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Countable Recognizability and Residual Properties of Groups

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Abstract – A class of groups $\mathfrak{X}$ is said to be countably recognizable if a group belongs to $\mathfrak{X}$ whenever all its countable subgroups lie in $\mathfrak{X}$. It is proved here that the class of groups whose subgroups are closed in the profinite topology is countably recognizable. Moreover, countably detectable properties of the finite residual of a group are studied.

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1. Introduction

A group class $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group. Countably recognizable classes of groups were introduced by R. Baer [1]. In his paper, Baer produced many interesting examples of countably recognizable group classes, and later many other relevant countably recognizable classes of groups were discovered (see for instance [3],[10],[11],[14] and the recent papers [4],[5],[6]). In particular, B.H. Neumann [9] proved that the class $\boldsymbol{R}\mathfrak{F}$ of residually finite groups is countably recognizable.

Let G be any group, and let $\mathcal{J}(G)$ be the set of all normal subgroups of finite index of G. The *profinite topology* on G can be defined by choosing the set $\mathcal{J}(G)$ as a base of neighbourhoods of the identity; if X is any subgroup of G, the closure $\hat{X}$ of X with respect to this topology is the intersection of all subgroups of finite

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index of G containing X, i.e.

$$\hat{X} = \bigcap_{H \in \mathcal{J}(G)} XH.$$

In particular, a subgroup X is *closed* (with respect to the profinite topology) if and only if it is the intersection of a collection of subgroups of finite index, and a group G is residually finite if and only if the trivial subgroup $\{1\}$ is closed. It is also well-known that every subgroup of an arbitrary polycyclic group is closed. The structure of nilpotent groups in which all subgroups are closed was studied by M. Menth [8], while D.J.S. Robinson, A. Russo and G. Vincenzi [13] recently characterized groups wi th the same property within the universe of groups with finite conjugacy classes, and B.A.F. Wehrfritz [15] investigated the case of linear groups.

The aim of this paper is to show that closure properties with respect to profinite topology can be detected from the behavior of countable subgroups. Our first main result is the following.

Theorem A *Let G be a group, and let X be a subgroup of G such that $X \cap K$ is closed in K for each countable subgroup K of G. Then X is a closed subgroup of G.*

This statement has a number of interesting consequences, the most striking being that the class of groups in which all subgroups are closed is countably recognizable. Notice also that Theorem A will be obtained as a special case of a more general result.

If $\mathfrak{X}$ is any class of groups, the $\mathfrak{X}$-*residual* of a group G is the intersection of all normal subgroups N such that G/N belongs to $\mathfrak{X}$, and G is *residually* $\mathfrak{X}$ if its $\mathfrak{X}$-residual is trivial. It follows from Theorem A that residually supersoluble groups form a countably recognizable class; moreover, if $\mathfrak{F}_\pi$ denotes the class of finite π-groups, it turns out that $\boldsymbol{R}\mathfrak{F}_\pi$ is countably recognizable for each set π of prime numbers.

The second part of the paper deals with properties of the finite residual of a group which can be countably detected. It is known that the class $\mathfrak{F}(\boldsymbol{R}\mathfrak{F})$, consisting of all groups G whose finite residual $J(G)$ is finite is countably recognizable (see [5], Theorem 3.6), and our next main result shows that in this statement the class $\mathfrak{F}$ can be replaced by any subgroup closed and countably recognizable group class.

Theorem B *Let $\mathfrak{X}$ be a subgroup closed countably recognizable class of groups. Then the class $\mathfrak{X}(\boldsymbol{R}\mathfrak{F})$, consisting of all groups whose finite residual belongs to $\mathfrak{X}$, is countably recognizable.*

Most of our notation is standard and can be found in [12]. In particular, we refer to the first chapter of [12] for definitions and properties of Philip Hall's operations on group classes.

