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Regularity Results for Quasilinear Degenerate Elliptic Obstacle Problems in Carnot Groups

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Abstract – Let $\{X_1,\dots,X_m\}$ be a basis of the space of horizontal vector fields on the Carnot group $\mathbb{G}=(\mathbb{R}^N,\circ)(m<N)$. We establish regularity results for solutions to the following quasilinear degenerate elliptic obstacle problem

$$\int_\Omega \langle\langle AXu,Xu\rangle^{\frac{p-2}{2}}AXu,X(v-u)\rangle dx \geq \int_\Omega B(x,u,Xu)(v-u)dx + \int_\Omega \langle f(x),X(v-u)\rangle dx,\quad \forall v\in\mathcal{K}^\theta_\psi(\Omega),$$

where $A=(a_{ij}(x))_{m\times m}$ is a symmetric positive-definite matrix with measurable coefficients, p is close to 2, $\mathcal{K}^\theta_\psi(\Omega)=\{v\in HW^{1,p}(\Omega): v\geq\psi \text{ a.e. in } \Omega, v-\theta\in HW^{1,p}_0(\Omega)\}$, ψ is a given obstacle function, θ is a boundary value function with $\theta\geq\psi$. We first prove the $C^{0,\alpha}_X$ regularity of solutions provided that the coefficients of A are of vanishing mean oscillation (VMO). Then the $C^{1,\alpha}_X$ regularity of solutions is obtained if the coefficients belong to the class BMO_ω which is a proper subset of VMO.

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1. Introduction

Let $\{X_1, \dots, X_m\}$ be a basis of the space of horizontal vector fields on the Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ)$. The main purpose of this paper is to study regularity for solutions to the quasilinear degenerate elliptic obstacle problems constructed by $X_1, \dots, X_m$, namely, we consider the following variational inequality

$$\begin{aligned}
&\int_\Omega \langle \langle AXu, Xu\rangle^{\frac{p-2}{2}} AXu, X(v-u)\rangle dx \\
&\geq \int_\Omega B(x,u,Xu)(v-u)dx + \int_\Omega \langle f(x), X(v-u)\rangle dx, \quad \forall v \in \mathcal{K}^\theta_\psi(\Omega).
\end{aligned} \tag{1.1}$$

Here $A = (a_{ij}(x))_{m\times m}$ is a symmetric positive-definite matrix with measurable coefficients, Xu is the horizontal gradient of u, ψ is a given obstacle function, θ is a boundary value function with $\theta \geq \psi$, and

$$\mathcal{K}^\theta_\psi(\Omega) = \left\{ v \in HW^{1,p}(\Omega) : v - \theta \in HW^{1,p}_0(\Omega), v \geq \psi \text{ a.e. in } \Omega \right\},$$

$HW^{1,p}(\Omega)$ and $HW^{1,p}_0(\Omega)$ are Sobolev spaces introduced in Section 2.

The obstacle problem is a classic topic in the mathematical study of variational inequalities and free boundary problems in the area of partial differential equations and their applications, with crucial implications in many contexts in Physics, in Financial Mathematics, in Biology, and so on. It is also closely related to the study of minimal surfaces and the capacity of a set in potential theory as geometrical problems. For further discussions and more details on the obstacle problem and its applications we refer to [6], [21], [27], [30], [40].

Regularity for solutions to elliptic obstacle problems involving p-Laplacian functions (in the case $X = \nabla$, $A = \mathbb{I}$) has been extensively studied, see [25, 26, 37, 39, 32, 22, 7, 8, 18]. For example, Choe in [7] obtained the $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity for solutions to the following obstacle problem

$$\begin{aligned}
\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla(v-u)\rangle dx \geq &\int_\Omega B(x,u,\nabla u)(v-u)dx \\
&+ \int_\Omega \langle f(x), \nabla(v-u)\rangle dx, \quad \forall v \in \mathcal{K}^\theta_\psi(\Omega),
\end{aligned}$$

under various restrictions on B, f and ψ. In [18], Eleuteri extended these results in a sharp way and obtained the $C^{1,\alpha}$ regularity for local minimizers of the integral functionals with obstacle under growth conditions of p-type. We also refer the readers to [29, 31, 11, 12, 46] for results of regularity for solutions to elliptic equations and systems. Huang in [29] obtained the gradient estimates in the generalized Morrey spaces $L^{2,\lambda}_\varphi$ and BMO_φ of weak solutions to the linear elliptic systems

$$-D_\alpha \left(a^{\alpha\beta}_{ij}(x) D_\beta u^j \right) = g_i(x) - \mathrm{div} f^i(x), \quad i = 1, 2, \dots, N,$$

under the assumptions that $a_{ij}^{\alpha\beta}(x) \in L^\infty \cap \mathrm{VMO}$ and $a_{ij}^{\alpha\beta}(x) \in L^\infty \cap \mathrm{VMO}_\omega$, respectively. In [12], Daněček and Viszus proved the $\mathcal{L}^{2,\Phi}$ regularity for a nonlinear elliptic systems of second order with the coefficients $a_{ij}^{\alpha\beta}(x) \in L^\infty \cap \mathcal{L}^{2,\Psi}$. The authors in [46] proved the Morrey regularity and Hölder continuity for weak solutions to the following nonlinear elliptic equation

$$\mathrm{div}(\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} A(x)\nabla u) = B(x, u, \nabla u), \tag{1.2}$$

where the coefficients of $A(x)$ belong to VMO(Ω). When $B(x, u, \nabla u) = \mathrm{div}F$, Kinnunen and Zhou in [31] established the L^q $(q > p)$ estimates of weak solutions. To the best of our knowledge, there is no literature considering the obstacle problems to (1.2) for the case of non-diagonal matrices A.

Since Hörmander's celebrated paper [28], there has been a tremendous amount of work on degenerate elliptic PDEs structured on non-commuting vector fields; see [20, 41, 38, 34, 4, 44, 5, 3] and references therein. Regularity of the degenerate elliptic systems has been studied and several important results have been proved, see for instance [42, 13, 23, 16, 17, 45, 15, 14, 43]. Here we just quote some results concerning this paper. Dong and Niu [16] established gradient estimates in Morrey spaces and Hölder continuity for weak solutions to a class of degenerate elliptic systems with VMO coefficients for $p = 2$ and then generalized these results to the nondiagonal quasilinear degenerate elliptic systems, see [17]. The $C_X^{1,\alpha}$ regularity results for subelliptic p-harmonic functions (i.e. weak solutions to subelliptic p-Laplace equations) in the Heisenberg group and Carnot group with p close to 2 were shown by Domokos and Manfredi in [15] and [14]. Recently, Zheng and Zhao in [45] proved the $C_X^{1,\alpha}$ regularity of subelliptic p-harmonic systems with subcritical growth in the Carnot group if p is not too far from 2. Based on [45], Yu and Zheng [43] derived the Morrey regularity for a class of quasilinear subelliptic p-Laplace type systems with VMO coefficients in Carnot group.

As we know, in the past decades a large amount of work has been devoted to the study of regularity for solutions to the degenerate elliptic obstacle problems [9, 24, 36, 1, 10] due to their important applications in mechanical engineering, mathematical finance, image reconstruction and neurophysiology. We mention here that, under some technical assumptions, Marchi in [36] proved the $C_X^{1,\alpha}$ regularity of solutions to the double obstacle problem related to the following operator on the Heisenberg group

$$\mathrm{div}_{\mathbb{H}} A(x, Xu) - B(x, u, Xu),$$

where $A(x, Xu) = g(x)|Xu|^{p-2}Xu$, $p > 1$. Moreover, for the case $A = \mathbb{I}$, Farnana in [19] obtained the Hölder regularity results for double obstacle problems in complete metric measure spaces. Motivated by [7], [29], [12] and [45], the aim of this paper is to establish the $C_X^{0,\alpha}$ and $C_X^{1,\alpha}$ regularity for solutions to the quasilinear degenerate elliptic obstacle problem (1.1) under various assumptions. In order to state our results, we make the following hypotheses:

(H1) the matrix $A = (a_{ij}(x))_{m\times m}$ of coefficients is symmetric, positive-definite and satisfies the uniform ellipticity conditions, i.e., there exists a constant $\Lambda > 0$

such that

$$\Lambda^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2, \tag{1.3}$$

for all $\xi \in \mathbb{R}^m$ and almost every $x \in \Omega$.

(H2) (controllable growth condition) for any $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^m$,

$$|B(x, u, \xi)| \le a(g(x) + |\xi|^{p-1}), \tag{1.4}$$

where a is a positive constant and $g \in L^{p/(p-1),Q}(\Omega)$. Here Q is the homogeneous dimension of $\mathbb{G}$.

(H3) the obstacle function $\psi \in C^{1,\gamma}_X(\Omega)$, $0 < \gamma < 1$.

The main results of this paper are the following.

THEOREM 1.1. *Suppose that (H1)-(H3) hold, $f \in \mathcal{L}^{p/(p-1),Q}(\Omega)$, the coefficients $a_{ij}(x) \in \mathrm{VMO}(\Omega)$ for $i, j = 1, \cdots, m$ and $u \in \mathcal{K}^{\theta}_{\psi}(\Omega)$ is a solution to (1.1). If p is close to 2, then for any $0 < \lambda < Q$, we have $Xu \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$. Moreover, there exists $0 < \alpha < 1$ such that $u \in C^{0,\alpha}_X(\Omega)$.*

THEOREM 1.2. *Suppose that (H1)-(H3) hold, $f \in C^{0,\gamma}_X(\Omega)(0 < \gamma < 1)$ and the coefficients $a_{ij}(x) \in \mathrm{BMO}_{\omega}(\Omega)$, where $\omega(R) = R^{\gamma}$. If $u \in \mathcal{K}^{\theta}_{\psi}(\Omega)$ is a solution to the obstacle problem (1.1) with p close to 2, then $Xu \in \mathcal{L}^{p,Q+\delta}_{\mathrm{loc}}(\Omega)$ for some $\delta > 0$. Moreover, there exists $0 < \alpha < 1$ such that $u \in C^{1,\alpha}_X(\Omega)$.*

REMARK 1.3. The above results in Theorem 1.1 and Theorem 1.2 are valid only under the assumption of p close to 2 for the degenerate elliptic obstacle problem (1.1) on Carnot group, because the local boundedness of subelliptic p-harmonic functions is not necessarily true if p is too far from 2. However, the results hold true for all $1 < p < \infty$ if $X = (D_1, D_2, \cdots, D_N)$ is the usual gradient in the Euclidian space $\mathbb{R}^N$. On the other hand, for $p \neq 2$ (even close to 2) the needed estimates (see (3.9) and (3.10) in Section 3) are not yet available in the literature for generic Hörmander's vector fields.

REMARK 1.4. The above results are even new in the case $A = \mathbb{I}$. Namely, if (H1)-(H3) hold, $f \in C^{0,\gamma}_X(\Omega)(0 < \gamma < 1)$ and $u \in \mathcal{K}^{\theta}_{\psi}(\Omega)$ satisfy the variational inequality

$$\int_{\Omega} \langle |Xu|^{p-2}Xu, X(v-u)\rangle dx \ge \int_{\Omega} B(x, u, Xu)(v-u)dx + \int_{\Omega} \langle f(x), X(v-u)\rangle dx$$

for all $v \in \mathcal{K}^{\theta}_{\psi}(\Omega)$, then $u \in C^{1,\alpha}_X(\Omega)$ for some $0 < \alpha < 1$.

