) if and only if for every endomorphism \(h : M \to M\), \(h(K) = \text{Im}(hi) \subseteq \text{Im}(i) = K\) if and only if \(K\) is a fully invariant submodule of \(M\).

Let \(\mathcal{A}\) be an abelian category. Recall that a preradical \(r\) of \(\mathcal{A}\) is a subfunctor of the identity functor on \(\mathcal{A}\), that is, \(r : \mathcal{A} \to \mathcal{A}\) is a functor which assigns to each object \(A\) of \(\mathcal{A}\) a subobject \(r(A)\) such that every morphism \(A \to B\) induces a morphism \(r(A) \to r(B)\) by restriction (e.g., see [2, I.1]). The following proposition relates fully invariant short exact sequences and preradicals, and will be implicitly used, without further reference. It is an immediate generalization of [2, Proposition I.6.2] from module categories to abelian categories. It shows that there are plenty of fully invariant short exact sequences.

Proposition 2.3. Let \(\mathcal{A}\) be an abelian category. Then a short exact sequence \(0 \to F \to M \to C \to 0\) in \(\mathcal{A}\) is fully invariant if and only if there is a preradical \(r\) of \(\mathcal{A}\) such that \(r(M) = F\).

The following known lemma will be frequently used.

Lemma 2.4. [7, Lemma 2.4] Let \(\mathcal{A}\) be an abelian category. Then the composition of two fully invariant kernels in \(\mathcal{A}\) is a fully invariant kernel.

Proposition 2.5. Let \(\mathcal{A}\) be an abelian category. Let \(0 \to F \xrightarrow{i} M \xrightarrow{d} M/F \to 0\) be a fully invariant short exact sequence in \(\mathcal{A}\) and \(M = M_1 \oplus M_2\). Then \(F \cong (F \cap M_1) \oplus (F \cap M_2)\).

Proof. For \(l = 1, 2\), denote by \(w_l : M_l \to M\) the canonical injection, and by \(p_l : M \to M_l\) the canonical projection. Also, for \(l = 1, 2\), denote
by \( j_l : F \cap M_l \to M_l \) and \( k_l : F \cap M_l \to F \) the inclusion monomorphisms. The universal property of the coproduct yields the monomorphism \([k_1, k_2] : (F \cap M_1) \oplus (F \cap M_2) \to F\).

Since \( i : F \to M \) is fully invariant, there exist morphisms \( \alpha_1, \alpha_2 : F \to F \) such that \( u_1 p_1 i = i \alpha_1 \) and \( u_2 p_2 i = i \alpha_2 \). Then \( i(\alpha_1 + \alpha_2) = u_1 p_1 i + u_2 p_2 i = i \), hence \( \alpha_1 + \alpha_2 = 1_F \), because \( i \) is a monomorphism. For \( l = 1, 2 \) the following square is a pullback:

\[
\begin{array}{ccc}
F \cap M_l & \xrightarrow{j_l} & M_l \\
\downarrow{k_l} & & \downarrow{u_l} \\
F & \xrightarrow{i} & M
\end{array}
\]

For \( l = 1, 2 \), by the pullback property, there exists a morphism \( \gamma_l : F \to F \cap M_l \) such that \( j_l \gamma_l = i_l \) and \( k_l \gamma_l = \alpha_l \). Then \( [k_1, k_2] [\gamma_1] = k_1 \gamma_1 + k_2 \gamma_2 = \alpha_1 + \alpha_2 = 1_F \). Hence \([k_1, k_2]\) is a retraction, and so it is an isomorphism. This shows that \( F \cong (F \cap M_1) \oplus (F \cap M_2) \). □

**Proposition 2.6.** Let \( A \) be an abelian category. Let \( 0 \to F \xrightarrow{i} M \xrightarrow{\delta} M/F \to 0 \) be a fully invariant short exact sequence and \( 0 \to G \xrightarrow{j} M \xrightarrow{\varphi} M/G \to 0 \) a short exact sequence in \( A \). Let \( u : F \cap G \to G \) be the inclusion monomorphism.

(i) Assume that the above second short exact sequence is also fully invariant.

Then the inclusion monomorphism \( j u : F \cap G \to M \) is fully invariant.

(ii) Assume that every morphism \( G \to G \) can be extended to a morphism \( M \to M \) in \( A \). Then \( u : F \cap G \to G \) is fully invariant.

**Proof.** Denote by \( k : F \cap G \to F \) the inclusion monomorphism.

(i) Let \( f : M \to M \) be a morphism in \( A \). Since \( i \) and \( j \) are fully invariant kernels, there exist morphisms \( \alpha : F \to F \) and \( \beta : G \to G \) such that \( i \alpha = f i \) and \( j \beta = f j \). The following commutative square is a pullback:

\[
\begin{array}{ccc}
F \cap G & \xrightarrow{u} & G \\
\downarrow{k} & & \downarrow{j} \\
F & \xrightarrow{i} & M
\end{array}
\]

We have \( i \alpha k = f i k = f j u = j \beta u \). By the pullback property, there exists a unique morphism \( \gamma : F \cap G \to F \cap G \) such that \( k \gamma = \alpha k \) and \( u \gamma = \beta u \). We have \( f j u = j \beta u = j u \gamma \). Hence \( j u : F \cap G \to M \) is fully invariant.
(ii) Let \( h : G \to G \) be a morphism in \( \mathcal{A} \). By hypothesis, \( h \) can be extended to a morphism \( f : M \to M \). Hence \( fj = jh \). Since \( i \) is fully invariant, there exists a morphism \( \alpha : F \to F \) such that \( i\alpha = fi \). Consider the pullback square from the proof of (i). We have \( iok = fik = fjhu \). By the pullback property, there exists a unique morphism \( \gamma : F \cap G \to F \cap G \) such that \( k\gamma = ok \) and \( u\gamma = hu \). This shows that \( u \) is fully invariant. \( \square \)

**Corollary 2.7.** Let \( \mathcal{A} \) be an abelian category. Let \( 0 \to F \to M \to M/F \to 0 \) be a fully invariant short exact sequence in \( \mathcal{A} \) and \( M = M_1 \oplus M_2 \). Then the inclusion monomorphism \( u : F \cap M_1 \to M_1 \) is fully invariant.

**Proof.** Note that every morphism \( M_1 \to M_1 \) can be extended to a morphism \( M \to M \) in \( \mathcal{A} \), and use Proposition 2.6 (ii). \( \square \)

**Proposition 2.8.** Let \( \mathcal{A} \) be an abelian category. Let \( g : B \to B' \) be a kernel and \( i : A \to B \) a morphism such that \( i' = gi : A \to B' \) is a fully invariant section. Then \( i \) is a fully invariant section.

**Proof.** Note that \( i \) is a section. One may construct the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{f} & \downarrow{d} & \downarrow{h} \\
A & \xrightarrow{i'} & B' \\
\end{array}
\]

where the rows are exact and the right square is a pushout. Then \( h \) is a kernel and \( d' \) is a fully coinvariant retraction. Now let \( f : B \to B \) be a morphism. There exists a morphism \( r : B' \to A \) such that \( ri' = 1_A \). Consider the morphism \( gfi' : B' \to B' \). Since \( d' \) is a fully coinvariant retraction, there exists a morphism \( \beta : C' \to C' \) such that \( d'gfi' = \beta d' \). Then \( hdfi = d'gfi = d'gfi'\beta = \beta d'i' = 0 \), whence we have \( dfi = 0 \), because \( h \) is a monomorphism. Then there exists a morphism \( \alpha : A \to A \) such that \( fi = i\alpha \). Hence \( i \) is a fully invariant section. \( \square \)

**Proposition 2.9.** Let \( \mathcal{A} \) be an abelian category. Let \( \begin{bmatrix} i & 0 \\ 0 & i' \end{bmatrix} : A \oplus A' \to B \oplus B' \) be a fully invariant kernel in \( \mathcal{A} \). Then \( i : A \to B \) and \( i' : A' \to B' \) are fully invariant kernels.

**Proof.** Clearly, \( i \) is a kernel. Let \( h : B \to B \) be a morphism. For the morphism \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : B \oplus B' \to B \oplus B' \) there exists a morphism \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} : A \oplus A' \to
$A \oplus A'$ such that
\[
\begin{bmatrix}
 h \\ 0 \\
 0 \\ i
\end{bmatrix}
\begin{bmatrix}
 i \\ 0 \\
 0 \\ i'
\end{bmatrix}
= \begin{bmatrix}
 a \\ b \\
 c \\ d
\end{bmatrix}.
\]
Then $hi = ia$, which shows that $i$ is fully invariant. □

**Proposition 2.10.** Let $\mathcal{A}$ be an $\mathcal{AB}4$ abelian category, i.e. an abelian category with exact coproducts. Let $(M_k)_{k \in K}$ be a family of objects of $\mathcal{A}$ such that $\text{Hom}_\mathcal{A}(M_k, M_l) = 0$ for every $k,l \in K$ with $k \neq l$. Then the short exact sequences $0 \to F_k \to M_k \to C_k \to 0$ are fully invariant for every $k \in K$ if and only if the induced short exact sequence $0 \to \bigoplus_{k \in K} F_k \to \bigoplus_{k \in K} M_k \to \bigoplus_{k \in K} C_k \to 0$ is fully invariant.

**Proof.** Straightforward. □

**F-split objects**

In the present section we introduce the most important concepts of the paper. They generalize (strong) relative Rickart objects and their duals [5], as shown in the introduction.

**Definition 3.1.** Let $\mathcal{A}$ be an abelian category. Let $M$ be an object of $\mathcal{A}$, and let $0 \to F \xrightarrow{j} N \xrightarrow{d} C \to 0$ be a fully invariant short exact sequence in $\mathcal{A}$. Then $N$ is called:

1. (strongly) $M$-$F$-split if for every morphism $g : M \to N$ the morphism $j : P \to M$ from the following pullback square is a (fully invariant) section:

\[
\begin{array}{ccc}
P & \xrightarrow{j} & M \\
\downarrow{f} & & \downarrow{g} \\
F & \xrightarrow{i} & N
\end{array}
\]

2. dual (strongly) $M$-$F$-split if for every morphism $g : N \to M$ the morphism $p : M \to Q$ from the following pushout square is a (fully coinvariant) retraction:

\[
\begin{array}{ccc}
N & \xrightarrow{d} & C \\
\downarrow{g} & & \downarrow{h} \\
M & \xrightarrow{p} & Q
\end{array}
\]

3. (strongly) self-$F$-split if $N$ is (strongly) $N$-$F$-split.
(4) **dual (strongly) self-$F$-split** if $N$ is dual (strongly) $N$-$F$-split.

The following lemma gives a useful characterization of (strong) relative $F$-splitness, which will be frequently used.

**Lemma 3.2.** Let $\mathcal{A}$ be an abelian category. Let $M$ be an object of $\mathcal{A}$, and let $0 \to F \xrightarrow{i} N \xrightarrow{d} C \to 0$ be a fully invariant short exact sequence in $\mathcal{A}$. Then:

(1) $N$ is (strongly) $M$-$F$-split if and only if for every morphism $g : M \to N$, $\ker(dg)$ is a (fully invariant) section.

(2) $N$ is dual (strongly) $M$-$F$-split if and only if for every morphism $g : N \to M$, $\coker(gi)$ is a (fully coinvariant) retraction.

**Proof.** (1) Consider the following diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{i} & M & \xrightarrow{dg} & C \\
\downarrow j & & \downarrow g & & \downarrow \\
F & \xrightarrow{i} & N & \xrightarrow{d} & C \\
\end{array}
\]

Since $i = \ker(d)$, the diagram may be completed to a pullback square $PMFN$ if and only if $j = \ker(dg)$ [28, Proposition 13.2]. Now the conclusion is clear.

**Remark 3.3.** Consider the module category $\text{Mod}(R)$, and let $M, N$ be right $R$-modules. We refer to notation and diagrams from Definition 3.1.

(1) Let $F$ be a fully invariant submodule of $N$. Then $N$ is (strongly) $M$-$F$-split if and only if for every homomorphism $g : M \to N$, $j : P \to M$ is a (fully invariant) section if and only if for every homomorphism $g : M \to N$, $P = g^{-1}(F)$ is a (fully invariant) direct summand of $M$.

(2) Let $F$ be a fully invariant submodule of $N$, or equivalently, let $C = N/F$ be a fully coinvariant factor module of $N$. Then $N$ is dual (strongly) $M$-$F$-split if and only if for every homomorphism $g : N \to M$, $p : M \to Q$ is a (fully coinvariant) retraction if and only if for every homomorphism $g : N \to M$, $\ker(p)$ is a (fully invariant) section if and only if for every homomorphism $g : N \to M$, $gi$ is a (fully invariant) section if and only if for every homomorphism $g : N \to M$, $g(F)$ is a (fully invariant) direct summand of $M$. 
Example 3.4. (1) Let $M$ and $N$ be objects of an abelian category $\mathcal{A}$. Obviously, $N$ is strongly $M$-$N$-split and $N$ is dual strongly $M$-$0$-split.

