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# CLEANNESS OF COHEN-MACAULAY MONOMIAL IDEAL GENERATED BY AT MOST FIVE ELEMENTS

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Abstract. In this paper, we prove that any Cohen-Macaulay monomial ideal generated by at most five elements is clean.

## 1. Introduction

Let $n$ be a positive integer, and put $[n] := \{1, 2, \dots, n\}$. A nonempty subset $\Delta \subset 2^{[n]}$ is called a *simplicial complex* on $[n]$ if the following conditions are satisfied: $(i)$ $F \in \Delta,\ G \subset F \Longrightarrow G \in \Delta$, $(ii)$ $\{v\} \in \Delta$ for every $v \in [n]$.

Let $\Delta$ be a simplicial complex on $[n]$. An element $F$ of $\Delta$ is called a *face* of $\Delta$. The dimension of a face $F$ is $|F|-1$, where $|F|$ denotes the cardinality of $F$. Set $\dim \Delta = \max\{\dim F \mid F \in \Delta\} = d-1$. The maximal face of $\Delta$ under inclusion is called a *facet*. A simplicial complex is called *pure* if all facets have the same dimension. A pure simplicial complex $\Delta$ (say, $d = \dim \Delta + 1$) is said to be *shellable* if there exists an order $F_1, \dots, F_t$ of the facets of $\Delta$ such that for each $2 \le i \le t$, $\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i \rangle$ is a pure $(d-2)$-dimensional simplicial complex, where $\langle G_1, \dots, G_r\rangle$ is a simplicial complex generated by $G_1, \dots, G_r$.

Let $K$ be a field, and let $\Delta$ be a simplicial complex on $[n]$. Put $S = K[X_1, \dots, X_n]$ a polynomial ring over $K$. Then the squarefree monomial ideal

$$I_\Delta = (X_{i_1} \cdots X_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n,\ \{i_1, \dots, i_r\} \notin \Delta)S$$

is called the *Stanley-Reisner ideal* of $\Delta$. Notice that any squarefree monomial ideal $I$ is given as the Stanley-Reisner ideal $I_\Delta$ for some simplicial complex $\Delta$. Moreover, $K[\Delta] = S/I_\Delta$ is called the *Stanley-Reisner ring* of $\Delta$. Many combinatorial properties of $\Delta$ are recognized as ring-theoretical properties of $K[\Delta]$. For instance, $\dim \Delta = \dim K[\Delta] - 1$, and $\Delta$ is pure if and only if $K[\Delta]$ is equidimensional. How about shellablity? For this question, Dress [2] introduced the notion of clean modules and answered this question: $I_\Delta$ is clean if and only if $\Delta$ is shellable provided that $\Delta$ is pure. We recall the notion of cleanness.

**Definition 1.1** (Dress [2])**.** For a monomial ideal $I$ of $S$, $I$ is *clean* (or $S/I$ is clean) if there exists a finite chain of monomial ideals

$$\mathcal{F}:\ I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

such that $I_i/I_{i-1} \cong S/P_i$ for minimal prime ideals $P_i$ of $I$ for each $i$.

The following question is natural.

**Question.** *When is a monomial ideal $I$ clean?*

It is known that any shellable complex is Cohen-Macaulay over any field $K$. In other words, $S/I$ is Cohen-Macaulay for any pure clean ideal $I$ of $S$. The converse is *not* true in general. For any homogeneous ideal $I \subset S$, $\operatorname{height} I \le \mu(I)$ holds true, where $\operatorname{height} I$ (resp. $\mu(I)$) denotes the height (resp. the minimal number of generators) of $I$. Then $d(I) = \mu(I) - \operatorname{height} I$ is called the *deviation* of $I$. One can expect that if $\mu(I)$ or $d(I)$ is small then any Cohen-Macaulay ideal is clean. In fact, Herzog et.al [4, Proposition 2.2] proved that any monomial complete intersection ideal (i.e.

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$d(I) = 0$) is clean. Furthermore, Bandari et al. proved that any almost complete intersection squarefree monomial ideal (i.e. $d(I) = 1$) is also clean, and proved that any squarefree monomial ideal $I$ with $\mu(I) \leq 3$ is clean; see [1, Theorem 2.5, Corollary 2.6].

The main theorem of the paper is the following:

**Theorem 1.2.** *Let $S$ be a polynomial ring over a field $K$, and let $I \subset S$ be a monomial ideal. If $S/I$ is Cohen-Macaulay and $\mu(I) \leq 5$, then $I$ is a clean ideal.*

Notice that the condition "$\mu(I) \leq 5$" is best possible because there exists a Cohen-Macaulay but *not* clean squarefree monomial ideal with $\mu(I) = 6$; see Example 2.8.

Let us explain the organization of the paper. In Section 2, we prove our Theorem 1.2 by using the following theorem, which is proved in Section 3.

**Theorem 1.3.** *Let $I \subset S$ be a squarefree monomial ideal with $d(I) = 2$. Then the Alexander dual $I^\vee$ of $I$ has linear quotients if $I$ is Cohen-Macaulay with $\mu(I) = 5$.*

Section 3 is devoted to the proof of the theorem above. A key idea is a classification theorem of Cohen-Macaulay squarefree monomial ideals of deviation 2; see Kimura et al. [8].

## 2. Proof of main theorem

Throughout this section, let $S$ be a polynomial ring over a field $K$, and let $I \subset S$ be a monomial ideal.

First we recall the definition of pretty clean ideals, which was introduced by Herzog and Popescu [3].

**Definition 2.1** (cf. [3])**.** A monomial ideal $I \subset S$ is *pretty clean* if there exists a chain of monomial ideals

$$I = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = S$$

such that

(i) For every $j = 1, \ldots, r$, $I_j/I_{j-1} \cong S/P_j$, where $P_j$ is a monomial prime ideal, which is generated by a subset of the variables.

(ii) $P_i = P_j$ holds true whenever $P_i \subset P_j$ for $i < j$.

If $I$ is clean, then it is pretty clean. The converse is not true in general. For instance, $(x^2, xy)$ is pretty clean but not clean in $S = K[x, y]$; see [3, Example 3.6].

Let $\mathrm{Ass}_S(M)$ (resp. $\mathrm{Min}_S(M)$) denote the set of associated prime ideals (resp. minimal prime ideals) of an $S$-module $M$.

**Lemma 2.2** (See [3, Corollary 3.4])**.** *Assume that $\mathrm{Ass}_S(S/I) = \mathrm{Min}_S(S/I)$. Then $I$ is clean if and only if it is pretty clean.*

*In particular,*

(1) *Any pretty clean squarefree monomial ideal is clean.*

(2) *Any pretty clean Cohen-Macaulay monomial ideal is clean.*

Let us recall the notion of polarization of monomial ideals. For a monomial $m = x_1^{a_1} \cdots x_r^{a_r}$, the polarization of $m$ is defined by

$$m^{\mathrm{pol}} = (x_{11} \cdots x_{1a_1})(x_{21} \cdots x_{2a_2}) \cdots (x_{r1} \cdots x_{ra_r}).$$

For a monomial ideal $I = (m_1, \ldots, m_\nu)$,

$$I^{\mathrm{pol}} = (m_1^{\mathrm{pol}}, \ldots, m_\nu^{\mathrm{pol}})$$

is called the *polarization* of $I$. Notice that $I^{\mathrm{pol}}$ can be regarded as a squarefree monomial ideal of the polynomial ring $S^{\mathrm{pol}}$.

**Lemma 2.3** (See [5, Theorem 3.10])**.** *Let $I^{\mathrm{pol}} \subset S^{\mathrm{pol}}$ be the polarization of $I$. Then $I$ is pretty clean if and only if $I^{\mathrm{pol}}$ is clean.*

**Proposition 2.4** (cf. [1, 4])**.** *The following monomial ideals are clean.*

(1) *$I$ is a complete intersection ideal, that is,* $d(I) = 0$.
(2) *$I$ is an almost complete intersection (that is,* $d(I) = 1$*) Cohen-Macaulay ideal.*
(3) *$I$ is a Cohen-Macaulay ideal with* $\mu(I) \leq 3$.
(4) *$I$ is a Cohen-Macaulay ideal with* $\operatorname{height} I = 2$.
(5) *$I$ is a Gorenstein ideal with* $\operatorname{height} I = 3$.

*Proof.* (1) See Herzog et.al [4, Proposition 2.2].

(2) By Bandari et.al [1, Theorem 2.5], we have that $I$ is pretty clean. Since $S/I$ is Cohen-Macaulay, $I$ is clean by Lemma 2.2(2).

(3) See (2) and Bandari et.al [1, Corollary 2.6].

(4) See Herzog et.al [4, Proposition 2.4].