The techniques in our proofs are a combination of those in [17], [45] and [7]. More precisely, we first consider the following homogeneous obstacle problem corresponding to (1.1):

$$\int_\Omega \langle \langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X(\bar{v}-\bar{u})\rangle dx \geq 0 \quad \forall \bar{v} \in \mathcal{K}_\psi^\theta(\Omega), \tag{1.5}$$

and prove the higher integrability for horizontal gradients of solutions to (1.5) by constructing a suitable test function and using the Gehring lemma on metric measure space. With the higher integrability and the local boundedness result of horizontal gradients of subelliptic p-harmonic functions with p close to 2 proved in [45], a Morrey type estimate (when A belongs to VMO) and a Campanato type estimate (when A belongs to BMO_ω) for horizontal gradients of solutions to (1.5) are obtained. Based on these estimates, we can prove the Morrey and Campanato type estimates for horizontal gradients of solutions to (1.1). Finally, the $C_X^{0,\alpha}$ and $C_X^{1,\alpha}$ regularity for solutions to (1.1) are proved by exploiting the integral characterization of Hölder continuous functions in the Carnot-Carathéodory space.

The remainder of the paper is organized as follows. In Section 2, we recall some basic facts of Carnot group and some preliminary results concerning the Carnot-Carathéodory metric. In Section 3, we prove a Morrey type estimate for solutions to (1.5) based on the higher integrability and the Morrey type estimate for solutions to a degenerate elliptic equation with constant coefficients. In Section 4, Theorem 1.1 is proved by applying the previous Morrey type estimate to (1.5). Section 5 is dedicated to the Campanato type estimate for solutions to (1.5). Finally, in Section 6 we get the Campanato type estimate for solutions to (1.1) and then prove Theorem 1.2.

2. Some Preliminaries

We start with some notations on Carnot groups. For the more details, we refer to the monograph [2].

Let $\mathbb{G} = (\mathbb{R}^N, \circ)$ be a Carnot group of step $r \geq 2$, that is, a simply connected Lie group whose Lie algebra $\mathfrak{g}$ has the dimension N, and admits a decomposition $\mathfrak{g} = \oplus_{i=1}^r V_j$ such that $[V_1, V_j] = V_{j+1}$ if $1 \leq j \leq r-1$ and $[V_1, V_r] = 0$. The homogeneous dimension of $\mathbb{G}$ is $Q = \sum\limits_{i=1}^r i m_i$, where $m_i = \dim V_i$ is the topological dimension with $m_1 = m$. Let $X = \{X_1, ..., X_m\}$ be an orthonormal basis of V_1. We call $X_1, ..., X_m$ horizontal vector fields, because they generate the horizontal distribution for the related subriemannian geometry. From the fact that V_1 generates $\mathfrak{g}$ as an algebra, we know that the family of vector fields $\{X_1, ..., X_m\}$ satisfies the Hörmander finite rank condition: $\operatorname{rank}(\mathrm{Lie}\{X_1, ..., X_m\}) = N$. Denote by $Xu = (X_1 u, ..., X_m u)$ the horizontal gradient of a function u and write $|Xu(x)| = \left(\sum_{j=1}^m |X_j u(x)|^2\right)^{\frac{1}{2}}$. The most important non-trivial example of Carnot group of step two is the Heisenberg group $\mathbb{H}^n$.

An absolutely continuous curve $\gamma : [a, b] \to \mathbb{G}$ is said to be X-subunit if there exist functions $c_i(t)$, $a \le t \le b$, satisfying

$$\sum_{i=1}^{m} c_i(t)^2 \le 1 \text{ and } \gamma'(t) = \sum_{i=1}^{m} c_i(t) X_i(\gamma(t)) \text{ a.e. } t \in [a, b].$$

The Carnot-Carathéodory distance $d_X(x, y)$ is defined by

$$d_X(x, y) = \inf \{T > 0 : \text{ there is a } X\text{-subunit curve } \gamma, \text{ with } \gamma(0) = x,\ \gamma(T) = y\}.$$

Due to the structure of Carnot group, it is known that d_X is left invariant and 1-homogeneous on $\mathbb{G}$. The metric ball is denoted by

$$B_r(x) = \{y \in \mathbb{G} : d_X(x, y) < r\}.$$

If $\sigma > 0$ and $B = B(x_0, r)$, we will write $\sigma B = B(x_0, \sigma r)$. Since the Haar measure on $\mathbb{G}$ is the Lebesgue measure in $\mathbb{R}^N$, we have

$$|B(x, r)| = c_{\mathbb{G}} r^Q. \tag{2.1}$$

Here $|B(x, r)|$ denotes the Lebesgue measure of $B(x, r)$, $c_{\mathbb{G}}$ is a positive constant.

Next, we define the Sobolev space with respect to the horizontal vector fields X. For any $1 < p < \infty$ and bounded domain $\Omega \subset \mathbb{G}$, we let

$$HW^{1,p}(\Omega) = \{u \in L^p(\Omega) : X_j u \in L^p(\Omega), j = 1, 2, \cdots, m\}$$

with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)}.$$

Here $X_j u$ is the distributional derivative of $u \in L^1_{\text{loc}}(\Omega)$ given by

$$\int_\Omega X_j u \phi dx = \int_\Omega u X_j^* \phi dx,\ \forall \phi \in C_0^\infty(\Omega),$$

where $X_j^* = -X_j$ is the formal adjoint of X_j. The closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ is denoted by $HW_0^{1,p}(\Omega)$. We will write $u \in HW^{1,p}_{\text{loc}}(\Omega)$ to mean $u \in HW^{1,p}(K)$ for every compact set $K \subset \Omega$.

The following Sobolev inequalities with respect to the horizontal vector fields can be found in [34], [42].

Lemma 2.1. *For any $1 \le p < \infty$ and $u \in HW^{1,p}(B_R)$, there exists a constant $C > 0$ such that*

$$\left(\fint_{B_R} |u - u_R|^{\kappa p} dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}, \tag{2.2}$$

where $u_R = \fint_{B_R} u dx$ is the integral average of u on B_R, and $1 \le \kappa \le Q/(Q - p)$ if $1 \le p < Q$; $1 \le \kappa < \infty$ if $p \ge Q$. Moreover, for any $u \in HW_0^{1,p}(B_R)$,

$$\left(\fint_{B_R} |u|^{\kappa p} dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}. \tag{2.3}$$

Now we define several function spaces with respect to the Carnot-Carathéodory metric. For convenience, we use the notations:

$$\Omega(x,R)=\Omega\cap B(x,R),\quad f_{x,R}=\frac{1}{|\Omega(x,R)|}\int_{\Omega(x,R)}f(x)dx.$$

DEFINITION 2.2. Let $1<p<\infty$ and $\lambda\geq 0$. We say that $f\in L^p_{\mathrm{loc}}(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|_{L^{p,\lambda}(\Omega)}=\sup_{x\in\Omega,0<\rho<\mathrm{diam}\Omega}\left(\rho^{-\lambda}\int_{\Omega(x,\rho)}|f(y)|^p dy\right)^{\frac{1}{p}}<\infty;$$

we say that $f\in L^p_{\mathrm{loc}}(\Omega)$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}=\sup_{x\in\Omega,0<\rho<\mathrm{diam}\Omega}\left(\rho^{-\lambda}\int_{\Omega(x,\rho)}|f(y)-f_{x,\rho}|^p dy\right)^{\frac{1}{p}}<\infty.$$

DEFINITION 2.3. Let $\alpha\in(0,1)$. The Hölder space $C^{0,\alpha}_X(\bar\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{C^{0,\alpha}_X(\bar\Omega)}=\sup_{\Omega}|f|+\sup_{\bar\Omega}\frac{|f(x)-f(y)|}{[d_X(x,y)]^\alpha}<\infty.$$

Furthermore,

$$C^{1,\alpha}_X(\bar\Omega)=\left\{u\in C^{0,\alpha}_X(\bar\Omega): X_j u\in C^{0,\alpha}_X(\bar\Omega),\ j=1,2,\dots,m\right\}.$$

We call that $f\in C^{0,\alpha}_X(\Omega)$, if $f\in C^{0,\alpha}_X(K)$ for every compact set $K\subset\Omega$.

DEFINITION 2.4. Let $\omega\geq 0$ be a nondecreasing continuous function on $[0,\mathrm{diam}\Omega]$. A function $f\in L^1_{\mathrm{loc}}(\Omega)$ is said to be in $\mathrm{BMO}_\omega(\Omega)$ if

$$\|f\|_{*,\omega,\Omega}=\sup_{x\in\Omega,0<\rho<\mathrm{diam}\Omega}\frac{1}{|\Omega(x,\rho)|\omega(\rho)}\int_{\Omega(x,\rho)}|f(y)-f_{x,\rho}|dy<\infty$$

and by $\mathrm{VMO}_\omega(\Omega)$ we denote the subspace of all $f\in\mathrm{BMO}_\omega(\Omega)$ such that

$$\eta(f,\omega,\Omega)(r)=\sup_{x\in\Omega,0<\rho<r}\frac{1}{|\Omega(x,\rho)|\omega(\rho)}\int_{\Omega(x,\rho)}|f(y)-f_{x,\rho}|dy\to 0,\ \text{as } r\to 0.$$

When $\omega(\rho)=1$, the spaces $\mathrm{BMO}_1(\Omega)$ and $\mathrm{VMO}_1(\Omega)$ will be denoted by $\mathrm{BMO}(\Omega)$ and $\mathrm{VMO}(\Omega)$ respectively. Here BMO and VMO stand for "Bounded Mean Oscillation" and "Vanishing Mean Oscillation". It is obvious that $\mathrm{BMO}_\omega(\Omega)\subseteq\mathrm{VMO}(\Omega)$ if $\omega(\rho)\to 0$ as $\rho\to 0$.

As in the Euclidian case, we have the following integral characterization for a Hölder continuous function (see [35], [42]).

Lemma 2.5. *If $u \in \mathcal{L}^{p,Q+p\alpha}(\Omega)$, $1 < p < \infty$, $0 < \alpha < 1$, then $u \in C^{0,\alpha}(\Omega)$.*

We end this section with a useful iteration lemma [26, p.86 Lemma 2.1].

Lemma 2.6. *Let $\Phi(\rho)$ be a nonnegative and nondecreasing function on $[0, R_0]$ satisfying*

$$\Phi(\rho) \leq A\left(\left(\frac{\rho}{R}\right)^a + \varepsilon\right)\Phi(R) + BR^b, \quad 0 < \rho \leq R \leq R_0,$$

where A, a, b and B are nonnegative constants, $b < a$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A, a, b)$ such that if $\varepsilon < \varepsilon_0$ we have

$$\Phi(\rho) \leq C\left(\left(\frac{\rho}{R}\right)^b \Phi(R) + B\rho^b\right), \quad 0 < \rho \leq R \leq R_0,$$

where C is a constant depending on A, a and b.

3. Higher integrability and Morrey type estimates for homogeneous obstacle problem

In this section, we consider the homogeneous $\mathcal{K}^\theta_\psi$-obstacle problem corresponding to (1.1), i.e., the problem of finding a fuction $\bar{u} \in \mathcal{K}^\theta_\psi(\Omega)$ satisfying the variational inequality

$$\int_\Omega \langle \langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X(\bar{v} - \bar{u})\rangle dx \geq 0, \ \bar{v} \in \mathcal{K}^\theta_\psi(\Omega), \tag{3.1}$$

where the coefficients $a_{ij}(x) \in \mathrm{VMO}(\Omega)$ satisfy (H1). We first recall the following Gehring lemma on the metric measure space (Y, d, μ), where d is a metric and μ is a doubling measure. Since the Carnot group is a special homogeneous metric measure space, we can apply this lemma in our setting.