(2) Consider the module category $\text{Mod}(R)$. Let $M$ and $N$ be right $R$-modules. Then $N$ is $M$-$F$-split if and only if $N$ is $M$-$F$-inverse split in the sense of [37]. For $F = Z_M^2(N)$ (see the notation from the last section of our paper), a module $N$ is $M$-$F$-split if and only if $N$ is $M$-$T$-Rickart in the sense of [15]. Let us note that our categorical dual notion of dual (relative) $F$-splitness does not coincide with dual (relative) $F$-inverse splitness in the sense of [38] (e.g., apart from their definitions, compare our forthcoming Theorem 4.3 and [38, Theorem 2.2]). But for $F = \mathbb{Z}_M^2(N)$ (see the notation from the last section of our paper), a module $N$ is dual $M$-$F$-split if and only if $N$ is $M$-$T$-dual Rickart in the sense of [17].

Strong self-$F$-splitness and self-$F$-splitness are related by the following result.

**Theorem 3.5.** Let $\mathcal{A}$ be an abelian category. Let $0 \to F \xrightarrow{i} M \xrightarrow{\delta} C \to 0$ be a fully invariant short exact sequence in $\mathcal{A}$. Then:

(1) $M$ is strongly self-$F$-split if and only if $M$ is self-$F$-split and every direct summand of $M$ which contains $F$ is fully invariant.

(2) $M$ is dual strongly self-$F$-split if and only if $M$ is dual self-$F$-split and every direct summand of $M$ which is contained in $F$ is fully invariant.

**Proof.** (1) Assume that $M$ is strongly self-$F$-split. Clearly, $M$ is self-$F$-split. Now let $X$ be a direct summand of $M$ such that $F \subseteq X$. Write $M = X \oplus Y$ for some subobject $Y$ of $M$, and denote by $u : F \to X$ the inclusion monomorphism. Then $i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} : F \to X \oplus Y$. We claim that the following commutative square is a pullback:

$$
\begin{array}{ccc}
X & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & X \oplus Y \\
0 & \downarrow & 0 \\
F & \xrightarrow{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} & X \oplus Y
\end{array}
$$

To this end, let $\alpha : Z \to F$ and $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} : Z \to X \oplus Y$ be morphisms such that $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$. Then $\beta_2 = 0$ and $u \alpha = 0$. Hence $\alpha = 0$, because $u$ is a monomorphism. It is easy to check that $\beta_1 : Z \to X$ is the unique
morphism such that $0\beta_1 = \alpha$ and $[\beta_1]\beta_1 = [\beta_1 \beta_2]$. Hence the required square is a pullback. Since $M$ is strongly self-$F$-split, it follows that the upper horizontal morphism is a fully invariant section, hence $X$ is a fully invariant direct summand of $M$.

Conversely, assume that $M$ is self-$F$-split and every direct summand of $M$ which contains $F$ is fully invariant. Consider a pullback square:

$$
\begin{array}{ccc}
P & \xrightarrow{i} & M \\
\downarrow f & & \downarrow g \\
F & \xrightarrow{i} & M
\end{array}
$$

Since $M$ is self-$F$-split, $P$ is a direct summand of $M$. Since $i$ is fully invariant, there is a morphism $\alpha : F \to F$ such that $gi = i\alpha$. The pullback property yields a unique morphism $\gamma : F \to P$ such that $f\gamma = \alpha$ and $j\gamma = i$. Then $\gamma$ is a monomorphism, hence $F \subseteq P$. By hypothesis, $P$ must be a fully invariant direct summand of $M$. Hence $M$ is strongly self-$F$-split. \qed

Let $\mathcal{A}$ be an abelian category. Following the corresponding module-theoretic concepts [18, 42], we say that an object $M$ of $\mathcal{A}$ has the (strong) \textit{summand intersection property}, briefly SIP (SSIP), for a class $C$ of direct summands of $M$ if the intersection of any finite family (any family) of objects from $C$ belongs to $C$. Dually, an object $M$ of $\mathcal{A}$ has the (strong) \textit{summand sum property}, briefly SSP (SSSP), for a class $C$ of direct summands of $M$ if the sum of any finite family (any family) of objects from $C$ belongs to $C$.

It is known that self-Rickart objects have SIP [5, Corollary 3.10]. In order to deduce a similar result for self-$F$-split objects (but only for some direct summands), we first show the following proposition, which generalizes [37, Proposition 2.17].

\textbf{Proposition 3.6.} Let $\mathcal{A}$ be an abelian category. Let $0 \to F \xrightarrow{j} M \xrightarrow{\delta} C \to 0$ be a fully invariant short exact sequence in $\mathcal{A}$. Assume that $M$ is (strongly) self-$F$-split. Let $N$ be a direct summand of $M$ with $F \subseteq N$. Then for every (fully invariant) direct summand $K$ of $M$, $K \cap N$ is a (fully invariant) direct summand of $M$.

\textbf{Proof.} Denote by $u : F \to N$ the inclusion monomorphism. Write $M = N \oplus X = K \oplus Y$ for some subobjects $X, Y$ of $M$. As in the proof of
Theorem 3.5, the following commutative square is a pullback:

\[
\begin{array}{ccc}
N & \xrightarrow{j=\begin{bmatrix} 0 & 1 \end{bmatrix}} & M = N \oplus X \\
\downarrow & & \downarrow \\
F & \xrightarrow{i=\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & M = N \oplus X
\end{array}
\]

Let \( k : K \cap N \to K \) and \( n : K \cap N \to N \) be the inclusion monomorphisms. Consider the following commutative diagram:

\[
\begin{array}{ccc}
(K \cap N) \oplus Y & \xrightarrow{l=\begin{bmatrix} k & 0 \\ 1 & 0 \end{bmatrix}} & M = K \oplus Y \\
\downarrow & & \downarrow \\
N & \xrightarrow{i=\begin{bmatrix} 1 \\ j_1 \end{bmatrix}} & M = K \oplus Y
\end{array}
\]

The upper and the lower squares are clearly pullbacks, hence so is the outer rectangle [22, Lemma 5.1]. Denote \( f_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} : K \oplus Y \to K \oplus Y \). Glueing together the above 3 pullbacks, one obtains the following pullback square [22, Lemma 5.1]:

\[
\begin{array}{ccc}
(K \cap N) \oplus Y & \xrightarrow{l} & M \\
\downarrow & & \downarrow \\
F & \xrightarrow{i} & M
\end{array}
\]

Since \( M \) is (strongly) self-\( F \)-split, \( l = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \) is a (fully invariant) section. It follows that \( K \cap N \) is a (fully invariant) direct summand of \( K \) (by Proposition 2.9), and so a (fully invariant) direct summand of \( M \) (by Lemma 2.4).

\[\square\]

Corollary 3.7. Let \( \mathcal{A} \) be an abelian category. Let \( 0 \to F \to M \to C \to 0 \) be a fully invariant short exact sequence in \( \mathcal{A} \) such that \( M \) is (strongly) self-\( F \)-split. Then for every (fully invariant) direct summand \( K \) of \( M \), \( F \cap K \) is a (fully invariant) direct summand of \( M \). In particular, \( M \) has SIP for (fully invariant) direct summands containing \( F \).
Next we study whether relative $F$-splitness is preserved by direct summands. It turns out that the answer is positive, as a consequence of the following more general property.

**Theorem 3.8.** Let $\mathcal{A}$ be an abelian category. Let $0 \to F \overset{i}{\to} N \overset{d}{\to} C \to 0$ be a fully invariant short exact sequence in $\mathcal{A}$. Let $e : M \to M'$ be an epimorphism and let $m : N' \to N$ be a monomorphism in $\mathcal{A}$.

1. Assume that the inclusion monomorphism $u : F \cap N' \to N'$ is fully invariant and $N$ is (strongly) $M$-$F$-split. Then $N'$ is (strongly) $M'$-$(F \cap N')$-split.

2. Assume that the induced epimorphism $q : N/N' \to ((F + N')/N')$ is fully coinvariant and $N$ is dual (strongly) $M$-$F$-split. Then $N/N'$ is dual (strongly) $M/M'$-$(F + N')/N')$-split.

**Proof.** (1) Let $g : M' \to N'$ be a morphism in $\mathcal{A}$. Let $G = F \cap N'$ and consider the pullback of $u$ and $g$ to get morphisms $l : Q \to M'$ and $q : Q \to G$. Let $t : G \to F$ be the inclusion monomorphism. Consider the pullback of $i$ and $mge$ to get morphisms $j : P \to M$ and $f : P \to F$. The pullback property of the square $GN'FN$ yields a unique morphism $h : P \to G$ such that $th = f$ and $uh = gej$. The pullback property of the square $QMGN'$ yields a unique morphism $p : P \to Q$ such that $qp = h$ and $lp = ej$.

In this way one constructs the following commutative diagram:

```
\begin{center}
\begin{tikzcd}
\text{P} \arrow{d}{p} \arrow{r}{j} & \text{M} \arrow{d}{c} \\
\text{Q} \arrow[hookrightarrow]{d}{q} \arrow[hookrightarrow]{r}{l} & \text{M}' \arrow{d}{s} \\
\text{G} \arrow[hookrightarrow]{d}{t} & \text{N}' \arrow{d}{m} & \text{F} \arrow[hookrightarrow]{d}{i} \\
\text{N} & & \\
\end{tikzcd}
\end{center}
```

The rectangle $PMFN$ and the square $GN'FN$ are pullbacks, hence so is the rectangle $PMGN'$ [22, Lemma 5.1]. Since the square $QM'GN'$ is a pullback, so is the square $PMQM'$ [22, Lemma 5.1]. Since $N$ is (strongly) $M$-$F$-split, $j$ is a (fully invariant) section. It is easy to check that the square $PMQM'$
is also a pushout, hence \( l \) is a section. We may construct the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
P & \xrightarrow{j} & M & \xrightarrow{b} & D \\
\downarrow{p} & & \downarrow{e} & & \downarrow{d} \\
Q & \xrightarrow{l} & M' & \xrightarrow{d'} & D
\end{array}
\]

If \( j \) is a fully invariant section, then \( b = d'e \) is a fully coinvariant retraction. Hence \( d' \) is a fully coinvariant retraction by the dual of Proposition 2.8, and so \( l \) is a fully invariant section. Hence \( N' \) is (strongly) \( M'(F\cap N') \)-split. □

The following corollary generalizes [37, Proposition 2.12 and Theorem 4.2].

**Corollary 3.9.** Let \( A \) be an abelian category. Let \( M \) be an object and let \( 0 \to F \to N \to C \to 0 \) be a fully invariant short exact sequence in \( A \). Then \( N \) is (strongly) \( M \)-\( F \)-split if and only if for every direct summand \( M_1 \) of \( M \) and for every direct summand \( N_1 \) of \( N \), \( N_1 \) is (strongly) \( M_1 \)-\( (F \cap N_1) \)-split.

**Proof.** Note that the inclusion monomorphism \( u : F \cap N_1 \to N_1 \) is fully invariant by Corollary 2.7. Then use Theorem 3.8. □

The following corollary generalizes [37, Lemma 2.10].

**Corollary 3.10.** Let \( A \) be an abelian category. Let \( 0 \to F \to M \to M/F \to 0 \) be a fully invariant short exact sequence and \( 0 \to N \to M \to M/N \to 0 \) a short exact sequence in \( A \). Assume that every morphism \( N \to M \) can be extended to a morphism \( M \to M \) in \( A \). If \( M \) is (strongly) self-\( F \)-split, then \( N \) is (strongly) self-\( (F \cap N) \)-split.

**Proof.** By Proposition 2.6, the inclusion monomorphism \( u : F \cap N \to N \) is fully invariant. Then use Theorem 3.8. □

Let \( A \) be an abelian category and let \( r \) be a preradical of \( A \). Then one may define a preradical \( r^{-1} \) of \( A^{\text{op}} \) by \( r^{-1}(A) = A/r(A) \) for every object \( A \) of \( A \). Recall that \( r \) is called hereditary if \( r \) is a left exact functor, and cohereditary if \( r^{-1} \) is a right exact functor (e.g., see [2]).

**Corollary 3.11.** Let \( A \) be an abelian category and let \( r \) be a preradical of \( A \). Let \( e : M \to M' \) be an epimorphism and let \( m : N' \to N \) be a monomorphism in \( A \).
(1) Assume that \( r \) is hereditary and \( N \) is (strongly) \( M-r(N) \)-split. Then \( N' \) is (strongly) \( M'-r(N') \)-split.

(2) Assume that \( r \) is cohereditary and \( N \) is dual (strongly) \( M-r(N) \)-split. Then \( N/N' \) is dual (strongly) \( M/M'-r(N/N') \)-split.

Proof. If \( r \) is hereditary, then \( r(N) \cap N' = r(N') \) is fully invariant in \( N' \) [2, I.2.1]. Also, if \( r \) is cohereditary, then \( (r(N) + N')/N' = r(N/N') \) is fully invariant in \( N/N' \) [2, I.2.8]. Then use Theorem 3.8. □

**F-split objects versus Rickart objects**

Let us note that (strongly) self-F-split objects generalize (strongly) self-Rickart objects. In this section we explore more connections between these notions. Let us first recall the definition of (strongly) relative Rickart objects and their duals.

