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$\mathbb{Z}R$ and rings of Witt vectors $W_S(R)$

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Abstract – Using λ operations, we give some results on the kernel of the natural map from the monoid algebra $\mathbb{Z}R$ of a commutative ring R to the ring of S-Witt vectors of R. As a byproduct we obtain a very natural interpretation of a power series used by Dwork in his proof of the rationality of zeta functions for varieties over finite fields.

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Contents

1. Introduction

For a commutative ring R, let $\mathbb{Z}R$ be the monoid algebra of $(R, \cdot)$. Let T be a divisor stable subset of the natural numbers $\mathbb{N}$ and consider the ring $W_T(R)$ of T-Witt vectors. The Teichmüller map $R \to W_T(R)$ is multiplicative and hence extends uniquely to a ring homomorphism $\alpha_T : \mathbb{Z}R \to W_T(R)$. We are interested in the kernel of this map. If R has no T-torsion, the ghost

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map $\mathcal{G}_T : W_T(R) \to R^T$ is injective and hence $\ker \alpha_T = \ker(\mathcal{G}_T \circ \alpha_T)$ consists of the elements $x = \sum_{r\in R} n_r[r] \in \mathbb{Z}R$ which satisfy the equations

$$\sum_{r\in R} n_r r^\nu = 0 \quad \text{for } \nu \in T\,.$$

If R is a perfect $\mathbb{F}_p$-algebra and $T = \{1, \ldots, p^{n-1}\}$ then $\ker \alpha_T = I^n$, where I is the kernel of the map $\mathbb{Z}R \to R$ sending x to $\sum n_r r$. In this case the induced map $\mathbb{Z}R/I^n \xrightarrow{\sim} W_T(R)$ is actually an isomorphism, see [CD14]. More generally, $\ker \alpha_T$ is known for all $\mathbb{F}_p$-algebras R with injective Frobenius map, see [CD15] Theorem 7.1.

In the present note we use λ-ring structures to describe $\ker \alpha_T$ for more general rings R and certain subsets $T = S_N$ obtained as follows. Fix a divisor stable subset S which is also multiplicatively closed. Thus S consists of all natural numbers whose prime divisors lie in a given set of prime numbers. Fix some $1 \le N \le \infty$ and set $S_N := \{\nu \in S \mid \nu < N\}$. For a $\mathbb{Z}_S = \mathbb{Z}[p^{-1}, p \notin S]$-algebra R all its (truncated) Witt rings are $\mathbb{Z}_S$-algebras as well, see [Hes15], Lemma 1.9. Thus the Teichmüller map $R \to W_{S_N}(R)$ induces a homomorphism of $\mathbb{Z}_S$-algebras:

$$\alpha_{S_N} : \mathbb{Z}_S R = \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}_S \longrightarrow W_{S_N}(R)\,. \tag{1}$$

Let $\pi : \mathbb{Z}_S R \to R$ be the map sending $\sum n_r[r]$ to $\sum n_r r$, and for an integer $n \ge 1$ write $n_S = \prod_{p\in S} p^{\mathrm{ord}_p(n)}$. Then $n_S \in S$ since S is multiplicatively closed. The following result holds:

Theorem 1.1. *Consider the unique (special) λ-ring structure (λ^n_S) on $\mathbb{Z}_S R$ whose associated Adams operators $\psi^n_S : \mathbb{Z}_S R \to \mathbb{Z}_S R$ are determined by the formula $\psi^n_S[r] = [r]^{n_S}$ for $r \in R$. Then we have*

$$\ker \alpha_{S_N} = \{x \in \mathbb{Z}_S R \mid \pi\lambda^n_S(x) = 0 \textit{ for } 1 \le n < N\}\,.$$

Existence and uniqueness of the special λ-ring structure are special cases of a classical result [Wil82], Proposition 1.2.

