CLEANNESS OF COHEN-MACAULAY MONOMIAL IDEAL GENERATED BY AT MOST FIVE ELEMENTS

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Abstract. In this paper, we prove that any Cohen-Macaulay monomial ideal generated by at most five elements is clean.

1. Introduction

Let \( n \) be a positive integer, and put \([n] := \{1, 2, \ldots, n\}\). A nonempty subset \( \Delta \subset 2^n \) is called a simplicial complex on \([n]\) if the following conditions are satisfied: (i) \( F \in \Delta \), \( G \subset F \implies G \in \Delta \), (ii) \( \{v\} \in \Delta \) for every \( v \in [n] \).

Let \( \Delta \) be a simplicial complex on \([n]\). An element \( F \) of \( \Delta \) is called a face of \( \Delta \). The dimension of a face \( F \) is \( |F| - 1 \), where \(|F|\) denotes the cardinality of \( F \). Set \( \dim \Delta = \max \{ \dim F : F \in \Delta \} = d - 1 \).

The maximal face of \( \Delta \) under inclusion is called a facet. A simplicial complex is called pure if all facets have the same dimension. A pure simplicial complex \( \Delta \) (say, \( d = \dim \Delta + 1 \)) is said to be shellable if there exists an order \( F_1, \ldots, F_t \) of the facets of \( \Delta \) such that for each \( 2 \leq i \leq t \), \( (F_1, \ldots, F_{i-1}) \cap \langle F_i \rangle \) is a pure \((d - 2)\)-dimensional simplicial complex, where \( \langle G_1, \ldots, G_r \rangle \) is a simplicial complex generated by \( G_1, \ldots, G_r \).

Let \( K \) be a field, and let \( \Delta \) be a simplicial complex on \([n]\). Put \( S = K[X_1, \ldots, X_n] \) a polynomial ring over \( K \). Then the squarefree monomial ideal \( I_\Delta = (X_{i_1} \cdots X_{i_r} : 1 \leq i_1 < i_2 < \cdots < i_r \leq n, \{i_1, \ldots, i_r\} \notin \Delta)S \)

is called the Stanley-Reisner ideal of \( \Delta \). Notice that any squarefree monomial ideal \( I \) is given as the Stanley-Reisner ideal \( I_\Delta \) for some simplicial complex \( \Delta \). Moreover, \( K[\Delta] = S/I_\Delta \) is called the Stanley-Reisner ring of \( \Delta \). Many combinatorial properties of \( \Delta \) are recognized as ring-theoretical properties of \( K[\Delta] \). For instance, \( \dim \Delta = \dim K[\Delta] - 1 \), and \( \Delta \) is pure if and only if \( K[\Delta] \) is equidimensional. How about shellability? For this question, Dress [2] introduced the notion of clean modules and answered this question: \( I_\Delta \) is clean if and only if \( \Delta \) is shellable provided that \( \Delta \) is pure. We recall the notion of cleanness.

Definition 1.1 (Dress [2]). For a monomial ideal \( I \) of \( S \), \( I \) is clean (or \( S/I \) is clean) if there exists a finite chain of monomial ideals \( F : I = I_0 \subset I_1 \subset \cdots \subset I_r = S \)

such that \( I_i/I_{i-1} \cong S/P_i \) for minimal prime ideals \( P_i \) of \( I \) for each \( i \).

The following question is natural.

Question. When is a monomial ideal \( I \) clean?

It is known that any shellable complex is Cohen-Macaulay over any field \( K \). In other words, \( S/I \) is Cohen-Macaulay for any pure clean ideal \( I \) of \( S \). The converse is not true in general. For any homogeneous ideal \( I \subset S \), height \( I \leq \mu(I) \) holds true, where height \( I \) (resp. \( \mu(I) \)) denotes the height (resp. the minimal number of generators) of \( I \). Then \( d(I) = \mu(I) - \text{height } I \) is called the deviation of \( I \). One can expect that if \( \mu(I) \) or \( d(I) \) is small then any Cohen-Macaulay ideal is clean. In fact, Herzog et.al [4, Proposition 2.2] proved that any monomial complete intersection ideal (i.e.

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\(d(I) = 0\) is clean. Furthermore, Bandari et al. proved that any almost complete intersection squarefree monomial ideal (i.e. \(d(I) = 1\)) is also clean, and proved that any squarefree monomial ideal \(I\) with \(\mu(I) \leq 3\) is clean; see [1, Theorem 2.5, Corollary 2.6].

The main theorem of the paper is the following:

**Theorem 1.2.** Let \(S\) be a polynomial ring over a field \(K\), and let \(I \subset S\) be a monomial ideal. If \(S/I\) is Cohen-Macaulay and \(\mu(I) \leq 5\), then \(I\) is a clean ideal.

Notice that the condition \(\mu(I) \leq 5\) is best possible because there exists a Cohen-Macaulay but not clean squarefree monomial ideal with \(\mu(I) = 6\); see Example 2.8.

Let us explain the organization of the paper. In Section 2, we prove our Theorem 1.2 by using the following theorem, which is proved in Section 3.

**Theorem 1.3.** Let \(I \subset S\) be a squarefree monomial ideal with \(d(I) = 2\). Then the Alexander dual \(I^\vee\) of \(I\) has linear quotients if \(I\) is Cohen-Macaulay with \(\mu(I) = 5\).

Section 3 is devoted to the proof of the theorem above. A key idea is a classification theorem of Cohen-Macaulay squarefree monomial ideals of deviation 2; see Kimura et al. [8].

### 2. Proof of main theorem

Throughout this section, let \(S\) be a polynomial ring over a field \(K\), and let \(I \subset S\) be a monomial ideal.

First we recall the definition of pretty clean ideals, which was introduced by Herzog and Popescu [3].

**Definition 2.1** (cf. [3]). A monomial ideal \(I \subset S\) is **pretty clean** if there exists a chain of monomial ideals

\[
I = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = S
\]

such that

(i) For every \(j = 1, \ldots, r\), \(I_j/I_{j-1} \cong S/P_j\), where \(P_j\) is a monomial prime ideal, which is generated by a subset of the variables.

(ii) \(P_i = P_j\) holds true whenever \(P_i \subset P_j\) for \(i < j\).

If \(I\) is clean, then it is pretty clean. The converse is not true in general. For instance, \((x^2, xy)\) is pretty clean but not clean in \(S = K[x, y]\); see [3, Example 3.6].

Let \(\text{Ass}_S(M)\) (resp. \(\text{Min}_S(M)\)) denote the set of associated prime ideals (resp. minimal prime ideals) of an \(S\)-module \(M\).

**Lemma 2.2** (See [3, Corollary 3.4]). Assume that \(\text{Ass}_S(S/I) = \text{Min}_S(S/I)\). Then \(I\) is clean if and only if it is pretty clean.

In particular,

(1) Any pretty clean squarefree monomial ideal is clean.

(2) Any pretty clean Cohen-Macaulay monomial ideal is clean.