**Definition 4.1** ([5, 7]). Let \( M \) and \( N \) be objects of an abelian category \( \mathcal{A} \). Then \( N \) is called:

1. (strongly) \( M \)-Rickart if for every morphism \( f : M \to N \), \( \ker(f) \) is a (fully invariant) section.
2. dual (strongly) \( M \)-Rickart if for every morphism \( f : M \to N \), \( \coker(f) \) is a (fully coinvariant) retraction.
3. (strongly) self-Rickart if \( N \) is (strongly) \( N \)-Rickart.
4. dual (strongly) self-Rickart if \( N \) is dual (strongly) \( N \)-Rickart.

**Remark 4.2.** Let \( M, N \) be objects of an abelian category \( \mathcal{A} \). Then \( N \) is (strongly) \( M \)-Rickart if and only if \( N \) is (strongly) \( M \)-0-split, while \( N \) is (strongly) dual \( M \)-Rickart if and only if \( N \) is dual (strongly) \( M \)-\( M \)-split. Also, \( N \) is (strongly) self-Rickart if and only if \( N \) is (strongly) self-0-split, while \( N \) is (strongly) dual self-Rickart if and only if \( N \) is dual (strongly) self-\( N \)-split.

Now we establish a characterization of (strongly) self-F-split objects in connection with self-Rickart objects, which allows us to give a variety of examples. This theorem generalizes [37, Theorem 2.3 and Proposition 2.4], and is one of the key results of the paper.

**Theorem 4.3.** Let \( \mathcal{A} \) be an abelian category. Let \( 0 \to F \overset{j}{\to} M \overset{q}{\to} C \to 0 \) be a fully invariant short exact sequence in \( \mathcal{A} \).
(1) The following are equivalent:

(i) \( M \) is (strongly) self-F-split.

(ii) \( M \cong F \oplus C \) and \( C \) is (strongly) self-Rickart.

(2) The following are equivalent:

(i) \( M \) is dual (strongly) self-F-split.

(ii) \( M \cong F \oplus C \) and \( F \) is dual (strongly) self-Rickart.

Proof. (1) (i)⇒(ii) Assume that \( M \) is (strongly) self-F-split. Then the morphism \( j : P \to M \) from the following pullback square is a (fully invariant) section:

\[
\begin{array}{ccc}
P & \to & M \\
\downarrow^f & & \downarrow_{1_M} \\
F & \to & M \\
\end{array}
\]

Since \( f \) must be an isomorphism, it follows that \( i \) is a section. Hence \( M \cong F \oplus C \).

Let \( g : C \to C \) be a morphism in \( \text{A} \) with kernel \( k : K \to C \). Consider the following commutative square:

\[
\begin{array}{ccc}
F \oplus K & \to & F \oplus C \\
\downarrow^{[1 0]} & & \downarrow^{[1 0]}
\end{array}
\]

We claim that it is a pullback square. To this end, let \( \alpha : Z \to F \) and \( \left[ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right] : Z \to F \oplus C \) be morphisms in \( \text{A} \) such that \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right] = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \alpha \), that is, \( \beta_1 = \alpha \) and \( g \beta_2 = 0 \). Then there exists a unique morphism \( \gamma : Z \to K \) such that \( \beta_2 = k \gamma \). Consider the morphism \( \left[ \begin{array}{c} \beta_1 \\ \gamma \end{array} \right] : Z \to F \oplus K \). Then \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} \beta_1 \\ \gamma \end{array} \right] = \alpha \) and \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} \beta_1 \\ \gamma \end{array} \right] = \left[ \begin{array}{c} \beta_1 \\ \gamma \end{array} \right] \). If \( \left[ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right] : Z \to F \oplus K \) is another morphism such that \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right] = \alpha \) and \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right] = \left[ \begin{array}{c} \beta_1 \\ \gamma \end{array} \right] \), then \( \gamma_1 = \alpha = \beta_1 \) and \( k \gamma_2 = \beta_2 = k \gamma \). This implies that \( \gamma_2 = \gamma \), because \( k \) is a monomorphism, and so \( \left[ \begin{array}{c} \gamma_2 \\ \gamma \end{array} \right] = \left[ \begin{array}{c} \beta_1 \\ \gamma \end{array} \right] \). Hence the square is a pullback.

Since \( M \cong F \oplus C \) is (strongly) self-F-split, \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \) is a (fully invariant) section. It follows that \( k \) is a (fully invariant) section (by Proposition 2.9). Hence \( C \) is (strongly) self-Rickart.
(ii)⇒(i) Assume that $M \cong F \oplus C$ and $C$ is (strongly) self-Rickart. Let $f = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : F \oplus C \to F \oplus C$ be a morphism. Since $c : F \to C$ and $F$ is a fully invariant subobject of $M \cong F \oplus C$, it follows that $c = 0$, hence $f = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. Denote $k = \ker(d) : K \to C$. We have $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} [\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}] = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} [\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}]$, hence the following square is commutative:

$$
\begin{array}{ccc}
F \oplus K & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}} & F \oplus C \\
\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} & \downarrow & \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \\
F & \xrightarrow{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}} & F \oplus C
\end{array}
$$

We claim that it is a pullback square. To this end, let $\alpha : Z \to F$ and $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} : Z \to F \oplus C$ be morphisms in $A$ such that $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, that is, $a\beta_1 + b\beta_2 = \alpha$ and $d\beta_2 = 0$. Then there exists a unique morphism $\gamma : Z \to K$ such that $\begin{bmatrix} \beta_2 = k\gamma \end{bmatrix}$. Consider the morphism $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} : Z \to F \oplus K$. Then $\begin{bmatrix} a & b \\ \gamma \end{bmatrix} = \begin{bmatrix} a \beta_1 + b\gamma \end{bmatrix} = a\beta_1 + b\beta_2 = \alpha$ and $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$.

If $\begin{bmatrix} \gamma_2 \end{bmatrix} : Z \to F \oplus K$ is another morphism such that $\begin{bmatrix} a & b \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma_2 \end{bmatrix}$, then $\gamma_1 = \gamma_1 + k\gamma_2 = \beta_1 = \beta_2 = k\gamma$. This implies that $\gamma_2 = \gamma$, because $k$ is a monomorphism, and so $\begin{bmatrix} \gamma_2 \end{bmatrix} = \begin{bmatrix} \beta_2 \end{bmatrix}$. This shows that the square is a pullback.

Since $C$ is (strongly) self-Rickart, $k$ is a (fully invariant) section. It follows that $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ is a section. Hence $M \cong F \oplus C$ is self-$F$-split.

If $k$ is fully invariant, then we claim that $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ is also fully invariant. To this end, let $\begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix} : F \oplus C \to F \oplus C$ be a morphism. As above, we must have $h_3 = 0$. For the morphism $h_4 : C \to C$, there exists a morphism $\alpha : K \to K$ such that $h_4k = k\alpha$. Consider the morphism $\begin{bmatrix} h_1 & h_2k \end{bmatrix} : F \oplus K \to F \oplus K$. Then $\begin{bmatrix} h_1 & h_2 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} h_1 & h_2k \\ 0 & k \end{bmatrix}$, which shows that $\begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}$ is fully invariant. Hence, if $C$ is strongly self-Rickart, then $M \cong F \oplus C$ is strongly self-$F$-split.

**Corollary 4.4.** Let $M$ be an object of an abelian category $A$. Then $M$ is (strongly) self-Rickart if and only if for every split fully invariant short exact sequence $0 \to F \to M \to C \to 0$, $M$ is (strongly) self-$F$-split.

**Proof.** For the direct implication, let $0 \to F \to M \to C \to 0$ be a split fully invariant short exact sequence. Then $M \cong F \oplus C$. By [5, Corollary 2.11] ([7, Corollary 2.18]), $C$ is (strongly) self-Rickart. Hence $M$ is (strongly) self-$F$-split by Theorem 4.3.
Conversely, for $F = 0$, $M$ is (strongly) self-0-split, that is, $M$ is (strongly) self-Rickart.

Using Theorem 4.3, now we are able to present several examples illustrating our theory.

**Example 4.5.** Consider the abelian group $G = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_q$, where $p$ and $q$ are distinct primes and we denote $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for every natural number $n$. The subgroups of $G$ are 0, $G$, $H_1$, $H_2$, $H_3$ and $H_4$, where the last 4 subgroups have orders $q$, $p$, $pq$ and $p^2$ respectively. Note that all subgroups of $G$ are fully invariant, while 0, $G$, $H_1$ and $H_4$ are direct summands of $G$. Clearly, $G$ is self-$G$-split. But $G$ is not self-0-split, i.e., self-Rickart, because its direct summand $H_4 \cong \mathbb{Z}_{pq}$ is not self-Rickart [24, Theorem 5.6]. By Theorem 4.3, $G$ cannot be either self-$H_2$-split or self-$H_3$-split. Again by Theorem 4.3, $G \cong H_1 \oplus H_4$ is not self-$H_1$-split, because $H_4$ is not self-Rickart, while $G \cong H_1 \oplus H_4$ is strongly self-$H_4$-split, because $H_1 \cong \mathbb{Z}_q$ is strongly self-Rickart [7, Corollary 3.9]. Hence $G$ is strongly self-$F$-split for $F \in \{H_4, G\}$ and not self-$F$-split for $F \in \{0, H_1, H_2, H_3\}$. Similarly, $G$ is dual strongly self-$F$-split for $F \in \{H_1, G\}$ and not dual self-$F$-split for $F \in \{0, H_2, H_3, H_4\}$. This also shows that there are abelian groups which are self-$F$-split but not dual self-$F$-split, and viceversa.

**Example 4.6.** Consider the abelian group $G = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z} \oplus \mathbb{Q}$ for some prime $p$, and its subgroup $F \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. By [31, Lemma 1.9], $F$ is a fully invariant subgroup of $G$, because $\text{Hom}_\mathbb{Z}(\mathbb{Z}_p \oplus \mathbb{Z}_p, \mathbb{Z} \oplus \mathbb{Q}) = 0$. By Theorem 4.3, $G \cong F \oplus \mathbb{Z} \oplus \mathbb{Q}$ is self-$F$-split, because $\mathbb{Z} \oplus \mathbb{Q}$ is self-Rickart [26, Example 2.15]. On the other hand, $G$ is not strongly self-$F$-split, because $\mathbb{Z} \oplus \mathbb{Q}$ is not strongly self-Rickart [7, Theorem 3.6]. By Theorem 4.3, $G \cong F \oplus \mathbb{Z} \oplus \mathbb{Q}$ is dual self-$F$-split, because $F$ is clearly dual self-Rickart being semisimple. On the other hand, $G$ is not dual strongly self-$F$-split, because $F$ is not dual strongly self-Rickart [7, Theorem 3.6].

Let $A$ be an abelian category and let $r$ be a preradical of $A$. Recall that $r$ is called *idempotent* if $rr = r$, and *radical* if $rr^{-1} = 0$. Note that every hereditary preradical is idempotent, and every cohereditary preradical is a radical (e.g., see [2]).

**Example 4.7.** For every abelian group $G$, denote by $t(G)$ the set of elements of $G$ having finite order (torsion elements), and by $d(G)$ the sum of its divisible (injective) subgroups. Then $t$ is a hereditary radical and $d$
is an idempotent radical of the category \( \text{Ab} \) of abelian groups. Hence both \( t(G) \) and \( d(G) \) are fully invariant subgroups of \( G \).

Let \( G \) be a finitely generated abelian group. Then there is a direct sum decomposition \( G = t(G) \oplus F \) for some (torsionfree) subgroup \( F \) of \( G \). Then \( F \cong \mathbb{Z}^n \) for some \( n \in \mathbb{N} \), hence \( F \) is projective. It follows that \( G \) is self-\( t(G) \)-split by Theorem 4.3 and \([5, \text{Corollary 4.8}]\). By \([7, \text{Corollary 3.9}]\), \( G \) is strongly self-\( t(G) \)-split if and only if \( F \cong \mathbb{Z} \).

Now let \( G \) be an arbitrary abelian group. Then there is a direct sum decomposition \( G = d(G) \oplus R \) for some (reduced) subgroup \( R \) of \( G \). Since \( d(G) \) is injective, \( G \) is dual self-\( d(G) \)-split by Theorem 4.3 and \([5, \text{Corollary 4.8}]\). By \([7, \text{Corollary 3.9}]\), \( G \) is strongly dual self-\( d(G) \)-split if and only if \( d(G) \cong \bigoplus_{i \in I} G_i \) for some distinct \( G_i \) which are either \( \mathbb{Q} \) or \( \mathbb{Z}_{p_i} \) (the Prüfer \( p_i \)-group) for some primes \( p_i \) (\( i \in I \)).

**Example 4.8.** Consider the ring \( R = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \) and \( M = R_R \). Then \( F = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \) is an ideal of \( R \), hence it is a fully invariant submodule of \( M \). Moreover, we have \( M = F \oplus C \), where \( C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \). Since \( C \cong \mathbb{Z} \) is strongly self-Rickart \([7, \text{Corollary 3.9}]\), \( M \) is strongly self-\( F \)-split by Theorem 4.3. Note that \( F \) is self-Rickart \([26, \text{Example 1.2}]\). If \( F \) were dual self-Rickart, then it would be self-regular, and so \( \text{End}_R(F) \cong \mathbb{Z} \) would be a von Neumann regular ring, contradiction. Hence \( F \) is not dual self-Rickart, and so \( M \) is not dual self-\( F \)-split by Theorem 4.3.