(5) See the proof of Herzog et.al [4, Theorem 3.1]. □

Let $I$ be a squarefree monomial ideal of $S$. We denote by $G(I)$ the minimal set of monomial generators of $I$. Set $G(I) = \{m_1, \ldots, m_\mu\}$, where

$$m_i = x_{t_{i1}} x_{t_{i2}} \cdots x_{t_{ij_i}}.$$

Then the *Alexander dual ideal* $I^\vee$ of $I$ is given by

$$I^\vee = \bigcap_{i=1}^{\mu} (x_{t_{i1}}, x_{t_{i2}}, \ldots, x_{t_{ij_i}}).$$

The Alexander dual complex $\Delta^\vee$ of a simplicial complex $\Delta$ is given by $\{[n] \setminus F \mid F \notin \Delta\}$. Notice that $I_{\Delta^\vee} = (I_\Delta)^\vee$.

A monomial ideal $I \subset S$ *has linear quotients* if there exists an order $m_1, \ldots, m_\mu$ of $G(I)$ such that for any $2 \leq i \leq \mu$, the ideal $(m_1, \ldots, m_{i-1}) \colon m_i$ is a monomial prime ideal.

**Lemma 2.5** ([1, Lemma 2.1])**.** *Let $I$ be a squarefree monomial ideal. Then $I$ is clean if and only if $I^\vee$ has linear quotients.*

In the next section, we prove Theorem 1.3. By using this, we prove the main theorem (Theorem 1.2).

*Proof of Theorem* 1.2. It suffices to show that if $I$ is a Cohen-Macaulay ideal with $\mu(I) \leq 5$ then $I$ is clean.

**Case 1:** The case where $I$ is a squarefree monomial ideal.

Assume that $I$ is a Cohen-Macaulay squarefree monomial ideal with $\mu(I) \leq 5$.

If $\mu(I) \leq 3$, then $I$ is clean by Proposition 2.4(3). If $\operatorname{height} I = 1$, then $I$ is a principal ideal and thus clean.

If $\operatorname{height} I = 2$, then $I$ is clean by Proposition 2.4(4). Hence, we consider the case $(\operatorname{height} I, \mu(I)) = (3, 4), (4, 4), (3, 5), (4, 5)$ and $(5, 5)$ only.

If $(\operatorname{height} I, \mu(I)) = (4, 4), (5, 5)$, then $I$ is complete intersection and thus it is clean by Proposition 2.4(1). If $(\operatorname{height} I, \mu(I)) = (3, 4)$, $(4, 5)$, then $I$ is an almost complete intersection ideal and thus it is clean by Proposition 2.4(2).

Suppose that $(\operatorname{height} I, \mu(I)) = (3, 5)$. Then Theorem 1.3 yields that $I^\vee$ has linear quotient. Therefore $I$ is clean by Lemma 2.5.

**Case 2:** The case where $I$ is not a squarefree monomial ideal.

Let $I^{\text{pol}} \subset S^{\text{pol}}$ be the polarization of $I$. Then $S^{\text{pol}}/I^{\text{pol}}$ is Cohen-Macaulay, $\mu(I^{\text{pol}}) = \mu(I) \leq 5$ and $I^{\text{pol}}$ is a squarefree monomial ideal. Hence Case 1 yields that $I^{\text{pol}}$ is clean. By Lemma 2.3, $I$ is pretty clean. Hence it is clean because $S/I$ is Cohen-Macaulay; see Lemma 2.2. □

Let us recall the characterization of clean ideals by Dress [2].

**Theorem 2.6** (Dress [2])**.** *Assume that $I = I_\Delta$ and $\Delta$ is pure. Then $I$ is a clean ideal if and only if $\Delta$ is shellable.*

An ideal $I$ of $S$ is called *pure* if $\dim S/P = \dim S/I$ for every minimal prime ideal $P$ of $I$. If a simplicial complex $\Delta$ is pure shellable, then it is Cohen-Macaulay over any field $K$. Thus we obtain the following.

**Corollary 2.7** (cf. Stanley, Dress [2])**.** *A squarefree monomial ideal $I = I_\Delta$ is a pure clean ideal, then it is Cohen-Macaulay over any field $K$.*

In Theorem 1.2, we cannot relax the assumption that $\mu(I) \le 5$. Indeed, the following exmaple shows that there exists a non-clean Cohen-Macaulay ideal with $\mu(I) = 6$.

**Example 2.8** (See Kimura et.al [8, Section 6])**.** Let $K$ be a field. Set

$$m_1 = x_1x_2x_8x_9x_{10}, \quad m_2 = x_2x_3x_4x_5x_{10}, \quad m_3 = x_5x_6x_7x_8x_{10},$$
$$m_4 = x_1x_4x_5x_6x_9, \quad m_5 = x_1x_2x_3x_6x_7, \quad m_6 = x_3x_4x_7x_8x_9,$$

and $I = (m_1, m_2, m_3, m_4, m_5, m_6)$. Then $\operatorname{height} I = 3$ and $\mu(I) = 6$. Moreover, since

$$\operatorname{pd}_S S/I = \begin{cases} 3 & \operatorname{char} K \neq 2, \\ 4 & \operatorname{char} K = 2. \end{cases}$$

Hence if $\operatorname{char} K \neq 2$, then $S/I$ is Cohen-Macaulay but *not* clean.

## 3. Proof of Theorem 1.3 (the case of $d(I) = 2$)

First recall the notion of hypergraph in order to represent monomial ideals of devitaion 2. See basic terminologies for [7, 8, 9]. Let $V = [\mu] := \{1, 2, \ldots, \mu\}$. A hypergraph $\mathcal{H}$ on the vertex set $V$ is a collection of subsets of $V$ with $\bigcup_{F \in \mathcal{H}} F = V$. A subhypergraph $\mathcal{H}'$ (on $V$) of a hypergraph $\mathcal{H}$ means that $\mathcal{H}'$ is a subset of $\mathcal{H}$ such that $\bigcup_{F \in \mathcal{H}'} F = V$.

For an arbitrary squarefree monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, let $G(I) = \{m_1, \ldots, m_\mu\}$ denote the minimal set of monomial generators of $I$. Then the hypergraph $\mathcal{H}(I)$ associated to $I$ on a vertex set $V = [\mu]$ is defined by

$$\mathcal{H}(I) := \big\{\{j \in V : m_j \text{ is divisible by } x_i\} \;:\; i = 1, 2, \ldots, n\big\}.$$

On the other hand, for a hypergraph $\mathcal{H}$, when $n$ is large enough, if we assign a variable $x_F$ to each $F \in \mathcal{H}$, then

$$I_{\mathcal{H}} := \Bigg( \prod_{j \in F \in \mathcal{H}} x_F \;:\; j = 1, 2, \ldots, \mu \Bigg)$$

gives a squarefree monomial ideal of $K[x_F : F \in \mathcal{H}]$. Thus we can construct a squarefree monomial ideal from a given hypergraph. Note that $\mathcal{H}(I_{\mathcal{H}}) = \mathcal{H}$, and that there exist many ideals I so that $\mathcal{H}(I) = \mathcal{H}$.

We now recall some basic properties of this correspondence (see [7]).

1. A subset $\mathcal{C} \subset \mathcal{H}$ is called a *cover* of $H$ if $\bigcup_{F \in \mathcal{C}} F = V$. A cover $\mathcal{C}$ is called a minimal cover if it has no proper subset that is a cover of $\mathcal{H}$.
2. $F \in \mathcal{H}$ is called a face of $\mathcal{H}$. A face in $\mathcal{H}$ which is maximal with respect to inclusion is called a *facet* of $\mathcal{H}$. The dimension of $F$ is defined by $\dim F := \sharp(F) - 1$. If $\dim F = i$, then $F$ is called an $i$-face. A 1-face is called an edge. The dimension of $\mathcal{H}$ is defined by $\dim \mathcal{H} := \max\{\dim F \;:\; F \in \mathcal{H}\}$.
3. $H$ is called *pure* if all minimal covers of $\mathcal{H}$ have the same cardinality.
4. $\mathcal{H}$ is called *disconnected* if there exist hypergraphs $\mathcal{H}_i \subsetneq \mathcal{H}$ and vertex sets $V_i$, $V$ $(i = 1, 2)$ such that $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$, $V_1 \cap V_2 = \emptyset$, and $V_1 \cup V_2 = V$. If $\mathcal{H}$ is not disconnected, then $\mathcal{H}$ is called *connected*.

Let $I$ be a squarefree monomial ideal of S. For all $i, j \in V$ $(i \neq j)$, there exist $F, G \in \mathcal{H}(I)$ such that $i \in F \setminus G$ and $j \in G \setminus F$. We call this condition "separability". Then $\mathcal{H}(I)$ satisfies the separability condition. Conversely, for a given separable hypergraph $\mathcal{H}$, we can construct a squarefree monomial ideal $I$ in a polynomial ring with enough variables so that $\mathcal{H} = \mathcal{H}(I)$.

