SIGN-CHANGING SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SEMILINEAR $\Delta_\gamma-$LAPLACE EQUATIONS

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Abstract. In this article, we study the existence of multiple solutions for the boundary value problem

$$\begin{aligned} -G_\alpha u &= g(x,y,u) + f(x,y,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 2), \alpha \in \mathbb{N}, g(x,y,\xi)$, $f(x,y,\xi)$ are Carathéodory functions and G_α is the Grushin operator. We use the lower bounds of eigenvalues and an abstract theory on sign-changing solutions.

1. Introduction

Boundary value problems for semilinear elliptic equations were studied in [1,27] (see also the references therein). Many publications [4–8, 10–12, 18, 26, 29, 31] are devoted to the study of the existence of sign-changing solutions of classical elliptic boundary value problems such as

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \; u = 0 \quad \text{on } \partial\Omega,$$

where $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. There have been several methods developed in studying sign-changing solutions of nonlinear elliptic equations, such as the invariant sets of descending flow method developed by Liu and Sun [5, 18, 31], and the minimax method which is established by Berestycki and Lions in the classical paper [8].

One of the classes of degenerate elliptic equations that has been studied widely in recent years is the class of equations involving an operator of the Grushin type (see [14])

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha \geq 0.$$

Note that $G_0 \equiv \Delta$ is the Laplacian operator, and G_α, when $\alpha > 0$, is not elliptic in domains intersecting the surface $x = 0$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [36, 37] (see also some recent results in [2, 13, 17, 19–23, 25, 33–35]).

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In this paper, we consider the existence of sign-changing solutions of the Dirichlet boundary value problem

$$-G_\alpha u = g(x,y,u) + f(x,y,u) \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} := \mathbb{R}^N, N_1, N_2, \alpha \in \mathbb{N}, \Omega \cap \{(x,y) \in \mathbb{R}^N : x = 0\} \neq \emptyset$, and

$$\Delta_x := \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2}, \Delta_y := \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2}, |x|^{2\alpha} := \left(\sum_{i=1}^{N_1} x_i^2\right)^\alpha,$$

and the nonlinearity f is a real Carathéodory function on $\Omega \times \mathbf{R}$ and satisfies the following conditions

- (A1) There exist $p \in (2, 2_\alpha^*)$, and constants $C_1, C_2 > 0$ such that
$$|f(x,y,\xi)| \leq C_1 + C_2 |\xi|^{p-1} \text{ almost everywhere } (x,y,\xi) \in \Omega \times \mathbb{R},$$
where $2_\alpha^* := \frac{2N_\alpha}{N_\alpha - 2}, N_\alpha := N_1 + (1+\alpha)N_2 > 2$;
- (A2) $f(x,y,\xi) = \circ(|\xi|)$, uniformly in $(x,y) \in \overline{\Omega}$, as $\xi \to 0$ and $f(x,y,\xi)\xi \geq 0$ for all $\xi \in \mathbb{R}$ and a.e. $(x,y) \in \Omega$;
- (A3) There exists a constant $\mu > 2$ such that
$$0 \leq \mu F(x,y,\xi) \leq \xi f(x,y,\xi), \quad \forall (x,y) \in \bar{\Omega},\ \xi \in \mathbb{R} \setminus \{0\},$$
where $F(x,y,\xi) = \int\limits_0^\xi f(x,y,\tau) d\tau$;
- (A4) $f(x,y,-\xi) = -f(x,y,\xi)$ for all $(x,y,\xi) \in \overline{\Omega} \times \mathbb{R}$;
- (A5) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. There exists $\sigma < \frac{\mu}{2}$ such that
$$|g(x,y,\xi)| \leq C(1 + |\xi|^\sigma), \text{ for all } \xi \in \mathbb{R}, \text{ and a.e. } (x,y) \in \Omega.$$
Moreover, $g(x,y,\xi) = \circ(|\xi|)$, uniformly in $(x,y) \in \overline{\Omega}$, as $\xi \to 0$ and $g(x,y,\xi)\xi > 0$ for all $\xi \in \mathbb{R} \setminus \{0\}$ and a.e. $(x,y) \in \Omega$.

Our main result is given by the following theorem.

Theorem 1.1. *Assume that f, g satisfies the conditions* (A1)–(A5) *and*

$$\frac{2p}{N_\alpha(p-2)} - 1 > \frac{\mu}{\mu - \sigma - 1}. \tag{1.3}$$

Then the problem (1.1)–(1.2) *has infinitely many sign-changing solutions.*

This article is organized as follows. In section 2, we present some definitions and preliminary results. Next, combining the lower bounds of eigenvalues and an abstract theory on sign-changing solutions, we give the proof of Theorem 1.1.

2. Preliminary results

Definition 2.1. By $S_1^2(\Omega)$ we will denote the set of all functions $u \in L^2(\Omega)$ such that $\dfrac{\partial u}{\partial x_i} \in L^2(\Omega), |x|^\alpha \dfrac{\partial u}{\partial y_j} \in L^2(\Omega), i = 1, 2, \ldots, N_1, j = 1, 2, \ldots, N_2$. We define the norm in this space as follows

$$\|u\|_{S_1^2(\Omega)} = \left\{ \int\limits_\Omega \left(|u|^2 + |\nabla_\alpha u|^2\right) \mathrm{d}X \right\}^{\frac{1}{2}},$$

where

$$\mathrm{d}X = \mathrm{d}x_1 \ldots \mathrm{d}x_{N_1} \mathrm{d}y_1 \ldots \mathrm{d}y_{N_2}, \nabla_\alpha u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{N_1}}, |x|^\alpha \frac{\partial u}{\partial y_1}, \ldots, |x|^\alpha \frac{\partial u}{\partial y_{N_2}}\right).$$

We can also define the scalar product in $S_1^2(\Omega)$ as follows

$$(u, v)_{S_1^2(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla_\alpha u, \nabla_\alpha v)_{L^2(\Omega)}.$$

The space $S_{1,0}^2(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in the space $S_1^2(\Omega)$.

The following embedding inequality was proved in [33, 37]

$$\left(\int_\Omega |u|^p \mathrm{d}X\right)^{\frac{1}{p}} \leq C(p, \Omega) \|u\|_{S_{1,0}^2(\Omega)},$$

where $1 \leq p \leq 2_\alpha^*, C(p, \Omega) > 0$. The number 2_α^* is the critical Sobolev exponent of the embedding $S_{1,0}^2(\Omega) \hookrightarrow L^p(\Omega)$ and when $1 \leq p < 2_\alpha^*$, the embedding is compact.

Definition 2.2. Let $\mathbb{V}$ be a real Banach space with its dual space $\mathbb{V}^*, \Phi \in C^1(\mathbb{V}, \mathbb{R})$. We say that Φ satisfies the Palais–Smale if for any sequence $\{u_n\}_{n=1}^{n=+\infty} \subset \mathbb{V}$ such that $\Phi(u_n)$ is bounded and

$$\|\Phi'(u_n)\|_{\mathbb{V}^*} \to 0 \text{ as } n \to \infty,$$

then there exists a subsequence $\{u_{n_k}\}_{k=1}^{k=+\infty}$ that converges strongly in $\mathbb{V}$.