2. Closed subgroups

Let $\mathfrak{X}$ be a class of groups, and for any group G let $\mathcal{J}_{\mathfrak{X}}(G)$ be the set of all normal subgroups N of G such that G/N belongs to $\mathfrak{X}$. A subgroup X of a group G is said to be $\mathfrak{X}$-*closed* in G if

$$X = \bigcap_{N \in \mathcal{J}_{\mathfrak{X}}(G)} XN.$$

In particular, if $\mathfrak{F}$ is the class of all finite groups, we have $\mathcal{J}_{\mathfrak{F}}(G) = \mathcal{J}(G)$ and so the subgroup X is $\mathfrak{F}$-closed in G if and only if it is the intersection of a collection of subgroups of finite index of G, i.e. if and only if it is a closed subgroup of G. Note also that, if the group class $\mathfrak{X}$ is closed under homomorphic images, then a normal subgroup H of a group G is $\mathfrak{X}$-closed if and only if the factor group G/H is residually $\mathfrak{X}$.

Lemma 2.1. *Let $\mathfrak{X}$ be a subgroup closed class of groups, and let X be an $\mathfrak{X}$-closed subgroup of a group G. Then $X \cap K$ is $\mathfrak{X}$-closed in K for each subgroup K of G. In particular, X is $\mathfrak{X}$-closed in H, whenever H is a subgroup of G containing X.*

Proof. As the class $\mathfrak{X}$ is subgroup closed, the intersection $N \cap K$ belongs to $\mathcal{J}_{\mathfrak{X}}(K)$ for every $N \in \mathcal{J}_{\mathfrak{X}}(G)$, so that

$$\begin{aligned} X \cap K &= \Big(\bigcap_{N \in \mathcal{J}_{\mathfrak{X}}(G)} XN \Big) \cap K = \bigcap_{N \in \mathcal{J}_{\mathfrak{X}}(G)} (XN \cap K) \\ &\geq \bigcap_{N \in \mathcal{J}_{\mathfrak{X}}(G)} (X \cap K)(N \cap K) \geq \bigcap_{L \in \mathcal{J}_{\mathfrak{X}}(K)} (X \cap K)L \end{aligned}$$

and hence

$$X \cap K = \bigcap_{L \in \mathcal{J}_{\mathfrak{X}}(K)} (X \cap K)L$$

is an $\mathfrak{X}$-closed subgroup of K. □

Theorem A is a special case of the main result of this section, that will be proved by using an inverse limit argument. To this purpose, we will need the following classical theorem of Kurosh on inverse systems of finite sets (see for instance [7], Theorem 1.K.1).

Lemma 2.2. *The inverse limit of an inverse system of finite non-empty sets is non-empty.*

Theorem 2.3. *Let $\mathfrak{X}$ be a subgroup closed class of finite groups, and let X be a subgroup of a group G. If $X \cap K$ is $\mathfrak{X}$-closed in K for every countable subgroup K of G, then X is $\mathfrak{X}$-closed in G.*

Proof. Let g be any element of $G \setminus X$, and for each countable subgroup K of G containing g, let $H(K)$ be a normal subgroup of K such that g does not belong to $(X \cap K)H(K)$ and $K/H(K)$ is an $\mathfrak{X}$-group whose order $h(K)$ is smallest possible under these conditions. Consider the set $\mathscr{E}$ of all finitely generated subgroups of G containing g, and assume that there exists an infinite sequence $(E_n)_{n\in\mathbb{N}}$ of elements of $\mathscr{E}$ such that

$$h(E_1) < h(E_2) < \ldots < h(E_n) < \ldots$$

Clearly,

$$U = \langle E_n \mid n \in \mathbb{N} \rangle$$

is a countable subgroup of G, and $|E_n : H(U) \cap E_n| \leq |U : H(U)|$, so that we have $h(E_n) \leq h(U)$ for all n. This contradiction shows that the set of positive integers

$$\{h(E) \mid E \in \mathscr{E}\}$$

is finite, and so it has a largest element m.