Lemma 3.1 ([44]). *Let $q \in [\bar{q}, 2Q]$, where $\bar{q} > 1$ is fixed and Q is the homogeneous dimension of Y. Assume that functions F and G are nonnegative and $G \in L^q_{\mathrm{loc}}(Y, \mu)$, $F \in L^{r_0}_{\mathrm{loc}}(Y, \mu)$, for some $r_0 > q$. If there exists a constant $b > 1$ such that for every ball $B \subset \sigma B \subset Y$,*

$$⨍_B G^q d\mu \leq b\left[\left(⨍_{\sigma B} G d\mu\right)^q + ⨍_{\sigma B} F^q d\mu\right],$$

then there exists a positive constant $\varepsilon_0 = \varepsilon_0(b, \bar{q}, Q, \sigma)$ such that $G \in L^r_{\mathrm{loc}}(Y, \mu)$, $r \in [q, q + \varepsilon_0]$, and moreover

$$\left(⨍_B G^r d\mu\right)^{\frac{1}{r}} \leq C\left[\left(⨍_{\sigma B} G^q d\mu\right)^{\frac{1}{q}} + \left(⨍_{\sigma B} F^r d\mu\right)^{\frac{1}{r}}\right],$$

for some positive constant $C = C(b, \bar{q}, Q, \sigma)$.

LEMMA 3.2 (Higher integrability). *Let $\bar{u} \in \mathcal{K}_\psi^\theta(\Omega)$ be a solution to* (3.1)*. Then there exists $t > p$ such that $\bar{u} \in HW_{\rm loc}^{1,t}(\Omega)$. Furthermore, for any $B_R \subset\subset \Omega$,*

$$\left(\fint_{B_{R/2}} |X\bar{u}|^t dx\right)^{\frac{1}{t}} \le c\left[\left(\fint_{B_R} |X\bar{u}|^p dx\right)^{\frac{1}{p}} + \left(\fint_{B_R} |X\psi|^t dx\right)^{\frac{1}{t}}\right], \tag{3.2}$$

where c does not depend on R.

PROOF. For $\bar{u} \in \mathcal{K}_\psi^\theta(\Omega)$ and $B_R \subset\subset \Omega$, consider the function

$$\bar{v} = \bar{u} - \bar{u}_R - \eta^p(\bar{u} - \psi - (\bar{u} - \psi)_R),$$

where $\eta \in C_0^\infty(B_R)$, $0 \le \eta \le 1$, $\eta = 1$ in $B_{R/2}$, and $|X\eta| \le c/R$. Since $\bar{u}_R \ge \psi_R$ a.e., we have

$$\begin{aligned}\bar{v} &= (1-\eta^p)(\bar{u} - \bar{u}_R) + \eta^p(\psi - \psi_R) \ge (1-\eta^p)(\bar{u} - \bar{u}_R) + \eta^p(\psi - \bar{u}_R)\\ &\ge (1-\eta^p)(\psi - \bar{u}_R) + \eta^p(\psi - \bar{u}_R) = \psi - \bar{u}_R, \text{ a.e. in } \Omega.\end{aligned}$$

On the other hand, we know that $\bar{v} - (\theta - \bar{u}_R) \in HW_0^{1,p}(\Omega)$ from $\bar{u} - \theta \in HW_0^{1,p}(\Omega)$. Thus $\bar{v} \in \mathcal{K}_{\psi-\bar{u}_R}^{\theta-\bar{u}_R}(\Omega)$. Noting $\bar{u} - \bar{u}_R \in \mathcal{K}_{\psi-\bar{u}_R}^{\theta-\bar{u}_R}(\Omega)$ and

$$\begin{aligned}X\bar{v} &= (1-\eta^p)X(\bar{u} - \bar{u}_R) + \eta^p X(\psi - \psi_R) - p\eta^{p-1}X\eta(\bar{u} - \psi - (\bar{u} - \psi)_R)\\ &= (1-\eta^p)X\bar{u} + \eta^p X\psi - p\eta^{p-1}X\eta(\bar{u} - \psi - (\bar{u} - \psi)_R),\end{aligned}$$

we have from (3.1) that

$$\begin{aligned}&\int_\Omega \langle\langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X\bar{u}\rangle dx\\ \le &\int_\Omega \langle\langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X\bar{v}\rangle dx\\ \le &\int_\Omega (1-\eta^p)\langle\langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X\bar{u}\rangle + c\int_\Omega \eta^p |X\bar{u}|^{p-1}|X\psi| dx\\ &+ c\int_\Omega \eta^{p-1}|\bar{u} - \psi - (\bar{u} - \psi)_R||X\bar{u}|^{p-1}|X\eta| dx.\end{aligned}$$

From (H1) and the Young inequality with ε, we have

$$\begin{aligned}\int_\Omega \eta^p |X\bar{u}|^p dx &\le c\int_\Omega \eta^p \langle AX\bar{u}, X\bar{u}\rangle^{\frac{p}{2}} dx\\ &\le \frac{\varepsilon}{2}\int_\Omega \eta^p |X\bar{u}|^p dx + c_\varepsilon \int_\Omega \eta^p |X\psi|^p dx + \frac{\varepsilon}{2}\int_\Omega \eta^p |X\bar{u}|^p dx\\ &\quad + c_\varepsilon \int_\Omega |\bar{u} - \bar{u}_R - (\psi - \psi_R)|^p |X\eta|^p dx.\end{aligned}$$

Using $\eta = 1$ on $B_{R/2}$ and taking $\varepsilon = \frac{1}{2}$, the Sobolev inequality (2.2) implies

$$\begin{aligned}\int_{B_{R/2}} |X\bar{u}|^p dx &\le c\int_{B_R} |X\psi|^p dx + \frac{c}{R^p}\int_{B_R} |\bar{u} - \bar{u}_R - (\psi - \psi_R)|^p dx \\ &\le c\int_{B_R} |X\psi|^p dx + \frac{c}{R^p}\left(\int_{B_R} |X\bar{u}|^{pQ/(p+Q)} dx\right)^{(p+Q)/Q}.\end{aligned}$$

Dividing by $|B_{R/2}|$ on both sides, we arrive at

$$\fint_{B_{R/2}} |X\bar{u}|^p dx \le c\left(\fint_{B_R} |X\bar{u}|^{pQ/(p+Q)} dx\right)^{(p+Q)/Q} + c\fint_{B_R} |X\psi|^p dx.$$

Now in Lemma 3.1 we set $q = \frac{p+Q}{Q}$, $G = |X\bar{u}|^{\frac{p}{q}}$ and $F = |X\psi|^{\frac{p}{q}}$. Then there exists $\varepsilon_0 > 0$ such that

$$|X\bar{u}|^{\frac{p}{q}} \in L^r_{\text{loc}}(\Omega), \quad r \in [q, q + \varepsilon_0),$$

and

$$\left(\fint_{B_{R/2}} |X\bar{u}|^{\frac{pr}{q}} dx\right)^{\frac{1}{r}} \le c\left(\fint_{B_R} |X\bar{u}|^p dx\right)^{\frac{1}{q}} + c\left(\fint_{B_R} |X\psi|^{\frac{pr}{q}} dx\right)^{\frac{1}{r}}.$$

We set $t = \frac{pr}{q}$, then $t \in [p, p + \frac{p\varepsilon_0}{q})$ and

$$\left(\fint_{B_{R/2}} |X\bar{u}|^t dx\right)^{\frac{p}{tq}} \le c\left(\fint_{B_R} |X\bar{u}|^p dx\right)^{\frac{1}{q}} + c\left(\fint_{B_R} |X\psi|^t dx\right)^{\frac{p}{tq}},$$

which implies (3.2).

□

For the fixed $x \in \Omega$ and a small $R > 0$, let $B_R = B_R(x) \subset\subset \Omega$. In order to prove the Morrey type estimate for solutions to (3.1), we first establish the Morrey type estimate for weak solutions to the following degenerate elliptic equation with constant coefficients

$$-X^*(\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw) = -X^*(\langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi), \tag{3.3}$$

where $A_{R/2} = \fint_{B_{R/2}} A(x)dx$ is the integral average of $A(x)$. Let $w \in HW^{1,p}(B_{R/2})$ be the weak solution to (3.3), i.e. for any $\phi \in C_0^\infty(B_{R/2})$,

$$\begin{aligned}&\int_{B_{R/2}} \langle\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, X\phi\rangle dx \\ &= \int_{B_{R/2}} \langle\langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi, X\phi\rangle dx.\end{aligned} \tag{3.4}$$

Recall that for any $\xi, \eta \in \mathbb{R}^m$, it holds (see [31])

$$\langle \langle A\xi, \xi \rangle^{\frac{p-2}{2}} A\xi - \langle A\eta, \eta \rangle^{\frac{p-2}{2}} A\eta, \xi - \eta \rangle \geq C(p)(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2; \tag{3.5}$$

particularly, for $p \geq 2$,

$$\langle \langle A\xi, \xi \rangle^{\frac{p-2}{2}} A\xi - \langle A\eta, \eta \rangle^{\frac{p-2}{2}} A\eta, \xi - \eta \rangle \geq C(p)|\xi - \eta|^p. \tag{3.6}$$

We now show that the Proposition 1 and Corollary 1 in [45] are still valid for weak solutions to

$$-X^*(\langle A_{R/2} X\bar{w}, X\bar{w} \rangle^{\frac{p-2}{2}} A_{R/2} X\bar{w}) = 0. \tag{3.7}$$

Set $A_{R/2} := (b_{ij})_{m \times m}$ and consider operators

$$\mathcal{L} = \sum_{i,j=1}^{m} b_{ij} X_i X_j \quad \text{and} \quad \mathcal{A} = \sum_{i,j=1}^{m} a_{ij}(x) X_i X_j,$$

where $a_{ij}(x) \in L^\infty(\Omega)$. According to Theorem 2 in [4], we know that for all $1 < s < \infty$ there exists $\mathcal{C} = \mathcal{C}(s, \Lambda, \mathbb{G})$ such that

$$\|X^2 v\|_{L^s} \leq \mathcal{C} \|\mathcal{L} v\|_{L^s}, \ \forall \ v \in HW_0^{2,s}(\Omega).$$

Let $0 < \varepsilon < 1$ such that $\varepsilon \cdot \mathcal{C} < 1$ and suppose that $v \in HW_0^{2,s}(\Omega)$ satisfies the Cordes condition

$$|\mathcal{A}v - \mathcal{L}v| \leq \varepsilon |X^2 v|, \ \text{a.e. } x \in \Omega. \tag{3.8}$$

By a similar argument to that in the proof of [14, Theorem 4.1], we know that there exists $C > 0$ only depending on ε such that

$$\|X^2 v\|_{L^s(\Omega)} \leq C \|\mathcal{A} v\|_{L^s(\Omega)}.$$

Next, we consider a sequence $\{\bar{w}_k\}_{k=1}^\infty \subset HW_0^{1,2}(\Omega)$ satisfying

$$X_i \left[\left(\frac{1}{k} + \langle A_{R/2} X\bar{w}_k, X\bar{w}_k \rangle \right)^{\frac{p-2}{2}} b_{ij} X_j \bar{w}_k \right] = 0, \quad x \in \Omega,$$

with indices $i, j = 1, \ldots, m$ and where we use the convention that repeated indices are summed. The differentiated version of the above equation has the form

$$\mathcal{A}_k v := \sum_{i,j=1}^{m} c_{ij}^k(x) X_i X_j v = 0, \quad x \in \Omega,$$

where $v(x) = \bar{w}_k(x)$ and

$$c_{ij}^k(x) = b_{ij} + \frac{(p-2)(b_{jt} b_{il} X_t \bar{w}_k X_l \bar{w}_k + b_{sj} b_{il} X_s \bar{w}_k X_l \bar{w}_k)}{2\left(\frac{1}{k} + \langle A_{R/2} X\bar{w}_k, X\bar{w}_k \rangle\right)}.$$

Since

$$\Lambda^{-1}|X\bar{w}_k|^2 \le \langle A_{R/2}X\bar{w}_k, X\bar{w}_k\rangle \le \Lambda|X\bar{w}_k|^2,$$

we conclude that $c_{ij}^k(x) \in L^\infty(\Omega)$ and

$$|\mathcal{A}_k\bar{w}_k - \mathcal{L}\bar{w}_k| \le C(\Lambda, Q)|p-2||X^2\bar{w}_k(x)|, \text{ a.e. } x \in \Omega.$$

Obviously, $\bar{w}_k$ satisfies the Cordes condition (3.8) if we take p in a neighborhood of 2. Then $\bar{w}_k \in HW_0^{2,s}(\Omega)$ and there exists a $C > 0$ independent of k such that

$$\|X^2\bar{w}_k\|_{L^s(\Omega)} \le C\|\mathcal{A}_k\bar{w}_k\|_{L^s(\Omega)}.$$

Then, using the same arguments as in the proof of Proposition 1 and Corollary 1 in [45], we know that the conclusions of Proposition 1 and Corollary 1 in [45] hold true for weak solutions to (3.7) if p is close to 2. Based on these results and with a similar proof of Lemma 7 in [45], we have the following Lemma.