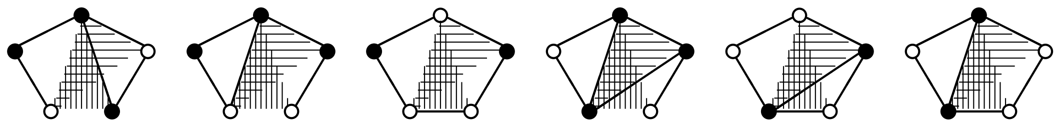
Let $I \subset S = K[x_1, \ldots, x_n]$ be a squarefree monomial Cohen-Macaulay ideal of $d(I) = 2$ and $\mu(I) = 5$. Then we must show that $I$ is clean (equivalently, $I^\vee$ has linear quotients). By applying [10, Theorem 2.1] and [4, Proposition 3.3] to our cases, we may assume that $I = I_{\mathcal{H}}$, where $\mathcal{H}$ is a Cohen-Macaulay hypergraph of deviation 2 which is one of [8, Theorem 4.9]. Notice that the cleanness of $I_{\mathcal{H}'}$ implies the cleanness of $I_{\mathcal{H}}$ if $\mathcal{H}$ is subhypergraph of $\mathcal{H}'$; see Example 3.2 for concrete examples. Hence it suffices to prove that $I_{\mathcal{H}}$ is clean for all hypergraph $\mathcal{H}$ appearing in the following theorem. (Note that $(H_2 9)^{+1}_E$ and $(H_2 9)^{+1}_F$ correspond to Cohen-Macaulay ideals of height 2).

In the classification theorem [8, Theorem 4.9], we focus on 0-extension. But for our proof of Theorem 1.2, it is useful to consider a hypergraph as contained in a maximal Cohen–Macaulay hypergraph of deviation 2. From this point of view, we have the following theorem:

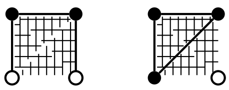
**Theorem 3.1** ([8, Theorem 4.9])**.** *Let $\mathcal{H}$ be a Cohen–Macaulay hypergraph of deviation 2 without isolated vertices. Then $\mathcal{H}$ satisfies one of (A) and (B) :*

(A) *$\mathcal{H}$ is a disjoint union of two Cohen–Macaulay hypergraphs of deviation 1.*
(B) *$\mathcal{H}$ is contained in one of the following maximal Cohen–Macaulay hypergraphs of deviation 2:*

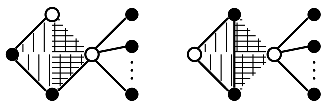
$(H_1 12)^{+1}_C$ $(H_1 12)^{+2}_E$ $(H_1 12)^{+3}_F$ $(H_1 12)^{+1}_G$ $(H_1 12)^{+1}_H$ $(H_1 12)^{+1}_I$



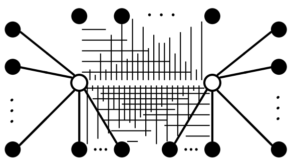
$(H_2 9)^{+1}_E$ $(H_2 9)^{+1}_F$



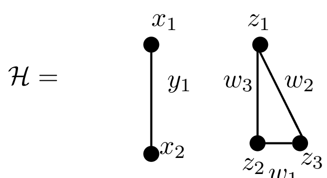
$(H_3 6)^{+1}_D$ $(H_3 6)^{+1}_E$



$(H_4 3)_C$



In the rest of the paper, we prove the main theorem. We first prove the case (A). It suffices to show that $I_{\mathcal{H}}$ is clean for the following hypergraph $\mathcal{H}$:



Let us show that $I = I_{\mathcal{H}} = (x_1y_1, x_2y_1, z_1w_2w_3, z_2w_1w_3, z_3w_1w_2)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (x_1, y_1) \cap (x_2, y_1) \cap (z_1, w_2, w_3) \cap (z_2, w_1, w_3) \cap (z_3, w_1, w_2) \\ &= (y_1w_1w_2, y_1w_1w_3, y_1w_2w_3, y_1z_1w_1, y_1z_2w_2, y_1z_3w_3, y_1z_1z_2z_3, \\ &\quad x_1x_2w_1w_2, x_1x_2w_1w_3, x_1x_2w_2w_3, x_1x_2z_1w_1, x_1x_2z_2w_2, x_1x_2z_3w_3, x_1x_2z_1z_2z_3). \end{aligned}$$

Set $M_1 = y_1w_1w_2$, $M_2 = y_1w_1w_3$, $M_3 = y_1w_2w_3$, $\ldots$, and $M_{14} = x_1x_2z_1z_2z_3$. Then we have

- $(M_1) : M_2 = (w_2)$.
- $(M_1) : M_3 = (w_1)$.
- $(M_1) : M_4 = (w_2)$, $(M_2) : M_4 = (w_3)$.
- $(M_1) : M_5 = (w_1)$, $(M_3) : M_5 = (w_3)$.
- $(M_2) : M_6 = (w_1)$, $(M_3) : M_6 = (w_2)$.
- $(M_4) : M_7 = (w_1)$, $(M_5) : M_7 = (w_2)$, $(M_6) : M_7 = (w_3)$.
- $(M_1) : M_8 = (y_1)$.
- $(M_2) : M_9 = (y_1)$, $(M_8) : M_9 = (w_2)$.
- $(M_3) : M_{10} = (y_1)$, $(M_8) : M_{10} = (w_1)$.
- $(M_4) : M_{11} = (y_1)$, $(M_8) : M_{11} = (w_2)$, $(M_9) : M_{11} = (w_3)$.
- $(M_5) : M_{12} = (y_1)$, $(M_8) : M_{12} = (w_1)$, $(M_{10}) : M_{12} = (w_3)$.
- $(M_6) : M_{13} = (y_1)$, $(M_9) : M_{13} = (w_1)$, $(M_{10}) : M_{13} = (w_2)$.
- $(M_7) : M_{14} = (y_1)$, $(M_{11}) : M_{14} = (w_1)$, $(M_{12}) : M_{14} = (w_2)$, $(M_{13}) : M_{14} = (w_3)$.

Since $(M_2)\colon M_3 \subseteq (w_1)$, we have

$$(M_1, M_2)\colon M_3 = (M_1)\colon M_3 + (M_2)\colon M_3 = (w_1).$$

Moreover, since $(M_3)\colon M_4 = (w_2w_3) \subseteq (w_2)$, we have

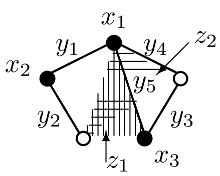
$$(M_1, M_2, M_3)\colon M_4 = (M_1)\colon M_4 + (M_2)\colon M_4 + (M_3)\colon M_4 = (w_2, w_3).$$

Similarly, we can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 14$. Thus $I^\vee$ has linear quotients.

When $\mathcal{H} = (H_29)^{+1}_E$ or $(H_29)^{+1}_F$, since $I_{\mathcal{H}}$ has height 2, the cleanness of $I_{\mathcal{H}}$ follows from Proposition 2.4. Hence it is enough to show that $I_{\mathcal{H}}$ is clean in the cases of $\mathcal{H} = (H_112)^{+1}_C$, $(H_112)^{+2}_E$, $(H_112)^{+3}_F$, $(H_112)^{+1}_G$, $(H_112)^{+1}_H$, $(H_112)^{+1}_I$, $(H_36)^{+1}_D$, $(H_36)^{+1}_E$, and $(H_43)_C$.

### 3.1. **The case where $\mathcal{H}$ is a subgraph of $(H_112)^{+1}_C$.**

$(H_112)^{+1}_C$




Let us show that $I = I_{\mathcal{H}} = (x_1y_1y_4y_5z_1z_2,\ x_2y_1y_2,\ y_2z_1z_2,\ x_3y_3y_5z_1,\ y_3y_4z_2)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (x_1, y_1, y_4, y_5, z_1, z_2) \cap (x_2, y_1, y_2) \cap (y_2, z_1, z_2) \cap (x_3, y_3, y_5, z_1) \cap (y_3, y_4, z_2) \\ &= (y_1y_2y_3, y_2y_3y_4, x_3y_2y_4, y_2y_4z_1, y_2y_3z_1, y_1y_3z_1, x_2y_3z_1, x_2y_4z_1, y_1y_4z_1, y_1z_1z_2, y_1y_3z_2, y_2y_3z_2, \\ &\quad x_2y_3z_2, x_2z_1z_2, y_2z_1z_2, x_3y_2z_2, x_3y_1z_2, x_2x_3z_2, x_1y_2y_3, y_2y_3y_5, y_2y_4y_5, y_2y_5z_2, y_1y_5z_2, x_2y_5z_2). \end{aligned}$$

Set $M_1 = y_1y_2y_3$, $M_2 = y_2y_3y_4$, $M_3 = x_3y_2y_4$, $\ldots$, and $M_{24} = x_2y_5z_2$. Then we have

- $(M_1) : M_2 = (y_1)$.
- $(M_2) : M_3 = (y_3)$.
- $(M_2) : M_4 = (y_3)$, $(M_3) : M_4 = (x_3)$.
- $(M_1) : M_5 = (y_1)$, $(M_4) : M_5 = (y_4)$.
- $(M_5) : M_6 = (y_2)$.