Lemma 1.2 (Wilkerson). *Let B be commutative ring without $\mathbb{Z}$-torsion and for $n \ge 1$ let ψ^n be a family of ring endomorphisms of B such that $\psi^1 = \mathrm{id}$ and $\psi^n \circ \psi^m = \psi^{nm}$ and such that $\psi^p(b) \equiv b^p \bmod pB$ for all $b \in B$ and all prime numbers p. Then there is a unique structure of a (special) λ-ring on B whose Adams operators are the given maps ψ^n.*

The main ingredient in the proof of Theorem 1.1 is a (unital) ring homomorphism $\overline{\varphi}_S : W_S(R) \to W(R)$ for $\mathbb{Z}_S$-algebras R which splits the canonical projection $W(R) \to W_S(R)$.

Adapting a method of Dwork in the theory of p-adic formal power series, we obtain explicit albeit complicated formulas for the operations λ^n_S in Theorem 1.1. They are given as follows. Let μ be the Moebius function and for $x \in \mathbb{Z}_S R$ and $k \in S$ set:

$$\tau_k(x) = k^{-1} \sum_{d|k} \mu(d) \psi_S^{k/d}(x) \; . \tag{2}$$

For an S-tuple $\nu = (\nu_k)_{k \in S}$ with all $\nu_k \geq 0$ we write

$$\binom{\tau(x)}{\nu} = \prod_{k \in S} \binom{\tau_k(x)}{\nu_k} \in \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Q} \; .$$

Set $|\nu| = \sum_{k \in S} \nu_k$ and $\|\nu\| = \sum_{k \in S} k\nu_k$.

Theorem 1.3. *In the situation of Theorem 1.1, for $n \geq 1$ and $x \in \mathbb{Z}_S R$ the following explicit formula holds in $\mathbb{Z}_S R$:*

$$(-1)^n \lambda^n_S(x) = \sum_{\|\nu\| = n} (-1)^{|\nu|} \binom{\tau(x)}{\nu} \; .$$

The methods work in the more general situation where instead of $\mathbb{Z}_S R$ we start with a $\mathbb{Z}$-torsionfree $\mathbb{Z}_S$-algebra B which is equipped with commuting Frobenius lifts ψ^p_S for all primes $p \in S$. Setting $\psi^p_S = \mathrm{id}$ for $p \notin S$, we show that the corresponding λ-ring structure is given by the same formula as in Theorem 1.3. Moreover any homomorphism $\pi : B \to R$ into a $\mathbb{Z}_S$-algebra R factors canonically over maps $\alpha_{S_N} : B \to W_{S_N}(R)$ and the kernel of α_{S_N} is obtained as in Theorem 1.1.

For background on the theory of Witt vectors we refer to [Hes15], [Rab14], [BW05] and [CD15]. The latter approaches avoid universal polynomials.

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2. A projector on Witt vector rings

All rings are associative, commutative and unital. All ring homomorphisms are unital. For any commutative ring A we give A the discrete topology and $A^{\mathbb{N}}$ the product topology. Then $W(A) \equiv A^{\mathbb{N}}$ is a topological ring and the ghost map $\mathcal{G} : W(A) \to A^{\mathbb{N}}$ is continuous.

For S as in the introduction, consider the ring homomorphism:

$$\varphi_S : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}} \; , \; \varphi_S((a_n)_{n\ge 1}) = (a_{n_S})_{n\ge 1} \; .$$

It maps $0 \times A^{\mathbb{N}\setminus S}$ to zero and therefore factors over $A^S \cong A^{\mathbb{N}}/(0 \times A^{\mathbb{N}\setminus S})$. Note that $\varphi_S^2 = \varphi_S$. For $k \ge 1$, Frobenius and Verschiebung maps from $A^{\mathbb{N}}$ to $A^{\mathbb{N}}$ are defined as follows:

$$F_k((a_n)_{n\ge 1}) = (a_{nk})_{n\ge 1} \quad \text{and} \quad V_k((a_n)_{n\ge 1}) = k(\delta_{k|n} a_{n/k})_{n\ge 1} \; .$$

Here, $\delta_{k|n} = 1$ if $k \mid n$ and $= 0$ if $k \nmid n$. Now assume that A is a $\mathbb{Z}_S$-algebra and for any prime $l \notin S$ consider the map

$$T_l = 1 + l^{-1} V_l (1 - F_l) : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}} \; .$$