**Example 4.9.** Consider the ring \( R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix} \) for some field \( K \) and \( M = R_R \). Then \( F = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \) is an ideal of \( R \), hence it is a fully invariant submodule of \( M \). Moreover, we have \( M = F \oplus C \), where \( C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \). Since \( C \cong K \) is strongly self-Rickart and the indecomposable \( F \) is strongly dual self-Rickart by \([25, \text{Example 3.9}]\), it follows that \( M \) is both strongly self-\( F \)-split and dual strongly self-\( F \)-split by Theorem 4.3.

Now we may give another result relating strong self-\( F \)-splitness and self-\( F \)-splitness. First recall that a ring \( R \) is called abelian if every idempotent element of \( R \) is central.

**Theorem 4.10.** Let \( \mathcal{A} \) be an abelian category. Let \( 0 \to F \to M \to C \to 0 \) be a fully invariant short exact sequence in \( \mathcal{A} \).

1. \( M \) is strongly self-\( F \)-split if and only if \( M \) is self-\( F \)-split and \( \text{End}_\mathcal{A}(C) \) is abelian.
2. \( M \) is dual strongly self-\( F \)-split if and only if \( M \) is dual self-\( F \)-split and \( \text{End}_\mathcal{A}(F) \) is abelian.
Proof. (1) Assume first that $M$ is strongly self-$F$-split. Then $M$ is self-$F$-split. Also, by Theorem 4.3, $M \cong F \oplus C$ and $C$ is strongly self-Rickart. Then $\text{End}_A(C)$ is abelian by [7, Proposition 2.14].

Conversely, assume that $M$ is self-$F$-split and $\text{End}_A(C)$ is abelian. Then $M \cong F \oplus C$ and $C$ is self-Rickart by Theorem 4.3. It follows that $C$ is strongly self-Rickart by [7, Proposition 2.14]. Hence $M$ is strongly self-$F$-split by Theorem 4.3. □

Example 4.11. Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$, $T = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \text{ is eventually constant}\}$ and $I = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \text{ is eventually zero}\} = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Consider the ring $R = \begin{bmatrix} T & I \\ 0 & T \\ I & T \end{bmatrix}$ and $M = R$. Then $F = \begin{bmatrix} T & I \\ 0 & T \\ I & T \end{bmatrix}$ is an ideal of $R$, hence it is a fully invariant submodule of $M$. Moreover, we have $M = F \oplus C$, where $C = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix}$. Since $C$ is projective, $C$ is self-Rickart by [5, Theorem 4.7], and so $M$ is self-$F$-split by Theorem 4.3. Also, $F$ is dual self-Rickart by [25, Example 4.1], and so $M$ is dual self-$F$-split by Theorem 4.3. Since $\text{End}_R(C)$ and $\text{End}_R(F) = \begin{bmatrix} T & I \\ 0 & T \end{bmatrix}$ are commutative, $M$ is strongly self-$F$-split and dual strongly self-$F$-split by Theorem 4.10.

We end this section with two more consequences of Theorem 4.3. The following result generalizes [37, Theorem 2.9].

Proposition 4.12. Let $A$ be an abelian category. Let $0 \to F \xrightarrow{i} M \xrightarrow{\phi} C \to 0$ be a fully invariant short exact sequence in $A$. Then the following are equivalent:

(i) $M$ is (strongly) self-$F$-split and for every pullback square

\[
\begin{array}{ccc}
P & \xrightarrow{j} & M \\
f \downarrow & & \downarrow g \\
F & \xrightarrow{i} & M
\end{array}
\]

the unique morphism $l : \text{Ker}(g) \to P$ such that $jl = \text{ker}(g)$ and $fl = 0$ is a (fully invariant) section.

(ii) $M$ is (strongly) self-Rickart and $M \cong F \oplus C$.

Proof. (i)⇒(ii) Assume that (i) holds. Let $g : M \to M$ be a morphism in $A$ with kernel $k : \text{Ker}(g) \to M$. Consider the pullback of $g$ and $i$ as in the above diagram. The pullback property yields a unique morphism $l : \text{Ker}(g) \to P$ such that $k = jl$ and $fl = 0$. By assumption, $j : P \to M$ is a
(fully invariant) section. It follows that \( k = jl \) is a (fully invariant) section (by Lemma 2.4). Hence \( M \) is (strongly) self-Rickart. Also, \( M \cong F \oplus C \) by Theorem 4.3.

(ii) \( \Rightarrow \) (i) Assume that (ii) holds. Consider a pullback square as above. Let \( k : \text{Ker}(g) \to M \) be the kernel of \( g : M \to M \). The pullback property yields a unique morphism \( l : \text{Ker}(g) \to P \) such that \( k = jl \) and \( fl = 0 \).

Since \( M \cong F \oplus C \) is (strongly) self-Rickart, so is \( C \) by [5, Corollary 2.11] ([7, Corollary 2.18]). Hence \( M \) is (strongly) self-\( F \)-split by Theorem 4.3. Since \( M \) is (strongly) self-Rickart, \( k = jl \) is a (fully invariant) section. It follows that \( l : \text{Ker}(g) \to P \) is a (fully invariant) section (by Proposition 2.8). \( \square \)

The next result generalizes [37, Proposition 2.16], without the quasi-projectivity condition.

**Proposition 4.13.** Let \( \mathcal{A} \) be an abelian category. Let \( 0 \to F \xrightarrow{i} M \xrightarrow{d} C \to 0 \) be a fully invariant short exact sequence in \( \mathcal{A} \). Then \( M \) is (strongly) self-\( F \)-split if and only if for every subobject \( K \) of \( M \) with \( K \subseteq F \), \( M/K \) is (strongly) self-(\( F/K \))-split.

**Proof.** Assume first that \( M \) is (strongly) self-\( F \)-split. By Theorem 4.3, \( M \cong F \oplus C \) and \( C \) is (strongly) self-Rickart. Let \( K \) be a subobject of \( M \) with \( K \subseteq F \). One may construct the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & F & \xrightarrow{i} & M & \xrightarrow{d} & C & \to & 0 \\
& & \downarrow{f} & & \downarrow{g} & & \downarrow{d} & & \downarrow{0} \\
0 & \to & F/K & \xrightarrow{i'} & M/K & \to & C & \to & 0
\end{array}
\]

where the rows are short exact sequences and \( f, g \) are the cokernels induced by the inclusions of \( K \) into \( F \) and \( M \). Since \( d = dg' \) is a fully coinvariant retraction, so is \( d' \) by the dual of Proposition 2.8. Then \( i' : F/K \to M/K \) is a fully invariant section, and so \( M/K \cong F/K \oplus C \). Hence \( M/K \) is (strongly) self-(\( F/K \))-split by Theorem 4.3.

The converse is clear. \( \square \)

**Coproducts of** \( F \)-**split objects**

In general arbitrary coproducts of relative split objects are not relative split objects, for instance, see [37, Example 3.1]. Nevertheless, we may give the following theorem if we impose some extra conditions on the coproducts.
THEOREM 5.1. Let $\mathcal{A}$ be an AB4 abelian category, and let $(M_k)_{k \in K}$ be a family of objects of $\mathcal{A}$. Let $0 \to F_k \xrightarrow{i_k} M_k \xrightarrow{d_k} C_k \to 0$ be fully invariant short exact sequences in $\mathcal{A}$ for every $k \in K$.

1. (i) Assume that $\text{Hom}_\mathcal{A}(M_k, M_l) = 0$ for every $k, l \in K$ with $k \neq l$. Then $\bigoplus_{k \in K} M_k$ is (strongly) self-$\bigoplus_{k \in K} F_k$-split if and only if $M_k$ is (strongly) self-$F_k$-split for every $k \in K$.

(ii) Then $\bigoplus_{k \in K} M_k$ is strongly self-$\bigoplus_{k \in K} F_k$-split if and only if $M_k$ is strongly self-$F_k$-split for every $k \in K$ and $\text{Hom}_\mathcal{A}(C_k, C_l) = 0$ for every $k, l \in K$ with $k \neq l$.

2. (i) Assume that $\text{Hom}_\mathcal{A}(M_k, M_l) = 0$ for every $k, l \in K$ with $k \neq l$. Then $\bigoplus_{k \in K} M_k$ is dual (strongly) self-$\bigoplus_{k \in K} F_k$-split if and only if $M_k$ is dual (strongly) self-$F_k$-split for every $k \in K$.

(ii) Then $\bigoplus_{k \in K} M_k$ is dual strongly self-$\bigoplus_{k \in K} F_k$-split if and only if $M_k$ is dual strongly self-$F_k$-split for every $k \in K$ and $\text{Hom}_\mathcal{A}(F_k, F_l) = 0$ for every $k, l \in K$ with $k \neq l$.

PROOF. (1) (i) The direct implication follows by Proposition 2.10 and Corollary 3.9.

Conversely, assume that $M_k$ is (strongly) self-$F_k$-split for every $k \in K$. Let $g : \bigoplus_{k \in K} M_k \to \bigoplus_{k \in K} M_k$ be a morphism. Since $\text{Hom}_\mathcal{A}(M_k, M_l) = 0$, for every $k, l \in K$ with $k \neq l$, the matrix of $g$ has zero entries except for the entries $(k, k)$ with $k \in K$, which are some morphisms $g_k : M_k \to M_k$. Hence $g = \bigoplus_{k \in K} g_k$. Then we have pullback squares as follows:

\[
\begin{array}{ccc}
P_k & \xrightarrow{j_k} & M_k \\
\downarrow j_k & & \downarrow g_k \\
F_k & \xrightarrow{f_k} & M_k \\
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
EXAMPLE 5.2. (a) Consider the abelian group $G = G_1 \oplus G_2$, where $G_1 = \mathbb{Z}_q \oplus \mathbb{Z}$ and $G_2 = \mathbb{Z}_p$ for some distinct primes $p$ and $q$. By [31, Lemma 1.9], $F_1 \cong \mathbb{Z}_q$ is a fully invariant subgroup of $G_1$, because $\text{Hom}_\mathbb{Z}(\mathbb{Z}_q, \mathbb{Z}) = 0$. Clearly, $F_2 = 0$ is a fully invariant subgroup of $G_2$. By Theorem 4.3, $G_1 \cong F_1 \oplus \mathbb{Z}$ is strongly self-$F_1$-split and $G_2 = \mathbb{Z}_p$ is strongly self-$F_2$-split, because $\mathbb{Z}$ and $\mathbb{Z}_p$ are strongly self-Rickart [7, Corollary 3.9]. Note that $F_1 \oplus F_2 = \mathbb{Z}_q$ is a fully invariant subgroup of $G = G_1 \oplus G_2$, because $\text{Hom}_\mathbb{Z}(\mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_p) = 0$. By Theorem 4.3, $G$ is not self-$F_1 \oplus F_2$-split, because $\mathbb{Z} \oplus \mathbb{Z}_p$ is not self-Rickart [24, Example 2.5]. Note that $\text{Hom}_\mathbb{Z}(G_1, G_2) \neq 0$.

(b) Consider the abelian group $G = G_1 \oplus G_2$, where $G_1 = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_q$ and $G_2 = \mathbb{Z}_p$ for some distinct primes $p$ and $q$. By [31, Lemma 1.9], $F_1 \cong \mathbb{Z}_{p^\infty}$ is a fully invariant subgroup of $G_1$, because $\text{Hom}_\mathbb{Z}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_q) = 0$. Clearly, $F_2 = \mathbb{Z}_p$ is a fully invariant subgroup of $G_2$. By Theorem 4.3, $G_1 \cong F_1 \oplus \mathbb{Z}_q$ is dual strongly self-$F_1$-split and $G_2 = \mathbb{Z}_p$ is dual strongly 0-split, because $F_1 \cong \mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_p$ are dual strongly self-Rickart [7, Corollary 3.9]. Note that $F_1 \oplus F_2 = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ is a fully invariant subgroup of $G = G_1 \oplus G_2$, because $\text{Hom}_\mathbb{Z}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p, \mathbb{Z}_q) = 0$. By Theorem 4.3, $G$ is not dual self-$F_1 \oplus F_2$-split, because $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ is not self-Rickart [25, Example 2.10]. Note that $\text{Hom}_\mathbb{Z}(G_2, G_1) \neq 0$.

When restricting to finite direct sum decompositions, we have the following result.

**Theorem 5.3.** Let $A$ be an abelian category. Let $M$ and $N$ be objects of $A$, $N = \bigoplus_{k=1}^n N_k$ a direct sum decomposition, and $0 \to F \to N \to C \to 0$ a fully invariant short exact sequence in $A$. Then:

1. $N$ is (strongly) $M$-$F$-split if and only if $N_k$ is (strongly) $M$-$(F \cap N_k)$-split for every $k \in \{1, \ldots, n\}$.

2. $N$ is dual (strongly) $M$-$F$-split if and only if $N_k$ is dual (strongly) $M$-$(F + N_k)/N_k$-split for every $k \in \{1, \ldots, n\}$.