- $(M_5) : M_7 = (y_2)$, $(M_6) : M_7 = (y_1)$.
- $(M_4) : M_8 = (y_2)$, $(M_7) : M_8 = (y_3)$.
- $(M_4) : M_9 = (y_2)$, $(M_6) : M_9 = (y_3)$, $(M_8) : M_9 = (x_2)$.
- $(M_6) : M_{10} = (y_3)$, $(M_9) : M_{10} = (y_4)$.
- $(M_1) : M_{11} = (y_2)$, $(M_{10}) : M_{11} = (z_1)$.
- $(M_2) : M_{12} = (y_4)$, $(M_5) : M_{12} = (z_1)$, $(M_{11}) : M_{12} = (y_1)$.
- $(M_7) : M_{13} = (z_1)$, $(M_{11}) : M_{13} = (y_1)$, $(M_{12}) : M_{13} = (y_2)$.
- $(M_8) : M_{14} = (y_4)$, $(M_{10}) : M_{14} = (y_1)$, $(M_{13}) : M_{14} = (y_3)$.
- $(M_4) : M_{15} = (y_4)$, $(M_{10}) : M_{15} = (y_1)$, $(M_{12}) : M_{15} = (y_3)$, $(M_{14}) : M_{15} = (x_2)$.
- $(M_3) : M_{16} = (y_4)$, $(M_{12}) : M_{16} = (y_3)$, $(M_{15}) : M_{16} = (z_1)$.
- $(M_{10}) : M_{17} = (z_1)$, $(M_{11}) : M_{17} = (y_3)$, $(M_{16}) : M_{17} = (y_2)$.
- $(M_{13}) : M_{18} = (y_3)$, $(M_{14}) : M_{18} = (z_1)$, $(M_{16}) : M_{18} = (y_2)$, $(M_{17}) : M_{18} = (y_1)$.
- $(M_1) : M_{19} = (y_1)$, $(M_2) : M_{19} = (y_4)$, $(M_5) : M_{19} = (z_1)$, $(M_{12}) : M_{19} = (z_2)$.
- $(M_1) : M_{20} = (y_1)$, $(M_2) : M_{20} = (y_4)$, $(M_5) : M_{20} = (z_1)$, $(M_{12}) : M_{20} = (z_2)$, $(M_{19}) : M_{20} = (x_1)$.
- $(M_3) : M_{21} = (x_3)$, $(M_4) : M_{21} = (z_1)$, $(M_{20}) : M_{21} = (y_3)$.
- $(M_{15}) : M_{22} = (z_1)$, $(M_{16}) : M_{22} = (x_3)$, $(M_{20}) : M_{22} = (y_3)$, $(M_{21}) : M_{22} = (y_4)$.
- $(M_{10}) : M_{23} = (z_1)$, $(M_{11}) : M_{23} = (y_3)$, $(M_{17}) : M_{23} = (x_3)$, $(M_{22}) : M_{23} = (y_2)$.
- $(M_{13}) : M_{24} = (y_3)$, $(M_{14}) : M_{24} = (z_1)$, $(M_{18}) : M_{24} = (x_3)$, $(M_{22}) : M_{24} = (y_2)$, $(M_{23}) : M_{24} = (y_1)$.

Since $(M_1)\colon M_3 = (y_1y_3) \subseteq (y_3)$, we have

$$(M_1, M_2)\colon M_3 = (M_1)\colon M_3 + (M_2)\colon M_3 = (y_3).$$

Moreover, since $(M_1)\colon M_4 = (y_1y_3) \subseteq (y_3)$, we have

$$(M_1, M_2, M_3)\colon M_4 = (M_1)\colon M_4 + (M_2)\colon M_4 + (M_3)\colon M_4 = (x_3, y_3).$$

Similarly, we can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 24$. Thus $I^\vee$ has linear quotients.

For example, we consider the case of $\mathcal{H}=(H_1 12)_B^{+3}$.

**Example 3.2.** Let $\mathcal{H}=(H_1 12)_B^{+3}$ and put

$$I = I_{\mathcal{H}} = (x_1y_1y_4z_1z_2,\ x_2y_1y_2,\ y_2z_1z_2, x_3y_3z_1,\ y_3y_4z_2).$$

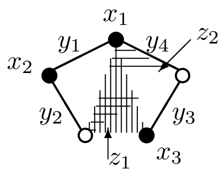
Set $T = S[y_5^{-1}]$ and

$$J = (x_1y_1y_4y_5z_1z_2,\ x_2y_1y_2,\ y_2z_1z_2, x_3y_3y_5z_1,\ y_3y_4z_2) = I_{(H_1 12)_C^{+1}}.$$

Then $IT = JT$. Since $J$ is a clean ideal of $S$, we can choose a finite chain of monomial ideals $J = I_0 \subset I_1 \subset \cdots \subset I_r = S$ such that $I_i/I_{i-1} \cong S/P_i$, where $P_i$ is a monomial prime ideal. Then $IT = JT = I_0T \subset I_1T \subset \cdots \subset I_rT = T$ and $I_iT/I_{i-1}T \cong T/P_iT$. Notice that $P_iT$ is a monomial prime ideal if $P_iT \neq T$. This implies that $IT$ is clean and so is $I$.

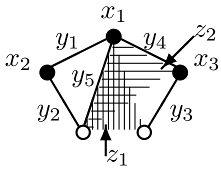
On the other hand, the cleanness of $I = (x_1y_1y_4y_5z_1z_2,\ x_2y_1y_2,\ y_2z_1z_2, x_3y_{31}y_{32}y_5z_1,\ y_{31}y_{32}y_4z_2)$ follows from that of $J$ and [4, Proposition 3.3].

$(H_1 12)_B^{+3}$



3.2. **The case where $\mathcal{H}$ is a subgraph of $(H_1 12)^{+2}_E$.**

$(H_1 12)^{+2}_E$




Let us show that $I = I_{\mathcal{H}} = (x_1y_1y_4y_5z_1z_2,\ x_2y_1y_2,\ y_2y_5z_1z_2,\ y_3z_1,\ x_3y_3y_4z_2)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (x_1,y_1,y_4,y_5,z_1,z_2)\cap(x_2,y_1,y_2)\cap(y_2,y_5,z_1,z_2)\cap(y_3,z_1)\cap(x_3,y_3,y_4,z_2)\\ &= (y_1y_2y_3, x_1y_2y_3, y_2y_3z_1, y_1y_3z_1, x_3y_1z_1, x_3y_2z_1, x_2x_3z_1, x_2y_3z_1, x_2y_4z_1, y_1y_4z_1, y_2y_4z_1, y_2y_3y_4,\\ &\quad y_2y_3y_5, x_2y_3y_5, y_1y_3y_5, y_1y_3z_2, y_2y_3z_2, x_2y_3z_2, x_2z_1z_2, y_1z_1z_2, y_2z_1z_2). \end{aligned}$$

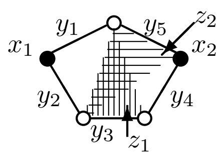
Set $M_1 = y_1y_2y_3$, $M_2 = x_1y_2y_3$, $M_3 = y_2y_3z_1$, $\ldots$, and $M_{21} = y_2z_1z_2$. Then we have

- $(M_1) : M_2 = (y_1)$.
- $(M_1) : M_3 = (y_1)$, $(M_2) : M_3 = (x_1)$.
- $(M_1) : M_4 = (y_2)$.
- $(M_4) : M_5 = (y_3)$.
- $(M_3) : M_6 = (y_3)$, $(M_5) : M_6 = (y_1)$.
- $(M_5) : M_7 = (y_1)$, $(M_6) : M_7 = (y_2)$.
- $(M_3) : M_8 = (y_2)$, $(M_4) : M_8 = (y_1)$, $(M_7) : M_8 = (x_3)$.
- $(M_7) : M_9 = (x_3)$, $(M_8) : M_9 = (y_3)$.
- $(M_4) : M_{10} = (y_3)$, $(M_5) : M_{10} = (x_3)$, $(M_9) : M_{10} = (x_2)$.
- $(M_3) : M_{11} = (y_3)$, $(M_6) : M_{11} = (x_3)$, $(M_9) : M_{11} = (x_2)$, $(M_{10}) : M_{11} = (y_1)$.
- $(M_1) : M_{12} = (y_1)$, $(M_2) : M_{12} = (x_1)$, $(M_{11}) : M_{12} = (z_1)$.
- $(M_1) : M_{13} = (y_1)$, $(M_2) : M_{13} = (x_1)$, $(M_3) : M_{13} = (z_1)$, $(M_{12}) : M_{13} = (y_4)$.
- $(M_8) : M_{14} = (z_1)$, $(M_{13}) : M_{14} = (y_2)$.
- $(M_4) : M_{15} = (z_1)$, $(M_{13}) : M_{15} = (y_2)$, $(M_{14}) : M_{15} = (x_2)$.
- $(M_1) : M_{16} = (y_2)$, $(M_4) : M_{16} = (z_1)$, $(M_{15}) : M_{16} = (y_5)$.
- $(M_2) : M_{17} = (x_1)$, $(M_3) : M_{17} = (z_1)$, $(M_{12}) : M_{17} = (y_4)$, $(M_{13}) : M_{17} = (y_5)$, $(M_{16}) : M_{17} = (y_1)$.
- $(M_8) : M_{18} = (z_1)$, $(M_{14}) : M_{18} = (y_5)$, $(M_{16}) : M_{18} = (y_1)$, $(M_{17}) : M_{18} = (y_2)$.
- $(M_7) : M_{19} = (x_3)$, $(M_9) : M_{19} = (y_4)$, $(M_{18}) : M_{19} = (y_3)$.
- $(M_5) : M_{20} = (x_3)$, $(M_{10}) : M_{20} = (y_4)$, $(M_{16}) : M_{20} = (y_3)$, $(M_{19}) : M_{20} = (x_2)$.
- $(M_6) : M_{21} = (x_3)$, $(M_{11}) : M_{21} = (y_4)$, $(M_{17}) : M_{21} = (y_3)$, $(M_{19}) : M_{21} = (x_2)$, $(M_{20}) : M_{21} = (y_1)$.