Let us recall the notion of polarization of monomial ideals. For a monomial \(m = x_1^{a_1} \cdots x_r^{a_r}\), the polarization of \(m\) is defined by

\[
m^{\text{pol}} = (x_{11} \cdots x_{1a_1})(x_{21} \cdots x_{2a_2}) \cdots (x_{r1} \cdots x_{ra_r}).
\]

For a monomial ideal \(I = (m_1, \ldots, m_\nu)\),

\[
I^{\text{pol}} = (m_1^{\text{pol}}, \ldots, m_\nu^{\text{pol}})
\]

is called the **polarization** of \(I\). Notice that \(I^{\text{pol}}\) can be regarded as a squarefree monomial ideal of the polynomial ring \(S^{\text{pol}}\).

**Lemma 2.3** (See [5, Theorem 3.10]). Let \(I^{\text{pol}} \subset S^{\text{pol}}\) be the polarization of \(I\). Then \(I\) is pretty clean if and only if \(I^{\text{pol}}\) is clean.

**Proposition 2.4** (cf. [1, 4]). The following monomial ideals are clean.
(1) $I$ is a complete intersection ideal, that is, $d(I) = 0$.

(2) $I$ is an almost complete intersection (that is, $d(I) = 1$) Cohen-Macaulay ideal.

(3) $I$ is a Cohen-Macaulay ideal with $\mu(I) \leq 3$.

(4) $I$ is a Cohen-Macaulay ideal with height $I = 2$.

(5) $I$ is a Gorenstein ideal with height $I = 3$.

Proof. (1) See Herzog et.al [4, Proposition 2.2].

(2) By Bandari et.al [1, Theorem 2.5], we have that $I$ is pretty clean. Since $S/I$ is Cohen-Macaulay, $I$ is clean by Lemma 2.2(2).

(3) See (2) and Bandari et.al [1, Corollary 2.6].

(4) See Herzog et.al [4, Proposition 2.4].

(5) See the proof of Herzog et.al [4, Theorem 3.1].

Let $I$ be a squarefree monomial ideal of $S$. We denote by $G(I)$ the minimal set of monomial generators of $I$. Set $G(I) = \{m_1, \ldots, m_\mu\}$, where

$$m_i = x_{t_1i}x_{t_2i} \cdots x_{t_{\mu}i}.$$ 

Then the Alexander dual ideal $I^\vee$ of $I$ is given by

$$I^\vee = \bigcap_{i=1}^{\mu} (x_{t_1i}, x_{t_2i}, \ldots, x_{t_{\mu}i}).$$

The Alexander dual complex $\Delta^\vee$ of a simplicial complex $\Delta$ is given by $\{[n] \setminus F \mid F \notin \Delta\}$. Notice that $I_{\Delta^\vee} = (I_{\Delta})^\vee$.

A monomial ideal $I \subset S$ has linear quotients if there exists an order $m_1, \ldots, m_\mu$ of $G(I)$ such that for any $2 \leq i \leq \mu$, the ideal $(m_1, \ldots, m_{i-1})$: $m_i$ is a monomial prime ideal.

Lemma 2.5 ([1, Lemma 2.1]). Let $I$ be a squarefree monomial ideal. Then $I$ is clean if and only if $I^\vee$ has linear quotients.

In the next section, we prove Theorem 1.3. By using this, we prove the main theorem (Theorem 1.2).

Proof of Theorem 1.2. It suffices to show that if $I$ is a Cohen-Macaulay ideal with $\mu(I) \leq 5$ then $I$ is clean.

Case 1: The case where $I$ is a squarefree monomial ideal.

Assume that $I$ is a Cohen-Macaulay squarefree monomial ideal with $\mu(I) \leq 5$.

If $\mu(I) \leq 3$, then $I$ is clean by Proposition 2.4(3). If height $I = 1$, then $I$ is a principal ideal and thus clean.

If height $I = 2$, then $I$ is clean by Proposition 2.4(4). Hence, we consider the case (height $I, \mu(I)) = (3,4), (4,4), (3,5), (4,5)$ and $(5,5)$ only.

If (height $I, \mu(I)) = (4,4), (5,5)$, then $I$ is complete intersection and thus it is clean by Proposition 2.4(1). If (height $I, \mu(I)) = (3,4), (4,5)$, then $I$ is an almost complete intersection ideal and thus it is clean by Proposition 2.4(2).

Suppose that (height $I, \mu(I)) = (3,5)$. Then Theorem 1.3 yields that $I^\vee$ has linear quotient. Therefore $I$ is clean by Lemma 2.5.

Case 2: The case where $I$ is not a squarefree monomial ideal.

Let $I^{pol} \subset S^{pol}$ be the polarization of $I$. Then $S^{pol}/I^{pol}$ is Cohen-Macaulay, $\mu(I^{pol}) = \mu(I) \leq 5$ and $I^{pol}$ is a squarefree monomial ideal. Hence Case 1 yields that $I^{pol}$ is clean. By Lemma 2.3, $I$ is pretty clean. Hence it is clean because $S/I$ is Cohen-Macaulay; see Lemma 2.2.

Let us recall the characterization of clean ideals by Dress [2].

Theorem 2.6 (Dress [2]). Assume that $I = I_\Delta$ and $\Delta$ is pure. Then $I$ is a clean ideal if and only if $\Delta$ is shellable.
An ideal \( I \) of \( S \) is called pure if \( \dim S/P = \dim S/I \) for every minimal prime ideal \( P \) of \( I \). If a simplicial complex \( \Delta \) is pure shellable, then it is Cohen-Macaulay over any field \( K \). Thus we obtain the following.

**Corollary 2.7** (cf. Stanley, Dress [2]). A squarefree monomial ideal \( I = I_\Delta \) is a pure clean ideal, then it is Cohen-Macaulay over any field \( K \).

In Theorem 1.2, we cannot relax the assumption that \( \mu(I) \leq 5 \). Indeed, the following example shows that there exists a non-clean Cohen-Macaulay ideal with \( \mu(I) = 6 \).

**Example 2.8** (See Kimura et.al [8, Section 6]). Let \( K \) be a field. Set

\[
\begin{align*}
    m_1 &= x_1x_2x_4x_9x_{10}, \\
    m_2 &= x_2x_3x_4x_5x_{10}, \\
    m_3 &= x_5x_6x_7x_9x_{10}, \\
    m_4 &= x_1x_4x_5x_6x_9, \\
    m_5 &= x_1x_2x_3x_6x_7, \\
    m_6 &= x_3x_4x_7x_8x_9,
\end{align*}
\]

and \( I = (m_1, m_2, m_3, m_4, m_5, m_6) \). Then height \( I = 3 \) and \( \mu(I) = 6 \). Moreover, since

\[
\operatorname{pd}_S S/I = \begin{cases} 3 & \text{char } K \neq 2, \\ 4 & \text{char } K = 2. \end{cases}
\]

Hence if \( \text{char } K \neq 2 \), then \( S/I \) is Cohen-Macaulay but not clean.

3. **Proof of Theorem 1.3 (The case of \( d(I) = 2 \))**

First recall the notion of hypergraph in order to represent monomial ideals of deviation 2. See basic terminologies for [7, 8, 9]. Let \( V = [\mu] := \{1, 2, \ldots, \mu\} \). A hypergraph \( \mathcal{H} \) on the vertex set \( V \) is a collection of subsets of \( V \) with \( \bigcup_{F \in \mathcal{H}} F = V \). A subhypergraph \( \mathcal{H}' \) (on \( V \)) of a hypergraph \( \mathcal{H} \) means that \( \mathcal{H}' \) is a subset of \( \mathcal{H} \) such that \( \bigcup_{F \in \mathcal{H}'} F = V \).