**Proof.** (1) The direct implication follows by Corollary 3.9.

For the sake of clarity, we prove the converse for $n = 2$, the general case following inductively. Assume that $N_k$ is (strongly) $M$-$(F \cap N_k)$-split for $k = 1, 2$. By Proposition 2.5 we have $F \cong (F \cap N_1) \oplus (F \cap N_2)$. Denote by $i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : (F \cap N_1) \oplus (F \cap N_2) \to N_1 \oplus N_2$ the inclusion morphism. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : M \to N = N_1 \oplus N_2$ be a morphism in $A$. Consider the following
pullback squares:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{j_1} & M \\
\downarrow{f_1} & & \downarrow{g_1} \\
F \cap N_1 & \xrightarrow{i_1} & N_1 \\
\end{array} \quad \quad \begin{array}{ccc}
P_2 & \xrightarrow{j_2} & M \\
\downarrow{f_2} & & \downarrow{g_2} \\
F \cap N_2 & \xrightarrow{i_2} & N_2 \\
\end{array}
\]

Since \(N_1\) is (strongly) \(M-(F \cap N_1)\)-split and \(N_2\) is (strongly) \(M-(F \cap N_2)\)-split, \(j_1\) and \(j_2\) are (fully invariant) sections. Then there exists an epimorphism \(p_2: M \rightarrow P_2\) such that \(p_2j_2 = 1_{P_2}\). Consider the pullback of \(g_1j_2: P_2 \rightarrow N_1\) and \(i_1\) in order to get the outer part of the following commutative diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{u} & P_2 \\
\downarrow{w} & & \downarrow{j_2} \\
F \cap N_1 & \xrightarrow{i_1} & N_1 \\
\end{array}
\]

By Theorem 3.8, \(N_1\) is \(P_2-(F \cap N_1)\)-split, hence \(u\) is a (fully invariant) section. Since the lower square is a pullback, so is the upper square by [22, Lemma 5.1]. Also, there exists a unique morphism \(w: P \rightarrow P_1\) such that \(f_1w = v\) and \(j_1w = j_2u\).

We claim that the following square is a pullback:

\[
\begin{array}{ccc}
P & \xrightarrow{j_2u} & M \\
\downarrow{[v \ f_2u]} & & \downarrow{[g_1 \ g_2]} \\
(F \cap N_1) \oplus (F \cap N_2) & \xrightarrow{\begin{bmatrix} i_1 & 0 \\ 0 & i_2 \end{bmatrix}} & N_1 \oplus N_2 \\
\end{array}
\]

It is commutative, because we have

\[
\begin{bmatrix} i_1 & 0 \\ 0 & i_2 \end{bmatrix} \begin{bmatrix} v \\ f_2u \end{bmatrix} = \begin{bmatrix} i_1v \\ i_2f_2u \end{bmatrix} = \begin{bmatrix} g_1j_2u \\ g_2j_2u \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} j_2u.
\]

Now let \(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}: X \rightarrow (F \cap N_1) \oplus (F \cap N_2)\) and \(\beta: X \rightarrow M\) be morphisms such that \(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \beta\). Hence \(i_1\alpha_1 = g_1\beta\) and \(i_2\alpha_2 = g_2\beta\). The pullback properties of the first two squares from the proof yield unique morphisms \(\gamma_1: X \rightarrow P_1\) such that \(f_1\gamma_1 = \alpha_1\) and \(j_1\gamma_1 = \beta\), and \(\gamma_2: X \rightarrow P_2\)
such that $f_2 \gamma_2 = \alpha_2$ and $j_2 \gamma_2 = \beta$. Hence $j_1 \gamma_1 = j_2 \gamma_2$. Since the square $PP_2 P_1 M$ is a pullback, there exists a unique morphism $\gamma : X \to P$ such that $w \gamma = \gamma_1$ and $u \gamma = \gamma_2$. It follows that $[f_2 u] \gamma = [f_1 w] = [f_2 \gamma_2] = [\alpha_1]$ and $j_2 u \gamma = j_2 \gamma_2 = \beta$. For uniqueness, if there exists a morphism $\gamma' : X \to P$ such that $[f_2 u] \gamma' = [\alpha_1]$ and $j_2 u \gamma' = \beta$, then we have $j_2 u \gamma = j_2 u \gamma'$. Then $\gamma = \gamma'$, because $j_2$ and $u$ are monomorphisms. Thus, the required square is a pullback.

Finally, since $j_2$ and $u$ are (fully invariant) sections, so is $j_2 u$ (by Lemma 2.4). This shows that $N$ is (strongly) $M$-$F$-split.

**Corollary 5.4.** Let $A$ be an abelian category. Let $M$ and $N$ be objects of $A$, $N = \bigoplus_{k \in K} N_k$ a direct sum decomposition, and $0 \to F \to N \to C \to 0$ a fully invariant short exact sequence in $A$. Assume that $M$ is finitely generated. Then $N$ is (strongly) $M$-$F$-split if and only if $N_k$ is (strongly) $M$-$(F \cap N_k)$-split for every $k \in K$.

**Proof.** The direct implication follows by Corollary 3.9.

Conversely, assume that $N_k$ is (strongly) $M$-$(F \cap N_k)$-split for every $k \in K$. Let $g : M \to N = \bigoplus_{k \in K} N_k$ be a morphism in $A$. Since $M$ is finitely generated, we may write $g = lg'$ for some morphism $g' : M \to \bigoplus_{k \in F} N_k$ and inclusion morphism $l : \bigoplus_{k \in A} N_k \to N$, where $A$ is a finite subset of $K$.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{j} & M \\
\downarrow{j'} & & \downarrow{g'} \\
F \cap (\bigoplus_{k \in A} N_k) & \xrightarrow{i'} & \bigoplus_{k \in A} N_k \\
\downarrow{v'} & & \downarrow{i} \\
F & \xrightarrow{i} & N
\end{array}
\]

in which the lower and the upper rectangles are pullbacks. Then the outer rectangle is a pullback [22, Lemma 5.1]. By Corollary 2.7, $i'$ is a fully invariant kernel. By Theorem 5.3, $\bigoplus_{k \in A} N_k$ is (strongly) $M$-$(F \cap (\bigoplus_{k \in A} N_k))$-split, hence $j$ is a (fully invariant) section. It follows that $N$ is (strongly) $M$-$F$-split.

**Remark 5.5.** Note that in general Theorem 5.3 does not hold for arbitrary coproducts. For an example in the case $F = 0$ see [5, Example 3.3].
Let \( \mathcal{A} \) be an abelian category and let \( r \) be a preradical of \( \mathcal{A} \). Recall that an object \( A \) of \( \mathcal{A} \) is called \( r \)-torsion if \( r(A) = A \), and \( r \)-torsionfree if \( r(A) = 0 \). The class of \( r \)-torsion objects is closed under factor objects and coproducts, while the class of \( r \)-torsionfree objects is closed under subobjects and products. The preradical \( r \) is called superhereditary if \( r \) is hereditary and the class of \( r \)-torsion objects of \( \mathcal{A} \) is closed under products (e.g., see [2]). Imposing conditions SSIP (SSSP) or SIP (SSP), we may give the following result involving coproducts (products).

**Theorem 5.6.** Let \( \mathcal{A} \) be an abelian category and let \( r \) be a preradical of \( \mathcal{A} \).

1. Let \( M \) be an object of \( \mathcal{A} \) and let \( (N_k)_{k \in K} \) be a family of objects of \( \mathcal{A} \) having a coproduct.

   (i) Assume that \( M \) has SSIP for (fully invariant) direct summands containing \( r(M) \). Then \( \bigoplus_{k \in K} N_k \) is (strongly) \( M \)-\( r(\bigoplus_{k \in K} N_k) \)-split if and only if \( N_k \) is (strongly) \( M \)-\( r(N_k) \)-split for every \( k \in K \).

   (ii) Assume that \( K \) is finite and \( M \) has SIP for (fully invariant) direct summands containing \( r(M) \). Then \( \bigoplus_{k \in K} N_k \) is (strongly) \( M \)-\( r(\bigoplus_{k \in K} N_k) \)-split if and only if \( N_k \) is (strongly) \( M \)-\( r(N_k) \)-split for every \( k \in K \).

2. Let \( N \) be an object of \( \mathcal{A} \) and let \( (M_k)_{k \in K} \) be a family of objects of \( \mathcal{A} \) having a product.

   (i) Assume that \( r \) is superhereditary and \( N \) has SSSP for (fully invariant) direct summands contained in \( r(N) \). Then \( N \) is dual (strongly) \( \prod_{k \in K} M_k \)-\( r(N) \)-split if and only if \( N \) is dual (strongly) \( M_k \)-\( r(N) \)-split for every \( k \in K \).

   (ii) Assume that \( K \) is finite and \( N \) has SSP for (fully invariant) direct summands contained in \( r(N) \). Then \( N \) is dual (strongly) \( \prod_{k \in K} M_k \)-\( r(N) \)-split if and only if \( N \) is dual (strongly) \( M_k \)-\( r(N) \)-split for every \( k \in K \).

**Proof.** (1) (i) The direct implication follows by Corollary 3.9.

Conversely, assume that \( N_k \) is (strongly) \( M \)-\( r(N_k) \)-split for every \( k \in K \). Denote by \( i : r(\bigoplus_{k \in K} N_k) \rightarrow \bigoplus_{k \in K} N_k \) the inclusion monomorphism and by \( d : \bigoplus_{k \in K} N_k \rightarrow C \) its cokernel. Let \( g : M \rightarrow \bigoplus_{k \in K} N_k \) be a morphism in \( \mathcal{A} \). Consider the pullback of \( g \) and \( i \) in order to get morphisms \( f : P \rightarrow r(\bigoplus_{k \in K} N_k) \) and \( j : P \rightarrow M \). For every \( k \in K \), denote by
Split objects in abelian categories

$p_k : \bigoplus_{k \in K} N_k \to N_k$ the canonical projection and $g_k = p_k g : M \to N_k$. For every $k \in K$, denote by $i_k : r(N_k) \to N_k$ the inclusion monomorphism and by $d_k : N_k \to C_k$ its cokernel. For every $k \in K$, consider the following pullback diagram:

\[
\begin{array}{ccc}
P_k & \xrightarrow{j_k} & M \\
\downarrow{f_k} & & \downarrow{g_k} \\
r(N_k) & \xrightarrow{i_k} & N_k
\end{array}
\]

Let $u : r(M) \to M$ be the inclusion monomorphism. Since $r$ is a preradical, each morphism $g_k : M \to N_k$ induces a morphism $l_k : r(M) \to r(N_k)$ such that $g_k u = i_k l_k$. By the pullback property of the upper square it follows that $r(M) \subseteq P_k$ for every $k \in K$. Since each $N_k$ is (strongly) $M$-$r(N_k)$-split, each $j_k : P_k \to M$ is a (fully invariant) section. Note that $r(\bigoplus_{k \in K} N_k) = \bigoplus_{k \in K} r(N_k)$ [2, I.1.2]. It follows that $\text{Ker}(dg) = P = \bigcap_{k \in K} P_k = \bigcap_{k \in K} \text{Ker}(d_k g_k)$ and $M$ has SSIP for (fully invariant) direct summands containing $r(M)$, it follows that $j = \text{ker}(dg) : P \to M$ is a (fully invariant) section. Hence $\bigoplus_{k \in K} N_k$ is (strongly) $M$-$r(\bigoplus_{k \in K} N_k)$-split by Lemma 3.2.

(ii) This is similar to (i).

(2) Let us only note that if $r$ is superhereditary, then $r(\prod_{k \in K} M_k) = \prod_{k \in K} r(M_k)$ by definition and [2, I.1.2]. Then the proof is dual to (1). \hfill \square

**Corollary 5.7.** Let $\mathcal{A}$ be an abelian category, let $r$ be a preradical of $\mathcal{A}$, let $M_1, \ldots, M_k$ be objects of $\mathcal{A}$ and $l \in \{1, \ldots, n\}$. Then $\bigoplus_{k=1}^n M_k$ is (strongly) $M_l$-$r(\bigoplus_{k=1}^n M_k)$-split if and only if $M_k$ is (strongly) $M_l$-$r(M_k)$-split for every $k \in \{1, \ldots, n\}$.

**Proof.** This follows by Corollary 3.7 and Theorem 5.6. \hfill \square

**Classes all of whose objects are $F$-split**

In this section we study classes all of whose objects are $F$-split. We begin with a characterization of spectral abelian categories in terms of relative $F$-splitness. Recall that an abelian category $\mathcal{A}$ is called *spectral* if every short exact sequence in $\mathcal{A}$ splits.

**Theorem 6.1.** Let $\mathcal{A}$ be an abelian category.

(1) The following are equivalent:
(i) \(A\) is spectral.

(ii) \(A\) has enough injectives and \(N\) is \(M\)-\(F\)-split for every objects \(M\) and \(N\) of \(A\) and for every fully invariant subobject \(F\) of \(N\).

(iii) \(A\) has enough injectives and every object \(N\) of \(A\) is self-\(F\)-split for every fully invariant subobject \(F\) of \(N\).

(iv) \(A\) has enough injectives and \(N\) is \(M\)-\(F\)-split for every objects \(M\) and \(N\) of \(A\) with \(N\) injective and for every fully invariant subobject \(F\) of \(N\).

(v) \(A\) has enough injectives and every injective object \(N\) of \(A\) is self-\(F\)-split for every fully invariant subobject \(F\) of \(N\).