It is a ring endomorphism because of the formula:

$$T_l((a_n)_{n\ge 1}) = (b_n)_{n\ge 1}$$

where $b_n = a_n$ if $l \nmid n$ and $b_n = a_{n/l}$ if $l \mid n$. For different primes l and l' not in S, the endomorphisms T_l and $T_{l'}$ commute with each other. For m prime to S, set

$$T_m = \prod_l T_l^{\mathrm{ord}_l m} \; .$$

Then the following limit formula holds in the pointwise topology:

$$\varphi_S = \lim_{\nu\to\infty} T_{m_\nu} \; . \tag{3}$$

Here (m_ν) is any sequence of positive integers prime to S such that $m_\nu \mid m_{\nu+1}$ for all ν and such that any number prime to S is a divisor of some m_ν. For example, if $l_1, l_2, \ldots$ are the primes not in S we could take $m_\nu = (l_1 \cdots l_\nu)^\nu$.

There are also other ways to express φ_S. Firstly, we have

$$\lim_{\nu\to\infty} T_{l^\nu} = \Big(\sum_{k=0}^{\infty} l^{-k} V_{l^k} \Big)(1 - l^{-1} V_l F_l) \; .$$

This can be either verified directly by looking at the action on sequences or deduced from the formula for T_l and the identity $F_k \circ V_k = k$. This leads to the following formula, where the sums are over all positive integers prime to S:

$$\varphi_S = \Big(\sum_{(n,S)=1} n^{-1} V_n \Big) \Big(\sum_{(n,S)=1} \mu(n) n^{-1} V_n F_n \Big) .$$

Frobenius and Verschiebung operators also exist on the ring $W(A)$ of (big) Witt vectors and they correspond to Frobenius and Verschiebung on $A^{\mathbb{N}}$ via the ghost map $\mathcal{G} : W(A) \to A^{\mathbb{N}}$. If the $\mathbb{Z}_S$-algebra A has no $\mathbb{Z}$-torsion, then $\mathcal{G}$ is injective and it follows from formula (3) that there is a unique ring homomorphism $\varphi_S : W(A) \to W(A)$ making the diagram

$$\begin{array}{ccc} W(A) & \overset{\mathcal{G}}{\hookrightarrow} & A^{\mathbb{N}} \\ {\scriptstyle \varphi_S}\downarrow & & \downarrow{\scriptstyle \varphi_S} \\ W(A) & \overset{\mathcal{G}}{\hookrightarrow} & A^{\mathbb{N}} \end{array}$$

commute. It also follows that φ_S factors uniquely over the canonical projection $\varphi_S : W(A) \xrightarrow{\mathrm{pr}_S} W_S(A) \xrightarrow{\overline{\varphi}_S} W(A)$.

We have

$$\mathrm{pr}_S \circ \overline{\varphi}_S = \mathrm{id} \, , \tag{4}$$

since this is true after applying the ghost map.

Now let R be *any* $\mathbb{Z}_S$-algebra and define $\varphi_S : W(R) \to W(R)$ by the pointwise limit (3). Convergence to a well defined ring homomorphism follows by comparison with the map φ_S for a $\mathbb{Z}$-torsion free $\mathbb{Z}_S$-algebra A surjecting onto R. In the same way we prove a unique factorization:

$$\varphi_S : W(R) \xrightarrow{\mathrm{pr}_S} W_S(R) \xrightarrow{\overline{\varphi}_S} W(R) \tag{5}$$

and the formula

$$\mathrm{pr}_S \circ \overline{\varphi}_S = \mathrm{id} \quad \text{on } W_S(R) \, . \tag{6}$$

In particular the (unital) ring homomorphism $\overline{\varphi}_S : W_S(R) \hookrightarrow W(R)$ is injective. By construction the maps φ_S and $\overline{\varphi}_S$ are functorial with respect to R. It is clear that $\varphi_S^2 = \varphi_S$.