Lemma 3.3. *Let $\bar{w} \in HW^{1,p}(\Omega)$ be a weak solution to (3.7). If p is close to 2, then there exist $C > 0$ and $0 < \beta < 1$ such that for any $B_{R/2} \subset\subset \Omega$ and $0 < \rho \le R/2$,*

$$\sup_{x\in B_{R/4}} |X\bar{w}|^p \le \frac{C}{|B_{R/2}|}\int_{B_{R/2}} |X\bar{w}|^p dx, \tag{3.9}$$

and

$$\int_{B_\rho} |X\bar{w} - (X\bar{w})_\rho|^p dx \le C\left(\frac{\rho}{R}\right)^{Q+2\beta}\int_{B_{R/2}} |X\bar{w} - (X\bar{w})_{R/2}|^p dx. \tag{3.10}$$

Lemma 3.4. *Let $w \in HW^{1,p}(B_{R/2})$ be a weak solution to (3.3) with p close to 2. Then for any $0 < \rho < R/2$ and $\varepsilon > 0$, we have*

$$\int_{B_\rho} |Xw|^p dx \le c(p,\Lambda)\left(\left(\frac{\rho}{R}\right)^Q + \varepsilon\right)\int_{B_{R/2}} |Xw|^p dx + c(p,\Lambda,\varepsilon,\|X\psi\|_{L^\infty})R^Q. \tag{3.11}$$

Proof. Let $\bar{w} \in HW^{1,p}(B_{R/2})$ be the weak solution to the following Dirichlet problem

$$\begin{cases} -X^*(\langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{w}) = 0, & \text{in } B_{R/2}, \\ \bar{w} - w \in HW_0^{1,p}(B_{R/2}). \end{cases}$$

Hence

$$\int_{B_{R/2}} \langle\langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{w}, X(\bar{w} - w)\rangle dx = 0. \tag{3.12}$$

From (H1) and the Young inequality, we obtain

$$\int_{B_{R/2}} |X\bar{w}|^p dx \le c\int_{B_{R/2}} \langle\langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{w}, X\bar{w}\rangle dx$$

$$
\begin{aligned}
&= c\int_{B_{R/2}} \langle\langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{w}, Xw\rangle dx \\
&\le c\int_{B_{R/2}} |X\bar{w}|^{p-1}|Xw|dx \\
&\le \varepsilon\int_{B_{R/2}} |X\bar{w}|^{p}dx + c_\varepsilon\int_{B_{R/2}} |Xw|^{p}dx.
\end{aligned}
$$

Taking $\varepsilon = \frac{1}{2}$, it follows

$$
\int_{B_{R/2}} |X\bar{w}|^{p}dx \le c\int_{B_{R/2}} |Xw|^{p}dx. \tag{3.13}
$$

On the other hand, we claim from (3.9) that for any $0 < \rho < R/2$,

$$
\int_{B_\rho} |X\bar{w}|^{p}dx \le c\left(\frac{\rho}{R}\right)^{Q}\int_{B_{R/2}} |X\bar{w}|^{p}dx. \tag{3.14}
$$

In fact, (3.14) is obviously true for $R/4 \le \rho < R/2$. If $0 < \rho < R/4$, we have from (3.9) that

$$
\begin{aligned}
\int_{B_\rho} |X\bar{w}|^{p}dx &\le |B_\rho|\sup_{B_\rho}|X\bar{w}|^{p} \le |B_\rho|\sup_{B_{R/4}}|X\bar{w}|^{p} \\
&\le c\frac{|B_\rho|}{|B_{R/2}|}\int_{B_{R/2}} |X\bar{w}|^{p}dx \le c\left(\frac{\rho}{R}\right)^{Q}\int_{B_{R/2}} |X\bar{w}|^{p}dx.
\end{aligned}
$$

Now (3.13) and (3.14) give

$$
\int_{B_\rho} |X\bar{w}|^{p}dx \le c\left(\frac{\rho}{R}\right)^{Q}\int_{B_{R/2}} |Xw|^{p}dx. \tag{3.15}
$$

Therefore

$$
\begin{aligned}
\int_{B_\rho} |Xw|^{p}dx &\le 2^{p}\int_{B_\rho} |X\bar{w}|^{p}dx + 2^{p}\int_{B_\rho} |Xw - X\bar{w}|^{p}dx \\
&\le c\left(\frac{\rho}{R}\right)^{Q}\int_{B_{R/2}} |Xw|^{p}dx + c\int_{B_{R/2}} |Xw - X\bar{w}|^{p}dx. \qquad (3.16)
\end{aligned}
$$

If $p \ge 2$, the last term in (3.16) can be estimated by using (3.6), (3.12) and (3.4) that

$$
\begin{aligned}
&\int_{B_{R/2}} |Xw - X\bar{w}|^{p}dx \\
&\le c\int_{B_{R/2}} \langle\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw - \langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{w}, Xw - X\bar{w}\rangle dx
\end{aligned}
$$

$$= c\int_{B_{R/2}} \langle\langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi, Xw - X\bar{w}\rangle dx$$
$$\le c\int_{B_{R/2}} |X\psi|^{p-1}|Xw - X\bar{w}|dx$$
$$\le \varepsilon\int_{B_{R/2}} |Xw - X\bar{w}|^p dx + c_\varepsilon\int_{B_{R/2}} |X\psi|^p dx.$$

Taking $\varepsilon = \frac{1}{2}$ and noting $X\psi \in C_X^{0,\gamma}(\Omega)$, it follows

$$\int_{B_{R/2}} |Xw - X\bar{w}|^p dx \le c\int_{B_{R/2}} |X\psi|^p dx \le cR^Q. \tag{3.17}$$

Putting (3.17) into (3.16) gives (3.11).

For the case $p < 2$, it follows by the Hölder inequality and (3.13) that

$$\int_{B_{R/2}} |Xw - X\bar{w}|^p dx$$
$$= \int_{B_{R/2}} \left(|Xw - X\bar{w}|^2 (|Xw| + |X\bar{w}|)^{p-2}\right)^{\frac{p}{2}} (|Xw| + |X\bar{w}|)^{\frac{p(2-p)}{2}} dx$$
$$\le c\left(\int_{B_{R/2}} |Xw - X\bar{w}|^2 (|Xw| + |X\bar{w}|)^{p-2} dx\right)^{\frac{p}{2}} \left(\int_{B_{R/2}} (|Xw| + |X\bar{w}|)^p dx\right)^{\frac{2-p}{2}}$$
$$\le c\left(\int_{B_{R/2}} |Xw - X\bar{w}|^2 \left(|Xw|^2 + |X\bar{w}|^2\right)^{\frac{p-2}{2}} dx\right)^{\frac{p}{2}} \left(\int_{B_{R/2}} |Xw|^p dx\right)^{\frac{2-p}{2}}$$
(3.18)
$$\le \varepsilon\int_{B_{R/2}} |Xw|^p dx + c_\varepsilon\int_{B_{R/2}} |Xw - X\bar{w}|^2 \left(|Xw|^2 + |X\bar{w}|^2\right)^{\frac{p-2}{2}} dx.$$

Thanks to (3.5), we have

$$\int_{B_{R/2}} |Xw - X\bar{w}|^2 \left(|Xw|^2 + |X\bar{w}|^2\right)^{\frac{p-2}{2}} dx$$
$$\le \int_{B_{R/2}} \langle\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw - \langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{w}, Xw - X\bar{w}\rangle dx$$
$$\le \int_{B_{R/2}} |X\psi|^{p-1}|Xw - X\bar{w}|dx$$
(3.19)
$$\le \sigma\int_{B_{R/2}} |Xw - X\bar{w}|^p dx + c_\sigma\int_{B_{R/2}} |X\psi|^p dx.$$

Inserting (3.19) into (3.18) and then taking $\sigma = \frac{1}{2}$, it shows

$$\int_{B_{R/2}} |Xw - X\bar{w}|^p dx \le \varepsilon\int_{B_{R/2}} |Xw|^p dx + cR^Q. \tag{3.20}$$

The proof is complete by combining (3.20) and (3.16). □

REMARK 3.5. Making use of Lemma 3.4 and Lemma 2.6, we know that for any $0 < \nu < Q$,

$$\int_{B_R} |Xw|^p dx \le cR^\nu,$$

where c is a constant independent of R.

Now we prove the Morrey type estimate for solutions to (3.1).

LEMMA 3.6. *Let* $\bar{u} \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a solution to* (3.1)*. If* p *is close to 2, then for any* $0 < \rho \le R$, $B_R \subset\subset \Omega$, *and* $\varepsilon > 0$ *it holds*

$$\int_{B_\rho} |X\bar{u}|^p dx \le c\left(\left(\frac{\rho}{R}\right)^Q + \vartheta(R,\varepsilon)\right)\int_{B_R} |X\bar{u}|^p dx + cR^Q, \tag{3.21}$$

where $\vartheta(R,\varepsilon) = \|A\|_{*,R/2}^{(t-p)/t} + \varepsilon$.

PROOF. Let $w \in HW^{1,p}(B_{R/2})$ be the weak solution to

$$\begin{cases} X^*(\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw) = X^*(\langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi), \text{ in } B_{R/2}, \\ w - \bar{u} \in HW^{1,p}_0(B_{R/2}), \end{cases}$$

hence

$$\begin{aligned} &\int_{B_{R/2}} \langle\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, Xw - X\bar{u}\rangle dx \\ &= \int_{B_{R/2}} \langle\langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi, Xw - X\bar{u}\rangle dx. \end{aligned} \tag{3.22}$$

Since $w - \bar{u} \in HW^{1,p}_0(B_{R/2})$ and $\bar{u} \ge \psi$ on $\partial B_{R/2}$, we have by the maximum principle that $w \ge \psi$ in $B_{R/2}$. Thereby, from (3.1)

$$\int_{B_{R/2}} \langle\langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X(\bar{u} - w)\rangle dx \le 0. \tag{3.23}$$

We may write the above inequality as follows

(3.24)

$$\begin{aligned} &\int_{B_{R/2}} \langle\langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u} - \langle A_{R/2}X\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{u}, X(\bar{u} - w)\rangle dx \\ &+ \int_{B_{R/2}} \langle\langle A_{R/2}X\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{u} - \langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, X(\bar{u} - w)\rangle dx \\ &+ \int_{B_{R/2}} \langle\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, X(\bar{u} - w)\rangle dx \le 0. \end{aligned}$$

For every $\xi \in \mathbb{R}^m$, we have (see [31])

$$|\langle A\xi,\xi\rangle^{\frac{p-2}{2}}A\xi - \langle A_{R/2}\xi,\xi\rangle^{\frac{p-2}{2}}A\xi| \le c(p,\Lambda)|A - A_{R/2}||\xi|^{p-1}. \tag{3.25}$$