For an arbitrary squarefree monomial ideal \( I \subset S = K[x_1, \ldots, x_n] \), let \( G(I) = \{m_1, \ldots, m_\mu\} \) denote the minimal set of monomial generators of \( I \). Then the hypergraph \( \mathcal{H}(I) \) associated to \( I \) on a vertex set \( V = [\mu] \) is defined by

\[
\mathcal{H}(I) := \{ \{j \in V : m_j \text{ is divisible by } x_i \} : i = 1, 2, \ldots, n \}.
\]

On the other hand, for a hypergraph \( \mathcal{H} \), when \( n \) is large enough, if we assign a variable \( x_F \) to each \( F \in \mathcal{H} \), then

\[
I_\mathcal{H} := \left( \prod_{j \in F \in \mathcal{H}} x_j : j = 1, 2, \ldots, \mu \right)
\]

gives a squarefree monomial ideal of \( K[x_F : F \in \mathcal{H}] \). Thus we can construct a squarefree monomial ideal from a given hypergraph. Note that \( \mathcal{H}(I_\mathcal{H}) = \mathcal{H} \), and that there exist many ideals \( I \) so that \( \mathcal{H}(I) = \mathcal{H} \).

We now recall some basic properties of this correspondence (see [7]).

1. A subset \( C \subset \mathcal{H} \) is called a cover of \( \mathcal{H} \) if \( \bigcup_{F \in C} F = V \). A cover \( C \) is called a minimal cover if it has no proper subset that is a cover of \( \mathcal{H} \).
2. \( F \in \mathcal{H} \) is called a face of \( \mathcal{H} \). A face in \( \mathcal{H} \) which is maximal with respect to inclusion is called a facet of \( \mathcal{H} \). The dimension of \( F \) is defined by \( \dim F := \sharp(F) - 1 \). If \( \dim F = i \), then \( F \) is called an \( i \)-face. A 1-face is called an edge. The dimension of \( \mathcal{H} \) is defined by \( \dim \mathcal{H} := \max \{ \dim F : F \in \mathcal{H} \} \).
3. \( \mathcal{H} \) is called pure if all minimal covers of \( \mathcal{H} \) have the same cardinality.
4. \( \mathcal{H} \) is called disconnected if there exist hypergraphs \( \mathcal{H}_1 \subseteq \mathcal{H} \) and vertex sets \( V_1, V_2 \) (\( i = 1, 2 \)) such that \( \mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}, V_1 \cap V_2 = \emptyset, \) and \( V_1 \cup V_2 = V \). If \( \mathcal{H} \) is not disconnected, then \( \mathcal{H} \) is called connected.

Let \( I \) be a squarefree monomial ideal of \( S \). For all \( i, j \in V \) (\( i \neq j \)), there exist \( F, G \in \mathcal{H}(I) \) such that \( i \in F \setminus G \) and \( j \in G \setminus F \). We call this condition "separability." Then \( \mathcal{H}(I) \) satisfies the separability condition. Conversely, for a given separable hypergraph \( \mathcal{H} \), we can construct a squarefree monomial ideal \( I \) in a polynomial ring with enough variables so that \( \mathcal{H} = \mathcal{H}(I) \).

Let \( I \subset S = K[x_1, \ldots, x_n] \) be a squarefree monomial Cohen-Macaulay ideal of \( d(I) = 2 \) and \( \mu(I) = 5 \). Then we must show that \( I \) is clean (equivalently, \( I' \) has linear quotients). By applying [10, Theorem 2.1] and [4, Proposition 3.3] to our cases, we may assume that \( I = I_\mathcal{H} \), where \( \mathcal{H} \) is
a Cohen-Macaulay hypergraph of deviation 2 which is one of [8, Theorem 4.9]. Notice that the cleanness of $I_{H'}$ implies the cleanness of $I_H$ if $H$ is subhypergraph of $H'$; see Example 3.2 for concrete examples. Hence it suffices to prove that $I_H$ is clean for all hypergraph $H$ appearing in the following theorem. (Note that $(H_29)^{+1}_E$ and $(H_29)^{+1}_F$ correspond to Cohen-Macaulay ideals of height 2).

In the classification theorem [8, Theorem 4.9], we focus on 0-extension. But for our proof of Theorem 1.2, it is useful to consider a hypergraph as contained in a maximal Cohen–Macaulay hypergraph of deviation 2. From this point of view, we have the following theorem:

**Theorem 3.1** ([8, Theorem 4.9]). Let $H$ be a Cohen–Macaulay hypergraph of deviation 2 without isolated vertices. Then $H$ satisfies one of (A) and (B):

(A) $H$ is a disjoint union of two Cohen–Macaulay hypergraphs of deviation 1.

(B) $H$ is contained in one of the following maximal Cohen–Macaulay hypergraphs of deviation 2:

\[
\begin{align*}
(H_{12})^{+1}_C & \quad (H_{12})^{+2}_E & \quad (H_{12})^{+3}_F & \quad (H_{12})^{+1}_L & \quad (H_{12})^{+1}_I \\
(H_{29})^{+1}_E & \quad (H_{29})^{+1}_F \\
(H_{36})^{+1}_D & \quad (H_{36})^{+1}_E \\
(H_{43}) & \quad \ldots
\end{align*}
\]

In the rest of the paper, we prove the main theorem. We first prove the case (A). It suffices to show that $I_H$ is clean for the following hypergraph $H$:
Let us show that \( I = I_H = (x_1y_1, x_2y_1, z_1w_2w_3, z_2w_1w_3, z_3w_1w_2) \) is clean. In order to do that, it is enough to prove that \( I^\vee \) has linear quotients. By definition, we have

\[
I^\vee = (x_1, y_1) \cap (x_2, y_1) \cap (z_1, w_2, w_3) \cap (z_2, w_1, w_3) \cap (z_3, w_1, w_2)
\]

\[
= (y_1w_1w_2, y_1w_1w_3, y_2w_3, y_1z_1w_1, y_1z_2w_2, y_1z_3w_3, y_1z_1z_2z_3, \\
x_1x_2w_1w_2, x_1x_2w_1w_3, x_1x_2w_2w_3, x_1x_2z_1w_1, x_1x_2z_2w_2, x_1x_2z_3w_3, x_1x_2z_1z_2z_3).
\]

Set \( M_1 = y_1w_1w_2, M_2 = y_1w_1w_3, M_3 = y_1w_2w_3, \ldots \), and \( M_{14} = x_1x_2z_1z_2z_3 \). Then we have

- \((M_1) : M_2 = (w_2)\).
- \((M_1) : M_3 = (w_1)\).
- \((M_1) : M_4 = (w_2), (M_2) : M_4 = (w_3)\).
- \((M_1) : M_5 = (w_1), (M_4) : M_5 = (w_3)\).
- \((M_2) : M_6 = (w_1), (M_4) : M_6 = (w_2)\).
- \((M_3) : M_7 = (w_1), (M_5) : M_7 = (w_3)\).
- \((M_1) : M_8 = (y_1)\).
- \((M_2) : M_9 = (y_1), (M_6) : M_9 = (w_2)\).
- \((M_1) : M_{10} = (y_1), (M_4) : M_{10} = (w_1)\).
- \((M_3) : M_{11} = (y_1), (M_5) : M_{11} = (w_2)\).
- \((M_3) : M_{12} = (y_1), (M_6) : M_{12} = (w_1)\).
- \((M_4) : M_{13} = (y_1), (M_6) : M_{13} = (w_1)\).
- \((M_5) : M_{14} = (y_1), (M_7) : M_{14} = (w_1), (M_8) : M_{14} = (w_2), (M_9) : M_{14} = (w_3)\).