For each element E of $\mathscr{E}$, let $\mathscr{L}(E)$ be the set of all normal subgroups L of E such that $g \notin (X \cap E)L$ and E/L is an $\mathfrak{X}$-group of order at most m. Clearly, the set $\mathscr{L}(E)$ is finite, because any finitely generated group contains only finitely many subgroups of a given finite index; moreover, it follows from the choice of m that every $\mathscr{L}(E)$ is non-empty. If E and F are elements of $\mathscr{E}$ such that $F \leq E$, the intersection $L \cap F$ belongs to $\mathscr{L}(F)$ for each subgroup $L \in \mathscr{L}(E)$, and so we may consider the intersection map $\alpha_{E,F}$ of $\mathscr{L}(E)$ into $\mathscr{L}(F)$. Then

$$\{\mathscr{L}(E), \alpha_{E,F} \mid E, F \in \mathscr{E},\ F \leq E\}$$

is an inverse system of finite non-empty sets, and so its inverse limit

$$\mathscr{L} = \varprojlim \mathscr{L}(E)$$

is not empty by Lemma 2.2. Let $(Y_E)_{E\in\mathscr{E}}$ be an element of $\mathscr{L}$. If E and F are arbitrary elements of $\mathscr{E}$, we have $\langle Y_E, Y_F\rangle \leq Y_{\langle E,F\rangle}$ and hence

$$Y = \bigcup_{E\in\mathscr{E}} Y_E$$

is a subgroup of G. Moreover, Y is normal in G, because if y is any element of Y and x is an arbitrary element of G, then y^x lies in $Y_{\langle g,y,x\rangle} \leq Y$.

If F and F^* are arbitrary elements of $\mathscr{E}$, we have $F \cap Y_{\langle F,F^*\rangle} = Y_F$, so that

$$F \cap Y = F \cap \Big(\bigcup_{E\in\mathscr{E}} Y_{\langle E,F\rangle}\Big) = \bigcup_{E\in\mathscr{E}} (F \cap Y_{\langle E,F\rangle}) = Y_F.$$

Assume now for a contradiction that $|G : Y|$ is infinite, and let $g_1, \ldots, g_m, g_{m+1}$ be $m+1$ different elements of a transversal to Y in G. Then

$$E = \langle g, g_1, \ldots, g_m, g_{m+1}\rangle$$

is an element of $\mathscr{E}$ and

$$|E : Y_E| = |E : Y \cap E| > m,$$

which is impossible because Y_E belongs to $\mathscr{L}(E)$. Therefore the index $|G : Y|$ is finite. Consider the element W of $\mathscr{E}$ generated by g and by a transversal to Y in G. Then $WY = G$ and hence

$$G/Y \simeq W/W \cap Y = W/Y_W$$

is an $\mathfrak{X}$-group.

Assume finally that g belongs to

$$XY = \bigcup_{E \in \mathscr{E}} XY_E,$$

so that there exist an element E of $\mathscr{E}$ and a finitely generated subgroup X_0 of X such that g lies in $X_0 Y_E$. Then the subgroup $F = \langle X_0, E \rangle$ is an element of $\mathscr{E}$ and g belongs to $(X \cap F)Y_F$. This contradiction proves that g is not in XY, so that X is $\mathfrak{X}$-closed in G because g is an arbitrary element of $G \setminus X$. □

The following result is an easy combination of Lemma 2.1 and Theorem 2.3.

Corollary 2.4. *Let $\mathfrak{X}$ be a subgroup closed class of finite groups, and let X be a subgroup of a group G. Then the following statements are equivalent:*

(a) *X is $\mathfrak{X}$-closed in G;*

(b) *X is $\mathfrak{X}$-closed in $\langle X, K \rangle$ for each countable subgroup K of G;*

(c) *$X \cap K$ is $\mathfrak{X}$-closed in K for each countable subgroup K of G.*

It follows from the above statement that if $\mathfrak{X}$ is any subgroup closed class of finite groups, then the class of groups in which all subgroups are $\mathfrak{X}$-closed is countably recognizable. In particular, groups all of whose subgroups are closed in the profinite topology form a countably recognizable class, although it is clear that such class is not local. Another special case is the following interesting fact.

Corollary 2.5. *Let G be a group whose countable subgroups are closed. Then all subgroups of G are closed.*

Actually, it can be remarked that for a single subgroup the embedding property of being closed is countably detectable.