On the basis of the inequality above, it follows from (3.24) and (3.22) that

$$\begin{aligned}
&\int_{B_{R/2}} \langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u} - \langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw, X(\bar{u}-w)\rangle dx\\
\le &\int_{B_{R/2}} \langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u} - \langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u}, X(\bar{u}-w)\rangle dx\\
&-\int_{B_{R/2}} \langle\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw, X(\bar{u}-w)\rangle dx\\
= &\int_{B_{R/2}} \langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u} - \langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u}, X(\bar{u}-w)\rangle dx\\
&-\int_{B_{R/2}} \langle\langle A_{R/2}X\psi,X\psi\rangle^{\frac{p-2}{2}}A_{R/2}X\psi, X(\bar{u}-w)\rangle dx\\
\le &\int_{B_{R/2}} |A_{R/2}-A||X\bar{u}|^{p-1}|X\bar{u}-Xw|dx + \int_{B_{R/2}} |X\psi|^{p-1}|X\bar{u}-Xw|dx.
\end{aligned} \tag{3.26}$$

On the other hand, from (H1) and (3.22)

$$\begin{aligned}
\int_{B_{R/2}} |Xw|^p dx &\le c\int_{B_{R/2}} \langle A_{R/2}Xw,Xw\rangle^{\frac{p}{2}}dx\\
&= c\int_{B_{R/2}} \langle\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw, X\bar{u}\rangle dx\\
&\quad + c\int_{B_{R/2}} \langle\langle A_{R/2}X\psi,X\psi\rangle^{\frac{p-2}{2}}A_{R/2}X\psi, Xw - X\bar{u}\rangle dx\\
&\le c\int_{B_{R/2}} |Xw|^{p-1}|X\bar{u}|dx + \int_{B_{R/2}} |X\psi|^{p-1}|Xw-X\bar{u}|dx\\
&\le \varepsilon\int_{B_{R/2}} |Xw|^p dx + c_\varepsilon\int_{B_{R/2}} |X\bar{u}|^p dx + c_\varepsilon\int_{B_{R/2}} |X\psi|^p dx
\end{aligned}$$

Taking ε small enough, we have

$$\int_{B_{R/2}} |Xw|^p dx \le c\int_{B_{R/2}} |X\bar{u}|^p dx + cR^Q. \tag{3.27}$$

Then it follows by (3.11) and (3.27) that for any $0<\rho<R/2$ and $\varepsilon>0$,

$$\int_{B_\rho} |X\bar{u}|^p dx \le 2^p\int_{B_\rho} |Xw|^p dx + 2^p\int_{B_\rho} |X\bar{u}-Xw|^p dx$$

$$\le c\left(\left(\frac{\rho}{R}\right)^Q+\varepsilon\right)\int_{B_{R/2}}|Xw|^p dx + cR^Q + 2^p\int_{B_\rho}|X\bar u - Xw|^p dx$$

$$\le c\left(\left(\frac{\rho}{R}\right)^Q+\varepsilon\right)\int_{B_{R/2}}|X\bar u|^p dx + cR^Q + 2^p\int_{B_{R/2}}|X\bar u - Xw|^p dx. \tag{3.28}$$

To estimate the last term in the right hand side of (3.28), we consider two cases: $p \ge 2$ and $p < 2$.

Assume $p \ge 2$. From (3.6) and (3.26), we have

$$\begin{aligned}
&\int_{B_{R/2}}|X\bar u - Xw|^p dx\\
&\le c\int_{B_{R/2}}\langle\langle A_{R/2}X\bar u, X\bar u\rangle^{\frac{p-2}{2}}A_{R/2}X\bar u - \langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw, X(\bar u - w)\rangle dx\\
&\le c\int_{B_{R/2}}|A_{R/2}-A||X\bar u|^{p-1}|X\bar u - Xw|dx + c\int_{B_{R/2}}|X\psi|^{p-1}|X\bar u - Xw|dx\\
&\le c\left[\left(\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar u|^p dx\right)^{\frac{p-1}{p}} + \left(\int_{B_{R/2}}|X\psi|^p dx\right)^{\frac{p-1}{p}}\right]\\
&\quad\times\left(\int_{B_{R/2}}|X\bar u - Xw|^p dx\right)^{\frac1p},
\end{aligned}$$

which implies

$$\int_{B_{R/2}}|X\bar u - Xw|^p dx \le c\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar u|^p dx + c\int_{B_{R/2}}|X\psi|^p dx. \tag{3.29}$$

As for the first integral in the right hand side of (3.29), we conclude by the Hölder inequality and Lemma 3.2 that there exists $t > p$ such that

$$\begin{aligned}
&c\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar u|^p dx\\
&\le c|B_{R/2}|\left(\fint_{B_{R/2}}|A_{R/2}-A|^{\frac{pt}{(p-1)(t-p)}}dx\right)^{\frac{t-p}{t}}\left(\fint_{B_{R/2}}|X\bar u|^t dx\right)^{\frac pt}\\
&\le c\|A\|_{*,R/2}^{(t-p)/t}|B_{R/2}|\left(\fint_{B_{R/2}}|X\bar u|^t dx\right)^{\frac pt}
\end{aligned}$$

$$\le c\|A\|_{*,R/2}^{(t-p)/t}\int_{B_R}|X\bar u|^p dx + c\|A\|_{*,R/2}^{(t-p)/t}|B_R|\left(\fint_{B_R}|X\psi|^t dx\right)^{\frac pt}. \tag{3.30}$$

Inserting (3.30) into (3.29), it follows

$$\int_{B_{R/2}}|X\bar u - Xw|^p dx \le c\|A\|_{*,R/2}^{(t-p)/t}\int_{B_R}|X\bar u|^p dx + cR^Q. \tag{3.31}$$

Taking (3.31) into (3.28) and then letting $\vartheta(R,\varepsilon)=\|A\|_{*,R/2}^{(t-p)/t}+\varepsilon$, we have that for any $0<\rho<R/2$,

$$\int_{B_\rho}|X\bar u|^p dx\le c\left[\left(\frac{\rho}{R}\right)^Q+\vartheta(R,\varepsilon)\right]\int_{B_R}|X\bar u|^p dx+cR^Q. \tag{3.32}$$

Now assume that $p<2$. With a similar argument as the one used to prove (3.18), we obtain from (3.5) and (3.27) that

$$\begin{aligned}
&\int_{B_{R/2}}|X\bar u-Xw|^p dx\\
&\le c\left(\int_{B_{R/2}}|X\bar u-Xw|^2\left(|X\bar u|^2+|Xw|^2\right)^{\frac{p-2}{2}}dx\right)^{\frac{p}{2}}\left(\int_{B_{R/2}}\left(|X\bar u|^p+|Xw|^p\right)dx\right)^{\frac{2-p}{2}}\\
&\le c\left(\int_{B_{R/2}}\langle\langle A_{R/2}X\bar u,X\bar u\rangle^{\frac{p-2}{2}}A_{R/2}X\bar u-\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw,X\bar u-Xw\rangle dx\right)^{\frac{p}{2}}\\
&\quad\times\left(\int_{B_{R/2}}\left(|X\bar u|^p+|X\psi|^p\right)dx\right)^{\frac{2-p}{2}}\\
&\le\varepsilon\int_{B_{R/2}}\left(|X\bar u|^p+|X\psi|^p\right)dx\\
&\quad+c_\varepsilon\int_{B_{R/2}}\langle\langle A_{R/2}X\bar u,X\bar u\rangle^{\frac{p-2}{2}}A_{R/2}X\bar u-\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw,X\bar u-Xw\rangle dx\\
&:=\varepsilon\int_{B_{R/2}}\left(|X\bar u|^p+|X\psi|^p\right)dx+I.
\end{aligned}\tag{3.33}$$

From (3.26) we have

$$\begin{aligned}
I&\le c_\varepsilon\int_{B_{R/2}}|A_{R/2}-A||X\bar u|^{p-1}|X\bar u-Xw|dx+c_\varepsilon\int_{B_{R/2}}|X\psi|^{p-1}|X\bar u-Xw|dx\\
&\le\sigma\int_{B_{R/2}}|X\bar u-Xw|^p dx+c(\varepsilon,\sigma)\int_{B_{R/2}}|X\psi|^p dx\\
&\quad+c(\varepsilon,\sigma)\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar u|^p dx.
\end{aligned}\tag{3.34}$$

Using (3.30) and taking $\sigma=\frac{1}{2}$, we obtain

$$\int_{B_{R/2}}|X\bar u-Xw|^p dx\le c\left(\|A\|_{*,R/2}^{(t-p)/t}+\varepsilon\right)\int_{B_R}|X\bar u|^p dx+cR^Q. \tag{3.35}$$

Now for $0 < \rho < R/2$, (3.21) holds by (3.35) and (3.28). It is obvious that for $R/2 \le \rho \le R$,

$$\int_{B_\rho} |X\bar{u}|^p dx \le \int_{B_R} |X\bar{u}|^p dx \le 2^Q \left(\frac{\rho}{R}\right)^Q \int_{B_R} |X\bar{u}|^p dx.$$

A combination of these inequalities shows that for any $0 < \rho \le R$, (3.21) holds. □

REMARK 3.7. Since the coefficients $a_{ij}(x) \in \mathrm{VMO}(\Omega)$, there exists $R_0 > 0$ such that $\vartheta(R, \varepsilon)$ is small enough for any $R \le R_0$. By Lemma 2.6 we know that for any $0 < \nu < Q$,

$$\int_{B_R} |X\bar{u}|^p dx \le cR^\nu,$$

where the constant c is is independent of R.

4. Proof of Theorem 1.1

We prove Theorem 1.1 based on the higher integrability and Morrey type estimate for (3.1).

Proof of Theorem 1.1. For the fixed $x \in \Omega$, let $B_R := B_R(x) \subset\subset \Omega$. Let $u \in \mathcal{K}_\psi^\theta(\Omega)$ be a solution to (1.1), namely for any $v \in \mathcal{K}_\psi^\theta(\Omega)$,

$$\begin{aligned} &\int_\Omega \langle \langle AXu, Xu\rangle^{\frac{p-2}{2}} AXu, X(v-u)\rangle dx \\ &\ge \int_\Omega B(x,u,Xu)(v-u)dx + \int_\Omega \langle f(x), X(v-u)\rangle dx. \end{aligned} \tag{4.1}$$

Let $\bar{u} \in \mathcal{K}_\psi^u(B_R)$ be a solution to the corresponding homogeneous obstacle problem for (1.1), i.e.

$$\int_{B_R} \langle \langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X(v-\bar{u})\rangle dx \ge 0, \quad \forall v \in \mathcal{K}_\psi^u(B_R). \tag{4.2}$$

Since $\bar{u} - u \in HW_0^{1,p}(B_R)$ and $\bar{u} \ge \psi$ a.e., we have from (4.1) that

$$\begin{aligned} &\int_{B_R} \langle \langle AXu, Xu\rangle^{\frac{p-2}{2}} AXu, X(u-\bar{u})\rangle dx \\ &\le \int_{B_R} B(x,u,Xu)(u-\bar{u})dx + \int_{B_R} \langle f - f_R, X(u-\bar{u})\rangle dx. \end{aligned} \tag{4.3}$$

We similarly obtain from (4.2) that

$$\int_{B_R} \langle \langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X(u-\bar{u})\rangle dx \ge 0. \tag{4.4}$$

Using (H1) and the Young inequality,

$$\begin{aligned}\int_{B_R}|X\bar{u}|^p dx &\le c\int_{B_R}\langle AX\bar{u}, X\bar{u}\rangle^{\frac{p}{2}}dx\\ &\le c\int_{B_R}\langle\langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u}, Xu\rangle dx\\ &\le c\int_{B_R}|X\bar{u}|^{p-1}|Xu|dx \le \varepsilon\int_{B_R}|X\bar{u}|^p dx + c_\varepsilon\int_{B_R}|Xu|^p dx,\end{aligned}$$

then

$$\int_{B_R}|X\bar{u}|^p dx \le c\int_{B_R}|Xu|^p dx. \tag{4.5}$$

As a consequence of (3.21) and (4.5), we deduce that for any $0<\rho\le R$,

$$\begin{aligned}\int_{B_\rho}|Xu|^p dx &\le 2^p\int_{B_\rho}|X\bar{u}|^p dx + 2^p\int_{B_\rho}|Xu - X\bar{u}|^p dx\\ &\le c\left[\left(\frac{\rho}{R}\right)^Q + \vartheta(R,\varepsilon)\right]\int_{B_R}|X\bar{u}|^p dx + cR^Q + 2^p\int_{B_\rho}|Xu-X\bar{u}|^p dx\\ &\le c\left[\left(\frac{\rho}{R}\right)^Q + \vartheta(R,\varepsilon)\right]\int_{B_R}|Xu|^p dx + cR^Q + c\int_{B_R}|Xu-X\bar{u}|^p dx. \qquad (4.6)\end{aligned}$$