Since \((M_2) : M_3 \subseteq (w_1)\), we have

\[ (M_1, M_2) : M_3 = (M_1) : M_3 + (M_2) : M_3 = (w_1). \]

Moreover, since \((M_3) : M_4 = (w_2w_3) \subseteq (w_2)\), we have

\[ (M_1, M_2, M_3) : M_4 = (M_1) : M_4 + (M_2) : M_4 + (M_3) : M_4 = (w_2, w_3). \]

Similarly, we can show that \((M_1, M_2, \ldots, M_{i-1}) : M_i \) is generated by variables for each \( i = 2, \ldots, 14 \). Thus \( I^\vee \) has linear quotients.

When \( H = (H_9)^{+1}_E \) or \( (H_9)^{+1}_F \), since \( I_H \) has height 2, the cleanness of \( I_H \) follows from Proposition 2.4. Hence it is enough to show that \( I_H \) is clean in the cases of \( H = (H_12)^{+1}_C \), \( (H_12)^{+2}_E \), \( (H_12)^{+3}_F \), \( (H_12)^{+1}_C \), \( (H_12)^{+1}_H \), \( (H_12)^{+1}_D \), \( (H_9)^{+1}_E \), and \( (H_43)^C \).

3.1. The case where \( H \) is a subgraph of \( (H_12)^{+1}_C \).

\[ (H_12)^{+1}_C \]

Let us show that \( I = I_H = (x_1y_1y_5z_1z_2, x_2y_1y_2, y_2z_1z_2, x_3y_3y_5z_1, y_3y_4z_2) \) is clean. In order to do that, it is enough to prove that \( I^\vee \) has linear quotients. By definition, we have

\[
I^\vee = (x_1, y_1, y_5, z_1, z_2) \cap (x_2, y_1, y_2) \cap (y_2, z_1, z_2) \cap (x_3, y_3, y_5, z_1) \cap (y_3, y_4, z_2)
\]

\[
= (y_1y_2y_3, y_2y_4y_5, y_3y_4z_1, y_4y_3z_1, y_1y_3z_2, y_2y_4z_1, y_4y_3z_1, y_1y_3z_2, y_1y_3z_2, y_2y_3z_2, \\
x_2y_3z_2, x_2y_3z_2, y_2z_1z_2, x_3y_2z_2, x_3y_2z_2, x_2y_3z_2, x_2y_3z_2, x_1y_3y_5, y_2y_3y_5, y_2y_3y_5, y_2y_3y_5, y_2y_3z_2, y_1y_3z_2, x_2y_3z_2).
\]

Set \( M_1 = y_1y_2y_3, M_2 = y_2y_3y_4, M_3 = x_3y_2y_4, \ldots \), and \( M_{24} = x_2y_5z_2 \). Then we have

- \((M_1) : M_2 = (y_1)\).
- \((M_2) : M_3 = (y_3)\).
- \((M_2) : M_4 = (y_3), (M_3) : M_4 = (x_3)\).
- \((M_1) : M_5 = (y_1), (M_4) : M_5 = (y_4)\).
- \((M_5) : M_6 = (y_2)\).
\( (M_7) : M_7 = (y_1), (M_6) : M_7 = (y_2), (M_5) : M_6 = (y_3), (M_4) : M_5 = (x_2), (M_3) : M_6 = (y_3), (M_2) : M_3 = (y_4), (M_1) : M_2 = (y_5) \).

Similarly, we can show that \( I_i \subset T \) for \( i = 2, \ldots, 24 \).

Thus \( I_i \) has linear quotients. Therefore, if \( i \neq j \), then \( I_i \cap I_j = (0) \).

For example, we consider the case of \( H=(H_{12})_{B}^{+3} \).

**Example 3.2.** Let \( H=(H_{12})_{B}^{+3} \) and put

\[
I = I_H = \langle x_1 y_1 y_4 z_1 z_2, x_2 y_1 y_2, y_2 z_1 z_2, x_3 y_3 z_1, y_4 y_4 z_2 \rangle.
\]

Set \( T = S[y_{12}^{-1}] \) and

\[
J = \langle x_1 y_1 y_4 z_1 z_2, x_2 y_1 y_2, y_2 z_1 z_2, x_3 y_3 y_5 z_1, y_3 y_4 z_2 \rangle = I_{(H_{12})_{C}^{+1}}.
\]

Then \( IT = JT \). Since \( J \) is a clean ideal of \( S \), we can choose a finite chain of monomial ideals \( J = J_0 \subset J_1 \subset \cdots \subset J_i = S \) such that \( I_i/I_{i-1} \cong S/P_i \), where \( P_i \) is a monomial prime ideal. Then \( IT = JT = I_0 T \subset I_1 T \subset \cdots \subset I_i T = T \) and \( I_i T / I_{i-1} T \cong T / P_i T \). Notice that \( P_i T \) is a monomial prime ideal if \( P_i T \neq T \). This implies that \( IT \) is clean and so is \( I \).

On the other hand, the cleanness of \( I = \langle x_1 y_1 y_4 z_1 z_2, x_2 y_1 y_2, y_2 z_1 z_2, x_3 y_3 y_5 z_1, y_3 y_4 y_4 z_2 \rangle \) follows from that of \( J \) and [4, Proposition 3.3].

![](image.png)
3.2. The case where \( \mathcal{H} \) is a subgraph of \((H_{112})_{E}^{+2}\).

\[
(H_{112})_{E}^{+2}
\]

Let us show that \( I = I_{\mathcal{H}} = (x_1y_1y_2y_3z_1z_2, x_2y_1y_2, y_2y_5z_1z_2, y_3z_1, x_3y_3y_4z_2) \) is clean. In order to do that, it is enough to prove that \( I' \) has linear quotients. By definition, we have

\[
I' = (x_1, y_1, y_4, y_5, z_1, z_2) \cap (x_2, y_1, y_2) \cap (y_2, y_5, z_1, z_2) \cap (y_3, z_1) \cap (x_3, y_3, y_4, z_2)
\]

\[
= (y_1y_2y_3, x_1y_2y_3, y_2y_3z_1, y_1y_3z_1, x_3y_1z_1, x_1y_2z_1, x_2y_3z_1, x_2y_4z_1, y_1y_4z_1, y_2y_4z_1, y_2y_3y_4, y_2y_3y_5, x_2y_3y_5, y_1y_3z_2, y_2y_3z_2, x_2y_3z_2, y_1z_1z_2, y_2z_1z_2).
\]

Set \( M_1 = y_1y_2y_3, M_2 = x_1y_2y_3, M_3 = y_2y_3z_1, \ldots, \) and \( M_{21} = y_2z_1z_2. \) Then we have