We can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 21$. Thus $I^\vee$ has linear quotients.

3.3. **The case where $\mathcal{H}$ is a subgraph of $(H_1 12)^{+3}_F$.**

$(H_1 12)^{+3}_F$



Let us show that $I = I_{\mathcal{H}} = (y_1y_5z_1z_2,\ x_1y_1y_2,\ y_2y_3z_1z_2,\ y_3y_4z_1,\ x_2y_4y_5z_2)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (y_1,y_5,z_1,z_2)\cap(x_1,y_1,y_2)\cap(y_2,y_3,z_1,z_2)\cap(y_3,y_4,z_1)\cap(x_2,y_4,y_5,z_2)\\ &= (y_1y_2y_4, y_1y_3y_4, y_1y_3y_5, y_2y_3y_5, y_2y_4y_5, x_1y_3y_5, x_2y_1y_3, y_1y_4z_1, y_1y_5z_1, y_2y_5z_1, y_2y_4z_1, x_1y_5z_1,\\ &\quad x_1y_4z_1, x_2y_1z_1, x_2y_2z_1, x_1x_2z_1, y_2z_1z_2, y_1z_1z_2, y_1y_3z_2, y_1y_4z_2, y_2y_4z_2, y_2y_3z_2, x_1z_1z_2, x_1y_3z_2, x_1y_4z_2). \end{aligned}$$

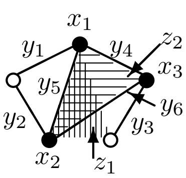
Set $M_1 = y_1y_2y_4$, $M_2 = y_1y_3y_4$, $M_3 = y_1y_3y_5$, …, and $M_{25} = x_1y_4z_2$. Then we have

- $(M_1) : M_2 = (y_2)$.
- $(M_2) : M_3 = (y_4)$.
- $(M_3) : M_4 = (y_1)$.
- $(M_1) : M_5 = (y_1)$, $(M_4) : M_5 = (y_3)$.
- $(M_3) : M_6 = (y_1)$, $(M_4) : M_6 = (y_2)$.
- $(M_2) : M_7 = (y_4)$, $(M_3) : M_7 = (y_5)$.
- $(M_1) : M_8 = (y_2)$, $(M_2) : M_8 = (y_3)$.
- $(M_3) : M_9 = (y_3)$, $(M_8) : M_9 = (y_4)$.
- $(M_4) : M_{10} = (y_3)$, $(M_5) : M_{10} = (y_4)$, $(M_9) : M_{10} = (y_1)$.
- $(M_8) : M_{11} = (y_1)$, $(M_{10}) : M_{11} = (y_5)$.
- $(M_6) : M_{12} = (y_3)$, $(M_9) : M_{12} = (y_1)$, $(M_{10}) : M_{12} = (y_2)$.
- $(M_8) : M_{13} = (y_1)$, $(M_{11}) : M_{13} = (y_2)$, $(M_{12}) : M_{13} = (y_5)$.
- $(M_7) : M_{14} = (y_3)$, $(M_8) : M_{14} = (y_4)$, $(M_9) : M_{14} = (y_5)$.
- $(M_{10}) : M_{15} = (y_5)$, $(M_{11}) : M_{15} = (y_4)$, $(M_{14}) : M_{15} = (y_1)$.
- $(M_{12}) : M_{16} = (y_5)$, $(M_{13}) : M_{16} = (y_4)$, $(M_{14}) : M_{16} = (y_1)$, $(M_{15}) : M_{16} = (y_2)$.
- $(M_{10}) : M_{17} = (y_5)$, $(M_{11}) : M_{17} = (y_4)$, $(M_{15}) : M_{17} = (x_2)$.
- $(M_8) : M_{18} = (y_4)$, $(M_9) : M_{18} = (y_5)$, $(M_{14}) : M_{18} = (x_2)$, $(M_{17}) : M_{18} = (y_2)$.
- $(M_2) : M_{19} = (y_4)$, $(M_3) : M_{19} = (y_5)$, $(M_7) : M_{19} = (x_2)$, $(M_{18}) : M_{19} = (z_1)$.
- $(M_1) : M_{20} = (y_2)$, $(M_{18}) : M_{20} = (z_1)$, $(M_{19}) : M_{20} = (y_3)$.
- $(M_5) : M_{21} = (y_5)$, $(M_{17}) : M_{21} = (z_1)$, $(M_{20}) : M_{21} = (y_1)$.
- $(M_4) : M_{22} = (y_5)$, $(M_{17}) : M_{22} = (z_1)$, $(M_{19}) : M_{22} = (y_1)$, $(M_{21}) : M_{22} = (y_4)$.
- $(M_{12}) : M_{23} = (y_5)$, $(M_{13}) : M_{23} = (y_4)$, $(M_{16}) : M_{23} = (x_2)$, $(M_{17}) : M_{23} = (y_2)$, $(M_{18}) : M_{23} = (y_1)$.
- $(M_6) : M_{24} = (y_5)$, $(M_{19}) : M_{24} = (y_1)$, $(M_{22}) : M_{24} = (y_2)$, $(M_{23}) : M_{24} = (z_1)$.
- $(M_{20}) : M_{25} = (y_1)$, $(M_{21}) : M_{25} = (y_2)$, $(M_{23}) : M_{25} = (z_1)$, $(M_{24}) : M_{25} = (y_3)$.

We can show that $(M_1, M_2, \ldots, M_{i-1}) \colon M_i$ is generated by variables for each $i = 2, \ldots, 25$. Thus $I^\vee$ has linear quotients.

### 3.4. **The case where $\mathcal{H}$ is a subgraph of $(H_1 12)^{+1}_G$.**

$(H_1 12)^{+1}_G$




Let us show that $I = I_{\mathcal{H}} = (x_1y_1y_4y_5z_1z_2,\ y_1y_2,\ x_2y_2y_5y_6z_1z_2,\ y_3z_1,\ x_3y_3y_4y_6z_2)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (x_1,y_1,y_4,y_5,z_1,z_2)\cap(y_1,y_2)\cap(x_2,y_2,y_5,y_6,z_1,z_2)\cap(y_3,z_1)\cap(x_3,y_3,y_4,y_6,z_2)\\ &= (y_1y_2y_3, x_1y_2y_3, x_2y_1y_3, y_1y_3z_1, y_2y_3z_1, x_3y_2z_1, x_3y_1z_1, y_1z_1z_2, y_2z_1z_2, y_2y_3z_2, y_1y_3z_2, y_1y_3y_5,\\ &\quad y_2y_3y_5, y_2y_3y_4, y_2y_4z_1, y_1y_4z_1, y_1y_6z_1, y_2y_6z_1, y_1y_3y_6). \end{aligned}$$

Set $M_1 = y_1y_2y_3$, $M_2 = x_1y_2y_3$, $M_3 = x_2y_1y_3$, …, and $M_{19} = y_1y_3y_6$. Then we have

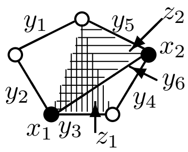
- $(M_1) : M_2 = (y_1)$.
- $(M_1) : M_3 = (y_2)$.
- $(M_1) : M_4 = (y_2)$, $(M_3) : M_4 = (x_2)$.