Assume $p\ge 2$. In view of (3.6), (4.3), (4.4) and the Sobolev inequality, one gets

$$\begin{aligned}&\int_{B_R}|Xu - X\bar{u}|^p dx\\ &\le \int_{B_R}\langle\langle AXu, Xu\rangle^{\frac{p-2}{2}}AXu - \langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u}, X(u-\bar{u})\rangle dx\\ &\le \int_{B_R}B(x,u,Xu)(u-\bar{u})dx + \int_{B_R}\langle f - f_R, X(u-\bar{u})\rangle dx\\ &\le \left(\int_{B_R}|B(x,u,Xu)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{B_R}|u-\bar{u}|^p dx\right)^{\frac{1}{p}} + \int_{B_R}\langle f-f_R, X(u-\bar{u})\rangle dx\\ &\le \left[cR\left(\int_{B_R}(|Xu|^p + |g|^{\frac{p}{p-1}})dx\right)^{\frac{p-1}{p}} + \left(\int_{B_R}|f-f_R|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\right]\\ &\quad\times\left(\int_{B_R}|Xu-X\bar{u}|^p dx\right)^{\frac{1}{p}}. \qquad (4.7)\end{aligned}$$

Then

$$\int_{B_R}|Xu - X\bar{u}|^p dx$$

$$\leq cR^{\frac{p}{p-1}}\int_{B_R}|Xu|^p dx + cR^{\frac{p}{p-1}}\int_{B_R}|g|^{\frac{p}{p-1}}dx + \int_{B_R}|f-f_R|^{\frac{p}{p-1}}dx$$

$$\leq cR^{\frac{p}{p-1}}\int_{B_R}|Xu|^p dx + cR^{Q+\frac{p}{p-1}}\|g\|^{p/(p-1)}_{L^{p/(p-1),Q}(\Omega)} + \int_{B_R}|f-f_R|^{\frac{p}{p-1}}dx \tag{4.8}$$

$$\leq cR^{\frac{p}{p-1}}\int_{B_R}|Xu|^p dx + c(\|g\|^{p/(p-1)}_{L^{p/(p-1),Q}}, \|f\|^{p/(p-1)}_{\mathcal{L}^{p/(p-1),Q}})R^Q. \tag{4.9}$$

Taking (4.9) into (4.6), we obtain, for any $0<\rho\leq R$,

$$\int_{B_\rho}|Xu|^p dx \leq c\left[\left(\frac{\rho}{R}\right)^Q + \vartheta(R,\varepsilon) + R^{\frac{p}{p-1}}\right]\int_{B_R}|Xu|^p dx + cR^Q. \tag{4.10}$$

When $p<2$, it follows from (4.5) and (3.5) that

$$\int_{B_R}|Xu-X\bar{u}|^p dx$$

$$\leq c\left(\int_{B_{R/2}}|Xu-X\bar{u}|^2\left(|Xu|+|X\bar{u}|\right)^{p-2}dx\right)^{\frac{p}{2}}\left(\int_{B_{R/2}}\left(|Xu|+|X\bar{u}|\right)^p dx\right)^{\frac{2-p}{2}}$$

(4.11)

$$\leq c\left(\int_{B_{R/2}}\langle\langle AXu,Xu\rangle^{\frac{p-2}{2}}AXu - \langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u}, Xu-X\bar{u}\rangle dx\right)^{\frac{p}{2}}$$

$$\times\left(\int_{B_{R/2}}|Xu|^p dx\right)^{\frac{2-p}{2}}$$

(4.12)

$$\leq \epsilon\int_{B_{R/2}}|Xu|^p dx + c_\epsilon\int_{B_{R/2}}\langle\langle AXu,Xu\rangle^{\frac{p-2}{2}}AXu - \langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u}, Xu-X\bar{u}\rangle dx.$$

To the last term in (4.12), by a similar argument as in (4.7), the Young inequality implies

$$\int_{B_{R/2}}\langle\langle AXu,Xu\rangle^{\frac{p-2}{2}}AXu - \langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u}, Xu-X\bar{u}\rangle dx$$

$$\leq \sigma\int_{B_R}|Xu-X\bar{u}|^p dx + c_\sigma R^{\frac{p}{p-1}}\int_{B_R}(|Xu|^p + |g|^{\frac{p}{p-1}})dx + c_\sigma\int_{B_R}|f-f_R|^{\frac{p}{p-1}}dx$$

(4.13)

$$\leq \sigma\int_{B_R}|Xu-X\bar{u}|^p dx + c_\sigma R^{\frac{p}{p-1}}\int_{B_R}|Xu|^p dx + c(\sigma, \|g\|^{p/(p-1)}_{L^{p/(p-1),Q}}, \|f\|^{p/(p-1)}_{\mathcal{L}^{p/(p-1),Q}})R^Q.$$

Taking (4.13) into (4.12) and then choosing $\sigma=\frac{1}{2}$, we have

$$\int_{B_R}|Xu-X\bar{u}|^p dx \leq c(\epsilon + R^{p/(p-1)})\int_{B_R}|Xu|^p dx + cR^Q \tag{4.14}$$

From (4.10), (4.14) and (4.6), we know that for any p, it holds

$$\int_{B_\rho} |Xu|^p dx \le c\left[\left(\frac{\rho}{R}\right)^Q + \vartheta(R,\varepsilon) + R^{\frac{p}{p-1}} + \epsilon\right]\int_{B_R} |Xu|^p dx + cR^Q.$$

Making use of the VMO assumptions on coefficients, we can choose R, ε and ϵ small enough such that $\vartheta(R,\varepsilon) + R^{\frac{p}{p-1}} + \epsilon$ is small enough. By virtue of Lemma 2.6 we find that for any $0 < \lambda < Q$,

$$\int_{B_\rho} |Xu|^p dx \le c\rho^\lambda, \tag{4.15}$$

which implies $Xu \in L^{p,\lambda}_{\text{loc}}(\Omega)$.

Next we prove the Hölder continuity of u. Note that $u \in HW^{1,p}(\Omega)$. If $p > Q$ then $u \in C^{0,\alpha}_X(\Omega)(0 < \alpha < 1 - \frac{Q}{p})$ is trivial in view of the Sobolev embedding theorem ([34]). If $p \le Q$ then we have by the Sobolev inequality that

$$\int_{B_\rho} |u - u_\rho|^p dx \le c\rho^p \int_{B_\rho} |Xu|^p dx.$$

Together with (4.15) (taking $\lambda = Q - p + p\alpha$) we see that

$$\int_{B_\rho} |u - u_\rho|^p dx \le c\rho^{Q+p\alpha},$$

where c is independent of the center x and the radius ρ of the ball B_ρ. This shows $u \in \mathcal{L}^{p,Q+p\alpha}_{X,\text{loc}}(\Omega)$. We immediately know from Lemma 2.5 that $u \in C^{0,\alpha}_X(\Omega)$, which completes the proof.

5. Campanato type estimate for homogeneous obstacle problem

In this section, we suppose $f \in C^{0,\gamma}_X(\Omega)(0 < \gamma < 1)$ and $a_{ij}(x) \in \text{BMO}_\omega(\Omega)$ $(\omega(R) = R^\gamma)$. We first recall the following inequalities in [33]: if $p \ge 2$, then

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \le (p-1)(|\xi|^{p-2} + |\eta|^{p-2})|\xi - \eta|; \tag{5.1}$$

if $1 < p < 2$, then there exists $C(p) > 0$ such that

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \le C(p)|\xi - \eta|^{p-1}. \tag{5.2}$$

Accordingly, we have

LEMMA 5.1. *For the symmetric, positive-definite matrix $A = (a_{ij}(x))_{m\times m}$ satisfying (H1), there exists a constant $C(p,\Lambda) > 0$ such that*

$$\left|\langle A_R\xi,\xi\rangle^{\frac{p-2}{2}} A_R\xi - \langle A_R\eta,\eta\rangle^{\frac{p-2}{2}} A_R\eta\right| \le C(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi - \eta|, \text{ if } p \ge 2;$$

$$\left|\langle A_R\xi,\xi\rangle^{\frac{p-2}{2}} A_R\xi - \langle A_R\eta,\eta\rangle^{\frac{p-2}{2}} A_R\eta\right| \le C|\xi - \eta|^{p-1}, \text{ if } 1 < p < 2.$$

PROOF. Since A_R is a symmetric and positive-definite constant matrix, there exists an orthogonal matrix P such that $A_R = P^T DP$, where D is a diagonal matrix whose diagonal elements are the eigenvalues $\lambda_1, \dots, \lambda_m$ of A_R with $0 < \Lambda^{-1} < \lambda_i < \Lambda$ for $i = 1, \dots, m$. Denote by $D^{\frac{1}{2}}$ the diagonal matrix with the diagonal elements $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_m}$, then for any $\xi \in \mathbb{R}^m$,

$$\begin{aligned}
&\left|\langle A_R\xi, \xi\rangle^{\frac{p-2}{2}} A_R\xi - \langle A_R\eta, \eta\rangle^{\frac{p-2}{2}} A_R\eta\right|^2 \\
&= \left|\langle P^T DP\xi, \xi\rangle^{\frac{p-2}{2}} P^T DP\xi - \langle P^T DP\eta, \eta\rangle^{\frac{p-2}{2}} P^T DP\eta\right|^2 \\
&= \left|\langle DP\xi, P\xi\rangle^{\frac{p-2}{2}} P^T DP\xi - \langle DP\eta, P\eta\rangle^{\frac{p-2}{2}} P^T DP\eta\right|^2 \\
&= \left|\langle D^{\frac{1}{2}}P\xi, D^{\frac{1}{2}}P\xi\rangle^{\frac{p-2}{2}} P^T DP\xi - \langle D^{\frac{1}{2}}P\eta, D^{\frac{1}{2}}P\eta\rangle^{\frac{p-2}{2}} P^T DP\eta\right|^2 \\
&= \left|P^T D^{\frac{1}{2}} \left(\langle D^{\frac{1}{2}}P\xi, D^{\frac{1}{2}}P\xi\rangle^{\frac{p-2}{2}} D^{\frac{1}{2}}P\xi - \langle D^{\frac{1}{2}}P\eta, D^{\frac{1}{2}}P\eta\rangle^{\frac{p-2}{2}} D^{\frac{1}{2}}P\eta\right)\right|^2 \\
&\le \Lambda \left|\langle D^{\frac{1}{2}}P\xi, D^{\frac{1}{2}}P\xi\rangle^{\frac{p-2}{2}} D^{\frac{1}{2}}P\xi - \langle D^{\frac{1}{2}}P\eta, D^{\frac{1}{2}}P\eta\rangle^{\frac{p-2}{2}} D^{\frac{1}{2}}P\eta\right|^2 \\
&= \Lambda \left||D^{\frac{1}{2}}P\xi|^{p-2} D^{\frac{1}{2}}P\xi - |D^{\frac{1}{2}}P\xi|^{p-2} D^{\frac{1}{2}}P\eta\right|^2.
\end{aligned}$$

