

A new characterization of some families of finite simple groups

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ABSTRACT – Let G be a finite group. A vanishing element of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G . Denote by $\text{Vo}(G)$ the set of the orders of vanishing elements of G . In this paper, we prove that if G is a finite group such that $\text{Vo}(G) = \text{Vo}(M)$ and $|G| = |M|$, then $G \cong M$, where M is a sporadic simple group, an alternating group, a projective special linear group $L_2(p)$, where p is an odd prime or a finite simple K_n -group, where $n \in \{3, 4\}$. These results confirm the conjecture posed in [17] for the simple groups under study.

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1. Introduction

Let G be a finite group. A *vanishing element* of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G . We will denote by $\text{Van}(G)$ the set of vanishing elements of G and by $\text{Vo}(G)$ the set of the orders of elements in $\text{Van}(G)$. According to [3] and [14], we know that the set $\text{Vo}(G)$ can release some information about the structure of a finite group G . For instance, Theorem C of [15] as a strengthening of (Corollary 3, [14]) states that if p is a

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prime divisor of $|G|$ and G does not have any vanishing element of order divisible by p , then G has a normal Sylow p -subgroup. It is also shown in [37] that if G is a finite group such that $\text{Vo}(G) = \text{Vo}(A_5)$, then $G \cong A_5$, i.e., the alternating group A_5 is characterizable by the set of orders of vanishing elements. According to this result, one may ask the following question:

Are all finite nonabelian simple groups characterizable by the set of orders of vanishing elements?

The answer to this question is not affirmative in general. For example, for the simple linear group $L_3(5)$, we have $\text{Vo}(L_3(5)) = \text{Vo}(\text{Aut}(L_3(5)))$ but $L_3(5) \not\cong \text{Aut}(L_3(5))$ because $|L_3(5)| \neq |\text{Aut}(L_3(5))|$. Therefore, in [17], the following conjecture was put forward:

Conjecture. Let G be a finite group and let M be a finite nonabelian simple group. If $\text{Vo}(G) = \text{Vo}(M)$ and $|G| = |M|$, then $G \cong M$.

Also, in [17], an affirmative answer was given to this conjecture for the simple groups $L_2(q)$, where $q \in \{5, 7, 8, 9, 17\}$, $L_3(4)$, A_7 , $Sz(8)$ and $Sz(32)$. In this paper, we first prove that the conjecture is confirmed for all sporadic simple groups, the alternating groups and projective special linear group $L_2(p)$, where p is an odd prime. So, we have the following result:

Theorem A. Let G be a finite group and M be a sporadic simple group, an alternating group or a projective special linear group $L_2(p)$, where p is an odd prime. If $|G| = |M|$ and $\text{Vo}(G) = \text{Vo}(M)$, then $G \cong M$.

The finite simple group G is called a K_n -group if its order has exactly n distinct prime divisors, where $n \in \mathbb{N}$. The following lemma determines all K_n -groups, where $n \in \{3, 4\}$:

LEMMA 1.1. [4], [18], [31], [36] *Let G be a finite simple K_n -group.*

- (1) *If $n = 3$, then G is isomorphic to one of the following groups:
 A_5 , A_6 , $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.*
- (2) *If $n = 4$, then G is isomorphic to one of the following groups:*
 - (a) A_7 , A_8 , A_9 , A_{10} , M_{11} , M_{12} , J_2 , $L_2(16)$, $L_2(25)$, $L_2(49)$,
 $L_2(81)$, $L_2(97)$, $L_2(243)$, $L_2(577)$, $L_3(4)$, $L_3(5)$, $L_3(7)$,
 $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$,
 $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$,
 $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;
 - (b) $L_2(r)$, where r is a prime, $r^2 - 1 = 2^a \cdot 3^b \cdot v$, $v > 3$ is a prime, $a, b \in \mathbb{N}$;
 - (c) $L_2(2^m)$, where m , $2^m - 1$ and $(2^m + 1)/3$ are primes greater than 3;
 - (d) $L_2(3^m)$, where m , $(3^m + 1)/4$ and $(3^m - 1)/2$ are odd primes.

As a second result of this paper, we show the validity of the conjecture for the groups listed in Lemma 1.1. In fact, we have the following result:

Theorem B. Let G be a finite group and let M be a simple K_3 -group or a simple K_4 -group. If $|G| = |M|$ and $\text{Vo}(G) = \text{Vo}(M)$, then $G \cong M$.

Throughout this paper, we use the following notation. Let G be a finite group, p be a prime number and m be a positive integer. The number of Sylow p -subgroups of G is denoted by $n_p(G)$. Also, $\text{Syl}_p(G)$ denotes the set of all Sylow p -subgroups of G . The notation $p^m \parallel |G|$ means that p^m divides $|G|$ but p^{m+1} does not divide $|G|$. Also, by $\omega(G)$ we denote the set of orders of elements of group G . All further notation is standard and can be found in [12], for instance.