We need a version of the maps $\overline{\varphi}$ for the truncation sets S_N: For any $\mathbb{Z}_S$-algebra R without $\mathbb{Z}$-torsion, it follows by comparing with the ghost side that there is a unique ring homomorphism $\overline{\varphi}_{S_N}$ such that the diagram

(7)
$$\begin{array}{ccc} W_S(R) & \twoheadrightarrow & W_{S_N}(R) \\ {\scriptstyle \overline{\varphi}_S}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}_{S_N}} \\ W(R) & \twoheadrightarrow & W_N(R) \end{array}$$

commutes. Here $W_N(R) = W_{\{1 \le \nu < N\}}(R)$. The point is that $n < N$ implies $n_S < N$. For the projection $\mathrm{pr}_{S_N} : W_N(R) \to W_{S_N}(R)$ we have $\mathrm{pr}_{S_N} \circ \overline{\varphi}_{S_N} = \mathrm{id}$. Similarly as before, it follows that unique functorial ring homomorphisms $\overline{\varphi}_{S_N}$ with the same properties exist for arbitrary $\mathbb{Z}_S$-algebras R.

3. Proofs of Theorems 1.1 and 1.3

For any ring A we give the set $\Lambda(A) = 1 + tA[[t]]$ the unique ring structure, for which the bijection

(8)
$$W(A) = A^{\mathbb{N}} \xrightarrow{\sim} \Lambda(A)\ ,\ (a_1, a_2, \ldots) \longmapsto \prod_{n=1}^{\infty} (1 - a_n t^n)$$

is an isomorphism. Then $\Lambda(A)$ is a topological ring for the t-adic topology whose multiplication is uniquely determined by the formula

$$(1 - a_1 t) \cdot (1 - a_2 t) = 1 - a_1 a_2 t \quad \text{for } a_1, a_2 \in A\ .$$

The addition in $\Lambda(A)$ is given by the multiplication of power series. We will usually view the topological isomorphism (8) as an identification. There is a commutative diagram of ring homomorphisms

(9)
$$\begin{array}{ccc} W(A) & \xrightarrow{\mathcal{G}} & A^{\mathbb{N}} \\ \Vert & & \Vert \\ \Lambda(A) & \xrightarrow{-t\partial_t \log} & tA[[t]]\ . \end{array}$$

Here we view $tA[[t]]$ as a commutative ring with the coefficientwise multiplication of power series, the Hadamard product.

As before let $S \subset \mathbb{N}$ be divisor stable and multiplicatively closed. Let B be a $\mathbb{Z}$-torsion free $\mathbb{Z}_S$-algebra with commuting Frobenius lifts ψ_S^p for all primes $p \in S$. For $n \ge 1$ we set

$$\psi_S^n = \prod_{p \in S} (\psi_S^p)^{\mathrm{ord}_p n}\ .$$

Let (λ_S^n) be the special λ-ring structure on B with Adams operators ψ_S^n, according to Lemma 1.2. Consider the ring homomorphism

$$\lambda_S : B \longrightarrow \Lambda(B)$$

defined by the formula

$$\lambda_S(x) = \sum_{i=0}^{\infty} (-1)^i \lambda_S^i(x) t^i \; .$$

Setting

$$\psi_S(x) = \sum_{n=1}^{\infty} \psi_S^n(x) t^n$$

we have

$$\psi_S(x) = -t\partial_t \log \lambda_S(x) \; .$$

Using diagram (9) for $A = B$ we may interpret λ_S as the unique ring homomorphism $\tilde{\alpha} : B \to W(B)$ such that $\mathcal{G} \circ \tilde{\alpha}$ maps $x \in B$ to $(\psi_S^n(x))_{n \geq 1} \in B^{\mathbb{N}}$. We have $\tilde{\alpha} = \varphi_S \circ \tilde{\alpha}$ since after applying the injective ghost map, this amounts to the equality $\psi_S^n = \psi_S^{ns}$ for $n \geq 1$. Hence we get

$$\tilde{\alpha} = \varphi_S \circ \tilde{\alpha} = \overline{\varphi}_S \circ \mathrm{pr}_S \circ \tilde{\alpha} = \overline{\varphi}_S \circ \tilde{\alpha}_S \; .$$

