SIGN-CHANGING SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SEMILINEAR \(\Delta_\gamma\)-LAPLACE EQUATIONS

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ABSTRACT. In this article, we study the existence of multiple solutions for the boundary value problem

\[-G_\alpha u = g(x, y, u) + f(x, y, u) \quad \text{in} \quad \Omega,\]

\[u = 0 \quad \text{on} \quad \partial\Omega,\]

where \(\Omega\) is a bounded domain with smooth boundary in \(\mathbb{R}^N (N \geq 2)\), \(\alpha \in \mathbb{N}\), \(g(x, y, \xi)\), \(f(x, y, \xi)\) are Carathéodory functions and \(G_\alpha\) is the Grushin operator. We use the lower bounds of eigenvalues and an abstract theory on sign-changing solutions.

1. INTRODUCTION

Boundary value problems for semilinear elliptic equations were studied in [1,27] (see also the references therein). Many publications [4–8, 10–12, 18, 26, 29, 31] are devoted to the study of the existence of sign-changing solutions of classical elliptic boundary value problems such as

\[-\Delta u = f(x, u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,\]

where \(f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})\), \(\Omega \subset \mathbb{R}^N (N \geq 2)\) is a bounded domain with smooth boundary \(\partial\Omega\). There have been several methods developed in studying sign-changing solutions of nonlinear elliptic equations, such as the invariant sets of descending flow method developed by Liu and Sun [5,18,31], and the minimax method which is established by Berestycki and Lions in the classical paper [8].

One of the classes of degenerate elliptic equations that has been studied widely in recent years is the class of equations involving an operator of the Grushin type (see [14])

\[G_\alpha := \Delta_x + |x|^{2\alpha} \Delta_y, \quad \alpha \geq 0.\]

Note that \(G_0 \equiv \Delta\) is the Laplacian operator, and \(G_\alpha\), when \(\alpha > 0\), is not elliptic in domains intersecting the surface \(x = 0\). Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [36,37] (see also some recent results in [2,13,17,19–23,25,33–35]).

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In this paper, we consider the existence of sign-changing solutions of the Dirichlet boundary value problem

\begin{align}
- G_\alpha u &= g(x, y, u) + f(x, y, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

where \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} := \mathbb{R}^N, N_1, N_2, \alpha \in \mathbb{N}, \Omega \cap \{(x, y) \in \mathbb{R}^N : x = 0\} \neq \emptyset, \) and

\[ \Delta_x := \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2}, \Delta_y := \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2}, |x|^{2\alpha} := \left( \sum_{i=1}^{N_1} x_i^2 \right)^\alpha, \]

and the nonlinearity \( f \) is a real Carathéodory function on \( \Omega \times \mathbb{R} \) and satisfies the following conditions

(A1) There exist \( p \in (2, 2^*_\alpha), \) and constants \( C_1, C_2 > 0 \) such that

\[ |f(x, y, \xi)| \leq C_1 + C_2 |\xi|^{p-1} \quad \text{almost everywhere } (x, y, \xi) \in \Omega \times \mathbb{R}, \]

where \( 2^*_\alpha := \frac{2N_\alpha}{N_\alpha - 2}, N_\alpha := N_1 + (1 + \alpha)N_2 > 2; \)

(A2) \( f(x, y, \xi) = o(|\xi|), \) uniformly in \( (x, y) \in \overline{\Omega}, \) as \( \xi \to 0 \) and \( f(x, y, \xi)\xi \geq 0 \) for all \( \xi \in \mathbb{R} \) and a.e. \( (x, y) \in \Omega; \)

(A3) There exists a constant \( \mu > 2 \) such that

\[ 0 \leq \mu F(x, y, \xi) \leq \xi f(x, y, \xi), \quad \forall (x, y) \in \overline{\Omega}, \xi \in \mathbb{R} \setminus \{0\}, \]

where \( F(x, y, \xi) = \int_0^\xi f(x, y, \tau) d\tau; \)

(A4) \( f(x, y, -\xi) = -f(x, y, \xi) \) for all \( (x, y, \xi) \in \overline{\Omega} \times \mathbb{R}; \)

(A5) \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function. There exists \( \sigma < \frac{\mu}{2} \) such that

\[ |g(x, y, \xi)| \leq C(1 + |\xi|^{\sigma}), \text{ for all } \xi \in \mathbb{R}, \text{ and a.e. } (x, y) \in \Omega. \]

Moreover, \( g(x, y, \xi) = o(|\xi|), \) uniformly in \( (x, y) \in \overline{\Omega}, \) as \( \xi \to 0 \) and \( g(x, y, \xi)\xi > 0 \) for all \( \xi \in \mathbb{R} \setminus \{0\} \) and a.e. \( (x, y) \in \Omega. \)

Our main result is given by the following theorem.

**Theorem 1.1.** Assume that \( f, g \) satisfies the conditions (A1)–(A5) and

\[ \frac{2p}{N_\alpha(p-2)} - 1 > \frac{\mu}{\mu - \sigma - 1}. \]

Then the problem (1.1)–(1.2) has infinitely many sign-changing solutions.

This article is organized as follows. In section 2, we present some definitions and preliminary results. Next, combining the lower bounds of eigenvalues and an abstract theory on sign-changing solutions, we give the proof of Theorem 1.1.
2. Preliminary results

Definition 2.1. By $S^2_1(\Omega)$ we will denote the set of all functions $u \in L^2(\Omega)$ such that $\frac{\partial u}{\partial x_i} \in L^2(\Omega), |x|^\alpha \frac{\partial u}{\partial y_j} \in L^2(\Omega), i = 1, 2, \ldots, N_1$, $j = 1, 2, \ldots, N_2$. We define the norm in this space as follows

$$\|u\|_{S^2_1(\Omega)} = \left\{ \int_\Omega \left( |u|^2 + |\nabla u|^2 \right) \, dX \right\}^{\frac{1}{2}},$$

where

$$dX = dx_1 \ldots dx_{N_1}dy_1 \ldots dy_{N_2}, \nabla u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{N_1}}, |x|^\alpha \frac{\partial u}{\partial y_1}, \ldots, |x|^\alpha \frac{\partial u}{\partial y_{N_2}} \right).$$

We can also define the scalar product in $S^2_1(\Omega)$ as follows

$$(u,v)_{S^2_1(\Omega)} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}.$$

The space $S^2_{1,0}(\Omega)$ is defined as the closure of $C^1_{0}(\Omega)$ in the space $S^2_1(\Omega)$.