If $p \ge 2$, we have from (5.1) that

$$\begin{aligned}
&\left|\langle A_R\xi, \xi\rangle^{\frac{p-2}{2}} A_R\xi - \langle A_R\eta, \eta\rangle^{\frac{p-2}{2}} A_R\eta\right|^2 \\
&\le (p-1)^2 (|D^{\frac{1}{2}}P\xi|^{p-2} + |D^{\frac{1}{2}}P\eta|^{p-2})^2 |D^{\frac{1}{2}}P\xi - D^{\frac{1}{2}}P\eta|^2.
\end{aligned} \tag{5.3}$$

Noting

$$|D^{\frac{1}{2}}P\eta|^2 = \eta^T P^T DP\eta = \eta^T A\eta = \langle A\eta, \eta\rangle \le \Lambda|\eta|^2,$$

we observe

$$\begin{aligned}
\left(|D^{\frac{1}{2}}P\xi|^{p-2} + |D^{\frac{1}{2}}P\eta|^{p-2}\right)^2 &\le c(p)(|D^{\frac{1}{2}}P\xi|^2 + |D^{\frac{1}{2}}P\eta|^2)^{p-2} \\
&\le c(p,\Lambda)(|\xi|^2 + |\eta|^2)^{p-2}
\end{aligned} \tag{5.4}$$

and

$$|D^{\frac{1}{2}}P\xi - D^{\frac{1}{2}}P\eta|^2 = |D^{\frac{1}{2}}P(\xi - \eta)|^2 \le \Lambda|\xi - \eta|^2. \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3) gives

$$\left|\langle A_R\xi, \xi\rangle^{\frac{p-2}{2}} A_R\xi - \langle A_R\eta, \eta\rangle^{\frac{p-2}{2}} A_R\eta\right|^2 \le c(p,\Lambda)(|\xi|^2 + |\eta|^2)^{p-2}|\xi - \eta|^2.$$

When $1 < p < 2$, we have by (5.2) that

$$\begin{aligned}
\left||D^{\frac{1}{2}}P\xi|^{p-2} D^{\frac{1}{2}}P\xi - |D^{\frac{1}{2}}P\xi|^{p-2} D^{\frac{1}{2}}P\eta\right|^2 &\le C(p)|D^{\frac{1}{2}}P\xi - D^{\frac{1}{2}}P\eta|^{2(p-1)} \\
&\le C(p,\Lambda)|\xi - \eta|^{2(p-1)}.
\end{aligned}$$

The proof is completed. □

In order to prove the $C_X^{1,\alpha}$ regularity for solutions to (1.1), we need to establish the Campanato type estimate for solutions to (3.1). To this end, we first study the Campanato type estimate for weak solutions to (3.3).

Lemma 5.2. *Let $w \in HW^{1,p}(B_{R/2})$ be a weak solution to* (3.3) *with p close to 2. Then for any $0 < \rho \le R/2$, there exists $\delta_1 > 0$ such that*

$$\int_{B_\rho} |Xw - (Xw)_\rho|^p dx \le c\left(\frac{\rho}{R}\right)^{Q+2\beta} \int_{B_{R/2}} |Xw - (Xw)_{R/2}|^p dx + cR^{Q+\delta_1}. \tag{5.6}$$

Proof. Let $\bar{w} \in HW^{1,p}(B_{R/2})$ be the weak solution to

$$\begin{cases} -X^*(\langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{w}) = 0, & \text{in } B_R, \\ \bar{w} - w \in HW_0^{1,p}(B_{R/2}). \end{cases}$$

It follows from (3.10) that for any $0 < \rho \le R/2$,

$$\int_{B_\rho} |X\bar{w} - (X\bar{w})_\rho|^p dx \le c\left(\frac{\rho}{R}\right)^{Q+2\beta} \int_{B_{R/2}} |X\bar{w} - (X\bar{w})_{R/2}|^p dx,$$

then

$$\begin{aligned} &\int_{B_\rho} |Xw - (Xw)_\rho|^p dx \\ &= \int_{B_\rho} |Xw - X\bar{w} + X\bar{w} - (X\bar{w})_\rho + (X\bar{w})_\rho - (Xw)_\rho|^p dx \\ &\le c\int_{B_\rho} |X\bar{w} - (X\bar{w})_\rho|^p dx + c\int_{B_\rho} |Xw - X\bar{w}|^p dx \\ &\le c\left(\frac{\rho}{R}\right)^{Q+2\beta} \int_{B_{R/2}} |X\bar{w} - (X\bar{w})_{R/2}|^p dx + c\int_{B_\rho} |Xw - X\bar{w}|^p dx \\ &\le c\left(\frac{\rho}{R}\right)^{Q+2\beta} \int_{B_{R/2}} |Xw - (Xw)_{R/2}|^p dx + c\int_{B_{R/2}} |Xw - X\bar{w}|^p dx. \end{aligned} \tag{5.7}$$

Next we estimate the second term in (5.7). Set

$$\Psi = \langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi.$$

If $p \ge 2$, it is easy to see from Lemma 5.1 that $\Psi \in C_X^{0,\gamma}$. Then from (3.6) and (3.3) we deduce

$$\begin{aligned} &\int_{B_{R/2}} |Xw - X\bar{w}|^p dx \\ &\le c\int_{B_{R/2}} \langle \langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw - \langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{w}, Xw - X\bar{w}\rangle dx \end{aligned}$$

$$= c\int_{B_{R/2}} \langle \langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi - (\langle A_{R/2}X\psi, X\psi\rangle^{\frac{p-2}{2}} A_{R/2}X\psi)_{R/2}, Xw - X\bar{w}\rangle dx$$
$$\le c\int_{B_{R/2}} |\Psi - \Psi_{R/2}||Xw - X\bar{w}|dx$$
$$\le c\left(\int_{B_{R/2}} |\Psi - \Psi_{R/2}|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} \left(\int_{B_{R/2}} |Xw - X\bar{w}|^p dx\right)^{\frac{1}{p}}.$$

Consequently,

$$\int_{B_{R/2}} |Xw - X\bar{w}|^p dx \le c\int_{B_{R/2}} |\Psi - \Psi_{R/2}|^{\frac{p}{p-1}}dx$$
$$\le c\int_{B_{R/2}} \left(⨍_{B_{R/2}} |\Psi(x) - \Psi(y)|dy\right)^{\frac{p}{p-1}} dx \le cR^{Q+\frac{p\gamma}{p-1}}. \tag{5.8}$$

Now (5.6) is immediately obtained by taking (5.8) into (5.7).

Next assume $p < 2$. In this case $\Psi \in C_X^{0,\gamma(p-1)}$ from Lemma 5.1. Hence

$$\int_{B_{R/2}} |\Psi - \Psi_{R/2}|^{\frac{p}{p-1}}dx \le cR^{Q+p\gamma}.$$

According to (3.13),

$$\int_{B_{R/2}} |Xw - X\bar{w}|^p dx$$
$$\le c\left(\int_{B_{R/2}} |Xw - X\bar{w}|^2 \left(|Xw| + |X\bar{w}|\right)^{p-2} dx\right)^{\frac{p}{2}} \left(\int_{B_{R/2}} \left(|Xw| + |X\bar{w}|\right)^p dx\right)^{\frac{2-p}{2}}$$
$$\le c\left(\int_{B_{R/2}} |Xw - X\bar{w}|^2 \left(|Xw|^2 + |X\bar{w}|^2\right)^{\frac{p-2}{2}} dx\right)^{\frac{p}{2}} \left(\int_{B_{R/2}} |Xw|^p dx\right)^{\frac{2-p}{2}}. \tag{5.9}$$

On the other hand, it follows from (3.5) that

$$\int_{B_{R/2}} |Xw - X\bar{w}|^2 \left(|Xw|^2 + |X\bar{w}|^2\right)^{\frac{p-2}{2}} dx$$
$$\le c\int_{B_{R/2}} \langle \langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw - \langle A_{R/2}X\bar{w}, X\bar{w}\rangle^{\frac{p-2}{2}} A_{R/2}X\bar{w}, Xw - X\bar{w}\rangle dx$$
$$= c\int_{B_{R/2}} \langle \Psi - \Psi_{R/2}, Xw - X\bar{w}\rangle dx$$
$$\le c\left(\int_{B_{R/2}} |\Psi - \Psi_{R/2}|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} \left(\int_{B_{R/2}} |Xw - X\bar{w}|^p dx\right)^{\frac{1}{p}}$$

(5.10)

$$\leq cR^{Q\frac{p-1}{p}+(p-1)\gamma}\left(\int_{B_{R/2}}|Xw-X\bar{w}|^p dx\right)^{\frac{1}{p}}.$$

By Remark 3.5, (5.9) and (5.10), we have

$$\int_{B_{R/2}}|Xw-X\bar{w}|^p dx\leq cR^{\frac{(2-p)\nu}{2}}\left(R^{Q\frac{p-1}{p}+(p-1)\gamma}\right)^{\frac{p}{2}}\left(\int_{B_{R/2}}|Xw-X\bar{w}|^p dx\right)^{\frac{1}{2}}.$$

Hence

$$\int_{B_{R/2}}|Xw-X\bar{w}|^p dx\leq cR^{Q+p(p-1)\gamma+(2-p)(\nu-Q)}. \tag{5.11}$$

The desired inequality follows from (5.7) and (5.11). □

Now we are in a position to prove the Campanato type estimate to (3.1).

Lemma 5.3. *Let $\bar{u}\in HW^{1,p}_{\rm loc}(\Omega)$ be a solution to* (3.1) *with p close to 2. Then for any $0<\rho\leq R$, there exists $\delta_2>0$ such that*

$$\int_{B_\rho}|X\bar{u}-(X\bar{u})_\rho|^p dx\leq c\left(\frac{\rho}{R}\right)^{Q+2\beta}\int_{B_R}|X\bar{u}-(X\bar{u})_R|^p dx+cR^{Q+\delta_2}. \tag{5.12}$$

Proof. Let $w\in HW^{1,p}(B_{R/2})$ be the weak solution to

$$\begin{cases}-X^*(\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw)=-X^*(\langle A_{R/2}X\psi,X\psi\rangle^{\frac{p-2}{2}}A_{R/2}X\psi), & \text{in } B_{R/2},\\ w-\bar{u}\in HW^{1,p}_0(B_{R/2}).\end{cases}$$

It follows from (3.27) and Remark 3.7 that for any $0<\nu<Q$,

$$\int_{B_{R/2}}|Xw|^p dx\leq c\int_{B_{R/2}}|X\bar{u}|^p dx+c\int_{B_{R/2}}|X\psi|^p dx\leq cR^\nu. \tag{5.13}$$

As before, we see from Lemma 5.2 that for any $0<\rho<R/2$,

$$\begin{aligned}&\int_{B_\rho}|X\bar{u}-(X\bar{u})_\rho|^p dx\\ &\leq c\int_{B_\rho}|Xw-(Xw)_\rho|^p dx+c\int_{B_\rho}|X\bar{u}-Xw|^p dx\\ &\leq c\left(\frac{\rho}{R}\right)^{Q+2\beta}\int_{B_{R/2}}|Xw-(Xw)_{R/2}|^p dx+cR^{Q+\delta_1}+c\int_{B_\rho}|X\bar{u}-Xw|^p dx\\ &\leq c\left(\frac{\rho}{R}\right)^{Q+2\beta}\int_{B_{R/2}}|X\bar{u}-(X\bar{u})_{R/2}|^p dx+cR^{Q+\delta_1}+c\int_{B_{R/2}}|X\bar{u}-Xw|^p dx\end{aligned}$$

(5.14)
$$\leq c\left(\frac{\rho}{R}\right)^{Q+2\beta}\int_{B_R}|X\bar{u}-(X\bar{u})_R|^p dx + cR^{Q+\delta_1} + c\int_{B_{R/2}}|X\bar{u}-Xw|^p dx.$$