From Theorem A in [30], we have

Proposition 2.3. Let $\mathbb{V}$ be a Hilbert space and $\Phi \in C^1(\mathbb{V}, \mathbb{R})$ be of the form $\Phi' = id - K_\Phi$ and satisfy the Palais-Smale condition, where K_Φ is a continuous operator. Assume that $K_\Phi(\pm\mathcal{D}_0) \subset \pm\mathcal{D}_0$ holds, where $\mathcal{D}_0 = \{u \in \mathbb{V} : \mathrm{dist}(u, \mathcal{P}) < \mu_0\}$ and $\mathcal{P} := \{u \in \mathbb{V}, u(x) \geq 0, \text{ for a.e. } x \in \Omega\}$ is the positive cone of $\mathbb{V}$. Let N, M be two closed subspaces of $\mathbb{V}$ with $\dim N < \infty, \dim N - \mathrm{codim} M \geq 1$. Suppose that

$$Q(\rho) := \{u \in M : \|u\|_{\mathbb{V}} = \rho\} \subset \mathcal{S} := \mathbb{V} \setminus (-\mathcal{D}_0 \cup \mathcal{D}_0).$$

Define

$$N^* = N \oplus \mathrm{span}\{u^*\} \quad u^* = \mathbb{V} \setminus N; \quad N_+^* = \{u + tu^* : u \in N, t \geq 0\}.$$

Assume that

(*i*) $\Phi(0)=0$;

(*ii*) there exists a $R_1>\rho$ such that $\Phi(u)\le 0$ for all $u\in N$ with $\|u\|_{\mathbb{V}}\ge R_1$;

(*iii*) there exists a $R_2\ge R_1$ such that $\Phi(u)\le 0$ for all $u\in N^*$ with $\|u\|_{\mathbb{V}}\ge R_2$;

Let

$$\Gamma=\{\phi\in C(\mathbb{V},\mathbb{V}):\phi \text{ is odd}, \ \phi(-\mathcal{D}\cup\mathcal{D})\subset(-\mathcal{D}\cup\mathcal{D});$$
$$\phi(u)=u \text{ if } \max\{\Phi(u),\Phi(-u)\}\le 0\}.$$

If

$$\gamma^*=\inf_{\phi\in\Gamma}\sup_{\phi(N_+^*)\cap\mathcal{S}}\Phi>\gamma^{**}=\inf_{\phi\in\Gamma}\sup_{\phi(N)\cap\mathcal{S}}\Phi>0,$$

then $\mathcal{K}[\gamma^{**},m_0+1]\cap(\mathbb{V}\backslash(-\mathcal{P}\cup\mathcal{P}))\neq\emptyset$, that is, there is a sign-changing critical point, where $m_0:=\{\sup_{N^*}\Phi<\infty,\}$ and $\mathcal{K}[\gamma^{**},m_0+1]$ denotes the set of critical points with critical values in $[\gamma^{**},m_0+1]$.

3. Proof of the main result

Define the Euler–Lagrange functional associated with the problem (1.1)–(1.2) as follows

$$\Phi(u):=\frac{1}{2}\int_\Omega|\nabla_\alpha u|^2\,\mathrm{d}X-\int_\Omega F(x,y,u)\,\mathrm{d}X,$$

and

$$\overline{\Phi}(u):=\frac{1}{2}\int_\Omega|\nabla_\alpha u|^2\,\mathrm{d}X-\int_\Omega F(x,y,u)\,\mathrm{d}X-\int_\Omega G(x,y,u)\,\mathrm{d}X=\Phi(u)-\int_\Omega G(x,y,u)\,\mathrm{d}X.$$

From Lemma in [25] and f satisfies (A1), g satisfies (A4), hence $\Phi,\overline{\Phi}\in C^1(S^2_{1,0}(\Omega),\mathbb{R})$ with

$$\langle\overline{\Phi}'(u),v\rangle=\int_\Omega\nabla_\alpha u\cdot\nabla_\alpha v\mathrm{d}X-\int_\Omega f(x,y,u)\,v\mathrm{d}X-\int_\Omega g(x,y,u)\,v\mathrm{d}X$$

for all $v\in S^2_{1,0}(\Omega)$.

Recall that a function $u\in S^2_{1,0}(\Omega)$ is called a weak solution of (1.1)–(1.2) if

$$\int_\Omega\nabla_\gamma u\cdot\nabla_\gamma v\mathrm{d}X=\int_\Omega f(x,y,u)v\mathrm{d}X+\int_\Omega g(x,y,u)v\mathrm{d}X,\quad\forall v\in S^2_{1,0}(\Omega).$$

One can also check that the critical points of $\overline{\Phi}$ are weak solutions of the problem (1.1)–(1.2).

From embedding theorems for weighted Sobolev spaces, it is not difficult to show that the Grushin type has discrete spectrum in $S^2_{1,0}(\Omega)$. Let $\lambda_1,\lambda_2,\lambda_3,\cdots$ be the eigenvalues of the problem

$$-G_\alpha u=\lambda u\quad\text{in }\ \Omega,\quad u=0\quad\text{on }\ \partial\Omega.$$

Then $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. Let $\mathbf{X}_j$ be the eigenspace associated to λ_j. We set for $k \ge 2$

$$\mathbf{Y}_k := \oplus_{j=1}^{k} \mathbf{X}_j \quad \text{and} \quad \mathbf{Z}_k = \overline{\oplus_{j=k}^{\infty} \mathbf{X}_j}.$$

Let

$$\mathcal{P} := \{u \in S^2_{1,0}(\Omega) : u(x,y) \ge 0 \text{ for a.e. } (x,y) \in \Omega\}$$

then $\mathcal{P}(-\mathcal{P})$ is the positive (negative) cone of $S^2_{1,0}(\Omega)$. We are going to consider an approximation for $S^2_{1,0}(\Omega) : \mathbf{Y}_1 \subset \mathbf{Y}_2 \subset \cdots$ and $\dim \mathbf{Y}_k < \infty$ for each $k > 2$, define

$$\Phi_k := \Phi\big|_{\mathbf{Y}_k} \quad \overline{\Phi}_k := \overline{\Phi}\big|_{\mathbf{Y}_k},$$

then $\Phi_k, \overline{\Phi}_k \in C^1(\mathbf{Y}_k, \mathbb{R})$.

Lemma 3.1. *Assume conditions* (A1)-(A5) *hold. Then $\overline{\Phi}_k$ (and hence Φ_k) satisfies the (PS) condition.*

Proof. The proof of this lemma is similar to the one of Lemmas 5 in [33] (or see [25]). We omit the details. □

Lemma 3.2. *Under the assumptions of Theorem 1.1, there exist $\rho_k > 0$ and $C_2 > 0$ such that*

$$\overline{\Phi}(u) \ge C_2 \lambda_k^{\frac{2(p_0-p)}{(p-2)(p_0-2)}} := \delta_k, \ \textit{for } u \in \mathbf{Q}(\rho_k) := \{u \in \mathbf{Y}_{k-1}^{\perp} : \|u\|_{S^2_{1,0}(\Omega)} = \rho_k\},$$

where $p < p_0 < 2^*_\alpha$, and C_2 is independent of k. Moreover, $\rho_k \to \infty$ as $k \to \infty$.