- \( (M_1) : M_2 = (y_1). \)
- \( (M_1) : M_3 = (y_1), (M_2) : M_3 = (x_1). \)
- \( (M_1) : M_4 = (y_2). \)
- \( (M_1) : M_5 = (y_3). \)
- \( (M_2) : M_6 = (y_3), (M_5) : M_6 = (y_1). \)
- \( (M_3) : M_7 = (y_1), (M_6) : M_7 = (y_2). \)
- \( (M_3) : M_8 = (y_2), (M_4) : M_8 = (y_1), (M_7) : M_8 = (x_3). \)
- \( (M_7) : M_9 = (x_3), (M_8) : M_9 = (y_3). \)
- \( (M_8) : M_{10} = (y_3), (M_5) : M_{10} = (x_3), (M_9) : M_{10} = (x_2). \)
- \( (M_3) : M_{11} = (y_3), (M_6) : M_{11} = (x_3), (M_9) : M_{11} = (x_2). \)
- \( (M_1) : M_{12} = (y_1), (M_3) : M_{12} = (x_1), (M_11) : M_{12} = (z_1). \)
- \( (M_1) : M_{13} = (y_1), (M_4) : M_{13} = (x_1), (M_3) : M_{13} = (z_1), (M_12) : M_{13} = (y_4). \)
- \( (M_8) : M_{14} = (z_1), (M_13) : M_{14} = (y_2). \)
- \( (M_4) : M_{15} = (z_1), (M_13) : M_{15} = (y_2), (M_14) : M_{15} = (x_2). \)
- \( (M_1) : M_{16} = (y_2), (M_4) : M_{16} = (z_1), (M_{15}) : M_{16} = (y_5). \)
- \( (M_1) : M_{17} = (z_1), (M_5) : M_{17} = (z_1), (M_{12}) : M_{17} = (y_4), (M_{13}) : M_{17} = (y_5), (M_{16}) : M_{17} = (y_1). \)
- \( (M_8) : M_{18} = (z_1), (M_{14}) : M_{18} = (y_5), (M_{16}) : M_{18} = (y_1), (M_{17}) : M_{18} = (y_2). \)
- \( (M_7) : M_{19} = (x_3), (M_9) : M_{19} = (y_1), (M_{18}) : M_{19} = (y_3). \)
- \( (M_8) : M_{20} = (x_3), (M_{10}) : M_{20} = (y_4), (M_{16}) : M_{20} = (y_3), (M_{19}) : M_{20} = (x_2). \)
- \( (M_1) : M_{21} = (x_3), (M_{11}) : M_{21} = (y_4), (M_{17}) : M_{21} = (y_3), (M_{19}) : M_{21} = (x_2), (M_{20}) : M_{21} = (y_1). \)

We can show that \( (M_1, M_2, \ldots, M_{i-1}) : M_i \) is generated by variables for each \( i = 2, \ldots, 21. \) Thus \( I' \) has linear quotients.

3.3. The case where \( \mathcal{H} \) is a subgraph of \((H_{112})_{F}^{+3}\).

\[
(H_{112})_{F}^{+3}
\]
Let us show that \( I = I_H = (y_1y_5z_1z_2, x_1y_1y_2, y_2y_3z_1z_2, y_3y_4z_1, x_2y_4y_5z_2) \) is clean. In order to do that, it is enough to prove that \( I^\vee \) has linear quotients. By definition, we have

\[
I^\vee = (y_1, y_5, z_1, z_2) \cap (x_1, y_1, y_2) \cap (y_2, y_3, z_1, z_2) \cap (y_3, y_4, z_1) \cap (x_2, y_4, y_5, z_2)
\]

\[
= (y_1y_2y_4, y_1y_3y_5, y_1y_3y_5, y_2y_4y_5, x_1y_1y_5, x_2y_1y_5, y_1y_4z_1, y_1y_5z_1, y_2y_5z_1, x_1y_1z_1, x_2y_2z_1, x_1y_2z_1, y_2y_3z_1, y_1y_3z_2, y_2y_3z_2, y_1y_4z_2, y_1y_4z_2, x_1y_3z_2, x_1y_3z_2, x_1y_4z_2).
\]

Set \( M_1 = y_1y_2y_4, M_2 = y_1y_3y_5, M_3 = y_1y_3y_5, \ldots, \) and \( M_{25} = x_1y_4z_2. \) Then we have

- \((M_1) : M_2 = (y_2), \)
- \((M_2) : M_3 = (y_4), \)
- \((M_3) : M_4 = (y_1), \)
- \((M_4) : M_5 = (y_1), (M_4) : M_6 = (y_3), \)
- \((M_5) : M_6 = (y_2), \)
- \((M_6) : M_7 = (y_4), (M_7) : M_8 = (y_5), \)
- \((M_7) : M_8 = (y_4), (M_7) : M_9 = (y_5), \)
- \((M_8) : M_9 = (y_3), (M_8) : M_{10} = (y_1), (M_9) : M_{10} = (y_1), \)
- \((M_9) : M_{11} = (y_1), (M_{10}) : M_{12} = (y_2), \)
- \((M_{10}) : M_{13} = (y_2), (M_{11}) : M_{13} = (y_2), (M_{12}) : M_{13} = (y_3), \)
- \((M_{11}) : M_{14} = (y_3), (M_{12}) : M_{14} = (y_4), (M_9) : M_{14} = (y_5), \)
- \((M_{12}) : M_{15} = (y_4), (M_{13}) : M_{15} = (y_4), (M_{14}) : M_{15} = (y_5), \)
- \((M_{13}) : M_{16} = (y_5), (M_{14}) : M_{16} = (y_4), (M_{15}) : M_{16} = (y_4), \)
- \((M_{14}) : M_{17} = (y_4), (M_{15}) : M_{17} = (y_4), (M_{16}) : M_{17} = (y_4), \)
- \((M_{15}) : M_{18} = (y_4), (M_{16}) : M_{18} = (y_4), (M_{17}) : M_{18} = (y_2), \)
- \((M_{16}) : M_{19} = (y_4), (M_{17}) : M_{19} = (y_5), (M_{18}) : M_{19} = (y_4), (M_{19}) : M_{20} = (y_3), \)
- \((M_{17}) : M_{20} = (y_4), (M_{18}) : M_{20} = (y_5), (M_{19}) : M_{20} = (y_3), \)
- \((M_{18}) : M_{21} = (y_5), (M_{19}) : M_{21} = (y_4), (M_{20}) : M_{21} = (y_5), \)
- \((M_{19}) : M_{22} = (y_4), (M_{20}) : M_{22} = (y_5), (M_{21}) : M_{22} = (y_5), (M_{22}) : M_{22} = (y_4), \)
- \((M_{20}) : M_{23} = (y_4), (M_{21}) : M_{23} = (y_4), (M_{22}) : M_{23} = (y_2), (M_{23}) : M_{24} = (y_2), \)
- \((M_{21}) : M_{25} = (y_1), (M_{22}) : M_{25} = (y_2), (M_{23}) : M_{25} = (y_3), (M_{24}) : M_{25} = (y_3). \)

We can show that \((M_1, M_2, \ldots, M_{t-1}) : M_t \) is generated by variables for each \( i = 2, \ldots, 25. \) Thus \( I^\vee \) has linear quotients.