(2) The following are equivalent:

(i) \(A\) is spectral.

(ii) \(A\) has enough projectives and \(N\) is dual \(M\)-\(F\)-split for every objects \(M\) and \(N\) of \(A\) and for every fully invariant subobject \(F\) of \(N\).

(iii) \(A\) has enough projectives and every object \(N\) of \(A\) is dual self-\(F\)-split for every fully invariant subobject \(F\) of \(N\).

(iv) \(A\) has enough projectives and \(N\) is dual \(M\)-\(F\)-split for every objects \(M\) and \(N\) of \(A\) with \(M\) projective and for every fully invariant subobject \(F\) of \(N\).

(v) \(A\) has enough projectives and every projective object \(N\) of \(A\) is dual self-\(F\)-split for every fully invariant subobject \(F\) of \(N\).

Proof. (1) (i)⇒(ii)⇒(iii)⇒(v) and (ii)⇒(iv)⇒(v) are clear.

(v)⇒(i) Assume that \(A\) has enough injectives and every injective object \(N\) of \(A\) is self-\(F\)-split for every fully invariant subobject \(F\) of \(N\). Taking \(F = 0\), it follows that every injective object \(N\) of \(A\) is self-Rickart, hence \(A\) is spectral by [5, Theorem 4.1]. \(\square\)

The category \(\text{Mod}(R)\) is a locally finitely generated (i.e., it has a family of finitely generated generators) Grothendieck category with enough injectives and enough projectives. It is spectral if and only if \(R\) is semisimple [34, Chapter V, Proposition 6.7].

Corollary 6.2. Let \(R\) be a unitary ring. Then the following are equivalent:

(i) \(R\) is semisimple.
(ii) Every right $R$-module $N$ is (strongly) self-$F$-split for every fully invariant submodule $F$ of $N$.

(iii) Every injective right $R$-module $N$ is (strongly) self-$F$-split for every fully invariant submodule $F$ of $N$.

(iv) Every right $R$-module $N$ is dual (strongly) self-$F$-split for every fully invariant submodule $F$ of $N$.

(v) Every projective right $R$-module $N$ is dual (strongly) self-$F$-split for every fully invariant submodule $F$ of $N$.

**Proof.** We only discuss the strong versions of the equivalences (i)$\iff$(ii) $\iff$(iii), the remaining part following by Theorem 6.1 and by duality.

(i)$\Rightarrow$(ii) Assume that $R$ is semisimple. Then every right $R$-module is semisimple, hence for every right $R$-module $N$, every submodule of $N$ is fully invariant. Thus every right $R$-module is (weak) duo [31]. By Theorem 6.1, every right $R$-module is self-0-split, i.e., self-Rickart. Then every right $R$-module is strongly self-Rickart [7, Corollary 2.10]. Now using Theorem 4.3, every right $R$-module $N$ is strongly self-$F$-split for every fully invariant submodule $F$ of $N$.

(ii)$\Rightarrow$(iii) This is clear.

(iii)$\Rightarrow$(i) This follows by Theorem 6.1. $\square$

Let $C\mathcal{M}$ be the category of left comodules over a coalgebra $C$ over a field [11]. Then $C\mathcal{M}$ is a locally finite (i.e., it has a family of generators of finite length) Grothendieck category with enough injectives. It has enough projectives if and only if $C$ is left semiperfect [11, Theorem 3.2.3]. It is spectral if and only if $C\mathcal{M}$ is semisimple if and only if $C$ is cosemisimple. Now one can deduce the following corollary similarly to Corollary 6.2.

**Corollary 6.3.** Let $C$ be a coalgebra over a field. Then the following are equivalent:

(i) $C$ is cosemisimple.

(ii) Every left $C$-comodule $N$ is (strongly) self-$F$-split for every fully invariant submodule $F$ of $N$.

(iii) Every injective left $C$-comodule $N$ is (strongly) self-$F$-split for every fully invariant submodule $F$ of $N$.

(iv) $C$ is left semiperfect and every left $C$-comodule $N$ is dual (strongly) self-$F$-split for every fully invariant submodule $F$ of $N$. 

(v) $C$ is left semiperfect and every projective left $C$-comodule $N$ is dual (strongly) self-$F$-split for every fully invariant subcomodule $F$ of $N$.

Recall that a Grothendieck category $\mathcal{A}$ is called a $V$-category if every simple object of $\mathcal{A}$ is injective [14], and a regular category if every object $B$ of $\mathcal{A}$ is regular in the sense that every short exact sequence of the form $0 \to A \to B \to C \to 0$ is pure in $\mathcal{A}$ (i.e., every finitely presented object is projective with respect to it) [43, p. 313]. In case $\mathcal{A}$ is also locally finitely generated, we may characterize when $\mathcal{A}$ is a $V$-category or a regular category as follows.

**Theorem 6.4.** Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. Then:

1. $\mathcal{A}$ is a $V$-category if and only if every finitely cogenerated (injective) object $N$ of $\mathcal{A}$ is self-$F$-split for every fully invariant subobject $F$ of $N$.

2. $\mathcal{A}$ is a regular category if and only if every finitely generated (projective) object $N$ of $\mathcal{A}$ is dual self-$F$-split for every finitely generated fully invariant subobject $F$ of $N$.

**Proof.** (1) If $\mathcal{A}$ is a $V$-category, then $\mathcal{A}$ has a semisimple cogenerator [14, Theorem 2.3]. Hence every finitely cogenerated (injective) object $N$ of $\mathcal{A}$ is semisimple, and so $N$ is self-$F$-split for every fully invariant subobject $F$ of $N$. The remaining part of the proof follows by [5, Theorem 4.4], using Remark 4.2.

(2) We give its proof, since it is not completely dual to (1). Assume that $\mathcal{A}$ is a regular category. Let $N$ be a finitely generated object of $\mathcal{A}$, $F$ a finitely generated fully invariant subobject of $N$ and $g : N \to N$ a morphism in $\mathcal{A}$. Then $\text{Im}(gi)$ is a finitely generated subobject of the regular object $N$, hence $\text{Im}(gi)$ is a direct summand of $N$ by [43, 37.4], whose proof works in locally finitely generated Grothendieck categories. Hence $\text{coker}(gi)$ is a retraction, and so $N$ is dual self-$F$-split by Lemma 3.2. The remaining part of the proof follows by [5, Theorem 4.4], using Remark 4.2.

**Corollary 6.5.** Let $R$ be a unitary ring. Then:

1. $R$ is a right $V$-ring if and only if every finitely cogenerated (injective) right $R$-module $N$ is self-$F$-split for every fully invariant submodule $F$ of $N$. 

(2) \( R \) is a von Neumann regular ring if and only if every finitely generated (projective) right \( R \)-module \( N \) is dual self-\( F \)-split for every finitely generated fully invariant submodule \( F \) of \( N \).

**Proof.** Note that \( \text{Mod}(R) \) is a \( V \)-category if and only if \( R \) is a right \( V \)-ring, and a regular category if and only if \( R \) is a von Neumann regular ring. Then use Theorem 6.4. \( \square \)

**Remark 6.6.** If \( R \) is a commutative unitary ring, then \( V \)-rings coincide with von Neumann regular rings \([43, 23.5]\), hence all statements from Corollary 6.5 are equivalent.

**Corollary 6.7.** Let \( C \) be a coalgebra over a field. Then the following are equivalent:

(i) \( C \) is cosemisimple.

(ii) Every finitely cogenerated (injective) left \( C \)-comodule \( N \) is self-\( F \)-split for every fully invariant subcomodule \( F \) of \( N \).

(iii) Every finitely generated (projective) left \( C \)-comodule \( N \) is dual self-\( F \)-split for every finitely generated fully invariant subcomodule \( F \) of \( N \).

**Proof.** Note that the comodule category \( C \mathcal{M} \) is a \( V \)-category if and only if \( C \mathcal{M} \) is a regular category if and only if \( C \) is cosemisimple by \([40, \text{Proposition 2.3}]\) and the proof of \([5, \text{Corollary 4.6}]\). Then use Theorem 6.4. \( \square \)

Recall that an abelian category \( \mathcal{A} \) is called (semi)hereditary if every (finitely generated) subobject of a projective object is projective, and co(semi)hereditary if every (finitely cogenerated) factor object of an injective object is injective. We may characterize such categories in terms of (dual) self-\( F \)-splitness of certain objects.

**Theorem 6.8.** Let \( \mathcal{A} \) be an abelian category.

(1) Assume that \( \mathcal{A} \) has enough projectives. Then \( \mathcal{A} \) is (semi)hereditary if and only if every (finitely generated) projective object \( N \) of \( \mathcal{A} \) is self-\( F \)-split for every fully invariant direct summand \( F \) of \( N \).

(2) Assume that \( \mathcal{A} \) has enough injectives. Then \( \mathcal{A} \) is co(semi)hereditary if and only if every (finitely cogenerated) injective object \( N \) of \( \mathcal{A} \) is dual self-\( F \)-split for every fully invariant direct summand \( F \) of \( N \).
Proof. (1) Assume that \( \mathcal{A} \) is (semi)hereditary. Let \( N \) be a (finitely generated) projective object of \( \mathcal{A} \), and \( 0 \to F \xrightarrow{i} N \xrightarrow{d} C \to 0 \) a fully invariant split short exact sequence in \( \mathcal{A} \). Let \( g : N \to N \) be a morphism in \( \mathcal{A} \). If \( N \) is finitely generated, then so is \( \text{Im}(dg) \). We have \( N \cong F \oplus C \), hence \( C \) is (finitely generated) projective. Since \( \mathcal{A} \) is (semi)hereditary, \( \text{Im}(dg) \subseteq C \) is projective. Then \( \text{Ker}(dg) \) is a direct summand of \( N \), and so \( N \) is self-F-split by Lemma 3.2. The remaining part of the proof follows by [5, Theorem 4.7], using Remark 4.2. \( \square \)

Corollary 6.9. Let \( R \) be a unitary ring. Then the following are equivalent:

(i) \( R \) is right hereditary.
(ii) Every projective right \( R \)-module \( N \) is self-F-split for every fully invariant direct summand \( F \) of \( N \).
(iii) Every injective right \( R \)-module \( N \) is dual self-F-split for every fully invariant direct summand \( F \) of \( N \).

Proof. Note that \( \text{Mod}(R) \) is (co)hereditary if and only if the ring \( R \) is right hereditary. Then use Theorem 6.8. \( \square \)

Corollary 6.10. Let \( C \) be a coalgebra over a field. Then:

(1) If \( C \) is left semiperfect, then \( C \) is hereditary if and only if every projective left \( C \)-comodule \( N \) is self-F-split for every fully invariant direct summand \( F \) of \( N \).

(2) \( C \) is hereditary if and only if every injective left \( C \)-comodule \( N \) is dual self-F-split for every fully invariant direct summand \( F \) of \( N \).

Proof. Note that the comodule category \( C \mathcal{M} \) is (co)hereditary if and only if \( C \) is a (left and right) hereditary coalgebra [30]. Then use Theorem 6.8. \( \square \)

Next we give some results in the case of abelian categories with a projective generator or an injective cogenerator. Recall that an object \( M \) of an abelian category is called (semi)hereditary if every (finitely generated) subobject of \( M \) is projective, and co(semi)hereditary if every (finitely cogenerated) factor object of \( M \) is injective.

Theorem 6.11. Let \( \mathcal{A} \) be an abelian category, and let \( r \) be a preradical of \( \mathcal{A} \).
(1) Assume that \( A \) has a projective generator \( G \) and enough injectives, and \( r \) is a radical. Then the following are equivalent:

(i) \( G/r(G) \) is a (semi)hereditary object of \( A \).

(ii) \( G = r(G) \oplus B \) for some (semi)hereditary object \( B \) of \( A \).

(iii) For every (finite) coproduct \( M = G^{(1)} \), \( M = r(M) \oplus D \) for some (semi)hereditary object \( D \) of \( A \).

(iv) For every (finitely generated) projective object \( M \), \( M = r(M) \oplus D \) for some (semi)hereditary object \( D \) of \( A \).

(v) \( r(G) \) is a direct summand of \( G \) and every (finitely generated) \( r \)-torsionfree projective object of \( A \) is (semi)hereditary.

(vi) \( r(G) \) is a direct summand of \( G \) and every (finitely generated) \( r \)-torsionfree projective object of \( A \) is self-Rickart.

(vii) Every (finite) coproduct \( M = G^{(1)} \) is self-\( r(M) \)-split.

(viii) Every (finitely generated) projective object \( M \) is self-\( r(M) \)-split.

(2) Assume that \( A \) has an injective cogenerator \( G \) and enough projectives, and \( r \) is a hereditary preradical. Then the following are equivalent:

(i) \( r(G) \) is a cosemihereditary object of \( A \).

(ii) \( r(G) \) is a cosemihereditary direct summand of \( G \).

(iii) For every finite product \( M = G^n \), \( r(M) \) is a cosemihereditary direct summand of \( M \).

(iv) For every finitely cogenerated injective object \( M \), \( r(M) \) is a cosemihereditary direct summand of \( M \).

(v) \( r(G) \) is a direct summand of \( G \) and every finitely cogenerated \( r \)-torsion injective object of \( A \) is cosemihereditary.