We also have from (3.24) and (3.22) that

$$\int_{B_{R/2}}\langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u}-\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw,X\bar{u}-Xw\rangle dx$$
$$\leq\int_{B_{R/2}}\langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u}-\langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u},X\bar{u}-Xw\rangle dx$$
$$-\int_{B_{R/2}}\langle\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw,X\bar{u}-Xw\rangle dx$$
$$=\int_{B_{R/2}}\langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u}-\langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u},X\bar{u}-Xw\rangle dx$$
$$-\int_{B_{R/2}}\langle\Psi-\Psi_{R/2},X\bar{u}-Xw)\rangle dx$$

(5.15)
$$\leq\int_{B_{R/2}}|A_{R/2}-A||X\bar{u}|^{p-1}|X\bar{u}-Xw|dx+\int_{B_{R/2}}|\Psi-\Psi_{R/2}||X\bar{u}-Xw|dx.$$

Case 1: $p\geq 2$. By (5.15), we have

$$\int_{B_{R/2}}|X\bar{u}-Xw|^p dx$$
$$\leq c\int_{B_{R/2}}\langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u}-\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw,X(\bar{u}-w)\rangle dx$$
$$\leq c\left[\left(\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar{u}|^p dx\right)^{\frac{p-1}{p}}+\left(\int_{B_{R/2}}|\Psi-\Psi_{R/2}|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\right]$$
$$\times\left(\int_{B_{R/2}}|X\bar{u}-Xw|^p dx\right)^{\frac{1}{p}},$$

and hence

$$\int_{B_{R/2}}|X\bar{u}-Xw|^p dx\leq\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar{u}|^p dx+c\int_{B_{R/2}}|\Psi-\Psi_{R/2}|^{\frac{p}{p-1}}dx.$$

Arguing similarly to (3.30), we have by using $A\in\mathrm{BMO}_\omega(\Omega)(\omega(R)=R^\gamma)$ and Remark 3.7 that there exists $\gamma_1>0$ such that

$$\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar{u}|^p dx\leq c\omega^{(t-p)/t}(R)\|A\|_{*,\omega,R/2}^{(t-p)/t}\left(\int_{B_R}|X\bar{u}|^p dx+R^Q\right)$$

$$\leq c\omega^{(t-p)/t}(R)\int_{B_R}|X\bar{u}|^p dx + c\omega^{(t-p)/t}(R)R^Q$$

$$\leq cR^{\nu}R^{(t-p)\gamma/t} \leq cR^{Q+\gamma_1}. \tag{5.16}$$

Note $\int_{B_{R/2}}|\Psi-\Psi_{R/2}|^{\frac{p}{p-1}}dx \leq cR^{Q+\frac{p\gamma}{p-1}}$ for $p \geq 2$. It follows

$$\int_{B_{R/2}}|X\bar{u}-Xw|^p dx \leq c(R^{Q+\gamma_1}+R^{Q+\frac{p\gamma}{p-1}}). \tag{5.17}$$

Case 2: $p<2$. In this case we deduce from (5.13), (5.15) and (5.16) that

$$\int_{B_{R/2}}|X\bar{u}-Xw|^p dx$$

$$\leq c\left(\int_{B_{R/2}}|X\bar{u}-Xw|^2\left(|X\bar{u}|+|Xw|\right)^{p-2}dx\right)^{\frac{p}{2}}\left(\int_{B_{R/2}}\left(|X\bar{u}|+|Xw|\right)^p dx\right)^{\frac{2-p}{2}}$$

$$\leq cR^{\frac{\nu(2-p)}{2}}\left(\int_{B_{R/2}}|X\bar{u}-Xw|^2\left(|X\bar{u}|^2+|Xw|^2\right)^{\frac{p-2}{2}}dx\right)^{\frac{p}{2}}$$

$$\leq cR^{\frac{\nu(2-p)}{2}}\times$$

$$\left(\int_{B_{R/2}}\langle\langle A_{R/2}X\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}A_{R/2}X\bar{u}-\langle A_{R/2}Xw,Xw\rangle^{\frac{p-2}{2}}A_{R/2}Xw, X\bar{u}-Xw\rangle dx\right)^{\frac{p}{2}}$$

$$\leq cR^{\frac{\nu(2-p)}{2}}\left[\left(\int_{B_{R/2}}|A_{R/2}-A|^{\frac{p}{p-1}}|X\bar{u}|^p dx\right)^{\frac{p-1}{p}}+\left(R^{Q+p\gamma}\right)^{\frac{p-1}{p}}\right]^{\frac{p}{2}}$$

$$\times\left(\int_{B_{R/2}}|X\bar{u}-Xw|^p dx\right)^{\frac{1}{2}}$$

$$\leq cR^{\frac{\nu(2-p)}{2}}\left(R^{(Q+\gamma_1)\frac{p-1}{p}}+R^{(Q+p\gamma)\frac{p-1}{p}}\right)^{\frac{p}{2}}\cdot\left(\int_{B_{R/2}}|X\bar{u}-Xw|^p dx\right)^{\frac{1}{2}}.$$

Thus

$$\int_{B_{R/2}}|X\bar{u}-Xw|^p dx \leq cR^{\nu(2-p)}\left(R^{(Q+\gamma_1)(p-1)}+R^{(Q+p\gamma)(p-1)}\right)$$

$$= cR^{Q+(\nu-Q)(2-p)+(p-1)\gamma_1}+R^{Q+(\nu-Q)(2-p)+p(p-1)\gamma}$$

$$\leq cR^{Q+\gamma_2}, \tag{5.18}$$

where $\gamma_2=\min\{(\nu-Q)(2-p)+(p-1)\gamma_1,(\nu-Q)(2-p)+p(p-1)\gamma\}>0$. Choosing $\delta_2=\min\{\delta_1,\gamma_1,\gamma_2,\frac{p\gamma}{p-1}\}>0$, it follows from (5.17), (5.18) and (5.14) that for any $0 < \rho < R/2$,

$$\int_{B_\rho} |X\bar{u} - (X\bar{u})_\rho|^p dx \le c \left(\frac{\rho}{R}\right)^{Q+2\beta} \int_{B_R} |X\bar{u} - (X\bar{u})_R|^p dx + cR^{Q+\delta_2}.$$

When $R/2 \le \rho \le R$, observing

$$\int_{B_\rho} |X\bar{u} - (X\bar{u})_\rho|^p dx \le 2^p \int_{B_\rho} |X\bar{u} - (X\bar{u})_R|^p dx,$$

we have

$$\begin{aligned}\int_{B_\rho} |X\bar{u} - (X\bar{u})_\rho|^p dx &\le 2^p \int_{B_R} |X\bar{u} - (X\bar{u})_R|^p dx \\ &\le 2^{p+Q+2\beta} \left(\frac{\rho}{R}\right)^{Q+2\beta} \int_{B_R} |X\bar{u} - (X\bar{u})_R|^p dx.\end{aligned}$$

Combining these two cases completes the proof of the lemma. □

6. Proof of Theorem 1.2

We prove Theorem 1.2 in this Section.

Proof of Theorem 1.2 Fix $x \in \Omega$ and let $B_R := B_R(x) \subset\subset \Omega$. Let $u \in \mathcal{K}^\theta_\psi(\Omega)$ be a solution to (1.1) and $\bar{u} \in \mathcal{K}^u_\psi(B_R)$ a solution to the corresponding homogeneous obstacle problem, namely, $\bar{u}$ satisfies

$$\int_\Omega \langle \langle AX\bar{u}, X\bar{u}\rangle^{\frac{p-2}{2}} AX\bar{u}, X(v-\bar{u})\rangle dx \ge 0, \quad \forall v \in \mathcal{K}^u_\psi(B_R).$$

Now we see from (5.12) that for any $0 < \rho \le R$,

$$\begin{aligned}&\int_{B_\rho} |Xu - (Xu)_\rho|^p dx \\ &\le c\int_{B_\rho} |X\bar{u} - (X\bar{u})_\rho|^p dx + c\int_{B_\rho} |Xu - X\bar{u}|^p dx \\ &\le c\left(\frac{\rho}{R}\right)^{Q+2\beta} \int_{B_R} |Xu - (Xu)_R|^p dx + cR^{Q+\delta_2} + c\int_{B_R} |Xu - X\bar{u}|^p dx. \qquad (6.1)\end{aligned}$$

If $p \ge 2$, it follows from (4.8) and Theorem 1.1 that there exits $\gamma_3 > 0$ such that

$$\begin{aligned}\int_{B_R} |Xu - X\bar{u}|^p dx &\le cR^{\frac{p}{p-1}} \int_{B_R} |Xu|^p dx + cR^{Q+\frac{p}{p-1}} + c\int_{B_R} |f - f_R|^{\frac{p}{p-1}} dx \\ &\le cR^{\lambda+\frac{p}{p-1}} + cR^{Q+\frac{p}{p-1}} + cR^{Q+\frac{p\gamma}{p-1}} \le cR^{Q+\gamma_3}. \qquad (6.2)\end{aligned}$$

Taking (6.2) into (6.1) yields
(6.3)

$$\int_{B_\rho} |Xu-(Xu)_\rho|^p dx \le c\left(\frac{\rho}{R}\right)^{Q+2\beta}\int_{B_R}|Xu-(Xu)_R|^p dx + cR^{Q+\delta_2}+cR^{Q+\gamma_3}.$$

If $1<p<2$, then from (4.11), (4.15) and (4.7)

$$\begin{aligned}&\int_{B_R}|Xu-X\bar{u}|^p dx\\ &\le cR^{\lambda\frac{2-p}{2}}\left(\int_{B_{R/2}}\langle\langle AXu,Xu\rangle^{\frac{p-2}{2}}AXu-\langle AX\bar{u},X\bar{u}\rangle^{\frac{p-2}{2}}AX\bar{u},Xu-X\bar{u}\rangle dx\right)^{\frac{p}{2}}\\ &\le cR^{\lambda\frac{2-p}{2}}\left[R\left(\int_{B_R}(|Xu|^p+|g|^{\frac{p}{p-1}})dx\right)^{\frac{p-1}{p}}+\left(\int_{B_R}|f-f_R|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\right]^{\frac{p}{2}}\\ &\quad\times\left(\int_{B_R}|Xu-X\bar{u}|^p dx\right)^{\frac{1}{2}}.\end{aligned}$$

Then

$$\begin{aligned}\int_{B_R}|Xu-X\bar{u}|^p dx &\le cR^{\lambda(2-p)}\left[R\left(R^{\lambda\frac{p-1}{p}}+R^{Q\frac{p-1}{p}}\right)+R^{\frac{p-1}{p}Q+\gamma}\right]^p\\ &=c(R^{\lambda(2-p)+p+\lambda(p-1)}+R^{\lambda(2-p)+p+Q(p-1)}+R^{\lambda(2-p)+Q(p-1)+p\gamma})\\ &=cR^{\lambda+p}+cR^{Q+(\lambda-Q)(2-p)+p}+cR^{Q+(\lambda-Q)(2-p)+p\gamma}.\end{aligned}$$

Since the above inequality holds for arbitrary $0<\lambda<Q$, there exists $\gamma_4>0$ such that

$$\int_{B_R}|Xu-X\bar{u}|^p dx\le cR^{Q+\gamma_4}. \tag{6.4}$$

Combining (6.4), (6.2) and (6.1), we know that there exists $0<\delta<2\beta$ such that

$$\int_{B_\rho}|Xu-(Xu)_\rho|^p dx\le c\left(\frac{\rho}{R}\right)^{Q+2\beta}\int_{B_R}|Xu-(Xu)_R|^p dx+cR^{Q+\delta}.$$

Now we can employ Lemma 2.6 to conclude

$$\int_{B_\rho}|Xu-(Xu)_\rho|^p dx\le c\rho^{Q+\delta},$$

which implies $Xu\in\mathcal{L}^{p,Q+\delta}_{\text{loc}}(\Omega)$. Then it follows from Lemma 2.5 that there exists $\alpha>0$ such that $Xu\in C^{0,\alpha}_X(\Omega)$, i.e. $u\in C^{1,\alpha}_X(\Omega)$.

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