3.4. The case where \( \mathcal{H} \) is a subgraph of \((H_{12})_{G}^{\pm 1}. \)

\[
(H_{12})_{G}^{\pm 1}
\]

Let us show that \( I = I_H = (x_1y_1y_4y_5z_1z_2, y_1y_2, x_2y_2y_5y_6z_1z_2, y_3z_1, x_3y_3y_4y_6z_2) \) is clean. In order to do that, it is enough to prove that \( I^\vee \) has linear quotients. By definition, we have

\[
I^\vee = (x_1, y_1, y_4, y_5, z_1, z_2) \cap (y_1, y_2) \cap (x_2, y_2, y_5, y_6, z_1, z_2) \cap (y_3, z_1) \cap (x_3, y_3, y_4, y_6, z_2)
\]

\[
= (y_1y_2y_3, x_1y_2y_3, x_2y_1y_3, y_1y_4z_1, y_2y_4z_1, x_3y_2z_1, x_3y_1z_1, y_1z_1z_2, y_2z_1z_2, y_2y_3z_2, y_1y_3z_2, y_1y_4z_5, y_2y_4z_5, y_2y_4z_5, y_2y_4z_5, y_2y_4z_5, y_2y_4z_5).
\]

Set \( M_1 = y_1y_2y_3, M_2 = x_1y_2y_3, M_3 = x_2y_1y_3, \ldots, \) and \( M_{19} = y_1y_3y_6. \) Then we have

- \((M_1) : M_2 = (y_1), \)
- \((M_1) : M_3 = (y_2), \)
- \((M_1) : M_4 = (y_2), (M_3) : M_4 = (x_2). \)
\textbf{3.5. The case where }\mathcal{H} \textit{ is a subgraph of } (H_{12})^+_H .

Let us show that \( I = I_\mathcal{H} = (y_1 y_5 z_1 z_2, y_1 y_2, x_1 y_2 y_3 y_4 z_1 z_2, y_3 y_4 z_1, x_2 y_4 y_5 y_6 z_2) \) is clean. In order to do that, it is enough to prove that \( I' \) has linear quotients. By definition, we have

\[
I' = (y_1, y_5, z_1, z_2) \cap (y_1, y_2, x_1, y_2, y_3, y_4, z_1, z_2) \cap (y_1, y_4, z_1) \cap (x_2, y_4, y_5, y_6, z_2)
\]

\[
= (y_1 y_3 y_4, y_1 y_3 y_5, y_2 y_3 y_5, y_2 y_4 y_5, y_1 y_2 y_4, y_1 y_4 z_1, y_2 y_4 z_1, y_2 y_5 z_1, y_1 y_5 z_1, x_2 y_1 z_1, x_2 y_2 z_1, x_2 y_3 z_1, x_1 y_1 y_4, y_1 y_3 y_4, y_1 y_3 y_5, y_2 y_3 y_5, y_2 y_4 y_5, y_1 y_2 y_4, y_1 y_4 z_1, y_2 y_4 z_1, y_2 y_5 z_1, y_1 y_5 z_1, x_2 y_1 z_1, x_2 y_2 z_1, x_2 y_3 z_1, x_1 y_1 y_4, y_1 y_3 y_4, y_1 y_3 y_5, y_2 y_3 y_5, y_2 y_4 y_5, y_1 y_2 y_4, y_1 y_4 z_1, y_2 y_4 z_1, y_2 y_5 z_1, y_1 y_5 z_1, x_2 y_1 z_1, x_2 y_2 z_1, x_2 y_3 z_1, y_1 y_3 y_4, y_1 y_3 y_5, y_2 y_3 y_5, y_2 y_4 y_5, y_1 y_2 y_4, y_1 y_4 z_1, y_2 y_4 z_1, y_2 y_5 z_1, y_1 y_5 z_1, x_2 y_1 z_1, x_2 y_2 z_1, x_2 y_3 z_1)
\]

Set \( M_1 = y_1 y_3 y_4, M_2 = y_1 y_3 y_5, M_3 = y_2 y_3 y_5, \ldots, \) and \( M_{23} = y_2 y_3 z_2. \) Then we have

- \( (M_1) : M_2 = (y_4), \)
- \( (M_2) : M_3 = (y_1), \)
- \( (M_3) : M_4 = (y_3), \)
- \( (M_1) : M_5 = (y_3), (M_4) : M_5 = (y_5), \)
- \( (M_1) : M_6 = (y_3), (M_5) : M_6 = (y_2), \)
- \( (M_1) : M_7 = (y_5), (M_6) : M_7 = (y_1), \)
- \( (M_1) : M_8 = (y_3), (M_7) : M_8 = (y_4), \)
- \( (M_2) : M_9 = (y_3), (M_8) : M_9 = (y_4), (M_7) : M_10 = (y_2). \)

We can show that \( (M_1, M_2, \ldots, M_{i-1}) : M_i \) is generated by variables for each \( i = 2, \ldots, 19. \) Thus \( I' \) has linear quotients.
3.6. The case where $\mathcal{H}$ is a subgraph of $(H_{12})_{I}^{\pm}$.  

Let us show that $I = I_{\mathcal{H}} = (x_1 y_1 y_5 y_6 z_1 z_2, y_1 y_2, x_2 y_2 y_3 y_6 z_1 z_2, y_3 y_4 z_1, y_4 y_5 z_2)$ is clean. In order to do that, it is easy to prove that $I^\vee$ has linear quotients. By definition, we have

$I^\vee = (x_1, y_1, y_5, y_6, z_1, z_2) \cap (y_1, y_2) \cap (x_2, y_2, y_3, y_6, z_1, z_2) \cap (y_3, y_4, z_1) \cap y_4, y_5, z_2) = (y_1 y_2 y_4, y_1 y_3 y_4, y_1 y_3 y_5, y_2 y_3 y_5, y_2 y_4 y_6, x_1 y_2 y_4, x_2 y_1 y_4, y_1 y_4 y_6, y_2 y_4 y_6, y_2 y_4 z_1, y_1 y_4 z_1, y_1 y_5 z_1, y_2 y_1 z_2, y_1 z_1 z_2, y_1 y_4 z_2, y_2 y_4 z_2, y_2 y_3 z_2, y_1 y_3 z_2).$

Set $M_1 = y_1 y_2 y_4, M_2 = y_1 y_5 y_3, M_3 = y_1 y_3 y_5, \ldots$, and $M_{19} = y_1 y_3 z_2$. Then we have