The following embedding inequality was proved in [33,37]

$$\left( \int_{\Omega} |u|^p \, dX \right)^{\frac{1}{p}} \leq C(p,\Omega) \|u\|_{S^2_{1,0}(\Omega)},$$

where $1 \leq p \leq 2^{*}_\alpha$, $C(p,\Omega) > 0$. The number $2^{*}_\alpha$ is the critical Sobolev exponent of the embedding $S^2_{1,0}(\Omega) \hookrightarrow L^p(\Omega)$ and when $1 \leq p < 2^{*}_\alpha$, the embedding is compact.

Definition 2.2. Let $V$ be a real Banach space with its dual space $V^*$, $\Phi \in C^1(V,\mathbb{R})$. We say that $\Phi$ satisfies the Palais–Smale if for any sequence $\{u_n\}_{n=1}^{+\infty} \subset V$ such that $\Phi(u_n)$ is bounded and

$$\|\Phi'(u_n)\|_{V^*} \to 0 \text{ as } n \to \infty,$$

then there exists a subsequence $\{u_{n_k}\}_{k=1}^{+\infty}$ that converges strongly in $V$.

From Theorem A in [30], we have

Proposition 2.3. Let $V$ be a Hilbert space and $\Phi \in C^1(V,\mathbb{R})$ be of the form $\Phi' = id - K_\Phi$ and satisfy the Palais-Smale condition, where $K_\Phi$ is a continuous operator. Assume that $K_\Phi(\pm D_0) \subset \pm D_0$ holds, where $D_0 = \{u \in V : \text{dist}(u,P) < \mu_0\}$ and $P = \{u \in V, u(x) \geq 0, \text{ for a.e. } x \in \Omega\}$ is the positive cone of $V$. Let $N, M$ be two closed subspaces of $V$ with $\text{dim}N < \infty$, $\text{dim}N - \text{codim}M \geq 1$. Suppose that

$$Q(\rho) := \{u \in M : \|u\|_V = \rho\} \subset S := V \setminus (-D_0 \cup D_0).$$

Define

$$N^* = N \oplus \text{span}\{u^*\} \quad u^* = V \setminus N; \quad N^*_+= \{u + tu^* : u \in N, t \geq 0\}.$$

Assume that
(i) \( \Phi(0) = 0; \)
(ii) there exists a \( R_1 > \rho \) such that \( \Phi(u) \leq 0 \) for all \( u \in N \) with \( \|u\|_V \geq R_1; \)
(iii) there exists a \( R_2 \geq R_1 \) such that \( \Phi(u) \leq 0 \) for all \( u \in N^* \) with \( \|u\|_V \geq R_2; \)

Let

\[
\Gamma = \{ \phi \in C(V, V) : \phi \text{ is odd, } \phi(-D \cup D) \subset (-D \cup D); \ 
\phi(u) = u \text{ if } \max\{\Phi(u), \Phi(-u)\} \leq 0 \}.
\]

If

\[
\gamma^* = \inf_{\phi \in \Gamma} \sup_{(N^*) \cap S} \phi > \gamma^{**} = \inf_{\phi \in \Gamma} \sup_{(N) \cap S} \phi > 0,
\]

then \( \mathcal{K}[\gamma^{**}, m_0 + 1] \cap (V \setminus (\mathcal{P} \cup \mathcal{P})) \neq \emptyset \), that is, there is a sign-changing critical point, where \( m_0 := \{ \sup \Phi < \infty \} \) and \( \mathcal{K}[\gamma^{**}, m_0 + 1] \) denotes the set of critical points with critical values in \( [\gamma^{**}, m_0 + 1] \).

3. PROOF OF THE MAIN RESULT

Define the Euler–Lagrange functional associated with the problem (1.1)–(1.2) as follows

\[
\Phi(u) := \frac{1}{2} \int_{\Omega} |\nabla_\alpha u|^2 \, dX - \int_{\Omega} F(x, y, u) \, dX,
\]

and

\[
\Phi(u) := \frac{1}{2} \int_{\Omega} |\nabla_\alpha u|^2 \, dX - \int_{\Omega} F(x, y, u) \, dX - \int_{\Omega} G(x, y, u) \, dX = \Phi(u) - \int_{\Omega} G(x, y, u) \, dX.
\]

From Lemma in [25] and \( f \) satisfies (A1), \( g \) satisfies (A4), hence \( \Phi, \Phi \in C^1(S_{1,0}(\Omega), \mathbb{R}) \) with

\[
\langle \Phi(u), v \rangle = \int_{\Omega} \nabla_\alpha u \cdot \nabla_\alpha v \, dX - \int_{\Omega} f(x, y, u) \, v \, dX - \int_{\Omega} g(x, y, u) \, v \, dX
\]

for all \( v \in S_{1,0}(\Omega) \).

Recall that a function \( u \in S_{1,0}(\Omega) \) is called a weak solution of (1.1)–(1.2) if

\[
\int_{\Omega} \nabla_\alpha u \cdot \nabla_\alpha v \, dX = \int_{\Omega} f(x, y, u) \, v \, dX + \int_{\Omega} g(x, y, u) \, v \, dX, \quad \forall v \in S_{1,0}(\Omega).
\]

One can also check that the critical points of \( \Phi \) are weak solutions of the problem (1.1)–(1.2).