Corollary 2.6. *Let X be a subgroup of a group G. If all countable subgroups of X are closed in G, then X itself is closed in G.*

Proof. Let K be any countable subgroup of G. Then the intersection $X \cap K$ is obviously countable and so it is closed in G. In particular, $X \cap K$ is closed in K, and hence X is a closed subgroup of G by Corollary 2.4. $\square$

Notice also that the proof of Corollary 2.6 can be used to prove that a corresponding more general statement holds for the property of being $\mathfrak{X}$-closed, where $\mathfrak{X}$ is any subgroup closed class of finite groups.

Corollary 2.7. *Let $\mathfrak{X}$ be a subgroup closed class of finite groups. Then the class $\boldsymbol{R}\mathfrak{X}$ of residually $\mathfrak{X}$ groups is countably recognizable.*

In particular, the latter statement improves Neumann's theorem on residually finite groups, showing for instance that the class $\boldsymbol{R}\mathfrak{F}_\pi$ is countably recognizable, for any set π of prime numbers.

Corollary 2.8. *Let $\mathfrak{X}$ be a group class which is closed with respect to subgroups and homomorphic images. If $\mathfrak{X}$ is contained in $\boldsymbol{R}\mathfrak{F}$, then the class $\boldsymbol{R}\mathfrak{X}$ is countably recognizable.*

Proof. Since the class $\mathfrak{X}$ is closed with respect to homomorphic images, we have

$$\boldsymbol{R}\mathfrak{X} = \boldsymbol{R}(\mathfrak{X} \cap \mathfrak{F}),$$

and hence the statement follows from Corollary 2.7. $\square$

Since any supersoluble group is residually finite, the above statement has the following special case.

Corollary 2.9. *The class of residually supersoluble groups is countably recognizable.*

Theorem 2.3 can be used to prove that also some other relevant group classes defined by closure properties in the profinite topology are countably recognizable. In fact, if Θ is any subgroup property such that $X \cap H$ is a Θ-subgroup of H whenever X is a Θ-subgroup of a group G and $H \leq G$, it follows that the class of groups whose Θ-subgroups are closed is countably recognizable. For instance, we have that the class of groups whose abelian subgroups are closed is countably recognizable, while if we apply this remark to the property of being a normal subgroup, we obtain the following interesting result.

Corollary 2.10. *The class of groups whose homomorphic images are residually finite is countably recognizable.*

Note that the above corollary can also be obtained as a special case of the following result.

Lemma 2.11. *Let $\mathfrak{X}$ be a countably recognizable class of groups. Then also the subclass $\mathfrak{X}^{\boldsymbol{H}}$ of $\mathfrak{X}$, consisting of all groups whose homomorphic images belong to $\mathfrak{X}$, is countable recognizable.*

Proof. Let G be any group whose countable subgroups belong to the class $\mathfrak{X}^{\boldsymbol{H}}$, and let N be any normal subgroup of G. If H/N is any countable subgroup of G/N, there exists a countable subgroup X of G such that $H = XN$, and so

$$H/N \simeq X/X \cap N$$

is an $\mathfrak{X}$-group. As $\mathfrak{X}$ is countably recognizable, it follows that G/N belongs to $\mathfrak{X}$. Therefore the class $\mathfrak{X}^{\boldsymbol{H}}$ is countably recognizable. □

Corollary 2.12. *Let $\mathfrak{X}$ be a subgroup closed class of finite groups. Then the class of groups whose homomorphic images are residually $\mathfrak{X}$ is countably recognizable.*

Proof. As the class of residually $\mathfrak{X}$ is countably recognizable by Corollary 2.7, the statement follows directly from Lemma 2.11. □

3. The finite residual

This section is devoted to the study of countably detectable properties of the finite residual. For any group G, we shall denote by $J(G)$ the finite residual of G.