Proof. By (A1)- (A5), for any $\epsilon > 0$ small enough, there exists a $C_\epsilon > 0$ such that

$$F(x,y,\xi) + G(x,y,\xi) \le \epsilon |\xi|^2 + C_\epsilon |\xi|^p, \ \text{ for all } \xi \in \mathbb{R} \text{ and a.e. } (x,y) \in \Omega.$$

Applying Sobolev's embedding $S^2_{1,0}(\Omega) \hookrightarrow L^{2^*_\alpha}(\Omega)$, and using the interpolation inequality, for any $u \in S^2_{1,0}(\Omega)$, we obtain

$$\begin{aligned}
\overline{\Phi}(u) &\ge \frac{1}{2} \int_\Omega |\nabla_\alpha u|^2 \,\mathrm{d}X - \int_\Omega (\epsilon |u|^2 + C_\epsilon |u|^p) \mathrm{d}X \\
&\ge \frac{1}{4} \|u\|^2_{S^2_{1,0}(\Omega)} - C_1 \|u\|^r_{L^2(\Omega)} \|u\|^{p-r}_{L^{p_0}(\Omega)} \\
&\ge \frac{1}{4} \|u\|^2_{S^2_{1,0}(\Omega)} - C_2 \|u\|^r_{L^2(\Omega)} \|u\|^{p-r}_{S^2_{1,0}(\Omega)}, \qquad (3.1)
\end{aligned}$$

where $\frac{r}{2} + \frac{p-r}{p_0} = 1, p_0 \in (p, 2^*_\alpha)$.
Moreover by $u \in \mathbf{Y}_{k-1}^{\perp}$, hence

$$\|u\|_{L^2(\Omega)} \le \lambda_k^{\frac{-1}{2}} \|u\|_{S^2_{1,0}(\Omega)}. \qquad (3.2)$$

Combining (3.1) and (3.2), for any $u \in \mathbf{Y}_{k-1}^{\perp}$, $\|u\|_{S^2_{1,0}(\Omega)} = \frac{\lambda_k^{\frac{r}{2(p-2)}}}{(2C_2p)^{\frac{1}{p-2}}} := \rho_k$, we have

$$\overline{\Phi}(u) \geq C_2\lambda_k^{\frac{2(p_0-p)}{(p-2)(p_0-2)}}.$$

□

For any $m > k+2$, let $\mathcal{P}_m := \mathcal{P} \cap \mathbf{Y}_m$ be the positive cone in $\mathbf{Y}_m$ and

$$Q(\rho_k, m) := \{u \in \mathbf{Y}_{k-1}^{\perp} \cap \mathbf{Y}_m : \|u\|_{S^2_{1,0}(\Omega)} = \rho_k\}.$$

Since $Q(\rho_k, m)$is compact in $\mathbf{Y}_m$ and includes only sign-changing elements, it is easy to check that

$$\mathrm{dist}(Q(\rho_k, m), \pm\mathcal{P}_m) := d_m > 0.$$

For any $\mu_m \in (0, \frac{d_m}{4})$. define

$$\mathcal{D}_0(m, \mu_m) := \{u \in \mathbf{Y}_m : \mathrm{dist}(u, \mathcal{P}_m) < \mu_m\},$$

then $\mathcal{D}_0(m, \mu_m)$ is open and convex in $\mathbf{Y}_m$, $\pm\mathcal{P}_m \subset \pm\mathcal{D}_0(m, \mu_m)$ and

(3.3) $$Q(\rho, m) \subset \mathcal{S}_m := \{\mathbf{Y}_m \backslash \mathcal{D}_m\}, \text{ where } \mathcal{D}_m := -\mathcal{D}_0(m, \mu_m) \cup \mathcal{D}_0(m, \mu_m).$$

Evidently, the gradient of $\overline{\Phi}_m$ can be expressed as $\overline{\Phi}' = \mathrm{id} - \mathrm{Proj}_m K_{\overline{\Phi}}$, where $K_{\overline{\Phi}} : S^2_{1,0}(\Omega) \to S^2_{1,0}(\Omega)$ is given by

$$K_{\overline{\Phi}}u = -G_\alpha^{-1}(f(\cdot, u(\cdot)) + g(\cdot, u(\cdot))) \text{ for all } u \in S^2_{1,0}(\Omega).$$

Proj_m is the projection on $\mathbf{Y}_m$ from $S^2_{1,0}(\Omega)$ and

$$\langle K_{\overline{\Phi}}u, v\rangle := \int_{\mathbb{R}^N} (f\,(x, y, u) + g\,(x, y, u))v\mathrm{d}X, \quad \forall u, v \in S^2_{1,0}(\Omega).$$

Lemma 3.3. *Assume conditions* (A1)-(A3) *and* (A5) *hold. Then there exists a* $\mu_m \in (0, d_m/4)$ *such that* $Proj_m K_{\overline{\Phi}}(\pm\mathcal{D}_0(m, \mu_m)) \subset \pm\mathcal{D}_0(m, \mu_m)$ *and* $Proj_m K_{\Phi}(\pm\mathcal{D}_0(m, \mu_m)) \subset \pm\mathcal{D}_0(m, \mu_m)$.

Proof. Write $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$. For any $u \in \mathbf{Y}_m, t \in [2, 2^*_\alpha)$, the exists a $C_t > 0$ such that

(3.4) $$\left\|u^\pm\right\|_{L^t(\Omega)} = \min_{\omega \in \mp\mathcal{P}_m} \|u - \omega\|_{L^t(\Omega)} \leq C_t \min_{\omega \in \mp\mathcal{P}_m} \|u - \omega\|_{S^2_{1,0}(\Omega)} = C_t \mathrm{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m).$$

By assumptions (A1), (A2) and (A5), for any $\epsilon > 0$ small enough, there exists a $C_\epsilon > 0$ such that

(3.5) $$f(x, y, \xi)\xi + g(x, y, \xi)\xi \leq \epsilon\,|\xi|^2 + C_\epsilon\,|\xi|^p\,, \text{ for all } \xi \in \mathbb{R} \text{ and a.e. } (x, y) \in \Omega.$$

Combining (3.4), (3.5) and $f(x,y,\xi)\xi \geq 0, g(x,y,\xi)\xi \geq 0$ for all $\xi \in \mathbb{R}$ and a.e. $(x,y) \in \Omega$, we have for $\epsilon > 0$ small enough

$$\begin{aligned}
\operatorname{dist}_{S^2_{1,0}(\Omega)}(v, \mp\mathcal{P}_m) \left\|v^{\pm}\right\|_{S^2_{1,0}(\Omega)} &\leq \left\|v^{\pm}\right\|^2_{S^2_{1,0}(\Omega)} = \langle v, v^{\pm}\rangle \\
&= \int\limits_{\mathbb{R}^N} \left(\left|f\left(x,y,u^{\pm}\right)\right| + \left|g\left(x,y,u^{\pm}\right)\right|\right) \left|v^{\pm}\right| \mathrm{d}X \\
&\leq \int\limits_{\mathbb{R}^N} \left(\epsilon \left|u^{\pm}\right| + C_\epsilon \left|u^{\pm}\right|^{p-1}\right) \left|v^{\pm}\right| \mathrm{d}X \\
&\leq \left[\frac{2}{5}\operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m) + C\operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m)^{p-1}\right] \left\|v^{\pm}\right\|_{S^2_{1,0}(\Omega)},
\end{aligned}$$

that is,

$$\operatorname{dist}_{S^2_{1,0}(\Omega)}(\operatorname{Proj}_m K_{\overline{\Phi}}, \mp\mathcal{P}_m) \leq \frac{2}{5}\operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m) + C_3\operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m)^{p-1}.$$

Therefore, there exists a $\mu_m < \frac{d_m}{4}$ such that $\operatorname{dist}_{S^2_{1,0}(\Omega)}(\operatorname{Proj}_m K_{\overline{\Phi}}, \mp\mathcal{P}_m) \leq \mu_m$ for every $u \in \mp\mathcal{D}_0(m, \mu_m)$. The conclusion follows. □