(vi) \( r(G) \) is a direct summand of \( G \) and every finitely cogenerated \( r \)-torsion injective object of \( A \) is dual self-Rickart.

(vii) Every finite product \( M = G^n \) is dual self-\( r(M) \)-split.

(viii) Every finitely cogenerated injective object \( M \) is dual self-\( r(M) \)-split.

Proof. (1) (i)\( \Rightarrow \) (ii) Assume that \( G/r(G) \) is a (semi)hereditary object of \( A \). It follows that \( G/r(G) \) is projective, and so \( G = r(G) \oplus B \) with \( B \cong G/r(G) \) (semi)hereditary.
(ii)⇒(iii) Assume that \( G = r(G) \oplus B \) for some (semi)hereditary object \( B \) of \( \mathcal{A} \). Let \( M = G(I) \) for some (finite) set \( I \). Then \( M = r(G(I)) \oplus B(I) = r(M) \oplus B(I) \). Since \( B \) is (semi)hereditary, so is \( B(I) \) by [43, 39.3, 39.7], whose proofs work in abelian categories with enough injectives.

(iii)⇒(iv) Assume that (iii) holds. Let \( M \) be a (finitely generated) projective object of \( \mathcal{A} \). Then \( M \) is a direct summand of a (finite) coproduct \( F = G(I) \), say \( F = M \oplus N \). It follows that \( r(F) = r(M) \oplus r(N) \) and \( F/r(F) \cong M/r(M) \oplus N/r(N) \). Since \( F/r(F) \) is (semi)hereditary, so is \( M/r(M) \). Since \( r(F) \) is a direct summand of \( F \), \( r(M) \) must be a direct summand of \( M \).

(iv)⇒(v) This is clear.

(v)⇒(vi) Assume that (v) holds. Let \( M \) be a (finitely generated) \( r \)-torsionfree projective object of \( \mathcal{A} \). Let \( f : M \to M \) be a morphism in \( \mathcal{A} \). Since \( M \) is (semi)hereditary, \( M/\text{Ker}(f) \cong \text{Im}(f) \) is projective, and so \( \text{Ker}(f) \) is a direct summand of \( M \). Hence \( M \) is self-Rickart.

(vi)⇒(vii) Assume that (vi) holds. Let \( M = G(I) \) be a (finite) coproduct. Since \( G = r(G) \oplus B \) for some object \( B \) of \( \mathcal{A} \), it follows that \( M = r(M) \oplus B(I) \). Since the (finitely generated) \( r \)-torsionfree projective object \( B(I) \) is self-Rickart, \( M \) is self-\( r(M) \)-split by Theorem 4.3.

(vii)⇒(viii) This follows by Corollary 3.9.

(viii)⇒(i) Assume that (viii) holds. We show that \( G/r(G) \) is (semi)hereditary. To this end, let \( A \) be a (finitely generated) subobject of \( G/r(G) \). Then \( A \cong M/K \) for some (finite) coproduct \( M = G(I) \) and subobject \( K \) of \( M \). Denote by \( \varphi : M/K \to A \) an isomorphism, by \( p : M \to M/K \) the natural epimorphism, by \( k : A \to M \) and \( i : r(M) \to M \) the inclusion monomorphisms and \( g = k\varphi p : M \to M \). Consider the pullback of \( i \) and \( g \) in order to get the following commutative square:

\[
\begin{array}{ccc}
P & \xrightarrow{j} & M \\
f \downarrow & & \downarrow g \\
r(M) & \xrightarrow{i} & M 
\end{array}
\]

Since \( r \) is a radical, \( G/r(G) \) is an \( r \)-torsionfree object, hence so is \( A \). This implies that \( if = gj = 0 \). Then \( pj = 0 \), and so there exists a unique morphism \( \alpha : P \to K \) such that \( k\alpha = j \), which implies that \( \alpha \) is a monomorphism. The pullback property yields a unique morphism \( \beta : K \to P \) such that \( f\beta = 0 \) and \( j\beta = k \). Then \( k\alpha\beta = k \), hence \( \alpha\beta = 1_K \), which shows that \( P \cong K \). Since \( M \) is projective, it is self-\( r(M) \)-split.
Then \( j : P \to M \) is a section, and so \( A \) is projective. Hence \( G/r(G) \) is (semi)hereditary.

(2) In general only the semihereditary case from (1) may be dualized, because \( r \) may not preserve products (unless it is superhereditary), and an arbitrary product of cohereditary objects may not be cohereditary (see [43, 18.2] and the proofs of [43, 39.3, 39.6, 39.7]). The hypothesis on \( r \) to be a cohereditary preradical is necessary for showing (viii) \( \Rightarrow \) (i).

**Corollary 6.12.** Let \( r \) be a radical of \( \text{Mod}(R) \). Then the following are equivalent:

(i) \( R/r(R) \) is a (semi)hereditary right \( R \)-module.

(ii) \( R = r(R) \oplus B \) for some (semi)hereditary right ideal \( B \) of \( R \).

(iii) For every (finitely generated) free right \( R \)-module \( M \), \( M = r(M) \oplus D \) for some (semi) hereditary right \( R \)-module \( D \).

(iv) For every (finitely generated) projective right \( R \)-module \( M \), \( M = r(M) \oplus D \) for some (semi)hereditary right \( R \)-module \( D \).

(v) \( r(R) \) is a direct summand of \( R \) and every (finitely generated) \( r \)-torsionfree projective right \( R \)-module is (semi)hereditary.

(vi) \( r(R) \) is a direct summand of \( R \) and every (finitely generated) \( r \)-torsionfree projective right \( R \)-module is self-Rickart.

(vii) Every (finitely generated) free right \( R \)-module \( M \) is self-\( r(M) \)-split.

(viii) Every (finitely generated) projective right \( R \)-module \( M \) is self-\( r(M) \)-split.

**Proof.** This follows by Theorem 6.11, noting that \( R \) is a projective generator of \( \text{Mod}(R) \).

**Corollary 6.13.** Let \( C \) be left semiperfect and let \( r \) be a cohereditary preradical of \( C \). Then the following are equivalent:

(i) \( r(C) \) is a cosemihereditary left \( C \)-comodule.

(ii) \( r(C) \) is a cosemihereditary direct summand of \( C \).

(iii) For every finitely cogenerated free left \( C \)-comodule \( M \), \( r(M) \) is a cosemihereditary direct summand of \( M \).

(iv) For every finitely cogenerated injective left \( C \)-comodule \( M \), \( r(M) \) is a cosemihereditary direct summand of \( M \).
(v) \( r(C) \) is a direct summand of \( C \) and every finitely cogenerated \( r \)-torsion injective left \( C \)-comodule is cosemihereditary.

(vi) \( r(C) \) is a direct summand of \( C \) and every finitely cogenerated \( r \)-torsion injective left \( C \)-comodule is dual self-Rickart.

(vii) Every finitely cogenerated free left \( C \)-comodule \( M \) is dual self-\( r(M) \)-split.

(viii) Every finitely cogenerated injective left \( C \)-comodule \( M \) is dual self-\( r(M) \)-split.

**Proof.** This follows by Theorem 6.11, noting that \( C \) is an injective cogenerator of \( CM \), and \( CM \) has enough projectives if and only if \( C \) is left semiperfect.

\[ \square \]

**Theorem 6.14.** Let \( A \) be an abelian category and let \( r \) be a preradical of \( A \).

(1) Assume that \( A \) has an injective cogenerator \( G \). Then the following are equivalent:

(i) \( G \) is a (strongly) self-\( r(G) \)-split object of \( A \).

(ii) \( G = r(G) \oplus D \) for some object \( D \) of \( A \) and \( \text{End}_A(D) \) is a von Neumann regular (strongly regular) ring.

(iii) Every finite product \( M = G^n \) is (strongly) self-\( r(M) \)-split.

(iv) Every finitely cogenerated injective object \( M \) of \( A \) is (strongly) self-\( r(M) \)-split.

(2) Assume that \( A \) has a projective generator \( G \). Then the following are equivalent:

(i) \( G \) is a dual (strongly) self-\( r(G) \)-split object.

(ii) \( r(G) \) is a direct summand of \( G \) and \( \text{End}_A(r(G)) \) is a von Neumann regular (strongly regular) ring.

(iii) Every finite coproduct \( M = G^n \) is dual (strongly) self-\( r(M) \)-split.

(iv) Every finitely generated projective object \( M \) of \( A \) is dual (strongly) self-\( r(M) \)-split.

**Proof.** (1) We first prove the theorem without strongness conditions.

(i)\(\Rightarrow\)(ii) Assume that (i) holds. Then \( G = r(G) \oplus D \) for some self-Rickart object \( D \) of \( A \) by Theorem 4.3. We claim that \( D \) is also dual self-Rickart. To this end, let \( f : D \to D \) be a morphism in \( A \). Then \( \text{Im}(f) \cong D/\text{Ker}(f) \)
is isomorphic to a direct summand of $D$. As $G$ is injective, so is $D$. Hence $D$ is direct injective, in the sense that every subobject of $D$ isomorphic to a direct summand of $D$ is a direct summand of $D$. It follows that $\text{Im}(f)$ is a direct summand of $D$. Hence $D$ is self-Rickart and dual self-Rickart. Then $\text{End}_A(D)$ is von Neumann regular by [5, Corollary 2.3].

(ii)⇒(iii) Assume that (ii) holds. Let $M = G^n$ for some $n \in \mathbb{N}$. Then $M = r(G^n) \oplus D^n = r(M) \oplus D^n$. Note that $\text{End}_A(D^n) \cong \text{End}_A(D)^n$ is von Neumann regular, hence $D^n$ is self-regular, and so $D^n$ is self-Rickart by [5, Corollary 2.3]. Finally, $M$ is self-$r(M)$-split by Theorem 4.3.

(iii)⇒(iv) Assume that (iii) holds. Note that every finitely cogenerated injective object of $\mathcal{A}$ is isomorphic to a direct summand of a product $G^n$. Then use Corollary 3.9.

(iv)⇒(i) This is clear.

Finally, the equivalence of the strong versions of the above conditions is immediately deduced by the above proof and the following properties: an object $M$ of $\mathcal{A}$ is strongly self-$r(M)$-split if and only if $M$ is self-$r(M)$-split and $\text{End}_A(M/r(M))$ is abelian (Theorem 4.10), while a ring is strongly regular if and only if it is von Neumann regular and abelian. \[\square\]

Corollary 6.15. Let $r$ be a preradical of $\mathcal{C}M$. Then the following are equivalent:

(i) $C$ is a (strongly) self-$r(C)$-split left $C$-comodule.

(ii) $C = r(C) \oplus D$ for some subcomodule $D$ of $C$ and $\text{End}_C(D)$ is a von Neumann regular (strongly regular) ring.

(iii) Every finitely cogenerated free left $C$-comodule $M$ is (strongly) self-$r(M)$-split.

(iv) Every finitely cogenerated injective left $C$-comodule $M$ is (strongly) self-$r(M)$-split.

Proof. This follows by Theorem 6.14, noting that $C$ is an injective cogenerator of $\mathcal{C}M$. \[\square\]

Corollary 6.16. Let $r$ be a preradical of $\text{Mod}(R)$. Then the following are equivalent:

(i) $R$ is a dual (strongly) self-$r(R)$-split right $R$-module.

(ii) $r(R)$ is a direct summand of $R$ and $\text{End}_R(r(R))$ is a von Neumann regular (strongly regular) ring.
(iii) Every finitely generated free right $R$-module $M$ is dual (strongly) self-$r(M)$-split.

(iv) Every finitely generated projective right $R$-module $M$ is dual (strongly) self-$r(M)$-split.

**Proof.** This follows by Theorem 6.14, noting that $R$ is a projective generator of $\text{Mod}(R)$. □

**Further applications**

Throughout the paper we have given several examples and consequences of our results. In this section we present some further applications to module and comodule categories.

**Module categories**

Throughout $R$ is a unitary ring.

**Proposition 7.1.** (1) Let $I$ be a right $T$-nilpotent ideal of $R$ and let $r$ be the cohereditary radical associated to $I$, given by $r(M) = MI$. Then a right $R$-module $M$ is (strongly) self-$r(M)$-split if and only if $M$ is $r$-torsionfree (strongly) self-Rickart.

(2) Let $I$ be a left $T$-nilpotent ideal of $R$ and let $r$ be the superhereditary preradical associated to $I$, given by $r(M) = \{ x \in M \mid xI = 0 \}$. Then a right $R$-module $M$ is dual (strongly) self-$r(M)$-split if and only if $M$ is $r$-torsion (strongly) dual self-Rickart.

**Proof.** (1) There is a bijective correspondence between the ideals $I$ of a ring $R$ and the cohereditary radicals $r$ of $\text{Mod}(R)$, where $r$ is given by $r(M) = MI$ [2, Corollary I.2.11]. If $M$ is (strongly) self-$r(M)$-split, then $r(M)$ is a superfluous direct summand of $M$ by [2, Proposition I.4.6] and Theorem 4.3, hence $r(M) = 0$, and so $M$ is $r$-torsionfree (strongly) self-Rickart. The converse is clear.