Proof of Theorem B − Let G be a group such that the finite residual of every countable subgroup of G belongs to $\mathfrak{X}$, and let $\mathcal{C}$ be the set of all countable subgroups of G. For each countable subgroup H of G, the set-theoretic union

$$L(H) = \bigcup_{C \in \mathcal{C}} \big(H \cap J(\langle H, C\rangle)\big)$$

is obviously a subgroup of H. If h is any element of $L(H)$, there exists a countable subgroup K_h of G containing H such that h lies in $H \cap J(K_h)$. Then

$$K = \langle K_h \mid h \in L(H)\rangle$$

is a countable subgroup of G and $L(H)$ is contained in the finite residual $J(K)$ of K. It follows that $L(H) = H \cap J(K)$ is the largest element of the set

$$\big\{H \cap J(\langle H, C\rangle) \mid C \in \mathcal{C}\big\}.$$

Moreover $L(H_1) \leq L(H_2)$ whenever H_1 and H_2 are countable subgroups of G such that $H_1 \leq H_2$, and so

$$L = \bigcup_{H \in \mathcal{C}} L(H)$$

is a subgroup of G. Let X be any countable subgroup of L, and let x be an arbitrary ele ment of X. Then there exist countable subgroups V_x and W_x of G such that $V_x \leq W_x$ and x lies in $L(V_x) = V_x \cap J(W_x)$, and hence $X \leq J(W)$, where

$$W = \langle W_x \mid x \in X \rangle$$

is countable. Therefore X belongs to the subgroup closed class $\mathfrak{X}$, and so L itself is an element of $\mathfrak{X}$, because $\mathfrak{X}$ is countably recognizable.

Let Y be any countable subgroup of G. Then $L \cap Y$ is a countable subgroup of L, and so there exists a countable subgroup E of G containing Y such that $L \cap Y \leq L(E)$. Then $L(E) = E \cap J(H)$ for some countable subgroup $H \geq E$, and hence

$$L(Y) \leq L \cap Y \leq L(E) \cap Y = E \cap J(H) \cap Y = J(H) \cap Y \leq L(Y).$$

It follows that $L \cap Y = J(H) \cap Y$ is a closed subgroup of Y, and an application of Theorem 2.3 yields that L is a closed subgroup of G. Therefore the finite residual $J(G)$ of G is contained in L, and hence it belongs to $\mathfrak{X}$. □

Corollary 3.1. *The class $\boldsymbol{P}(\boldsymbol{R}\mathfrak{F})$ of all groups admitting a finite series with residually finite factors is countably recognizable.*

Proof. For each positive integer n, let $(\boldsymbol{R}\mathfrak{F})^n$ be the class of all groups admitting a finite series of length at most n whose factors are residually finite. An obvious induction argument and Theorem B yield that $(\boldsymbol{R}\mathfrak{F})^n$ is countably recognizable for every n. Since it is also clear that each $(\boldsymbol{R}\mathfrak{F})^n$ is subgroup closed, it follows that also the class

$$\boldsymbol{P}(\boldsymbol{R}\mathfrak{F}) = \bigcup_{n \in \mathbb{N}} (\boldsymbol{R}\mathfrak{F})^n$$

is countably recognizable (see for instance [5], Lemma 2.1). □

It is known that the class of imperfect groups is countably recognizable (see for instance [5], Lemma 4.10). Our next result shows that also the class of groups which are not $\mathfrak{F}$-perfect can be countably detected; recall here that a group is called *$\mathfrak{F}$-perfect* if it has no proper subgroups of finite index.

Theorem 3.2. *Let G be a non-trivial group in which every countable non-trivial subgroup has a proper subgroup of finite index. Then G itself contains a proper subgroup of finite index.*

Proof. Assume that the statement is false, and suppose first that each element of G belongs to the finite residual of some countable subgroup of G. Fix a non-trivial element x of G, and let X_1 be a countable subgroup of G such that x belongs to the finite residual $J(X_1)$ of X_1. Assume now that a countable subgroup X_n has been chosen for some positive integer n. If y is any element of X_n, there exists a countable subgroup H_y of G such that $X_n \leq H_y$ and y lies in $J(H_y)$, and

$$X_{n+1} = \langle H_y \mid y \in X_n \rangle$$

is a countable subgroup of G such that $X_n \leq J(X_{n+1})$. It follows that

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

is a countable subgroup of G which coincides with its finite residual, i.e. which has no proper subgroups of finite index. This contradiction shows that there exists a non-trivial element g of G such that every countable subgroup K of G contains a subgroup of finite index $H(K)$ such that $g \notin K$ and the index $h(K) = |K : H(K)|$ is smallest possible.