Lemma 3.4. *Assume conditions* (A1)-(A3) *and* (A5) *hold. Then there exists a locally Lipschitz continuous map* $B_0 : \widetilde{S^2_{1,0}(\Omega)} \to S^2_{1,0}(\Omega)$ *such that*

$$B_0((\pm\mathcal{D}_0(m,\mu_m)) \cap \widetilde{S^2_{1,0}(\Omega)}) \subset \pm\mathcal{D}_0(m,\mu_m)$$

and $V_m(u) := i(u)u - B_0(u)$ *is a pseudo-gradient vector field of* Φ_m, *where* $\widetilde{S^2_{1,0}(\Omega)} := S^2_{1,0}(\Omega)\backslash\mathcal{K}, \mathcal{K} := \{u \in S^2_{1,0}(\Omega) : \Phi'(u) = 0\}$. *Moreover, since* Φ_m *and* i *are even functionals,* B_0 *(and hence* V_m*) can be choose to be odd.*

Proof. By Lemma 3.3, the proof of Lemma 3.4 is the same as the proof of Lemma 2.1 in [30], so we omit it. □

Lemma 3.5. *Suppose that* f *satisfies* (A1) *and* g *satisfies* (A5). *Then*

$$\lim_{u\in\mathbf{Y}_{k+1}, \|u\|_{S^2_{1,0}(\Omega)}\to\infty} \overline{\Phi}(u) = -\infty, \quad \lim_{u\in\mathbf{Y}_{k+1}, \|u\|_{S^2_{1,0}(\Omega)}\to\infty} \Phi(u) = -\infty.$$

Proof. By the definition of $\mathbf{Y}_{k+1}$, (A1) and (A5), it is easy to verify that

$$\lim_{u\in\mathbf{Y}_{k+1}, \|u\|_{S^2_{1,0}(\Omega)}\to\infty} \frac{\int\limits_\Omega F(x,y,u)\mathrm{d}X}{\|u\|^2_{S^2_{1,0}(\Omega)}} = \infty, \quad \lim_{u\in\mathbf{Y}_{k+1}, \|u\|_{S^2_{1,0}(\Omega)}\to\infty} \frac{\int\limits_\Omega G(x,y,u)\mathrm{d}X}{\|u\|^2_{S^2_{1,0}(\Omega)}} \leq \infty.$$

Then the conclusions of this lemma follow immediately. □

Lemma 3.6. *Suppose that* f *satisfies* (A1) - (A4) *and* g *satisfies* (A5). *Then for each fixed* $m > 0$, *there exists a* $C_4 > 0$ *such that*

$$\|u\|_{L^{\sigma+1}(\Omega)} \leq C_4 d^{\frac{1}{\mu}},$$

for all $u \in \pm U_\delta \cap \{u \in \mathbf{Y}_m : \overline{\Phi}_m(u) \le d\}$, *where* C_4 *is independent of* $m, d > 0$ *and*

$$U_\delta := \left\{ u \in \mathbf{Y}_m : \left\| \overline{\Phi}'_m(u) - \Phi'_m(u) \right\|_{(S^2_{1,0}(\Omega))^*} > \frac{\left\| \overline{\Phi}'_m(u) \right\|_{(S^2_{1,0}(\Omega))^*}}{\delta} \right\}.$$

Proof. We consider two cases.

Case 1. $u \in U_\delta \cap \{u \in \mathbf{Y}_m : \overline{\Phi}_m(u) \le d\}$. We have that

$$\frac{1}{2} \|u\|^2_{S^2_{1,0}(\Omega)} - \int_\Omega F(x,y,u)\mathrm{d}X - \int_\Omega G(x,y,u)\mathrm{d}X \le d. \tag{3.6}$$

$$\left\| \overline{\Phi}'_m(u) \right\|_{(S^2_{1,0}(\Omega))^*} < \delta \left\| \overline{\Phi}'_m(u) - \Phi'_m(u) \right\|_{(S^2_{1,0}(\Omega))^*} \tag{3.7}$$

and

$$\begin{aligned} \left| \langle \overline{\Phi}'_m(u), u \rangle \right| &= \left| \|u\|^2_{S^2_{1,0}(\Omega)} - \int_\Omega f(x,y,u)u\mathrm{d}X - \int_\Omega g(x,y,u)u\mathrm{d}X \right| \\ &\le \left\| \overline{\Phi}'_m(u) \right\|_{(S^2_{1,0}(\Omega))^*} \|u\|^2_{S^2_{1,0}(\Omega)} \le \delta \left\| \overline{\Phi}'_m(u) - \Phi'_m(u) \right\|_{(S^2_{1,0}(\Omega))^*} \|u\|^2_{S^2_{1,0}(\Omega)}. \end{aligned} \tag{3.8}$$

From (A5) hence

$$\delta \left\| \overline{\Phi}'_m(u) - \Phi'_m(u) \right\|_{(S^2_{1,0}(\Omega))^*} \|u\|^2_{S^2_{1,0}(\Omega)} \le C_5 \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)} \right),$$

we get by (3.8) that

(3.9)

$$- \|u\|^2_{S^2_{1,0}(\Omega)} \le - \int_\Omega f(x,y,u)u\mathrm{d}X - \int_\Omega g(x,y,u)u\mathrm{d}X + C_5 \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)} \right) \|u\|_{S^2_{1,0}(\Omega)}.$$

Choose $\mu_0 \in (2, \mu)$. By (3.6) and (3.9), we know that

$$\begin{aligned} \left(\frac{\mu_0}{2} - 1 \right) \|u\|^2_{S^2_{1,0}(\Omega)} \le \int_\Omega \left[\mu_0 F(x,y,u) - f(x,y,u)u \right] \mathrm{d}X + \int_\Omega \left[\mu_0 G(x,y,u) - g(x,y,u)u \right] \mathrm{d}X \\ + C_5 \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)} \right) \|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d. \end{aligned}$$

This and (A3), (A5) hence

$$\begin{aligned} \left(\frac{\mu_0}{2} - 1 \right) \|u\|^2_{S^2_{1,0}(\Omega)} + C_6 \|u\|^\mu_{L^\mu(\Omega)} \le \left(\frac{\mu_0}{2} - 1 \right) \|u\|^2_{S^2_{1,0}(\Omega)} + \\ \int_\Omega \left[f(x,y,u)u - \mu_0 F(x,y,u) \right] \mathrm{d}X + C_7 \le \int_\Omega \left[\mu_0 G(x,y,u) - g(x,y,u)u \right] \mathrm{d}X \\ + C_5 \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)} \right) \|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7 \\ \le C_8 \|u\|^2_{L^2(\Omega)} + C_8 \|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)} \\ + C_5 \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)} \right) \|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7. \end{aligned}$$

By $\mu > 2, \mu > \sigma + 1$, applying Young's inequalities and Cauchy's inequalities, we have

$$C_8 \|u\|^2_{S^2_{1,0}(\Omega)} + C_9 \|u\|^\mu_{L^\mu(\Omega)} \leq C_5 \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)}\right) \|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7$$
$$\leq C_\epsilon \|u\|^2_{L^2(\Omega)} + \epsilon \|u\|^2_{S^2_{1,0}(\Omega)} + C_{\epsilon'} \|u\|^{2\sigma}_{L^2(\Omega)} + \epsilon' \|u\|^2_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7,$$

for all $\epsilon, \epsilon' > 0$ small enough. By the fact that $2\sigma < \mu$, we can obtain

$$\|u\|_{L^{\sigma+1}(\Omega)} \leq C_{10} \|u\|_{L^\mu(\Omega)} \leq C_4 d^{\frac{1}{\mu}}.$$