From embedding theorems for weighted Sobolev spaces, it is not difficult to show that the Grushin type has discrete spectrum in \( S_{1,0}(\Omega) \). Let \( \lambda_1, \lambda_2, \lambda_3, \cdots \) be the eigenvalues of the problem

\[
-G_\alpha u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]
Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$. Let $X_j$ be the eigenspace associated to $\lambda_j$. We set for $k \geq 2$

$$Y_k := \oplus_{j=1}^k X_j \quad \text{and} \quad Z_k = \oplus_{j=k}^\infty X_j.$$  

Let

$$\mathcal{P} := \{ u \in S_{1,0}^2(\Omega) : u(x,y) \geq 0 \text{ for a.e. } (x,y) \in \Omega \}$$

then $\mathcal{P}(-\mathcal{P})$ is the positive (negative) cone of $S_{1,0}^2(\Omega)$. We are going to consider an approximation for $S_{1,0}^2(\Omega)$: $Y_1 \subset Y_2 \subset \cdots$ and $\dim Y_k < \infty$ for each $k > 2$, define

$$\Phi_k := \Phi|_{Y_k} \quad \overline{\Phi}_k := \overline{\Phi}|_{Y_k},$$

then $\Phi_k, \overline{\Phi}_k \in C^1(Y_k, \mathbb{R})$.

**Lemma 3.1.** Assume conditions (A1)-(A5) hold. Then $\overline{\Phi}_k$ (and hence $\Phi_k$) satisfies the $(PS)$ condition.

**Proof.** The proof of this lemma is similar to the one of Lemmas 5 in [33] (or see [25]). We omit the details. \hfill \Box

**Lemma 3.2.** Under the assumptions of Theorem 1.1, there exist $\rho_k > 0$ and $C_2 > 0$ such that

$$\overline{\Phi}(u) \geq C_2 \lambda_2^{\frac{2(p_0-p)}{p_0-p(\alpha-1)}} := \delta_k, \text{ for } u \in Q(\rho_k) := \{ u \in Y_{k-1}^+ : \| u \|_{S_{1,0}^2(\Omega)} = \rho_k \},$$

where $p < p_0 < 2_\alpha^*$, and $C_2$ is independent of $k$. Moreover, $\rho_k \to \infty$ as $k \to \infty$.

**Proof.** By (A1)–(A5), for any $\epsilon > 0$ small enough, there exists a $C_\epsilon > 0$ such that

$$F(x,y,\xi) + G(x,y,\xi) \leq \epsilon |\xi|^2 + C_\epsilon |\xi|^p, \text{ for all } \xi \in \mathbb{R} \text{ and a.e. } (x,y) \in \Omega.$$  

Applying Sobolev’s embedding $S_{1,0}^2(\Omega) \hookrightarrow L^{2_\alpha^*}(\Omega)$, and using the interpolation inequality, for any $u \in S_{1,0}^2(\Omega)$, we obtain

$$\overline{\Phi}(u) \geq \frac{1}{2} \int_\Omega |\nabla \alpha u|^2 \, dX - \int_\Omega (\epsilon |u|^2 + C_\epsilon |u|^p) \, dX$$

$$\geq \frac{1}{4} \| u \|_{S_{1,0}^2(\Omega)}^2 - C_1 \| u \|_{L^2(\Omega)}^{r-1} \| u \|_{L^{p_0}(\Omega)}^{p-r}$$

$$\geq \frac{1}{4} \| u \|_{S_{1,0}^2(\Omega)}^2 - C_2 \| u \|_{L^2(\Omega)}^{r-1} \| u \|_{S_{1,0}^2(\Omega)}^{p-r},$$

where $r + \frac{p-r}{p_0} = 1$, $p_0 \in (p, 2_\alpha^*)$.

Moreover by $u \in Y_{k-1}^+$, hence

$$\| u \|_{L^2(\Omega)} \leq \lambda_k^{\frac{1}{2}} \| u \|_{S_{1,0}^2(\Omega)}.$$  

(3.2)
Proof. Write \( a \subset C \). Evidently, the gradient of (3.3) then
\[
D \Phi(u) \geq C_2 \lambda_k^{\frac{2(p_0-p)}{p_0}}.
\]

For any \( m > k + 2 \), let \( \mathcal{P}_m := \mathcal{P} \cap Y_m \) be the positive cone in \( Y_m \) and
\[
Q(\rho_k, m) := \{ u \in Y_{k-1}^+ \cap Y_m : \| u \|_{S^2_0(\Omega)} = \rho_k \}.
\]
Since \( Q(\rho_k, m) \) is compact in \( Y_m \) and includes only sign-changing elements, it is easy to check that
\[
dist(Q(\rho_k, m), \pm \mathcal{P}_m) := d_m > 0.
\]
For any \( \mu_m \in (0, \frac{d_m}{4}) \), define
\[
D_0(m, \mu_m) := \{ u \in Y_m : \text{dist}(u, \mathcal{P}_m) < \mu_m \},
\]
then \( D_0(m, \mu_m) \) is open and convex in \( Y_m \), \( \pm \mathcal{P}_m \subset \pm D_0(m, \mu_m) \) and
\[
Q(\rho, m) \subset S_m := \{ Y_m \backslash D_m \}, \quad \text{where} \quad D_m := -D_0(m, \mu_m) \cup D_0(m, \mu_m).
\]
Evidently, the gradient of \( \Phi_m \) can be expressed as \( \Phi = \text{id} - \text{Proj}_m K_{\Phi} \), where \( K_{\Phi} : S^2_1(\Omega) \rightarrow S^2_1(\Omega) \) is given by
\[
K_{\Phi} u = -G^{-1}_\alpha(f(\cdot, u(\cdot)) + g(\cdot, u(\cdot))) \quad \text{for all} \quad u \in S^2_1(\Omega).
\]
\( \text{Proj}_m \) is the projection on \( Y_m \) from \( S^2_1(\Omega) \) and
\[
\langle K_{\Phi} u, v \rangle := \int_{\mathbb{R}^N} (f(x, y, u) + g(x, y, u)) v dX, \quad \forall u, v \in S^2_1(\Omega).
\]

**Lemma 3.3.** Assume conditions (A1)-(A3) and (A5) hold. Then there exists a \( \mu_m \in (0, d_m/4) \) such that \( \text{Proj}_m K_{\Phi}(\pm D_0(m, \mu_m)) \subset \pm D_0(m, \mu_m) \) and \( \text{Proj}_m K_{\Phi}(\pm D_0(m, \mu_m)) \subset \pm D_0(m, \mu_m) \).