Here $\tilde{\alpha}_S = \mathrm{pr}_S \circ \tilde{\alpha} : B \to W_S(B)$ is the unique ring homomorphism such that $\mathcal{G}_S \circ \tilde{\alpha}_S$ maps $x \in B$ to $(\psi_S^n(x))_{n \in S} \in B^S$. In conclusion, we have a commutative diagram:

(10)
$$\begin{array}{ccc} B & \xrightarrow{\tilde{\alpha}_S} & W_S(B) \\ {\scriptstyle \lambda_S}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}_S} \\ \Lambda(B) & = & W(B) \; . \end{array}$$

Remark. In the case $B = \mathbb{Z}_S R$ considered in Theorem 1.1, the map $\tilde{\alpha}_S$ is the unique $\mathbb{Z}_S$-algebra homomorphism extending the multiplicative map $R \to W_S(\mathbb{Z}_S R)$ which sends r to the Teichmüller representative of $[r]$. This follows by comparing ghost components.

Let $\pi : B \to R$ be a map of $\mathbb{Z}_S$-algebras. For $1 \leq N \leq \infty$ consider the composition

$$\alpha_{S_N} : B \xrightarrow{\tilde{\alpha}_S} W_S(B) \xrightarrow{W_S(\pi)} W_S(R) \longrightarrow W_{S_N}(R) \; .$$

For $B = \mathbb{Z}_S R \xrightarrow{\pi} R$, by the remark on $\tilde{\alpha}_S$ above, α_{S_N} agrees with the map (1) in the introduction. Hence Theorems 1.1 and 1.3 are special cases of the following result:

THEOREM 3.1. *With notations as above, we have*

$$\operatorname{Ker} \alpha_{S_N} = \{x \in B \mid \pi \lambda^n_S(x) = 0 \quad \textit{for } 1 \le n < N\} \; .$$

Moreover, with $\tau_k(x) \in B \otimes \mathbb{Q}$ as in (2), the following formula holds in $\Lambda(B)$:

$$\lambda_S(x) = \prod_{k \in S} (1 - t^k)^{\tau_k(x)} \; . \tag{11}$$

Equivalently, with notations as in Theorem 1.3 we have:

$$(-1)^n \lambda^n_S(x) = \sum_{\|\nu\| = n} (-1)^{|\nu|} \binom{\tau(x)}{\nu} \quad \textit{in } B, \textit{ for all } n \ge 1 \; .$$

PROOF. Using diagrams (7), (10) and the functoriality of $\overline{\varphi}_S$ we get a commutative diagram

$$\begin{array}{ccccccc}
B & \xrightarrow{\tilde{\alpha}_S} & W_S(B) & \xrightarrow{W_S(\pi)} & W_S(R) & \longrightarrow & W_{S_N}(R) \\
\Big\downarrow{\scriptstyle \lambda_S} & & \Big\downarrow{\scriptstyle \overline{\varphi}_S} & & \Big\downarrow{\scriptstyle \overline{\varphi}_S} & & \Big\downarrow{\scriptstyle \overline{\varphi}_{S_N}} \\
\Lambda(B) & = & W(B) & \xrightarrow{W(\pi)} & W(R) & \longrightarrow & W_N(R) \; .
\end{array}$$

Identifying $W_N(R)$ with $\Lambda_N(R) = \Lambda(R)/(1 + t^N R[[t]])$, the outer square becomes

$$\begin{array}{ccc}
B & \xrightarrow{\alpha_{S_N}} & W_{S_N}(R) \\
\Big\downarrow{\scriptstyle \lambda_S} & & \Big\downarrow{\scriptstyle \overline{\varphi}_{S_N}} \\
\Lambda(B) & \xrightarrow{\Lambda(\pi)} & \Lambda_N(R) \; .
\end{array}$$