Case 2. $u \in -U_\delta \cap \{u \in \mathbf{Y}_m : \overline{\Phi}_m(u) \leq d\}$, that is,

$$\left\|\overline{\Phi}'_m(-u)\right\|_{(S^2_{1,0}(\Omega))^*} < \delta \left\|\overline{\Phi}'_m(-u) - \Phi'_m(-u)\right\|_{(S^2_{1,0}(\Omega))^*} \text{ and } \overline{\Phi}_m(u) \leq d.$$

Then

$$\overline{\Phi}_m(-u) = \overline{\Phi}_m(u) + \int_\Omega [G(x,y,u) - G(x,y,-u)]\mathrm{d}X \leq d + C_{11} \|u\|^2_{L^2(\Omega)} + C_{11} \|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)}$$

(3.10)
$$\frac{1}{2} \|u\|^2_{S^2_{1,0}(\Omega)} - \int_\Omega F(x,y,-u)\mathrm{d}X - \int_\Omega G(x,y,-u)\mathrm{d}X \leq d + C_{11} \|u\|^2_{L^2(\Omega)} + C_{11} \|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)}$$

$$\left|\langle\overline{\Phi}'_m(-u), -u\rangle\right| = \left|\|u\|^2_{S^2_{1,0}(\Omega)} + \int_\Omega f(x,y,-u)u\mathrm{d}X + \int_\Omega g(x,y,-u)u\mathrm{d}X\right|$$
$$\leq \left\|\overline{\Phi}'_m(-u)\right\|_{(S^2_{1,0}(\Omega))^*} \|u\|^2_{S^2_{1,0}(\Omega)} \leq \delta \left\|\overline{\Phi}'_m(-u) - \Phi'_m(-u)\right\|_{(S^2_{1,0}(\Omega))^*} \|u\|^2_{S^2_{1,0}(\Omega)}. \tag{3.11}$$

Note that

$$\delta \left\|\overline{\Phi}'_m(-u) - \Phi'_m(-u)\right\|_{(S^2_{1,0}(\Omega))^*} \|u\|^2_{S^2_{1,0}(\Omega)} \leq C_{12} \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)}\right).$$

Then we get

(3.12)
$$-\|u\|^2_{S^2_{1,0}(\Omega)} \leq \int_\Omega f(x,y,-u)u\mathrm{d}X + \int_\Omega g(x,y,-u)u\mathrm{d}X + C_{12} \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)}\right) \|u\|_{S^2_{1,0}(\Omega)}.$$

Therefore, by (3.10)-(3.12),(A3) and (A5), we obtain

$$\left(\frac{\mu_0}{2} - 1\right) \|u\|^2_{S^2_{1,0}(\Omega)} + C_6 \|u\|^\mu_{L^\mu(\Omega)} \leq \left(\frac{\mu_0}{2} - 1\right) \|u\|^2_{S^2_{1,0}(\Omega)}$$
$$+ \int_\Omega \left[-f(x,y,-u)u - \mu_0 F(x,y,u)\right] \mathrm{d}X + C_7 \leq \int_\Omega \left[\mu_0 G(x,y,-u) + g(x,y,-u)u\right] \mathrm{d}X$$
$$+C_{12} \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)}\right) \|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7 \leq C_8 \|u\|^2_{L^2(\Omega)} + C_8 \|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)}$$
$$+C_{12} \left(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)}\right) \|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7.$$

This gives the desired result. □

Lemma 3.7. *Assume conditions* (A1)-(A5) *hold and assume that* $u_m \in \mathbf{Y}_m$ *is sign-changing and satises*

$$\overline{\Phi}'_m(u_m) = 0, \quad \sup_{m\geq 1} \left|\overline{\Phi}_m(u_m)\right| < \infty.$$

Then $\{u_m\}$ *has a convergent subsequence whose limit is a sign-changing critical point of* $\overline{\Phi}$.

Proof. From Lemma 3.1, we have $\{u_m\}$ has a convergent subsequence in $S^2_{1,0}(\Omega)$. We just prove that the limit of the subsequence is also sign-changing. Let $u_m^{\pm} := \max\{\pm u_m, 0\}$. Then

$$\left\|u_m^{\pm}\right\|^2_{S^2_{1,0}(\Omega)} = \int_\Omega [f(x,y,u_m^{\pm})u_m^{\pm} + g(x,y,u_m^{\pm})u_m^{\pm}]\mathrm{d}X.$$

By (A1), (A2) and (A5), we have for any $\epsilon > 0$, there exists a C_ϵ such that

$$f(x,y,\xi)\xi + g(x,y,\xi)\xi \leq \epsilon \left|\xi\right|^2 + C_\epsilon \left|\xi\right|^p, \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } (x,y) \in \Omega.$$

It follows that

$$\left\|u_m^{\pm}\right\|^2_{S^2_{1,0}(\Omega)} \leq \epsilon \left\|u\right\|^2_{S^2_{1,0}(\Omega)} + C_\epsilon \left\|\xi\right\|^p_{L^p(\Omega)}.$$

Hence, $\|u_m^{\pm}\|^2_{S^2_{1,0}(\Omega)} \geq C_{13}$, where C_{13} is a constant independent of m. This implies that the limit of the subsequence is also sign-changing. □

Proof of Theorem 1.1. Assume that there exists a $C_{14} > 0$ such that $\overline{\Phi}$ has no sign-changing critical point with critical value greater than C_{14}. Choose $k_0 > 0$ such that $\delta_k > C_{14}$ for all $k > k_0$, where δ_k comes from Lemma 3.2. Let $m > k+2 > k_0+2$. Then $\mathbf{Y}_k \subset \mathbf{Y}_m$. Let

$$N := \mathbf{Y}_k,\ M(m) = \mathbf{Y}^{\perp}_{k-1} \cap \mathbf{Y}_m,\ Q(\rho_k, m) := \{u \in M(m) : \|u\|_{S^2_{1,0}(\Omega)} = \rho_k\}.$$

Then by (3.3), we obtain

$$Q(\rho_k, m) \subset \mathcal{S}_m.$$

Define

$$N^* = N \oplus \operatorname{span}\{u^*\} \quad u^* \in \mathbf{Y}_{k+1}, u^* \notin \mathbf{Y}_k; \quad N^*_+ = \{u + tu^* : u \in N, t \geq 0\}.$$

Then $N^* \cap \mathbf{Y}_{k+1} \neq \{0\}$, and both N^* and N^*_+ are independent of m. Clearly, by Lemma 3.5, we have

(i) $\overline{\Phi}_m(0) = 0$;

(ii) there exists a $R_1 > \rho_k$ such that $\overline{\Phi}_m(u) \leq 0$ for all $u \in N$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_1$;

(iii) there exists a $R_2 \geq R_1 > 0$ such that $\overline{\Phi}_m(u) \leq 0$ for all $u \in N^*$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_2$;

Let

$$\Gamma_m := \{\phi \in C(\mathbf{Y}_m, \mathbf{Y}_m) : \phi \text{ is odd}, \ \phi(\mathcal{D}_m) \subset (\mathcal{D}_m);$$
$$\phi(u) = u \text{ if } \max\{\overline{\Phi}_m(u), \overline{\Phi}_m(-u)\} \leq 0\}.$$