**Proof.** Write \( u^+ = \max\{u, 0\}, u^- = \min\{u, 0\} \). For any \( u \in Y_m, t \in [2, 2^*_\alpha) \), the exists a \( C_t > 0 \) such that
\[
\| u^\pm \|_{L^t(\Omega)} = \min_{\omega \in \mathcal{P}_m} \| u - \omega \|_{L^t(\Omega)} \leq C_t \min_{\omega \in \mathcal{P}_m} \| u - \omega \|_{S^2_1(\Omega)} = C_t \text{dist}_{S^2_1(\Omega)}(u, \mathcal{P}_m).
\]
By assumptions (A1), (A2) and (A5), for any \( \epsilon > 0 \) small enough, there exists a \( C_\epsilon > 0 \) such that
\[
f(x, y, \xi)\xi + g(x, y, \xi)\xi \leq \epsilon \| \xi \|^p + C_\epsilon |\xi|^p, \quad \text{for all} \quad \xi \in \mathbb{R} \text{ and a.e.} \ (x, y) \in \Omega.
\]
Combining (3.4), (3.5) and \( f(x, y, \xi) \xi \geq 0, g(x, y, \xi) \xi \geq 0 \) for all \( \xi \in \mathbb{R} \) and a.e. \( (x, y) \in \Omega \), we have for \( \epsilon > 0 \) small enough
\[
\text{dist}_{S_{1,0}^2(\Omega)}(v, \mp \mathcal{P}_m) \|v^\pm\|_{S_{1,0}^2(\Omega)}^2 \leq \|v^\pm\|_{S_{1,0}^2(\Omega)}^2 = \langle v, v^\pm \rangle
\]
\[
= \int_{\mathbb{R}^N} \left( |f(x, y, u^\pm)| + |g(x, y, u^\pm)| \right) |v^\pm| \, dX
\]
\[
\leq \int_{\mathbb{R}^N} \left( \epsilon |u^\pm| + C_\epsilon |u^\pm|^{p-1} \right) |v^\pm| \, dX
\]
\[
\leq \left( \frac{2}{5} \text{dist}_{S_{1,0}^2(\Omega)}(u, \mp \mathcal{P}_m) + C\text{dist}_{S_{1,0}^2(\Omega)}(u, \mp \mathcal{P}_m)^{p-1} \right) \|v^\pm\|_{S_{1,0}^2(\Omega)},
\]
that is,
\[
\text{dist}_{S_{1,0}^2(\Omega)}(\text{Proj}_m \Phi \mp \mathcal{P}_m) \leq \frac{2}{5} \text{dist}_{S_{1,0}^2(\Omega)}(u, \mp \mathcal{P}_m) + C_3 \text{dist}_{S_{1,0}^2(\Omega)}(u, \mp \mathcal{P}_m)^{p-1}.
\]
Therefore, there exists a \( \mu_m < \frac{4}{5} \) such that \( \text{dist}_{S_{1,0}^2(\Omega)}(\text{Proj}_m \Phi \mp \mathcal{P}_m) \leq \mu_m \) for every \( u \in \mp \mathcal{D}_0(m, \mu_m) \). The conclusion follows.

**Lemma 3.4.** Assume conditions (A1)-(A3) and (A5) hold. Then there exists a locally Lipschitz continuous map \( B_0 : S_{1,0}^2(\Omega) \to S_{1,0}^2(\Omega) \) such that
\[
B_0((\pm \mathcal{D}_0(m, \mu_m)) \cap S_{1,0}^2(\Omega)) \subset \pm \mathcal{D}_0(m, \mu_m)
\]
and \( V_m(u) := i(u)u - B_0(u) \) is a pseudo-gradient vector field of \( \Phi_m \), where \( \widetilde{S_{1,0}^2(\Omega)} := S_{1,0}^2(\Omega) \setminus \mathcal{K}, \mathcal{K} := \{ u \in S_{1,0}^2(\Omega) : \Phi(u) = 0 \} \). Moreover, since \( \Phi_m \) and \( i \) are even functionals, \( B_0 \) (and hence \( V_m \)) can be chosen to be odd.

**Proof.** By Lemma 3.3, the proof of Lemma 3.4 is the same as the proof of Lemma 2.1 in [30], so we omit it.

**Lemma 3.5.** Suppose that \( f \) satisfies (A1) and \( g \) satisfies (A5). Then
\[
\lim_{u \in Y_{k+1, [\|u\|_{S_{1,0}^2(\Omega)}] \to \infty}} \overline{\Phi}(u) = -\infty, \quad \lim_{u \in Y_{k+1, [\|u\|_{S_{1,0}^2(\Omega)}] \to \infty}} \Phi(u) = -\infty.
\]

**Proof.** By the definition of \( Y_{k+1} \), (A1) and (A5), it is easy to verify that
\[
\lim_{u \in Y_{k+1, [\|u\|_{S_{1,0}^2(\Omega)}] \to \infty}} \frac{\int_{\Omega} F(x, y, u) \, dX}{\|u\|^2_{S_{1,0}^2(\Omega)}} = \infty, \quad \lim_{u \in Y_{k+1, [\|u\|_{S_{1,0}^2(\Omega)}] \to \infty}} \frac{\int_{\Omega} G(x, y, u) \, dX}{\|u\|^2_{S_{1,0}^2(\Omega)}} \leq \infty.
\]
Then the conclusions of this lemma follow immediately.

**Lemma 3.6.** Suppose that \( f \) satisfies (A1) - (A4) and \( g \) satisfies (A5). Then for each fixed \( m > 0 \), there exists a \( C_4 > 0 \) such that
\[
\|u\|_{L^{p+1}(\Omega)} \leq C_4 d^\frac{1}{q},
\]
for all \( u \in \pm U_\delta \cap \{ u \in Y_m : \Phi_m(u) \leq d \} \), where \( C_4 \) is independent of \( m, d > 0 \) and
\[
U_\delta := \left\{ u \in Y_m : \left\| \Phi_m'(u) - \Phi_m'(u) \right\|_{(S^2_{1,\sigma}(\Omega))^*} > \frac{\left\| \Phi_m'(u) \right\|_{(S^2_{1,\sigma}(\Omega))^*}}{\delta} \right\}.
\]