- $(M_2) : M_5 = (x_1)$, $(M_4) : M_5 = (y_1)$.
- $(M_5) : M_6 = (y_3)$.
- $(M_4) : M_7 = (y_3)$, $(M_6) : M_7 = (y_2)$.
- $(M_4) : M_8 = (y_3)$, $(M_7) : M_8 = (x_3)$.
- $(M_5) : M_9 = (y_3)$, $(M_6) : M_9 = (x_3)$, $(M_8) : M_9 = (y_1)$.
- $(M_1) : M_{10} = (y_1)$, $(M_2) : M_{10} = (x_1)$, $(M_9) : M_{10} = (z_1)$.
- $(M_3) : M_{11} = (x_2)$, $(M_8) : M_{11} = (z_1)$, $(M_{10}) : M_{11} = (y_2)$.
- $(M_1) : M_{12} = (y_2)$, $(M_3) : M_{12} = (x_2)$, $(M_4) : M_{12} = (z_1)$, $(M_{11}) : M_{12} = (z_2)$.
- $(M_2) : M_{13} = (x_1)$, $(M_5) : M_{13} = (z_1)$, $(M_{10}) : M_{13} = (z_2)$, $(M_{12}) : M_{13} = (y_1)$.
- $(M_1) : M_{14} = (y_1)$, $(M_2) : M_{14} = (x_1)$, $(M_5) : M_{14} = (z_1)$, $(M_{10}) : M_{14} = (z_2)$, $(M_{13}) : M_{14} = (y_5)$.
- $(M_6) : M_{15} = (x_3)$, $(M_9) : M_{15} = (z_2)$, $(M_{14}) : M_{15} = (y_3)$.
- $(M_4) : M_{16} = (y_3)$, $(M_7) : M_{16} = (x_3)$, $(M_8) : M_{16} = (z_2)$, $(M_{15}) : M_{16} = (y_2)$.
- $(M_4) : M_{17} = (y_3)$, $(M_7) : M_{17} = (x_3)$, $(M_8) : M_{17} = (z_2)$, $(M_{16}) : M_{17} = (y_4)$.
- $(M_5) : M_{18} = (y_3)$, $(M_6) : M_{18} = (x_3)$, $(M_9) : M_{18} = (z_2)$, $(M_{15}) : M_{18} = (y_4)$, $(M_{17}) : M_{18} = (y_1)$.
- $(M_1) : M_{19} = (y_2)$, $(M_3) : M_{19} = (x_2)$, $(M_{11}) : M_{19} = (z_2)$, $(M_{12}) : M_{19} = (y_5)$, $(M_{17}) : M_{19} = (z_1)$.

We can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 19$. Thus $I^\vee$ has linear quotients.

### 3.5. **The case where $\mathcal{H}$ is a subgraph of $(H_1 12)^{+1}_H$.**

$(H_1 12)^{+1}_H$




Let us show that $I = I_{\mathcal{H}} = (y_1y_5z_1z_2,\ y_1y_2,\ x_1y_2y_3y_6z_1z_2,\ y_3y_4z_1,\ x_2y_4y_5y_6z_2)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (y_1, y_5, z_1, z_2) \cap (y_1, y_2) \cap (x_1, y_2, y_3, y_6, z_1, z_2) \cap (y_3, y_4, z_1) \cap (x_2, y_4, y_5, y_6, z_2) \\ &= (y_1y_3y_4, y_1y_3y_5, y_2y_3y_5, y_2y_4y_5, y_1y_2y_4, y_1y_4z_1, y_2y_4z_1, y_2y_5z_1, y_1y_5z_1, x_2y_1z_1, x_2y_2z_1, \\ &\quad\ x_2y_1y_3, x_1y_1y_4, y_1y_4y_6, y_1y_3y_6, y_1y_6z_1, y_2y_6z_1, y_2y_4z_2, y_1y_4z_2, y_1y_3z_2, y_1z_1z_2, y_2z_1z_2, y_2y_3z_2). \end{aligned}$$

Set $M_1 = y_1y_3y_4$, $M_2 = y_1y_3y_5$, $M_3 = y_2y_3y_5$, …, and $M_{23} = y_2y_3z_2$. Then we have

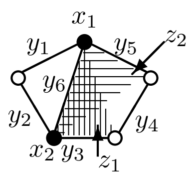
- $(M_1) : M_2 = (y_4)$.
- $(M_2) : M_3 = (y_1)$.
- $(M_3) : M_4 = (y_3)$.
- $(M_1) : M_5 = (y_3)$, $(M_4) : M_5 = (y_5)$.
- $(M_1) : M_6 = (y_3)$, $(M_5) : M_6 = (y_2)$.
- $(M_4) : M_7 = (y_5)$, $(M_6) : M_7 = (y_1)$.
- $(M_3) : M_8 = (y_3)$, $(M_7) : M_8 = (y_4)$.
- $(M_2) : M_9 = (y_3)$, $(M_6) : M_9 = (y_4)$, $(M_8) : M_9 = (y_2)$.
- $(M_6) : M_{10} = (y_4)$, $(M_9) : M_{10} = (y_5)$.
- $(M_7) : M_{11} = (y_4)$, $(M_8) : M_{11} = (y_5)$, $(M_{10}) : M_{11} = (y_1)$.
- $(M_1) : M_{12} = (y_4)$, $(M_2) : M_{12} = (y_5)$, $(M_{10}) : M_{12} = (z_1)$.
- $(M_1) : M_{13} = (y_3)$, $(M_5) : M_{13} = (y_2)$, $(M_6) : M_{13} = (z_1)$.
- $(M_1) : M_{14} = (y_3)$, $(M_5) : M_{14} = (y_2)$, $(M_6) : M_{14} = (z_1)$, $(M_{13}) : M_{14} = (x_1)$.
- $(M_2) : M_{15} = (y_5)$, $(M_{12}) : M_{15} = (x_2)$, $(M_{14}) : M_{15} = (y_4)$.
- $(M_9) : M_{16} = (y_5)$, $(M_{10}) : M_{16} = (x_2)$, $(M_{14}) : M_{16} = (y_4)$, $(M_{15}) : M_{16} = (y_3)$.
- $(M_7) : M_{17} = (y_4)$, $(M_8) : M_{17} = (y_5)$, $(M_{11}) : M_{17} = (x_2)$, $(M_{16}) : M_{17} = (y_1)$.

- $(M_4) : M_{18} = (y_5)$, $(M_5) : M_{18} = (y_1)$, $(M_7) : M_{18} = (z_1)$.
- $(M_1) : M_{19} = (y_3)$, $(M_6) : M_{19} = (z_1)$, $(M_{13}) : M_{19} = (x_1)$, $(M_{14}) : M_{19} = (y_6)$, $(M_{18}) : M_{19} = (y_2)$.
- $(M_2) : M_{20} = (y_5)$, $(M_{12}) : M_{20} = (x_2)$, $(M_{15}) : M_{20} = (y_6)$, $(M_{19}) : M_{20} = (y_4)$.
- $(M_9) : M_{21} = (y_5)$, $(M_{10}) : M_{21} = (x_2)$, $(M_{16}) : M_{21} = (y_6)$, $(M_{19}) : M_{21} = (y_4)$, $(M_{20}) : M_{21} = (y_3)$.
- $(M_8) : M_{22} = (y_5)$, $(M_{11}) : M_{22} = (x_2)$, $(M_{17}) : M_{22} = (y_6)$, $(M_{18}) : M_{22} = (y_4)$, $(M_{21}) : M_{22} = (y_1)$.
- $(M_3) : M_{23} = (y_5)$, $(M_{18}) : M_{23} = (y_4)$, $(M_{20}) : M_{23} = (y_1)$, $(M_{22}) : M_{23} = (z_1)$.

We can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 23$. Thus $I^\vee$ has linear quotients.

### 3.6. The case where $\mathcal{H}$ is a subgraph of $(H_1 12)_I^{+1}$.

$(H_1 12)_I^{+1}$




Let us show that $I = I_{\mathcal{H}} = (x_1y_1y_5y_6z_1z_2,\ y_1y_2,\ x_2y_2y_3y_6z_1z_2,\ y_3y_4z_1,\ y_4y_5z_2)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (x_1, y_1, y_5, y_6, z_1, z_2) \cap (y_1, y_2) \cap (x_2, y_2, y_3, y_6, z_1, z_2) \cap (y_3, y_4, z_1) \cap y_4, y_5, z_2) \\ &= (y_1y_2y_4, y_1y_3y_4, y_1y_3y_5, y_2y_3y_5, y_2y_4y_5, x_1y_2y_4, x_2y_1y_4, y_1y_4y_6, y_2y_4y_6, y_2y_4z_1, y_1y_4z_1, \\ &\quad y_1y_5z_1, y_2y_5z_1, y_2z_1z_2, y_1z_1z_2, y_1y_4z_2, y_2y_4z_2, y_2y_3z_2, y_1y_3z_2). \end{aligned}$$

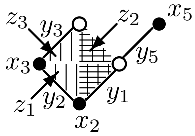
Set $M_1 = y_1y_2y_4$, $M_2 = y_1y_3y_4$, $M_3 = y_1y_3y_5$, $\ldots$, and $M_{19} = y_1y_3z_2$. Then we have