(2) There is a bijective correspondence between the ideals $I$ of a ring $R$ and the superhereditary preradicals $r$ of $\text{Mod}(R)$, where $r$ is given by $r(M) = \{ x \in M \mid xI = 0 \}$ [2, I.2.E6]. If $M$ is dual (strongly) self-$r(M)$-split, then $r(M)$ is an essential direct summand of $M$ by [2, Proposition I.4.12] and Theorem 4.3, hence $r(M) = M$, and so $M$ is $r$-torsion (strongly) dual self-Rickart. The converse is clear. □
For a right $R$-module $M$, denote by $\text{Soc}(M)$ the socle of $M$ (i.e., the sum of its simple submodules), and by $\text{Rad}(M)$ the Jacobson radical of $M$ (i.e., the intersection of its maximal submodules). Then $\text{Soc}$ is a hereditary preradical and $\text{Rad}$ is a cohereditary preradical of $\text{Mod}(R)$.

**Proposition 7.2.** (1) A finitely generated right $R$-module $M$ is (strongly) self-$\text{Rad}(M)$-split if and only if $M$ is semiprimitive (strongly) self-Rickart.

(2) A finitely cogenerated right $R$-module $M$ is dual (strongly) self-$\text{Soc}(M)$-split if and only if $M$ is semisimple.

**Proof.** (1) If $M$ is finitely generated (strongly) self-$\text{Rad}(M)$-split, then $\text{Rad}(M)$ is a superfluous direct summand of $M$ by Theorem 4.3, hence $\text{Rad}(M) = 0$, and so $M$ is semiprimitive (strongly) self-Rickart. The converse is clear.

(2) If $M$ is finitely cogenerated dual (strongly) self-$\text{Soc}(M)$-split, then $\text{Soc}(M)$ is an essential direct summand of $M$ by Theorem 4.3, hence $M = \text{Soc}(M)$ is semisimple. The converse is clear. □

**Corollary 7.3.** (1) Every finitely generated right $R$-module $M$ is (strongly) self-$\text{Rad}(M)$-split if and only if every finitely generated right $R$-module $M$ is semiprimitive (strongly) self-Rickart.

(2) Every finitely cogenerated right $R$-module $M$ is dual (strongly) self-$\text{Soc}(M)$-split if and only if $R$ is a right $V$-ring.

**Proof.** (2) Note that every finitely cogenerated right $R$-module is semisimple if and only if $R$ is a right $V$-ring [43, 23.1]. □

Recall that an object $M$ of an abelian category $\mathcal{A}$ is called (strongly) extending if every subobject of $M$ is essential in a (fully invariant) direct summand of $M$, and (strongly) lifting if every subobject $L$ of $M$ contains a (fully invariant) direct summand $K$ of $M$ such that $L/K$ is superfluous in $M/K$ [5, 16, 41].

For a right $R$-module $M$, consider the full subcategory $\sigma[M]$ of $\text{Mod}(R)$ consisting of $M$-subgenerated right $R$-modules. Note that if $M = R_R$, then $\sigma[M] = \text{Mod}(R)$. For every module $N \in \sigma[M]$, denote by:

- $Z_M(N) = \text{Tr}(\mathcal{U}, N) = \sum \{\text{Im}(f) \mid f \in \text{Hom}(\mathcal{U}, N)\}$, where $\mathcal{U}$ is the class of $M$-singular modules. Recall that a module $A$ is called $M$-singular (or singular in $\sigma[M]$) if $A \cong L/K$ for some $L \in \sigma[M]$ and essential
submodule $K$ of $L$. Also, $A$ is called non-$M$-singular (or non-singular in $\sigma[M]$) if $Z_M(N) = 0$.

- $Z^2_M(N)$ the second $M$-singular submodule of $N$, which is determined by the equality $Z^2_M(N)/Z_M(N) = Z_M(N/Z_M(N))$.

- $Z_M(N) = \text{Re}(N, U) = \bigcap \{ \text{Ker}(f) \mid f \in \text{Hom}(N, U) \}$, where $U$ is the class of $M$-small modules. Recall that a module $A$ is called $M$-small if $A$ is superfluous in some $L \in \sigma[M]$. Also, $N$ is called $M$-cosingular (or cosingular in $\sigma[M]$) if $Z_M(N) = 0$, and non-$M$-cosingular (or non-cosingular in $\sigma[M]$) if $Z_M(N) = N$.

- $Z^2_M(N) = Z_M(Z_M(N))$.

Then $Z_M$ is a hereditary preradical, $Z^2_M$ is a hereditary radical, while $Z_M$ and $Z^2_M$ are radicals of $\sigma[M]$.

**Proposition 7.4.** (i) Assume that $R$ is non-singular (i.e., $Z(R_R) = 0$) and every finitely generated non-singular right $R$-module is projective. Then every finitely generated right $R$-module $M$ is self-$Z^2(M)$-split.

(ii) Assume that every finitely cogenerated singular right $R$-module is injective. Then every finitely cogenerated right $R$-module $M$ is dual self-$Z(M)$-split.

**Proof.** (i) Let $M$ be a finitely generated right $R$-module. Since the finitely generated non-singular (see [39, Lemma 3.2]) right $R$-module $M/Z^2(M)$ is projective, $M \cong Z^2(M) \oplus M/Z^2(M)$. Since every ring with the property from hypothesis is right semihereditary [19, Theorem 5.18], $M/Z^2(M)$ is self-Rickart [5, Theorem 4.7]. Hence $M$ is self-$Z^2(M)$-split by Theorem 4.3.

(ii) Let $M$ be a finitely cogenerated right $R$-module. Since the finitely cogenerated singular right $R$-module $Z(M)$ is injective, it is a direct summand of $M$. We claim that $R$ is right cosemihereditary. To this end, let $E$ be an injective right $R$-module and $K$ a submodule of $E$ such that $E/K$ is finitely cogenerated. Consider the injective hull $E(K)$ of $K$ and the induced epimorphism $\gamma : E/K \to E/E(K)$. Then $\text{Ker}(\gamma) \cong E(K)/K$ is finitely cogenerated singular, and so it is injective. Also, we have $E \cong E(K) \oplus E/E(K)$, and so $E/E(K)$ is injective. It follows that $E/K$ is injective, which shows that $R$ is right cosemihereditary. Then $Z(M)$ is dual self-Rickart [5, Theorem 4.7]. Hence $M$ is dual self-$Z(M)$-split by Theorem 4.3. □

**Proposition 7.5.** Let $M$ be a right $R$-module. Then:
(1) Every (strongly) extending module \( N \in \sigma[M] \) is (strongly) self-\( \mathbb{Z}^2_M(N) \)-split.

(2) Every (strongly) lifting module \( N \in \sigma[M] \) is dual (strongly) self-\( \mathbb{Z}^2_M(N) \)-split.

Proof. (1) If \( N \) is (strongly) extending, then \( N = \mathbb{Z}^2_M(N) \oplus N' \) for some non-\( M \)-singular (strongly) extending submodule \( N' \) of \( N \) [39, Theorem 3.4] ([16, Theorem 2.24]). Note that a module \( X \in \sigma[M] \) is non-\( M \)-singular if and only if for every \( K \in \sigma[M] \) and for every \( 0 \neq f \in \text{Hom}(K,N') \), \( \text{Ker}(f) \) is not essential in \( K \). Clearly, every non-\( M \)-singular module \( X \in \sigma[M] \) is \( X \)-\( \mathcal{K} \)-nonsingular, in the sense that for every \( 0 \neq f \in \text{End}(X) \), \( \text{Ker}(f) \) is not essential in \( X \) [5, Definition 9.4]. It follows that \( N' \) is (strongly) self-Baer by [5, Theorem 9.5], and so \( N' \) is (strongly) self-Rickart. Hence \( N \) is (strongly) self-\( \mathbb{Z}^2_M(N) \)-split by Theorem 4.3.

(2) If \( N \) is (strongly) lifting, then \( N = \mathbb{Z}^2_M(N) \oplus N' \) for some submodule \( N' \) of \( N \) such that \( \mathbb{Z}^2_M(N) \) is non-\( M \)-cosingular (strongly) lifting [35, Theorem 4.1] ([41, Corollary 3.6]). Note that a module \( Y \in \sigma[M] \) is non-\( M \)-cosingular if and only if every non-zero factor module of \( Y \) is not \( M \)-small, that is, it does not exist any module \( L \in \sigma[M] \) such that \( Y \) is superfluous in \( L \). Clearly, every non-\( M \)-cosingular module \( Y \in \sigma[M] \) is \( Y \)-\( T \)-nonsingular, in the sense that for every \( 0 \neq f \in \text{End}(Y) \), \( \text{Im}(f) \) is not superfluous in \( Y \) [5, Definition 9.4]. It follows that \( \mathbb{Z}^2_M(N) \) is dual (strongly) self-Baer by [5, Theorem 9.5], and so \( \mathbb{Z}^2_M(N) \) is dual (strongly) self-Rickart. Hence \( N \) is dual (strongly) self-\( \mathbb{Z}^2_M(N) \)-split by Theorem 4.3. □

Comodule categories

Throughout \( C \) is a coalgebra over a field \( k \). Let \( C^* = \text{Hom}_k(C,k) \). Then \( \sigma[C_\cdot] = \text{Rat}(\text{Mod}(C^*)) \cong \mathcal{C} \mathcal{M} \), where \( \text{Rat}(\text{Mod}(C^*)) \) is the full subcategory of \( \text{Mod}(C^*) \) consisting of rational right \( C^* \)-modules and \( \mathcal{C} \mathcal{M} \) is the category of left \( C \)-comodules [11, Chapter 2]. Hence one has the (pre)radicals from categories of the form \( \sigma[M] \) for some module \( M \) available in comodule categories. Also, for every right \( C^* \)-module \( M \), denote by \( \text{Rat}(M) \) the largest rational submodule of \( M \). Then \( \text{Rat} \) is a hereditary preradical of \( \text{Mod}(C^*) \).

Proposition 7.6. Let \( M \) be a left \( C \)-comodule. Then the following are equivalent:

(1) Every (strongly) extending module \( N \in \sigma[M] \) is (strongly) self-\( \mathbb{Z}^2_M(N) \)-split.

(2) Every (strongly) lifting module \( N \in \sigma[M] \) is dual (strongly) self-\( \mathbb{Z}^2_M(N) \)-split.
(i) $M$ is semisimple.
(ii) $M$ is (strongly) self-Soc($M$)-split.
(iii) $M$ is dual (strongly) self-Soc($M$)-split.

Proof. If $M$ is non-zero (strongly) self-Soc($M$)-split or dual (strongly)
self-Soc($M$)-split, then Soc($M$) is a direct summand of $M$ by Theorem 4.3
and an essential subcomodule of $M$ by [11, Corollary 2.4.12], which implies
that $M = \text{Soc}(M)$ is semisimple by [11, Proposition 2.4.11]. The rest of
the proof is clear. □

Corollary 7.7. The following are equivalent:
(i) $C$ is cosemisimple.
(ii) Every left $C$-comodule $M$ is (strongly) self-Soc($M$)-split.
(iii) Every left $C$-comodule $M$ is dual (strongly) self-Soc($M$)-split.

Proposition 7.8. Let $C$ be such that $C^*$ is semisimple. Then the following
are equivalent:
(i) $C$ is cosemisimple.
(ii) Every right $C^*$-module $M$ is (strongly) self-Rat($M$)-split.
(iii) Every right $C^*$-module $M$ is dual (strongly) self-Rat($M$)-split.

Proof. If every right $C^*$-module $M$ is (strongly) self-Rat($M$)-split or
dual (strongly) self-Rat($M$)-split, then Rat($M$) is a direct summand of $M$
for every every right $C^*$-module $M$ by Theorem 4.3. Then $C$ is finite
dimensional by [29, Theorem 3.4]. Since $C^*$ is semisimple, it follows that $C$
is cosemisimple [11, Exercise 3.1.7]. The rest of the proof is clear. □

Proposition 7.9. Let $C$ be hereditary and let $P$ be a projective left $C$-
comodule. Then $P^* = \text{Hom}_k(P,k)$ is a dual self-Rat($C^*P^*$)-split right $C$-
comodule.

Proof. By [11, Corollary 2.4.18], Rat($C^*P^*$) is an injective right $C$-
comodule. Then Rat($C^*P^*$) is a direct summand of $P^*$ and dual self-Rickart
by [5, Corollary 4.9]. Hence $P^*$ is a dual self-Rat($C^*P^*$)-split right $C$-
comodule by Theorem 4.3. □

Proposition 7.10. Let $C$ be left semiperfect hereditary and let $F$ be a flat
left $C$-comodule. Then $F^* = \text{Hom}_k(F,k)$ is a dual self-Rat($C^*F^*$)-split right $C$-comodule.
Proof. By [10, Theorem 4.6] and [9, Corollary 3.4], $C$ is left semiperfect if and only if $\text{Rat}(C^*F^*)$ is an injective right $C$-comodule for every every flat left $C$-comodule $F$. Then $\text{Rat}(C^*F^*)$ is a direct summand of $F^*$ and dual self-Rickart by [5, Corollary 4.9]. Hence $F^*$ is a dual self-$\text{Rat}(C^*F^*)$-split right $C$-comodule by Theorem 4.3.

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References