**Proof.** We consider two cases.

**Case 1.** \( u \in \pm U_\delta \cap \{ u \in Y_m : \Phi_m(u) \leq d \} \). We have that
\[
\frac{1}{2} \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2 - \int_\Omega F(x, y, u) dX - \int_\Omega G(x, y, u) dX \leq d.
\]

This and (A3), (A5) hence
\[
\frac{1}{2} \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2 - \int_\Omega f(x, y, u) dX - \int_\Omega g(x, y, u) dX \leq \delta \left\| \Phi_m'(u) - \Phi_m'(u) \right\|_{(S^2_{1,\sigma}(\Omega))^*} \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2.
\]

From (A5) hence
\[
\delta \left\| \Phi_m'(u) - \Phi_m'(u) \right\|_{(S^2_{1,\sigma}(\Omega))^*} \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2 \leq C_5 \left( \left\| u \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{L^\sigma(\Omega)}^\sigma \right),
\]

we get by (3.8) that
\[
\delta \left\| \Phi_m'(u) - \Phi_m'(u) \right\|_{(S^2_{1,\sigma}(\Omega))^*} \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2 \leq C_5 \left( \left\| u \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{L^\sigma(\Omega)}^\sigma \right) \left\| u \right\|_{S^2_{1,\sigma}(\Omega)} + \mu_0 d.
\]

Choose \( \mu_0 \in (2, \mu) \). By (3.6) and (3.9), we know that
\[
\left( \frac{\mu_0}{2} - 1 \right) \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2 \leq \int_\Omega \left[ \mu_0 F(x, y, u) - f(x, y, u) \right] dX + \int_\Omega \left[ \mu_0 G(x, y, u) - g(x, y, u) \right] dX + C_5 \left( \left\| u \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{L^\sigma(\Omega)}^\sigma \right) \left\| u \right\|_{S^2_{1,\sigma}(\Omega)} + \mu_0 d.
\]

This and (A3), (A5) hence
\[
\left( \frac{\mu_0}{2} - 1 \right) \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2 + C_6 \left\| u \right\|_{L^\sigma(\Omega)}^\sigma \leq \left( \frac{\mu_0}{2} - 1 \right) \left\| u \right\|_{S^2_{1,\sigma}(\Omega)}^2 + C_7 \left( \left\| u \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{L^\sigma(\Omega)}^\sigma \right) \left\| u \right\|_{S^2_{1,\sigma}(\Omega)} + \mu_0 d + C_7
\]
\[
+ C_5 \left( \left\| u \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{L^\sigma(\Omega)}^\sigma \right) \left\| u \right\|_{S^2_{1,\sigma}(\Omega)} + \mu_0 d + \mu_0 d + C_7.
\]
By $\mu > 2, \mu > \sigma + 1$, applying Young’s inequalities and Cauchy’s inequalities, we have

$$C_8 \|u\|_{S^2_1,0(\Omega)}^2 + C_9 \|u\|_{L^\sigma(\Omega)}^\mu \leq C_5 \left(\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^\sigma(\Omega)}^\sigma\right) \|u\|_{S^2_1,0(\Omega)} + \mu_0 d + C_7$$

$$\leq C_\epsilon \|u\|_{L^2(\Omega)} + \epsilon \|u\|_{S^2_1,0(\Omega)} + C_\epsilon' \|u\|_{L^2(\Omega)} + \epsilon' \|u\|_{S^2_1,0(\Omega)} + \mu_0 d + C_7,$$

for all $\epsilon, \epsilon' > 0$ small enough. By the fact that $2\sigma < \mu$, we can obtain

$$\|u\|_{L^{\sigma+1}(\Omega)} \leq C_{10} \|u\|_{L^{\mu}(\Omega)} \leq C_{14}\frac{d}{\mu}.$$

Case 2. $u \in -U_\delta \cap \{u \in Y_m : \Phi_m(u) \leq d\}$, that is,

$$\|\Phi_m(-u)\|_{(S^2_1,0(\Omega))} < \delta \|\Phi_m(-u) - \Phi_m'(-u)\|_{(S^2_1,0(\Omega))},$$

and $\Phi_m(u) \leq d$.

Then

$$\Phi_m(-u) = \Phi_m(u) + \int_\Omega [G(x, y, u) - G(x, y, -u)]dX \leq d + C_{11} \|u\|_{L^2(\Omega)}^2 + C_{11} \|u\|_{L^{\sigma+1}(\Omega)}^{\sigma+1} \tag{3.10}$$

$$\frac{1}{2} \|u\|_{S^2_1,0(\Omega)}^2 - \int_\Omega F(x, y, -u)dX - \int_\Omega G(x, y, -u)dX \leq d + C_{11} \|u\|_{L^2(\Omega)}^2 + C_{11} \|u\|_{L^{\sigma+1}(\Omega)}^{\sigma+1} \tag{3.11}$$

$$\left|\Phi_m(-u), -u\right| = \|u\|_{S^2_1,0(\Omega)}^2 + \int_\Omega f(x, y, -u)dX + \int_\Omega g(x, y, -u)dX \leq \delta \|\Phi_m(-u) - \Phi_m'(-u)\|_{(S^2_1,0(\Omega))}, \|u\|_{S^2_1,0(\Omega)}^2 \tag{3.12}$$

Note that

$$\delta \|\Phi_m'(-u) - \Phi_m'(-u)\|_{(S^2_1,0(\Omega))}, \|u\|_{S^2_1,0(\Omega)}^2 \leq C_{12} \left(\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^{\mu}(\Omega)}^\mu\right).$$

Then we get

$$-\|u\|_{S^2_1,0(\Omega)}^2 \geq \int_\Omega f(x, y, -u)dX + \int_\Omega g(x, y, -u)dX + C_{12} \left(\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^{\mu}(\Omega)}^\mu\right) \|u\|_{S^2_1,0(\Omega)}^2.$$