- $(M_1) : M_2 = (y_2)$.
- $(M_2) : M_3 = (y_4)$.
- $(M_3) : M_4 = (y_1)$.
- $(M_1) : M_5 = (y_1)$, $(M_4) : M_5 = (y_3)$.
- $(M_1) : M_6 = (y_1)$, $(M_5) : M_6 = (y_5)$.
- $(M_1) : M_7 = (y_2)$, $(M_2) : M_7 = (y_3)$.
- $(M_1) : M_8 = (y_2)$, $(M_2) : M_8 = (y_3)$, $(M_7) : M_8 = (x_2)$.
- $(M_5) : M_9 = (y_5)$, $(M_6) : M_9 = (x_1)$, $(M_8) : M_9 = (y_1)$.
- $(M_1) : M_{10} = (y_1)$, $(M_5) : M_{10} = (y_5)$, $(M_6) : M_{10} = (x_1)$, $(M_9) : M_{10} = (y_6)$.
- $(M_2) : M_{11} = (y_3)$, $(M_7) : M_{11} = (x_2)$, $(M_8) : M_{11} = (y_6)$, $(M_{10}) : M_{11} = (y_2)$.
- $(M_3) : M_{12} = (y_3)$, $(M_{11}) : M_{12} = (y_4)$.
- $(M_4) : M_{13} = (y_3)$, $(M_{10}) : M_{13} = (y_4)$, $(M_{12}) : M_{13} = (y_1)$.
- $(M_{10}) : M_{14} = (y_4)$, $(M_{13}) : M_{14} = (y_5)$.
- $(M_{11}) : M_{15} = (y_4)$, $(M_{12}) : M_{15} = (y_5)$, $(M_{14}) : M_{15} = (y_2)$.
- $(M_1) : M_{16} = (y_2)$, $(M_2) : M_{16} = (y_3)$, $(M_7) : M_{16} = (x_2)$, $(M_8) : M_{16} = (y_6)$, $(M_{15}) : M_{16} = (z_1)$.
- $(M_5) : M_{17} = (y_5)$, $(M_6) : M_{17} = (x_1)$, $(M_9) : M_{17} = (y_6)$, $(M_{14}) : M_{17} = (z_1)$, $(M_{16}) : M_{17} = (y_1)$.
- $(M_4) : M_{18} = (y_5)$, $(M_{14}) : M_{18} = (z_1)$, $(M_{17}) : M_{18} = (y_4)$.
- $(M_3) : M_{19} = (y_5)$, $(M_{15}) : M_{19} = (z_1)$, $(M_{16}) : M_{19} = (y_4)$, $(M_{18}) : M_{19} = (y_2)$.

We can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 19$. Thus $I^\vee$ has linear quotients.

### 3.7. **The case where $\mathcal{H}$ is a subgraph of $(H_36)^{+1}_D$.**

$(H_36)^{+1}_D$




Let us show that $I = I_{\mathcal{H}} = (y_1y_5z_1z_2z_3,\ x_2y_1y_2z_1z_2,\ x_3y_2y_3z_1z_3,\ y_3z_2z_3,\ x_5y_5)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (y_1,y_5,z_1,z_2,z_3)\cap(x_2,y_1,y_2,z_1,z_2)\cap(x_3,y_2,y_3,z_1,z_3)\cap(y_3,z_2,z_3)\cap(x_5,y_5) \\ &= (x_2y_3y_5, y_3y_5z_2, x_5y_3z_2, x_3x_5z_2, x_3y_5z_2, y_5z_1z_2, x_5z_1z_2, y_3y_5z_1, x_5y_3z_1, y_1y_3y_5, x_5y_1y_3, \\ &\quad x_5z_2z_3, y_5z_2z_3, y_5z_1z_3, x_5z_1z_3, x_5y_1z_3, y_1y_5z_3, x_2y_5z_3, x_2x_5z_3, x_5y_2z_3, y_2y_5z_3, y_2y_3y_5, y_2y_5z_2, x_5y_2z_2). \end{aligned}$$

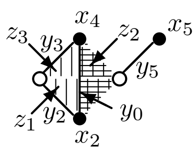
Set $M_1 = x_2y_3y_5$, $M_2 = y_3y_5z_2$, $M_3 = x_5y_3z_2$, $\ldots$, and $M_{24} = x_5y_2z_2$. Then we have

- $(M_1) : M_2 = (x_2)$.
- $(M_2) : M_3 = (y_5)$.
- $(M_3) : M_4 = (y_3)$.
- $(M_2) : M_5 = (y_3)$, $(M_4) : M_5 = (x_5)$.
- $(M_2) : M_6 = (y_3)$, $(M_5) : M_6 = (x_3)$.
- $(M_3) : M_7 = (y_3)$, $(M_4) : M_7 = (x_3)$, $(M_6) : M_7 = (y_5)$.
- $(M_1) : M_8 = (x_2)$, $(M_6) : M_8 = (z_2)$.
- $(M_7) : M_9 = (z_2)$, $(M_8) : M_9 = (y_5)$.
- $(M_1) : M_{10} = (x_2)$, $(M_2) : M_{10} = (z_2)$, $(M_8) : M_{10} = (z_1)$.
- $(M_3) : M_{11} = (z_2)$, $(M_9) : M_{11} = (z_1)$, $(M_{10}) : M_{11} = (y_5)$.
- $(M_3) : M_{12} = (y_3)$, $(M_4) : M_{12} = (x_3)$, $(M_7) : M_{12} = (z_1)$.
- $(M_2) : M_{13} = (y_3)$, $(M_5) : M_{13} = (x_3)$, $(M_6) : M_{13} = (z_1)$, $(M_{12}) : M_{13} = (x_5)$.
- $(M_8) : M_{14} = (y_3)$, $(M_{13}) : M_{14} = (z_2)$.
- $(M_9) : M_{15} = (y_3)$, $(M_{12}) : M_{15} = (z_2)$, $(M_{14}) : M_{15} = (y_5)$.
- $(M_{11}) : M_{16} = (y_3)$, $(M_{12}) : M_{16} = (z_2)$, $(M_{15}) : M_{16} = (z_1)$.
- $(M_{10}) : M_{17} = (y_3)$, $(M_{13}) : M_{17} = (z_2)$, $(M_{14}) : M_{17} = (z_1)$, $(M_{16}) : M_{17} = (x_5)$.
- $(M_1) : M_{18} = (y_3)$, $(M_{13}) : M_{18} = (z_2)$, $(M_{14}) : M_{18} = (z_1)$, $(M_{17}) : M_{18} = (y_1)$.
- $(M_{12}) : M_{19} = (z_2)$, $(M_{15}) : M_{19} = (z_1)$, $(M_{16}) : M_{19} = (y_1)$, $(M_{18}) : M_{19} = (y_5)$.
- $(M_{12}) : M_{20} = (z_2)$, $(M_{15}) : M_{20} = (z_1)$, $(M_{16}) : M_{20} = (y_1)$, $(M_{19}) : M_{20} = (x_2)$.
- $(M_{13}) : M_{21} = (z_2)$, $(M_{14}) : M_{21} = (z_1)$, $(M_{17}) : M_{21} = (y_1)$, $(M_{18}) : M_{21} = (x_2)$, $(M_{20}) : M_{21} = (x_5)$.
- $(M_1) : M_{22} = (x_2)$, $(M_2) : M_{22} = (z_2)$, $(M_8) : M_{22} = (z_1)$, $(M_{10}) : M_{22} = (y_1)$, $(M_{21}) : M_{22} = (z_3)$.
- $(M_5) : M_{23} = (x_3)$, $(M_6) : M_{23} = (z_1)$, $(M_{21}) : M_{23} = (z_3)$, $(M_{22}) : M_{23} = (y_3)$.
- $(M_3) : M_{24} = (y_3)$, $(M_4) : M_{24} = (x_3)$, $(M_7) : M_{24} = (z_1)$, $(M_{20}) : M_{24} = (z_3)$, $(M_{23}) : M_{24} = (y_5)$.

We can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 24$. Thus $I^\vee$ has linear quotients.

### 3.8. **The case where $\mathcal{H}$ is a subgraph of $(H_36)^{+1}_E$.**

$(H_36)^{+1}_E$



Let us show that $I = I_{\mathcal{H}} = (y_5z_1z_2z_3,\ x_2y_0y_2z_1z_2,\ y_2y_3z_1z_3,\ x_4y_0y_3z_2z_3,\ x_5y_5)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (y_5, z_1, z_2, z_3) \cap (x_2, y_0, y_2, z_1, z_2) \cap (y_2, y_3, z_1, z_3) \cap (x_4, y_0, y_3, z_2, z_3) \cap (x_5, y_5) \\ &= (x_4y_2y_5, y_0y_2y_5, y_2y_5z_3, x_5y_2z_3, x_5y_0z_3, y_0y_5z_3, y_0y_3y_5, y_2y_3y_5, x_2y_3y_5, x_2y_5z_3, x_2x_5z_3, x_5z_1z_3, \\ &\quad y_5z_1z_3, y_3y_5z_1, x_5y_3z_1, y_0y_5z_1, x_5y_0z_1, x_4y_5z_1, x_4x_5z_1, x_5z_1z_2, y_5z_1z_2, y_5z_2z_3, x_5z_2z_3, y_3y_5z_2, \\ &\quad x_5y_3z_2, y_2y_5z_2, x_5y_2z_2). \end{aligned}$$

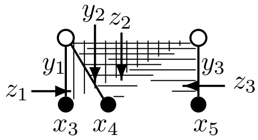
Set $M_1 = x_4y_2y_5$, $M_2 = y_0y_2y_5$, $M_3 = y_2y_5z_3$, …, and $M_{27} = x_5y_2z_2$. Then we have