- $(M_1) : M_2 = (y_2).$
- $(M_2) : M_3 = (y_4).$
- $(M_3) : M_4 = (y_1).$
- $(M_1) : M_5 = (y_1), (M_4) : M_5 = (y_3).$
- $(M_1) : M_6 = (y_1), (M_5) : M_6 = (y_5).$
- $(M_1) : M_7 = (y_2), (M_2) : M_7 = (y_3).$
- $(M_1) : M_8 = (y_2), (M_2) : M_8 = (y_3), (M_7) : M_8 = (x_2).$
- $(M_1) : M_9 = (y_5), (M_6) : M_9 = (x_1), (M_8) : M_9 = (y_1).$
- $(M_1) : M_{10} = (y_1), (M_5) : M_{10} = (y_5), (M_6) : M_{10} = (x_1), (M_9) : M_{10} = (y_6).$
- $(M_2) : M_{11} = (y_3), (M_7) : M_{11} = (x_2), (M_8) : M_{11} = (y_6), (M_{10}) : M_{11} = (y_2).$
- $(M_3) : M_{12} = (y_3), (M_{11}) : M_{12} = (y_4).$
- $(M_4) : M_{13} = (y_3), (M_{10}) : M_{13} = (y_4), (M_{12}) : M_{13} = (y_1).$
- $(M_{10}) : M_{14} = (y_4), (M_{13}) : M_{14} = (y_3).$
- $(M_{11}) : M_{15} = (y_4), (M_{12}) : M_{15} = (y_5), (M_{14}) : M_{15} = (y_2).$
- $(M_1) : M_{16} = (y_2), (M_2) : M_{16} = (y_3), (M_7) : M_{16} = (x_2), (M_8) : M_{16} = (y_6), (M_{15}) : M_{16} = (z_1).$
- $(M_5) : M_{17} = (y_5), (M_6) : M_{17} = (x_1), (M_9) : M_{17} = (y_6), (M_{14}) : M_{17} = (z_1), (M_{16}) : M_{17} = (y_1).$
- $(M_4) : M_{18} = (y_3), (M_{14}) : M_{18} = (z_1), (M_{17}) : M_{18} = (y_4).$
- $(M_3) : M_{19} = (y_5), (M_{15}) : M_{19} = (z_1), (M_{16}) : M_{19} = (y_4), (M_{18}) : M_{19} = (y_2).$

We can show that $(M_1, M_2, \ldots, M_{i-1}) : M_i$ is generated by variables for each $i = 2, \ldots, 19$. Thus $I^\vee$ has linear quotients.
3.7. The case where \( \mathcal{H} \) is a subgraph of \( (H_{36})^{+1}_D \).

Let us show that \( I = I_\mathcal{H} = (y_1y_5z_1z_2z_3, x_2y_1y_2z_1z_2, x_3y_2y_3z_1z_3, y_3z_2z_3, x_5y_5) \) is clean. In order to do that, it is enough to prove that \( I' \) has linear quotients. By definition, we have

\[
I' = (y_1, y_5, z_1, z_2, z_3) \cap (x_2, y_1, y_2, z_1, z_2) \cap (x_3, y_2, y_3, z_1, z_3) \cap (y_3, z_2, z_3) \cap (x_5, y_5)
\]

\[
= (x_2y_3y_5, y_3y_5z_2, x_5y_3z_2, x_3x_5y_2, x_3x_5y_5z_2, y_5z_1z_2, x_5z_1z_2, y_3y_5z_1, x_5y_3z_1, y_1y_3y_5, x_5y_1y_3, x_5y_2z_3, y_5z_2z_3, y_5z_1z_3, x_5y_1z_3, y_1y_5z_3, x_2y_5z_3, x_2x_5z_1, x_5y_2z_3, y_2y_5z_3, y_2y_3y_5, y_2y_5z_2, x_5y_2z_2, x_5y_2z_2).
\]

Set \( M_1 = x_2y_3y_5, M_2 = y_3y_5z_2, M_3 = x_5y_3z_2, \ldots, \) and \( M_{24} = x_5y_2z_2 \). Then we have

- \( M_1 : M_2 = (x_2) \),
- \( M_2 : M_3 = (y_5) \),
- \( M_3 : M_4 = (y_3) \),
- \( M_4 : M_5 = (x_3) \),
- \( M_5 : M_6 = (y_3) \),
- \( M_6 : M_7 = (x_3) \),
- \( M_7 : M_8 = (y_3) \),
- \( M_8 : M_9 = (z_2) \),
- \( M_9 : M_{10} = (x_2) \),
- \( M_{10} : M_{11} = (z_2) \),
- \( M_{11} : M_{12} = (x_2) \),
- \( M_{12} : M_{13} = (y_3) \),
- \( M_{13} : M_{14} = (z_2) \),
- \( M_{14} : M_{15} = (y_3) \),
- \( M_{15} : M_{16} = (y_3) \),
- \( M_{16} : M_{17} = (y_3) \),
- \( M_{17} : M_{18} = (y_3) \),
- \( M_{18} : M_{19} = (y_3) \),
- \( M_{19} : M_{20} = (y_3) \),
- \( M_{20} : M_{21} = (y_3) \),
- \( M_{21} : M_{22} = (y_3) \),
- \( M_{22} : M_{23} = (y_3) \),
- \( M_{23} : M_{24} = (y_3) \),
- \( M_{24} : M_{25} = (y_3) \).

We can show that \( (M_1, M_2, \ldots, M_{24}) : M_i \) is generated by variables for each \( i = 2, \ldots, 24 \). Thus \( I' \) has linear quotients.

3.8. The case where \( \mathcal{H} \) is a subgraph of \( (H_{36})^{+1}_E \).
Let us show that \( I = I_H = (y_5z_1z_2z_3, x_2y_0y_2z_1z_2, y_2y_3z_1z_3, x_4y_0y_3z_2z_3, x_5y_5) \) is clean. In order to do that, it is enough to prove that \( I^\vee \) has linear quotients. By definition, we have

\[
I^\vee = (y_5, z_1, z_2, z_3) \cap (x_2, y_0, y_2, z_1, z_2) \cap (y_2, y_3, z_1, z_3) \cap (x_4, y_0, y_3, z_2, z_3) \cap (x_5, y_5)
\]

\[
= (x_4y_2y_5, y_0y_2y_5, y_2y_3z_3, x_4y_3z_3, y_0y_3y_5, y_0y_3y_5, y_2y_3y_5, x_2y_5z_3, x_2x_5z_3, x_5z_1z_3, y_5z_1z_3, y_3y_5z_1, y_3y_5z_1, x_5y_0z_1, x_5y_0z_1, x_4y_5z_1, x_4y_5z_1, x_5z_1z_2, y_5z_1z_2, y_5z_2z_3, x_5z_2z_3, y_4y_5z_2, x_5y_5z_2, y_2y_5z_2, x_5y_2z_2).
\]

Set \( M_1 = x_4y_2y_5, M_2 = y_0y_2y_5, M_3 = y_2y_3z_3, \ldots, \) and \( M_{27} = x_5y_2z_2. \) Then we have