Therefore, by (3.10)-(3.12), (A3) and (A5), we obtain

$$\left(\mu_0 \frac{1}{2} - 1\right) \|u\|_{S^2_1,0(\Omega)}^2 + C_6 \|u\|_{L^{\mu}(\Omega)}^\mu \leq \left(\mu_0 \frac{1}{2} - 1\right) \|u\|_{S^2_1,0(\Omega)}^2$$

$$+ \int_\Omega [-f(x, y, -u) - \mu_0 F(x, y, u)]dX + C_7 \leq \int_\Omega \left[\mu_0 G(x, y, -u) + g(x, y, -u)u\right]dX$$

$$+ C_{12} \left(\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^{\mu}(\Omega)}^\mu\right) \|u\|_{S^2_1,0(\Omega)}^2 + \mu_0d + C_7 \leq C_8 \|u\|_{L^2(\Omega)}^2 + C_8 \|u\|_{L^{\mu}(\Omega)}^\mu + \mu_0d + C_7.$$

This gives the desired result. \qed
Lemma 3.7. Assume conditions (A1)-(A5) hold and assume that $u_m \in Y_m$ is sign-changing and satisfies

$$ \Phi_m'(u_m) = 0, \quad \sup_{m \geq 1} |\Phi_m(u_m)| < \infty. $$

Then $\{u_m\}$ has a convergent subsequence whose limit is a sign-changing critical point of $\Phi$.

Proof. From Lemma 3.1, we have $\{u_m\}$ has a convergent subsequence in $S^2_{1,0}(\Omega)$. We just prove that the limit of the subsequence is also sign-changing. Let $u^\pm_m := \max\{\pm u_m, 0\}$. Then

$$ \|u^\pm_m\|^2_{S^2_{1,0}(\Omega)} = \int_\Omega [f(x, y, u^\pm_m)u^\pm_m + g(x, y, u^\pm_m)u^\pm_m]dX. $$

By (A1), (A2) and (A5), we have for any $\epsilon > 0$, there exists a $C_\epsilon$ such that

$$ f(x, y, \xi)\xi + g(x, y, \xi)\xi \leq \epsilon |\xi|^2 + C_\epsilon |\xi|^p, \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } (x, y) \in \Omega. $$

It follows that

$$ \|u^\pm_m\|^2_{S^2_{1,0}(\Omega)} \leq \epsilon \|u\|^2_{S^2_{1,0}(\Omega)} + C_\epsilon \|\xi\|^p_{L^p(\Omega)}. $$

Hence, $\|u^\pm_m\|^2_{S^2_{1,0}(\Omega)} \geq C_{13}$, where $C_{13}$ is a constant independent of $m$. This implies that the limit of the subsequence is also sign-changing. \qed

Proof of Theorem 1.1. Assume that there exists a $C_{14} > 0$ such that $\Phi$ has no sign-changing critical point with critical value greater than $C_{14}$. Choose $k_0 > 0$ such that $\delta_k > C_{14}$ for all $k > k_0$, where $\delta_k$ comes from Lemma 3.2. Let $m > k + 2 > k_0 + 2$. Then $Y_k \subset Y_m$. Let

$$ N := Y_k, \quad M(m) = Y^\perp_{k-1} \cap Y_m, \quad Q(\rho_k, m) := \{u \in M(m) : \|u\|_{S^2_{1,0}(\Omega)} = \rho_k\}. $$

Then by (3.3), we obtain

$$ Q(\rho_k, m) \subset S_m. $$

Define

$$ N^* = N \oplus \text{span}\{u^*\} \quad u^* \in Y_{k+1}, \quad u^* \notin Y_k; \quad N^*_+ = \{u + tu^* : u \in N, t \geq 0\}. $$

Then $N^* \cap Y_{k+1} \neq \{0\}$, and both $N^*$ and $N^*_+$ are independent of $m$. Clearly, by Lemma 3.5, we have

(i) $\Phi_m(0) = 0$;

(ii) there exists a $R_1 > \rho_k$ such that $\Phi_m(u) \leq 0$ for all $u \in N$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_1$;

(iii) there exists a $R_2 \geq R_1 > 0$ such that $\Phi_m(u) \leq 0$ for all $u \in N^*$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_2$.

Let

$$ \Gamma_m := \{\phi \in \mathcal{C}(Y_m, Y_m) : \phi \text{ is odd, } \phi(\mathcal{D}_m) \subset (\mathcal{D}_m); \phi(u) = u \text{ if } \max\{\Phi_m(u), \Phi_m(-u)\} \leq 0\}. $$
Defining
\[ \gamma_k^*(m) := \inf_{\phi \in \Gamma_m} \sup_{\phi(N^*) \cap S_m} \Phi, \quad \gamma_k^{**}(m) := \inf_{\phi \in \Gamma_m} \sup_{\phi(N^*) \cap S_m} \Phi > 0, \]

For any \( \phi \in \Gamma_m \), by Lemma 1.44 in [28] (or see [3,32]), we have \( \phi(N \cap B_{R_1}) \cap Q(\rho_k, m) \neq \emptyset \), and by Lemma 3.2 we can obtain
\[ \sup_{\phi \in (N \cap B_{R_1}) \cap S_m} \Phi_m \leq \inf_{Q(\rho_k, m)} \Phi_m \leq 2\lambda_k^{(p-2)(p-2)} := \delta_k. \]

Therefore, we get
\[ (3.13) \quad \gamma_k^{**}(m) \geq 2\lambda_k^{(p-2)(p-2)} := \delta_k \rightarrow \infty, \quad \text{as } k \rightarrow \infty. \]

We consider two cases:

Case 1. For \( k \geq k_0 \), if there exists a sequence \( m_i \rightarrow \infty \) as \( i \rightarrow \infty \) such that
\[ \gamma_k^*(m_i) > \gamma_k^{**}(m_i) \quad \text{for all } i > 1, \]

then by Proposition 2.3, there exists a sign-changing critical point \( u_{m_i} \) such that
\[ \overline{\Phi}_{m_i}(u_{m_i}) = 0 \quad \text{and} \quad C_0 < \delta_k \leq \gamma_k^{**}(m_i) \leq \overline{\Phi}(u_{m_i}) \leq \sup_{N^*} \Phi + 1 \]

Here \( \sup_{N^*} \Phi \) is a constant depending on \( k \) and independent of \( m_i \). By Lemma 3.7, \( \{u_{m_i}\} \) has a convergent subsequence whose limit \( u \) is a sign-changing critical point of \( \Phi \), and \( \overline{\Phi}(u) \geq \delta_k > C_0 \). This contradicts the assumption.