Define

$$\gamma_k^*(m) := \inf_{\phi\in\Gamma_m} \sup_{\phi(N_+^*)\cap\mathcal{S}_m} \overline{\Phi}_m, \quad \gamma_k^{**}(m) := \inf_{\phi\in\Gamma_m} \sup_{\phi(N)\cap\mathcal{S}_m} \overline{\Phi}_m > 0,$$

For any $\phi \in \Gamma_m$, by Lemma 1.44 in [28] (or see [3,32]), we have $\phi(N\cap B_{R_1})\cap Q(\rho_k, m) \neq \emptyset$, and by Lemma 3.2 we can obtain

$$\sup_{\phi(N\cap B_{R_1})\cap\mathcal{S}_m} \overline{\Phi}_m \leq \inf_{Q(\rho_k,m)} \overline{\Phi}_m \leq C_2\lambda_k^{\frac{2(p_0-p)}{(p-2)(p_0-2)}} := \delta_k.$$

Therefore, we get

$$\gamma_k^{**}(m) \geq C_2\lambda_k^{\frac{2(p_0-p)}{(p-2)(p_0-2)}} := \delta_k \to \infty, \text{ as } k\to\infty. \tag{3.13}$$

We consider two cases.

Case 1. For $k \geq k_0$, if there exists a sequence $m_i \to \infty$ as $i \to \infty$ such that

$$\gamma_k^*(m_i) > \gamma_k^{**}(m_i) \text{ for all } i > 1,$$

then by Proposition 2.3, there exists a sign-changing critical point u_{m_i} such that

$$\overline{\Phi}'_{m_i}(u_{m_i}) = 0 \text{ and } C_0 < \delta_k \leq \gamma_k^{**}(m_i) \leq \overline{\Phi}(u_{m_i}) \leq \sup_{N^*}\overline{\Phi} + 1$$

Here $\sup_{N^*}\overline{\Phi}$ is a constant depending on k and independent of m_i. By Lemma 3.7, $\{u_{m_i}\}$ has a convergent subsequence whose limit u is a sign-changing critical point of $\overline{\Phi}$, and $\overline{\Phi}(u) \geq \delta_k > C_0$. This contradicts the assumption.

Case 2. For all $k \geq k_0$, there exists a m_k such that

$$\gamma_k^*(m) = \gamma_k^{**}(m) \text{ for all } m > m_k. \tag{3.14}$$

Let $K_{com}(m)$ denote the set of common critical points of Φ_m and $\overline{\Phi}_m$. By (A5), $K_{com}(m) = \{0\}$. Define

$$V_\delta := \{u \in \mathbf{Y}_m : \|u\|_{S^2_{1,0}(\Omega)} \leq \delta\},$$

and let U_δ be as in Lemma 3.6, which contains all non-common critical points of Φ_m and $\overline{\Phi}_m$. By Lemma 3.5, there exists a $R_1 > \rho_k$ such that $\overline{\Phi}_m(u) \leq 0$ for all $u \in N$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_1$. Here R_1 is independent of m. Combining the definition of $\gamma_k^*(m)$ and (3.13), we find a $\phi_0 \in \Gamma_m$ such that

$$\sup_{\phi_0(N_+^*)\cap\mathcal{S}_m} \overline{\Phi} = \sup_{\phi(N_+^*\cap B_{R_1})\cap\mathcal{S}_m} \overline{\Phi} \leq \gamma_k^*(m) + \frac{1}{2}. \tag{3.15}$$

Let

$$U_\delta^*(m) := V_\delta \cup U_\delta \cup (-U_\delta).$$

Then U_δ is a symmetric set and contains all critical points of Φ_m and $\overline{\Phi}_m$. Define two non-negative continuous functions:

$$\zeta_1(u) = \begin{cases} 0, & \text{if } \ u \in U_{10}^*(m), \\ 1, & \text{if } \ u \notin U_{20}^*(m), \end{cases} \quad \text{(is even)}, \quad \zeta_2(u) = \begin{cases} 0, & \text{if } \ u \leq 0, \\ 1, & \text{if } \ u \geq 1, \end{cases}$$

and a vector field

$$V_m^* := -\zeta_2(\max\{\overline{\Phi}_m(u), \overline{\Phi}_m(-u)\})\zeta_1(u)V_m(u),$$

where the pseudo gradient vector field $V_m$comes from Lemma 3.4 obtained for Φ_m. Since V_m can be choose to be odd, then V_m^* is odd.

Let $\Theta(t,u)$ denote the unique (odd in u) solution of the Cauchy initial value problem:

$$\frac{\mathrm{d}\Theta(t,u)}{\mathrm{d}t} = V_m^*(\Theta(t,u)), \quad \Theta(0,u) = u \in \mathbf{Y}_m.$$

Then $\Theta(t,u)$ is also a pseudo-gradient ow for $\overline{\Phi}_m$ and

$$\frac{\mathrm{d}\overline{\Phi}_m(\Theta(t,u))}{\mathrm{d}t} \le 0. \tag{3.16}$$

For any $u \notin U_\delta^*(m)$, we have $u \notin \pm U_\delta$ and by Lemma 3.6, we obtain

$$\|\Phi_m'(u)\|_{(S^2_{1,0}(\Omega))^*} \le \frac{\delta+1}{\delta}\left\|\overline{\Phi}_m'(u)\right\|_{(S^2_{1,0}(\Omega))^*}, \quad \left\|\overline{\Phi}_m'(u)\right\|_{(S^2_{1,0}(\Omega))^*} \le \frac{\delta}{\delta-1}\|\Phi_m'(u)\|_{(S^2_{1,0}(\Omega))^*}.$$

Further, for all $u \notin U_\delta^*(m)$, we get

$$\begin{aligned}\langle\overline{\Phi}_m'(u), V_m(u)\rangle &= \langle\Phi_m'(u), V_m(u)\rangle - \langle\Phi_m'(u) - \overline{\Phi}_m'(u), V_m(u)\rangle\\ &\ge \frac{1}{2}\|\Phi_m'(u)\|^2_{(S^2_{1,0}(\Omega))^*} - 2\|\Phi_m'(u)\|^2_{(S^2_{1,0}(\Omega))^*}\|\Phi_m'(u) - V_m(u)\|^2_{(S^2_{1,0}(\Omega))^*}\\ &\ge \frac{(\delta-1)^2 - 4(\delta+1)}{2\delta^2}\left\|\overline{\Phi}_m'(u)\right\|^2_{(S^2_{1,0}(\Omega))^*},\end{aligned}$$

and

$$\|V_m(u)\|_{(S^2_{1,0}(\Omega))^*} \le 2\|\Phi_m'(u)\|^2_{(S^2_{1,0}(\Omega))^*} \le \frac{2(\delta+1)}{\delta}\left\|\overline{\Phi}_m'(u)\right\|^2_{(S^2_{1,0}(\Omega))^*}.$$

Moreover, since $u \notin U_{20}^*(m)$ implies $\zeta_1(u) = 1$, we see that $\overline{\Phi}_m(u) > 1$ implies

$$\zeta_2(\max\{\overline{\Phi}_m(u), \overline{\Phi}_m(-u)\}) = 1.$$

We obtain

$$\begin{aligned}&\frac{\mathrm{d}\overline{\Phi}_m(\Theta(t,u))}{\mathrm{d}t}\Big|_{t=0} = \langle\overline{\Phi}_m'(\Theta(t,u)), \frac{\mathrm{d}\Theta}{\mathrm{d}t}\rangle\Big|_{t=0}\\ &= \langle\overline{\Phi}_m'(\Theta(t,u)), V_m^*(\Theta(t,u))\rangle\Big|_{t=0}\\ &= \langle\overline{\Phi}_m'(u), -\zeta_2(\max\{\overline{\Phi}_m(\Theta(t,u)), \overline{\Phi}_m(-\Theta(t,u))\})\zeta_1(\Theta(t,u))V_m(\Theta(t,u)))\rangle\Big|_{t=0}\\ &= \langle\overline{\Phi}_m'(u), -\zeta_2(\max\{\overline{\Phi}_m(\Theta(t,u)), \overline{\Phi}_m(-\Theta(t,u))\})\zeta_1(u)V_m(u))\rangle\\ &= -\langle\overline{\Phi}_m'(u), V_m(u)\rangle \le -\frac{277}{800}\left\|\overline{\Phi}_m'(u)\right\|^2_{(S^2_{1,0}(\Omega))^*}.\end{aligned} \tag{3.17}$$

for all $u \notin U_{20}^*(m)$ safisfying $\overline{\Phi}_m(u) > 1$.