- \( (M_1) : M_2 = (x_4), \)
- \( (M_1) : M_3 = (x_4), (M_2) : M_3 = (y_0), \)
- \( (M_3) : M_1 = (y_5), \)
- \( (M_4) : M_3 = (y_2), \)
- \( (M_5) : M_6 = (y_2), (M_6) : M_6 = (x_5), \)
- \( (M_7) : M_7 = (y_2), (M_6) : M_7 = (z_3), \)
- \( (M_1) : M_8 = (x_4), (M_3) : M_8 = (z_3), (M_7) : M_8 = (y_0), \)
- \( (M_7) : M_9 = (y_0), (M_8) : M_9 = (y_2), \)
- \( (M_3) : M_{10} = (y_2), (M_6) : M_{10} = (y_0), (M_9) : M_{10} = (y_3), \)
- \( (M_4) : M_{11} = (y_2), (M_5) : M_{11} = (y_0), (M_{10}) : M_{11} = (y_5), \)
- \( (M_4) : M_{12} = (y_2), (M_5) : M_{12} = (y_0), (M_{11}) : M_{12} = (x_2), \)
- \( (M_3) : M_{13} = (y_2), (M_6) : M_{13} = (y_0), (M_{10}) : M_{13} = (x_2), (M_{12}) : M_{13} = (x_5), \)
- \( (M_7) : M_{14} = (y_0), (M_6) : M_{14} = (y_2), (M_8) : M_{14} = (x_2), (M_{13}) : M_{14} = (z_3), \)
- \( (M_{12}) : M_{15} = (z_3), (M_{14}) : M_{15} = (y_5), \)
- \( (M_2) : M_{16} = (y_2), (M_3) : M_{16} = (z_3), (M_{14}) : M_{16} = (y_5), \)
- \( (M_{12}) : M_{17} = (z_3), (M_{15}) : M_{17} = (y_3), (M_{16}) : M_{17} = (y_5), \)
- \( (M_1) : M_{18} = (y_2), (M_{13}) : M_{18} = (z_3), (M_{14}) : M_{18} = (y_5), (M_{16}) : M_{18} = (y_0), \)
- \( (M_{12}) : M_{19} = (z_3), (M_{15}) : M_{19} = (y_3), (M_{17}) : M_{19} = (y_0), (M_{18}) : M_{19} = (y_5), \)
- \( (M_{12}) : M_{20} = (z_3), (M_{15}) : M_{20} = (y_3), (M_{17}) : M_{20} = (y_0), (M_{19}) : M_{20} = (x_4), \)
- \( (M_{14}) : M_{21} = (z_3), (M_{14}) : M_{21} = (y_3), (M_{16}) : M_{21} = (y_0), (M_{18}) : M_{21} = (x_4), (M_{20}) : M_{21} = (x_5), \)
- \( (M_3) : M_{22} = (y_2), (M_6) : M_{22} = (y_0), (M_{10}) : M_{22} = (x_2), (M_{21}) : M_{22} = (z_1), \)
- \( (M_3) : M_{23} = (y_2), (M_5) : M_{23} = (y_0), (M_{11}) : M_{23} = (x_2), (M_{20}) : M_{23} = (z_1), (M_{22}) : M_{23} = (y_5), \)
- \( (M_7) : M_{24} = (y_0), (M_8) : M_{24} = (y_2), (M_6) : M_{24} = (x_2), (M_{21}) : M_{24} = (z_1), (M_{22}) : M_{24} = (z_3), \)
- \( (M_{20}) : M_{25} = (z_1), (M_{23}) : M_{25} = (z_3), (M_{24}) : M_{25} = (y_5), \)
- \( (M_1) : M_{26} = (x_4), (M_{22}) : M_{26} = (y_0), (M_{21}) : M_{26} = (z_1), (M_{22}) : M_{26} = (z_3), (M_{24}) : M_{26} = (y_3), \)
- \( (M_{20}) : M_{27} = (z_1), (M_{23}) : M_{27} = (z_3), (M_{25}) : M_{27} = (y_3), (M_{26}) : M_{27} = (y_5), \)

We can show that \( (M_1, M_2, \ldots, M_{i-1}) : M_i \) is generated by variables for each \( i = 2, \ldots, 27. \) Thus \( I^\vee \) has linear quotients.

3.9. The case where \( \mathcal{H} \) is a subgraph of \( (H_43)_C. \)

\[
(H_43)_C
\]
Let us show that $I = I_H = (y_1y_2z_1z_2z_3, y_3z_1z_2z_3, x_3y_1z_1, x_4y_2z_2, x_5y_3z_3)$ is clean. In order to do that, it is enough to prove that $I^\vee$ has linear quotients. By definition, we have

$$I^\vee = (x_3y_1z_2, x_3y_2z_3, x_3y_3z_2, y_1y_2z_1, y_1y_2z_3, y_1y_3z_2, x_4y_2z_1, x_4y_3z_1, x_5y_1z_3, x_5y_2z_3, x_5y_3z_1),$$

Set $M_1 = x_5y_1z_2$, $M_2 = x_3y_5z_2$, $M_3 = x_3y_3z_2$, . . . , and $M_{22} = x_3y_2y_3$. Then we have

- $(M_1) : M_2 = (y_1),$
- $(M_2) : M_3 = (x_5),$
- $(M_1) : M_4 = (x_3), (M_3) : M_4 = (x_3),$
- $(M_1) : M_5 = (z_2),$
- $(M_3) : M_6 = (x_3), (M_4) : M_6 = (y_1),$
- $(M_1) : M_7 = (y_1), (M_2) : M_7 = (x_3), (M_6) : M_7 = (y_3),$
- $(M_3) : M_8 = (y_1), (M_6) : M_8 = (z_2),$
- $(M_7) : M_9 = (z_2), (M_4) : M_9 = (y_3),$
- $(M_8) : M_{10} = (y_1), (M_9) : M_{10} = (x_5),$
- $(M_4) : M_{11} = (y_1), (M_5) : M_{11} = (x_3), (M_{10}) : M_{11} = (x_4),$
- $(M_1) : M_{12} = (x_3), (M_4) : M_{12} = (y_3), (M_{11}) : M_{12} = (z_1),$
- $(M_2) : M_{13} = (x_3), (M_3) : M_{13} = (y_3), (M_{11}) : M_{13} = (z_1), (M_{12}) : M_{13} = (y_1),$
- $(M_5) : M_{14} = (y_3), (M_{10}) : M_{14} = (z_1), (M_{12}) : M_{14} = (z_2),$
- $(M_{10}) : M_{15} = (z_1), (M_{13}) : M_{15} = (z_2), (M_{14}) : M_{15} = (y_1),$
- $(M_{13}) : M_{16} = (z_2), (M_{15}) : M_{16} = (x_4),$
- $(M_{12}) : M_{17} = (z_2), (M_{14}) : M_{17} = (x_4), (M_{16}) : M_{17} = (x_3),$
- $(M_{10}) : M_{18} = (x_3), (M_{11}) : M_{18} = (x_2), (M_{16}) : M_{18} = (y_3), (M_{17}) : M_{18} = (y_1),$
- $(M_{17}) : M_{19} = (x_2), (M_{18}) : M_{19} = (x_4), (M_{18}) : M_{19} = (z_3),$
- $(M_{16}) : M_{20} = (x_2), (M_{19}) : M_{20} = (x_4), (M_{18}) : M_{20} = (z_3), (M_{19}) : M_{20} = (x_5),$
- $(M_{14}) : M_{21} = (x_2), (M_{16}) : M_{21} = (x_4), (M_{17}) : M_{21} = (z_3), (M_{20}) : M_{21} = (z_1),$
- $(M_{13}) : M_{22} = (x_2), (M_{16}) : M_{22} = (z_3), (M_{20}) : M_{22} = (z_1), (M_{21}) : M_{22} = (y_1).$

We can show that $(M_1, M_2, . . . , M_{i-1}) : M_i$ is generated by variables for each $i = 2, . . . , 22$. Thus $I^\vee$ has linear quotients.

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References

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