Case 2. For all \( k \geq k_0 \), there exists a \( m_k \) such that
\[ (3.14) \quad \gamma_k^*(m) = \gamma_k^{**}(m) \quad \text{for all } m > m_k. \]

Let \( K_{com}(m) \) denote the set of common critical points of \( \Phi_m \) and \( \overline{\Phi}_m \). By (A5), \( K_{com}(m) = \{0\} \). Define
\[ V_\delta := \{u \in Y_m : \|u\|_{s^2_{l,0}(\Omega)} \leq \delta\}, \]

and let \( U_\delta \) be as in Lemma 3.6, which contains all non-common critical points of \( \Phi_m \) and \( \overline{\Phi}_m \). By Lemma 3.5, there exists a \( R_1 > \rho_k \) such that \( \overline{\Phi}_m(u) \leq 0 \) for all \( u \in N \) with \( \|u\|_{s^2_{l,0}(\Omega)} \geq R_1 \). Here \( R_1 \) is independent of \( m \). Combining the definition of \( \gamma_k^*(m) \) and (3.13), we find a \( \phi_0 \in \Gamma_m \) such that
\[ (3.15) \quad \sup_{\phi_0(N^*) \cap S_m} \overline{\Phi} = \sup_{\phi_0(N^*) \cap B_{R_1} \cap S_m} \overline{\Phi} \leq \gamma_k^*(m) + \frac{1}{2}. \]

Let
\[ U_\delta^*(m) := V_\delta \cup U_\delta \cup (-U_\delta). \]

Then \( U_\delta \) is a symmetric set and contains all critical points of \( \Phi_m \) and \( \overline{\Phi}_m \). Define two non-negative continuous functions:
\[ \zeta_1(u) = \begin{cases} 0, & \text{if } u \in U_{10}^*(m), \\ 1, & \text{if } u \notin U_{20}^*(m), \end{cases} \quad \text{(is even)}, \quad \zeta_2(u) = \begin{cases} 0, & \text{if } u \leq 0, \\ 1, & \text{if } u \geq 1, \end{cases} \]
and a vector field
\[ V^*_m := -\zeta_2(\max\{\Phi_m(u), \Phi_m(-u)\})\zeta_1(u)V_m(u), \]
where the pseudo gradient vector field \( V_m \) comes from Lemma 3.4 obtained for \( \Phi_m \).
Since \( V_m \) can be chosen to be odd, then \( V^*_m \) is odd.

Let \( \Theta(t, u) \) denote the unique (odd in \( u \)) solution of the Cauchy initial value problem:
\[ \frac{d\Theta(t, u)}{dt} = V^*_m(\Theta(t, u)), \quad \Theta(0, u) = u \in Y_m. \]

Then \( \Theta(t, u) \) is also a pseudo-gradient ow for \( \Phi_m \) and
\[ \frac{d\Phi_m(\Theta(t, u))}{dt} \leq 0. \]

For any \( u \not\in U^*_\delta(m) \), we have \( u \not\in \pm U_\delta \) and by Lemma 3.6, we obtain
\[ \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))} \leq \frac{\delta + 1}{\delta} \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))}, \quad \|\Phi_m(u)\|_{(S^2_{\delta,0}(\Omega))} \leq \frac{\delta}{\delta - 1} \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))}. \]

Further, for all \( u \not\in U^*_\delta(m) \), we get
\[ \langle \Phi'_m(u), V_m(u) \rangle - \langle \Phi'_m(u) - \Phi'_m(u), V_m(u) \rangle \leq \frac{1}{2} \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))}^2 - 2 \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))} \|\Phi_m(u)\|_{(S^2_{\delta,0}(\Omega))}, \]
and
\[ \|V_m(u)\|_{(S^2_{\delta,0}(\Omega))} \leq 2 \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))} \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))} \leq \frac{2(\delta + 1)}{\delta} \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))}. \]

Moreover, since \( u \not\in U^*_\delta(m) \) implies \( \zeta_1(u) = 1 \), we see that \( \Phi_m(u) \geq 1 \) implies
\[ \zeta_2(\max\{\Phi_m(u), \Phi_m(-u)\}) = 1. \]

We obtain
\[ \frac{d\Phi_m(\Theta(t, u))}{dt} \bigg|_{t=0} = \langle \Phi'_m(\Theta(t, u)), \frac{d\Theta}{dt} \rangle \bigg|_{t=0} \]
\[ = \langle \Phi'_m(\Theta(t, u)), V^*_m(\Theta(t, u)) \rangle \bigg|_{t=0} \]
\[ = \langle \Phi'_m(u), -\zeta_2(\max\{\Phi_m(\Theta(t, u)), \Phi_m(-\Theta(t, u))\})\zeta_1(\Theta(t, u))V_m(\Theta(t, u)) \rangle \bigg|_{t=0} \]
\[ = \langle \Phi'_m(u), -\zeta_2(\max\{\Phi_m(\Theta(t, u)), \Phi_m(-\Theta(t, u))\})\zeta_1(u)V_m(u) \rangle \]
\[ = -\langle \Phi'_m(u), V_m(u) \rangle \leq -\frac{277}{800} \|\Phi'_m(u)\|_{(S^2_{\delta,0}(\Omega))}. \]

for all \( u \not\in U^*_\delta(m) \) satisfying \( \Phi_m(u) \geq 1. \)

We claim that \( \Theta(t, \phi_0(\cdot)) \in \Gamma_m \) for any \( t \geq 0 \). In fact, \( \Theta(t, \phi_0(\cdot)) \) is odd in \( u \) since \( \phi_0(\cdot) \) and \( V^*_m \) are odd. Recall that \( \phi_0 \in \Gamma_m \). Then, \( \phi_0 = u \) for \( u \) with \( \max\{\Phi(u), \Phi(-u)\} \leq 0 \). Hence, \( \Theta(t, \phi_0(u)) = \Theta(t, u) \) and \( V^*_m(u) = 0 \) for \( u \) with \( \max\{\Phi(u), \Phi(-u)\} \leq 0 \). It follows that \( \Theta(t, u) = u \) and then \( \Theta(t, \phi_0(u)) = u \) for \( u \).
with $\max\{\Phi(u), \Phi(-u)\} \leq 0$. Using Theorem 1 in [9] and Lemma 3.4 similar to the one of Theorem 2.1 in [30], we obtain

$$\Theta(t, \phi_0(\mathcal{D}_m)) \subset \Theta(t, \mathcal{D}_m) \subset \mathcal{D}_m, \quad \forall t \geq 0.$$ 