We claim that $\Theta(t, \phi_0(\cdot)) \in \Gamma_m$ for any $t \ge 0$. In fact, $\Theta(t, \phi_0(u))$ is odd in u since $\phi_0(u)$ and V_m^* are odd. Recall that $\phi_0 \in \Gamma_m$. Then, $\phi_0(u) = u$ for u with $\max\{\overline{\Phi}(u), \overline{\Phi}(-u)\} \le 0$. Hence, $\Theta(t, \phi_0(u)) = \Theta(t,u)$ and $V_m^*(u) = 0$ for u with $\max\{\overline{\Phi}(u), \overline{\Phi}(-u)\} \le 0$. It follows that $\Theta(t,u) = u$ and then $\Theta(t, \phi_0(u)) = u$ for u

with $\max\{\overline{\Phi}(u), \overline{\Phi}(-u)\} \leq 0$. Using Theorem 1 in [9] and Lemma 3.4 similar to the one of Theorem 2.1 in [30], we obtain

$$\Theta(t, \phi_0(\mathcal{D}_m)) \subset \Theta(t, \mathcal{D}_m) \subset \mathcal{D}_m, \quad \forall t \geq 0.$$

Therefore, $\Theta(t, \phi_0(u)) \in \Gamma_m$ for any $t \geq 0$. For any $t \geq 0$, we can deduce the following estimates which lead to a contradiction. In fact, by the fact that $\Theta(t, \phi_0(u)) \in \Gamma_m$ is odd, $\Theta(t, \phi_0(N_+^*)) \cap \mathcal{S}_m \subset \{u \in \mathbf{Y}_m : \overline{\Phi}_m(u) \leq \gamma_k^{**}(m) + \frac{1}{2}\}, V_{20}$ is bounded, (3.15) - (3.17) and Lemma 3.6, we have

$$\begin{aligned}
\gamma_k^*(m) + \frac{1}{2} = \gamma_k^{**}(m) + \frac{1}{2} &\geq \sup_{\phi_0(N_+^*)\cap \mathcal{S}_m} \overline{\Phi} \geq \sup_{\Theta(t,\phi_0(N_+^*))\cap \mathcal{S}_m} \overline{\Phi} \\
&\geq \sup_{\Theta(t,\phi_0(N^*))\cap \mathcal{S}_m} \overline{\Phi} - \left(\sup_{\Theta(t,\phi_0(N^*))\cap \mathcal{S}_m} \overline{\Phi} - \sup_{\Theta(t,\phi_0(N_+^*))\cap \mathcal{S}_m} \overline{\Phi} \right) \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in\Theta(t,\phi_0(N_+^*))\cap \mathcal{S}_m} \left(\overline{\Phi}(-u) - \overline{\Phi}(u)\right) \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in\Theta(t,\phi_0(N_+^*))\cap \mathcal{S}_m \cap\{u\in\mathbf{Y}_m:\overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}} \left(\overline{\Phi}(-u) - \overline{\Phi}(u)\right) \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in\Theta(t,\phi_0(N_+^*))\cap \mathcal{S}_m \cap\{u\in\mathbf{Y}_m:\overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}} \left|\overline{\Phi}(-u) - \overline{\Phi}(u)\right| \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in U_{20}^*\cap\{u\in\mathbf{Y}_m:\overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}} \left|\overline{\Phi}(-u) - \overline{\Phi}(u)\right| \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in(-U_{20}\cup U_{20})\cap\{u\in\mathbf{Y}_m:\overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}} \left|\overline{\Phi}(-u) - \overline{\Phi}(u)\right| - C_{15} \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in(-U_{20}\cup U_{20})\cap\{u\in\mathbf{Y}_m:\overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}} C_8 \left|u\right|_{L^{\sigma+1}(\Omega)}^{\sigma+1} - C_{15} \\
&\geq \gamma_{k+1}^{**}(m) - C_8(\gamma_k^{**}(m))^{\frac{1+\sigma}{\mu}} - C_{15}.
\end{aligned}$$

Therefore, we get the inequality

$$\gamma_{k+1}^{**}(m) \leq \gamma_k^{**}(m) \left(1 + C_8(\gamma_k^{**}(m))^{\frac{1+\sigma-\mu}{\mu}}\right), \tag{3.18}$$

for all $k \geq k_0$.

Then from the condition (1.3), we can take some $p_0 \in (2, 2_\alpha^*)$ and $q \in (\frac{2N_\alpha}{N_\alpha+2}, 2)$ such that

$$\frac{1+\sigma-\mu}{\mu} \frac{2(p_0-p)}{(p-2)(p_0-2)} \left(\frac{2}{N_\alpha} - \frac{2-q}{q}\right) < -1. \tag{3.19}$$

From Theorem 1.3 in [15] (or see [16]), we obtain

$$\lambda_k \geq C_{16} k^{\frac{2}{N_\alpha}}. \tag{3.20}$$

From (3.18) -(3.20), using iteration, we get that

$$\begin{aligned}\gamma^{**}_{k_0+\ell}(m) &\leq \gamma^{**}_{k_0}(m)\prod_{k=k_0}^{k_0+\ell-1}\left(1+C_8(\gamma^{**}_k(m))^{\frac{1+\sigma-\mu}{\mu}}\right)\\ &\leq \gamma^{**}_{k_0}(m)\exp\left(\sum_{k=k_0}^{k_0+\ell-1}\ln(1+C_8(\gamma^{**}_k(m))^{\frac{1+\sigma-\mu}{\mu}})\right)\\ &\leq \gamma^{**}_{k_0}(m)\exp C_8\left(\sum_{k=k_0}^{k_0+\ell-1}(\gamma^{**}_k(m))^{\frac{1+\sigma-\mu}{\mu}}\right)\\ &\leq \gamma^{**}_{k_0}(m)\exp C_8\left(\sum_{k=k_0}^{k_0+\ell-1}k^{\frac{1+\sigma-\mu}{\mu}\frac{2(p_0-p)}{(p-2)(p_0-2)}\left(\frac{2}{N_\alpha}-\frac{2-q}{q}\right)}\right)<\infty,\end{aligned}$$

for all $\ell\in\mathbb{N}$, which yields the desired contradiction. Thus, $\overline{\Phi}$ possesses an unbounded sign-changing sequence of critical values. $\square$