Since $\overline{\varphi}_{S_N}$ is injective being a splitting of the projection $\mathrm{pr}_{S_N} : W_N(R) \to W_{S_N}(R)$, the first assertion of Theorem 3.1 follows. In order to prove formula (11) it suffices to show the equality after applying $-t\partial_t \log$, i.e. the formula

$$\psi_S(x) = \sum_{k=1}^{\infty} \tau_k(x) \frac{kt^k}{1 - t^k} \; . \tag{12}$$

Generally, we have the identity of formal power series

$$\sum_{k=1}^{\infty} a_k \frac{t^k}{1 - t^k} = \sum_{n=1}^{\infty} A_n t^n \; .$$

where $A_n = \sum_{\nu|n} a_\nu$. In our case $a_k = k\tau_k(x)$, we obtain

$$A_n = \sum_{\nu|n}\sum_{d|\nu} \mu(d)\psi_S^{\nu/d}(x) = \psi_S^n(x) ,$$

by Moebius inversion. Thus formula (12) and hence the Theorem are proved. □

A priori the product $\prod_{k\in S}(1-t^k)^{\tau_k(x)}$ lies in $\Lambda(B\otimes\mathbb{Q})$ but its equality with $\lambda_S(x)$ shows that it lies in $\Lambda(B)$. The required integrality comes from Wilkerson's Lemma 1.2 and the congruences used in its proof. For $S = \{1, p, p^2, \ldots\}$ such products were considered by Dwork in his proof of Weil's rationality conjecture for zeta functions of varieties over finite fields. For $p^i \in S$ we have:

$$\tau_1(x) = x \quad \text{and} \quad \tau_{p^i}(x) = p^{-i}(\psi_S^{p^i}(x) - \psi_S^{p^{i-1}}(x)) \text{ for } i \geq 1 .$$

Thus equation (11) asserts:

$$\lambda_S(x) = (1-t)^x \prod_{i=1}^{\infty}(1-t^{p^i})^{\tau_{p^i}(x)} . \tag{13}$$

In [Dwo67] p. 2, using slightly different notation, Dwork considers the following product in the formal power series ring $\mathbb{Q}[[t, X]]$:

$$F(X,t) = (1+t)^X \prod_{i=1}^{\infty}(1+t^{p^i})^{p^{-i}(X^{p^i}-X^{p-1})} . \tag{14}$$

Using his well-known criterion [Dwo67] Lemma 1, he shows that the coefficients of $F(X,t)$ are p-integral. Our sign conventions concerning Λ- and Witt rings are not quite compatible with Dwork's. However, for odd p we can relate $F(X,t)$ to λ_S and φ_S as follows. For S as above, we have $\mathbb{Z}_S = \mathbb{Z}[l^{-1} \mid l \neq p]$. Equip the $\mathbb{Z}$-torsion free $\mathbb{Z}_S$-algebra $B = \mathbb{Z}_S[X]$ with the Frobenius lift ψ_S^p defined by $\psi_S^p(X) = X^p$. Then, as in the beginning of this section the corresponding λ-ring structure on $\mathbb{Z}_S[X]$ is encoded in a ring homomorphism

$$\lambda_S : \mathbb{Z}_S[X] \longrightarrow \Lambda(\mathbb{Z}_S[X]) = 1 + t\mathbb{Z}_S[X][[t]] \subset \mathbb{Z}_S[[t, X]] .$$

Comparing (13) and (14), we see that for $p \neq 2$ we have:

$$F(X,-t) = \lambda_S(X) . \tag{15}$$

In particular the p-integrality of (14) follows. In terms of the map

$$\overline{\varphi}_S : W_S(\mathbb{Z}_S[X]) \to W(\mathbb{Z}_S[X]) \equiv \Lambda(\mathbb{Z}_S[X])$$

we have

$$F(X, -t) = \overline{\varphi}_S(\langle X \rangle) \; .$$

Here $\langle X \rangle$ is the Teichmüller representative of X. This follows from diagram (10) and formula (15) noting that $\tilde{\alpha}_S(X) = \langle X \rangle$. The latter equality holds because $(\mathcal{G}_S \circ \tilde{\alpha}_S)(X) = (\psi_S^{p^i}(X)) = (X^{p^i})$ by the characterization of $\tilde{\alpha}_S$ and since $\mathcal{G}_S(\langle X \rangle) = (X^{p^i})$.

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