Therefore, $\Theta(t, \phi_0(u)) \in \Gamma_m$ for any $t \geq 0$. For any $t \geq 0$, we can deduce the following estimates which lead to a contradiction. In fact, by the fact that $\Theta(t, \phi_0(u)) \in \Gamma_m$ is odd, $\Theta(t, \phi_0(N^+)) \cap S_m \subset \{u \in Y_m : \Phi_m(u) \leq \gamma^*_k(m) + \frac{1}{2}\}$, $V_{20}$ is bounded, (3.15) - (3.17) and Lemma 3.6, we have

$$\gamma_k^*(m) + \frac{1}{2} = \gamma^*_k(m) + \frac{1}{2} \geq \sup_{\Theta(t, \phi_0(\mathcal{N}^+)) \cap S_m} \Phi - \sup_{\Theta(t, \phi_0(\mathcal{N}^+)) \cap S_m} \Phi$$

$$\geq \sup_{\Theta(t, \phi_0(\mathcal{N}^+)) \cap S_m} \Phi - \left(\sup_{\Theta(t, \phi_0(\mathcal{N}^+)) \cap S_m} \Phi - \sup_{\Theta(t, \phi_0(\mathcal{N}^+)) \cap S_m} \Phi\right)$$

$$\geq \gamma^*_k(m) - \sup_{u \in \Theta(t, \phi_0(\mathcal{N}^+)) \cap S_m} (\Phi(-u) - \Phi(u))$$

$$\geq \gamma^*_k(m) - \sup_{u \in \Theta(t, \phi_0(\mathcal{N}^+)) \cap S_m \cap \{u \in Y_m : \Phi_m(u) \leq \gamma^*_k(m) + \frac{1}{2}\}} |\Phi(-u) - \Phi(u)|$$

$$\geq \gamma^*_k(m) - \sup_{u \in U_{20} \cap \{u \in Y_m : \Phi_m(u) \leq \gamma^*_k(m) + \frac{1}{2}\}} |\Phi(-u) - \Phi(u)| - C_{15}$$

$$\geq \gamma^*_k(m) - \sup_{u \in (-U_{20}, U_{20}) \cap \{u \in Y_m : \Phi_m(u) \leq \gamma^*_k(m) + \frac{1}{2}\}} C_{8} |u|_{L^{\sigma+1}((\Omega))}^{\sigma+1} - C_{15}$$

$$\geq \gamma^*_k(m) - C_{8} (\gamma^*_k(m))^{\frac{1+\sigma}{\mu}} - C_{15}.$$ 

Therefore, we get the inequality

$$\gamma^*_k(m) \leq \gamma^*_k(m) \left(1 + C_{8} (\gamma^*_k(m))^{\frac{1+\sigma-\mu}{\mu}}\right),$$

for all $k \geq k_0$.

Then from the condition (1.3), we can take some $p_0 \in (2, 2_\alpha^*)$ and $q \in (\frac{2N_{\alpha}}{N_{\alpha}+2}, 2)$ such that

$$\frac{1+\sigma-\mu}{\mu} \frac{2(p_0-p)}{(p-2)(p_0-2)} \left(\frac{2}{N_{\alpha}} - \frac{2-q}{q}\right) < -1.$$ 

From Theorem 1.3 in [15] (or see [16]), we obtain

$$\lambda_k \geq C_{16} k^{\frac{2}{N_{\alpha}}}.$$
From (3.18) - (3.20), using iteration, we get that
\[
\gamma_{k_0+\ell}^{**}(m) \leq \gamma_{k_0}^{**}(m) \prod_{k=k_0}^{k_0+\ell-1} \left( 1 + C_8(\gamma_k^{**}(m)) \frac{1+\sigma-\mu}{\mu} \right)
\]
\[
\leq \gamma_{k_0}^{**}(m) \exp \left( \sum_{k=k_0}^{k_0+\ell-1} \ln \left( 1 + C_8(\gamma_k^{**}(m)) \frac{1+\sigma-\mu}{\mu} \right) \right)
\]
\[
\leq \gamma_{k_0}^{**}(m) \exp C_8 \left( \sum_{k=k_0}^{k_0+\ell-1} \gamma_k^{**}(m) \frac{1+\sigma-\mu}{\mu} \right)
\]
\[
\leq \gamma_{k_0}^{**}(m) \exp C_8 \left( \sum_{k=k_0}^{k_0+\ell-1} k \frac{1+\sigma-\mu}{\mu} \frac{2(p-2)}{(p-2)(p-2)} \left( \frac{2-2}{N_{\alpha} - \frac{2-2}{q}} \right) \right) < \infty,
\]
for all \(\ell \in \mathbb{N}\), which yields the desired contradiction. Thus, \(\Phi\) possesses an unbounded sign-changing sequence of critical values. \[\square\]

**Corollary 3.8.** Assume that \(f\) satisfies the conditions (A1)–(A3) and there is a \(\sigma < \frac{\mu}{2}\) such that
\[
|f(x,y,u) - f(x,y,-u)| \leq C(1 + |u|^\sigma) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } (x,y) \in \Omega.
\]
Then the problem
(3.21) \[-G_{\alpha}u = f(x,y,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,
\]
has an infinite sequence of sign-changing solutions provided that (1.3) holds.

**Corollary 3.9.** Assume that \(f\) satisfies the conditions (A1)–(A3), there exists a \(R > 0\) such that
\[
f(x,y,-u) = -f(x,y,u) \text{ for a.e. } (x,y) \in \Omega, |u| \geq R,
\]
and
\[
\frac{2p}{N_{\alpha}(p-2)} - 1 > \frac{\mu}{\mu - 1}.
\]
Then the problem (3.21) has an infinite sequence of sign-changing solutions in \(S_{1,0}^2(\Omega)\).

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**References**


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