Corollary 3.8. *Assume that f satisfies the conditions* (A1)–(A3) *and there is a $\sigma<\frac{\mu}{2}$ such that*

$$|f(x,y,u)-f(x,y,-u)|\leq C\left(1+|u|^\sigma\right)\ \text{for all } u\in\mathbb{R} \text{ and a.e. } (x,y)\in\Omega.$$

Then the problem

$$-G_\alpha u=f(x,y,u)\ \text{ in }\ \Omega,\quad u=0\ \text{ on }\ \partial\Omega, \tag{3.21}$$

has an infinite sequence of sign-changing solutions provided that (1.3) *holds.*

Corollary 3.9. *Assume that f satisfies the conditions* (A1)–(A3), *there exists a $R>0$ such that*

$$f(x,y,-u)=-f(x,y,u)\ \text{for a.e. } (x,y)\in\Omega,|u|\geq R,$$

and

$$\frac{2p}{N_\alpha(p-2)}-1>\frac{\mu}{\mu-1}.$$

Then the problem (3.21) *has an infinite sequence of sign-changing solutions in $S^2_{1,0}(\Omega)$.*

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References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] C. T. Anh and B. K. My, *Existence of solutions to $\Delta_\lambda-$Laplace equations without the Ambrosetti–Rabinowitz condition*, Complex Var. Elliptic Equ. **61** (1) (2016), 137–150.

[3] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications,* Trans. Amer. Math. Soc. **267** (1981), no. 1, 1–32.

[4] S. Barile and G. M. Figueiredo, *Existence of least energy positive, negative and nodal solutions for a class of pq−problems with potentials vanishing at infinity*, J. Math. Anal. Appl., **427** (2015), 1205–1233.

[5] T. Bartsch, Z. Liu and T. Weth *Sign changing solutions of superlinear Schrödinger equations*, Comm. Partial Differential Equations, **29** (2004), no. 1-2, 25–42.

[6] T. Bartsch and T. Weth, *Three nodal solutions of singularly perturbed elliptic equations on domains without topology*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **22** (2005), no. 3, 259–281.

[7] T. Bartsch, T. Weth and M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math., **96** (2005), 1–18.

[8] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations, II, Existence of infinitely many solutions*, Arch. Rational Mech. Anal., **82** (1983), no. 4, 313–345.

[9] H. Brezis, *On a characterization of flow-invariant sets.* Comm. Pure Appl. Math. **23** (1970), 261–263.

[10] A. Castro, J. Cossio and J. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math., **27** (1997), no. 4, 1041–1053.

[11] S. T. Chen, Y. B. Li and X. H. Tang, *Sign-changing solutions for asymptotically linear Schrodinger equation in bounded domains*, Electron. J. Differential Equations **317** (2016), 1–9.

[12] S. Dipierro, M. Medina and E. Valdinoci, Fractional Elliptic Problems with Critical Growth in the Whole of $\mathbb{R}^n$, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], **15**. Edizioni della Normale, Pisa, 2017. viii+152 pp.

[13] B. Franchi and E. Lanconelli, *An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality*, Comm. Partial Differential Equations **9** (1984), no. 13, 1237–1264.

[14] V. V. Grushin, *A certain class of hypoelliptic operators,* Mat. Sb. (N.S.) **83** (1970), no. 125, 456–473 [in Russian].

[15] C. Hua and L. Peng, *Lower bounds of Dirichlet eigenvalues for some degenerate elliptic operators,* Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2831–2852.

[16] C. Hua and Z. Yifu, *Lower bounds of eigenvalues for a class of bi-subelliptic operators.* J. Differential Equations **262** (2017), no. 12, 5860–5879.

[17] A. E. Kogoj and E. Lanconelli, *On semilinear* $\Delta_\lambda-$*Laplace equation,* Nonlinear Analysis. **75** (2012), no. 12, 4637–4649.

[18] Z. L. Liu and J. X. Sun, *Invariant Sets of Descending Flow in Critical Point Theory with Applications to Nonlinear Differential Equations*, J. Differential Equations. **172** (2001), no. 2, 257–299.

[19] D. Lupo and K. Payne, *Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types,* Comm. Pure Appl. Math. **56** (2003), no. 3, 403–424.

[20] D. T. Luyen, *Two nontrivial solutions of boundary value problems for semilinear* Δ_γ *differential equations,* Math. Notes **101** (2017), no. 5, 815–823.

[21] D. T. Luyen and N. M. Tri, *Existence of solutions to boundary value problems for semilinear* Δ_γ *differential equations,* Math. Notes **97** (2015), no. 1, 73–84.

[22] D. T. Luyen and N. M. Tri, *Large–time behavior of solutions to damped hyperbolic equation involving strongly degenerate elliptic differential operators,* (Russian) Sibirsk. Mat. Zh. **57** (2016), no. 4, 809–829; translation in Sib. Math. J. **57** (2016), no. 4, 632–649.

[23] D. T. Luyen and N. M. Tri, *Global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator,* Ann. Pol. Math. **117** (2) (2016), 141–162.

[24] D. T. Luyen and N. M. Tri, *Existence of infinitely many solutions for semilinear degenerate Schrödinger equations,* J. Math. Anal. Appl. **461** (2018), no. 2, 1271–1286.

[25] D. T. Luyen and N. M. Tri, *On the existence of multiple solutions to boundary value problems for semilinear elliptic degenerate operators.* Complex Var. Elliptic Equ. **64** (2019), no. 6, 1050–1066.

[26] E. S. Noussair and J. Wei, *On the effect of domain geometry on the existence of nodal solutions in singular perturbations problems*, Indiana Univ. Math. J., **46** (1997), 1255–1271.

[27] S. I. Pokhozaev, *Eigenfunctions for the equation* $\Delta u + \lambda f(u) = 0$, Russian Dokl. Akad. Nauk SSSR 165, 33–36 (1965).

[28] P. H. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. Amer. Math. Soc. **272** (1982), no. 2, 753–769.
[29] M. Schechter, Z. Q. Wang and W. Zou, *New Linking Theorem and Sign-Changing Solutions*, Comm. Partial Differential Equations **29** (2004), no. 3-4, 471–488.
[30] M. Schechter and W. Zou, *Infinitely many solutions to perturbed elliptic equations,* J. Funct. Anal. 228 (2005), no. 1, 1–38.
[31] M. Schechter and W. Zou, *Sign-changing critical points from linking type theorems*, Trans. Amer. Math. Soc. **358** (2006), no. 12, 5293–5318.
[32] M. Struwe, Variational Methods, Applications to nonlinear partial differential equations and Hamiltonian systems. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **34** (1996). Springer-Verlag, Berlin. xvi+272pp.
[33] P. T. Thuy and N. M. Tri, *Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations*, NoDEA Nonlinear Differential Equations Appl. **19** (3) (2012), 279–298.
[34] P. T. Thuy and N. M. Tri, *Long time behavior of solutions to semilinear parabolic equations involving strongly degenerate elliptic differential operators,* NoDEA Nonlinear Differential Equations Appl. **20** (3) (2013), 1213–1224.
[35] N. M. Tri, *Critical Sobolev exponent for hypoelliptic operators,* Acta Math. Vietnam. **23** (1) (1998), 83–94.
[36] N. M. Tri, Semilinear Degenerate Elliptic Differential Equations, Local and global theories, Lambert Academic Publishing, 2010, 271p.
[37] N. M. Tri, Recent Progress in the Theory of Semilinear Equations Involving Degenerate Elliptic Differential Operators, Publishing House for Science and Technology of the Vietnam Academy of Science and Technology, 2014, 380p.

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