- $(M_1) : M_2 = (x_4)$.
- $(M_1) : M_3 = (x_4)$, $(M_2) : M_3 = (y_0)$.
- $(M_3) : M_4 = (y_5)$.
- $(M_4) : M_5 = (y_2)$.
- $(M_3) : M_6 = (y_2)$, $(M_5) : M_6 = (x_5)$.
- $(M_2) : M_7 = (y_2)$, $(M_6) : M_7 = (z_3)$.
- $(M_1) : M_8 = (x_4)$, $(M_3) : M_8 = (z_3)$, $(M_7) : M_8 = (y_0)$.
- $(M_7) : M_9 = (y_0)$, $(M_8) : M_9 = (y_2)$.
- $(M_3) : M_{10} = (y_2)$, $(M_6) : M_{10} = (y_0)$, $(M_9) : M_{10} = (y_3)$.
- $(M_4) : M_{11} = (y_2)$, $(M_5) : M_{11} = (y_0)$, $(M_{10}) : M_{11} = (y_5)$.
- $(M_4) : M_{12} = (y_2)$, $(M_5) : M_{12} = (y_0)$, $(M_{11}) : M_{12} = (x_2)$.
- $(M_3) : M_{13} = (y_2)$, $(M_6) : M_{13} = (y_0)$, $(M_{10}) : M_{13} = (x_2)$, $(M_{12}) : M_{13} = (x_5)$.
- $(M_7) : M_{14} = (y_0)$, $(M_8) : M_{14} = (y_2)$, $(M_9) : M_{14} = (x_2)$, $(M_{13}) : M_{14} = (z_3)$.
- $(M_{12}) : M_{15} = (z_3)$, $(M_{14}) : M_{15} = (y_5)$.
- $(M_2) : M_{16} = (y_2)$, $(M_{13}) : M_{16} = (z_3)$, $(M_{14}) : M_{16} = (y_3)$.
- $(M_{12}) : M_{17} = (z_3)$, $(M_{15}) : M_{17} = (y_3)$, $(M_{16}) : M_{17} = (y_5)$.
- $(M_1) : M_{18} = (y_2)$, $(M_{13}) : M_{18} = (z_3)$, $(M_{14}) : M_{18} = (y_3)$, $(M_{16}) : M_{18} = (y_0)$.
- $(M_{12}) : M_{19} = (z_3)$, $(M_{15}) : M_{19} = (y_3)$, $(M_{17}) : M_{19} = (y_0)$, $(M_{18}) : M_{19} = (y_5)$.
- $(M_{12}) : M_{20} = (z_3)$, $(M_{15}) : M_{20} = (y_3)$, $(M_{17}) : M_{20} = (y_0)$, $(M_{19}) : M_{20} = (x_4)$.
- $(M_{13}) : M_{21} = (z_3)$, $(M_{14}) : M_{21} = (y_3)$, $(M_{16}) : M_{21} = (y_0)$, $(M_{18}) : M_{21} = (x_4)$, $(M_{20}) : M_{21} = (x_5)$.
- $(M_3) : M_{22} = (y_2)$, $(M_6) : M_{22} = (y_0)$, $(M_{10}) : M_{22} = (x_2)$, $(M_{21}) : M_{22} = (z_1)$.
- $(M_4) : M_{23} = (y_2)$, $(M_5) : M_{23} = (y_0)$, $(M_{11}) : M_{23} = (x_2)$, $(M_{20}) : M_{23} = (z_1)$, $(M_{22}) : M_{23} = (y_5)$.
- $(M_7) : M_{24} = (y_0)$, $(M_8) : M_{24} = (y_2)$, $(M_9) : M_{24} = (x_2)$, $(M_{21}) : M_{24} = (z_1)$, $(M_{22}) : M_{24} = (z_3)$.
- $(M_{20}) : M_{25} = (z_1)$, $(M_{23}) : M_{25} = (z_3)$, $(M_{24}) : M_{25} = (y_5)$.
- $(M_1) : M_{26} = (x_4)$, $(M_2) : M_{26} = (y_0)$, $(M_{21}) : M_{26} = (z_1)$, $(M_{22}) : M_{26} = (z_3)$, $(M_{24}) : M_{26} = (y_3)$.
- $(M_{20}) : M_{27} = (z_1)$, $(M_{23}) : M_{27} = (z_3)$, $(M_{25}) : M_{27} = (y_3)$, $(M_{26}) : M_{27} = (y_5)$.

We can show that $(M_1, M_2, \dots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \dots, 27$. Thus $I^\vee$ has linear quotients.

3.9. **The case where $\mathcal{H}$ is a subgraph of $(H_43)_C$.**

$(H_43)_C$



Let us show that $I = I_{\mathcal{H}} = (y_1y_2z_1z_2z_3,\ y_3z_1z_2z_3,\ x_3y_1z_1,\ x_4y_2z_2,\ x_5y_3z_3)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$\begin{aligned} I^\vee &= (y_1, y_2, z_1, z_2, z_3) \cap (y_3, z_1, z_2, z_3) \cap (x_3, y_1, z_1) \cap (x_4, y_2, z_2) \cap (x_5, y_3, z_3) \\ &= (x_5y_1z_2, x_3x_5z_2, x_3y_3z_2, y_1y_3z_2, x_4y_1y_3, y_3z_1z_2, x_5z_1z_2, x_4y_3z_1, x_4x_5z_1, x_4z_1z_3, z_1z_2z_3, \\ &\quad\ y_1z_2z_3, x_3z_2z_3, x_4y_1z_3, x_3x_4z_3, x_3y_2z_3, y_1y_2z_3, y_2z_1z_3, x_5y_2z_1, y_2y_3z_1, y_1y_2y_3, x_3y_2y_3). \end{aligned}$$

Set $M_1 = x_5y_1z_2$, $M_2 = x_3x_5z_2$, $M_3 = x_3y_3z_2$, $\ldots$, and $M_{22} = x_3y_2y_3$. Then we have

- $(M_1) : M_2 = (y_1)$.
- $(M_2) : M_3 = (x_5)$.
- $(M_1) : M_4 = (x_5)$, $(M_3) : M_4 = (x_3)$.
- $(M_4) : M_5 = (z_2)$.
- $(M_3) : M_6 = (x_3)$, $(M_4) : M_6 = (y_1)$.
- $(M_1) : M_7 = (y_1)$, $(M_2) : M_7 = (x_3)$, $(M_6) : M_7 = (y_3)$.
- $(M_5) : M_8 = (y_1)$, $(M_6) : M_8 = (z_2)$.
- $(M_7) : M_9 = (z_2)$, $(M_8) : M_9 = (y_3)$.
- $(M_8) : M_{10} = (y_3)$, $(M_9) : M_{10} = (x_5)$.
- $(M_6) : M_{11} = (y_3)$, $(M_7) : M_{11} = (x_5)$, $(M_{10}) : M_{11} = (x_4)$.
- $(M_1) : M_{12} = (x_5)$, $(M_4) : M_{12} = (y_3)$, $(M_{11}) : M_{12} = (z_1)$.
- $(M_2) : M_{13} = (x_5)$, $(M_3) : M_{13} = (y_3)$, $(M_{11}) : M_{13} = (z_1)$, $(M_{12}) : M_{13} = (y_1)$.
- $(M_5) : M_{14} = (y_3)$, $(M_{10}) : M_{14} = (z_1)$, $(M_{12}) : M_{14} = (z_2)$.
- $(M_{10}) : M_{15} = (z_1)$, $(M_{13}) : M_{15} = (z_2)$, $(M_{14}) : M_{15} = (y_1)$.
- $(M_{13}) : M_{16} = (z_2)$, $(M_{15}) : M_{16} = (x_4)$.
- $(M_{12}) : M_{17} = (z_2)$, $(M_{14}) : M_{17} = (x_4)$, $(M_{16}) : M_{17} = (x_3)$.
- $(M_{10}) : M_{18} = (x_4)$, $(M_{11}) : M_{18} = (z_2)$, $(M_{16}) : M_{18} = (x_3)$, $(M_{17}) : M_{18} = (y_1)$.
- $(M_7) : M_{19} = (z_2)$, $(M_9) : M_{19} = (x_4)$, $(M_{18}) : M_{19} = (z_3)$.
- $(M_6) : M_{20} = (z_2)$, $(M_8) : M_{20} = (x_4)$, $(M_{18}) : M_{20} = (z_3)$, $(M_{19}) : M_{20} = (x_5)$.
- $(M_4) : M_{21} = (z_2)$, $(M_5) : M_{21} = (x_4)$, $(M_{17}) : M_{21} = (z_3)$, $(M_{20}) : M_{21} = (z_1)$.
- $(M_3) : M_{22} = (z_2)$, $(M_{16}) : M_{22} = (z_3)$, $(M_{20}) : M_{22} = (z_1)$, $(M_{21}) : M_{22} = (y_1)$.

We can show that $(M_1, M_2, \ldots, M_{i-1})\colon M_i$ is generated by variables for each $i = 2, \ldots, 22$. Thus $I^\vee$ has linear quotients.

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