Let $\mathscr{E}$ be the set of all finitely generated subgroups of G containing g, and let $\mathscr{F}$ be any countable subset of $\mathscr{E}$. Then

$$F = \langle X \mid X \in \mathscr{F} \rangle$$

is a countable subgroup of G and $h(X) \leq h(F)$ for all $X \in \mathscr{F}$. It follows that there exists a positive integer m such that $h(E) \leq m$ for all $E \in \mathscr{E}$. For each element E of $\mathscr{E}$, consider the set $\mathscr{L}(E)$ of all subgroups L of E such that $g \notin L$ and $|E : L| \leq m$. Then each $\mathscr{L}(E)$ is a finite non-empty set, and if E and F are elements such that $F \leq E$, the intersection map $\alpha_{E,F}$ goes from $\mathscr{L}(E)$ into $\mathscr{L}(F)$. Therefore

$$\{\mathscr{L}(E), \alpha_{E,F} \mid E, F \in \mathscr{E},\ F \leq E\}$$

is an inverse system, and its inverse limit

$$\mathscr{L} = \varprojlim \mathscr{L}(E)$$

is not empty by Lemma 2.2. If $(Y_E)_{E \in \mathscr{E}}$ is an element of $\mathscr{L}$, it is easy to prove that

$$Y = \bigcup_{E \in \mathscr{E}} Y_E$$

is a subgroup of finite index of G, and $g \notin Y$. This contradiction completes the proof. $\square$

Observe that the argument of the above proof can also be used to show that, for any set π of prime numbers, the class of groups admitting a homomorphic image which is a finite non-trivial π-group is countably detectable.

Since a group G has a descending series with finite factors if and only if every non-trivial subgroup of G contains a proper subgroup of finite index, Theorem 3.2 has the following consequence.

Corollary 3.3. *The class $\grave{P}\mathfrak{F}$ of groups admitting a descending series with finite factors is countably recognizable.*

Notice that $\grave{P}_n(R\mathfrak{F}) = \grave{P}(R\mathfrak{F}) = \grave{P}\mathfrak{F}$, because the finite residual of any group is a characteristic subgroup, and so the above corollary should also be seen in relation to Corollary 3.1.

We shall say that an arbitrary group class $\mathfrak{X}$ has *countable character* (or that $\mathfrak{X}$ is $L_{\aleph_0}$*-closed*) if a group G belongs to $\mathfrak{X}$ whenever each countable subgroup of G is contained in some $\mathfrak{X}$-subgroup. Of course, every class of countable character is countably recognizable, and for subgroup closed group classes these two concepts coincide. On the other hand, a countable recognizable class need not have countable character: to see this, it is enough to consider the class $\mathfrak{A}_0^*$ formed by the trivial group and by all countable non-abelian groups, and observe that if G is any uncountable non-abelian group, then each countable subgroup of G lies in some countable non-abelian subgroup.

Recall that a topological group is said to be *profinite* if it is isomorphic to the inverse limit of an inverse system of finite groups endowed with discrete topologies. It is well known that a topological group is profinite if and only if it is compact and totally disconnected (see [16], Corollary 1.2.4). Of course, any profinite group is residually finite. The following example shows that the class of profinite groups does not have countable character.

Let $C = \{0, 1\}$ be the group with two elements, and in the cartesian power C^I of C over a set I of cardinality $\aleph_1$ consider the subgroup G consisting of all elements with countable support, endowed with the topology induced by the product topology of C^I. If X is any countable subgroup of G, there exists a countable subset I_0 of I such that $X \leq C^{I_0} \leq G$, and C^{I_0} is compact by Tychonoff's theorem. Since G is totally disconnected (see for instance [2], Theorem 1.8), it follows that every countable subgroup of G is contained in a profinite subgroup. On the other hand, the group G is not profinite because it is not compact.

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