2. Preliminaries

One of the main keys for the proof of our results is a result by Dolfi, et al. in [15] on the vanishing prime graph of a finite group and its relationship with the Gruenberg-Kegel graph. For this reason, we will recall the required definitions in the following.

Given a finite set of positive integers X , the prime graph $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes p such that there exists an element of X divisible by p , and two distinct vertices p, q are adjacent if and only if there exists an element of X divisible by pq . For a finite group G , the graph $\Pi(\omega(G))$, which we denote by $GK(G)$ is also known as the Gruenberg-Kegel graph of G . Also, the prime graph $\Pi(\text{Vo}(G))$, which in this paper we denote by $\Gamma(G)$, is called the vanishing prime graph of G .

We denote by $t(G)$ the number of connected components of $GK(G)$ and by $\pi_i(G)$, $i = 1, 2, \dots, t(G)$, the i th connected component of $GK(G)$. If the order of G is even, we set $2 \in \pi_1(G)$. We also, denote by $\pi(n)$ the set of all primes dividing n , where n is a natural number. Now $|G|$ can be expressed as the product of $m_1, m_2, \dots, m_{t(G)}$, where m_i 's are positive integers with $\pi(m_i) = \pi_i(G)$. We call $m_1, m_2, \dots, m_{t(G)}$ the order components of G and we write $OC(G) = \{m_1, m_2, \dots, m_{t(G)}\}$, the set of order components of G . A finite simple group S is said to be characterizable by its order components, if $S \cong G$ for each finite group G such that $OC(G) = OC(S)$.

A 2-Frobenius group is a group G that has proper normal subgroups K and L such that L is a Frobenius group with kernel K and G/K is a Frobenius group with kernel L/K . The following lemma determines the structure of the finite group with disconnected Gruenberg-Kegel graph:

LEMMA 2.1. [32] *Let G be a finite group. If $t(G) \geq 2$, then the structure of G is as follows.*

- (1) G is either a Frobenius group or a 2-Frobenius group.

- (2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(|H|) \cup \pi(|G/K|) \subseteq \pi_1(G)$, H is nilpotent and K/H is a nonabelian simple group.

LEMMA 2.2. [8] *Let G be a Frobenius group of even order with kernel F and complement H . Then*

- (1) $t(G) = 2$, $\{\pi_1(G), \pi_2(G)\} = \{\pi(|H|), \pi(|F|)\}$;
- (2) if H is a nonsolvable group, then there exists $H_0 \leq H$ such that $H_0 = L_2(5) \times Z$, where $(2 \cdot 3 \cdot 5, |Z|) = 1$ and the Sylow subgroups of Z are cyclic.

LEMMA 2.3. [5] *If G is a 2-Frobenius group with normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, then*

- (1) $t(G) = 2$, $\pi_1(G) = \pi(|G/K|) \cup \pi(|H|)$ and $\pi_2(G) = \pi(|K/H|)$;
- (2) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \text{Aut}(K/H)$;
- (3) G is solvable.

A group G is said to be a nearly 2-Frobenius group if there exist two normal subgroups F and L of G with the following properties: $F = F_1 \times F_2$ is nilpotent, where F_1 and F_2 are normal subgroups of G , furthermore G/F is a Frobenius group with kernel L/F , G/F_1 is a Frobenius group with kernel L/F_1 , and G/F_2 is a 2-Frobenius group.

LEMMA 2.4. [15], [16], [24]

- (1) If G is a finite nonabelian simple group, then $GK(G) = \Gamma(G)$, unless $G \cong A_7$.
- (2) If G is a solvable Frobenius group with Frobenius kernel F and Frobenius complement H , then either $GK(G) = \Gamma(G)$ or $\Gamma(G)$ coincides with the connected component of $GK(G)$ with vertex set $\pi(|H|)$.
- (3) If G is a solvable group, then $\Gamma(G)$ has at most two connected components. Moreover, if $\Gamma(G)$ is disconnected, then G is either a Frobenius group or a nearly 2-Frobenius group.
- (4) Let G be a solvable group with a Fitting subgroup $F(G)$. If x is a non-vanishing element of G , then $xF(G)$ is a 2-element of $G/F(G)$.
- (5) Let N be a normal subgroup of G . If $xN \in \text{Van}(G/N)$, then $xN \subseteq \text{Van}(G)$.

LEMMA 2.5. (1) *Let S be a simple group with disconnected Gruenberg-Kegel graph, except $U_4(2), U_5(2)$. If G is a finite group with $OC(G) = OC(S)$, then G is neither Frobenius nor 2-Frobenius.*

- (2) *Let $S \in \{U_4(2), U_5(2)\}$. If G is a finite group with $OC(G) = OC(S)$, then G is a 2-Frobenius group or $G \cong S$.*

PROOF. (1) is Main Theorem of [29]. Also, according to [29], there are 2-Frobenius groups U and W with $OC(U) = OC(U_4(2))$ and $OC(W) = OC(U_4(2))$. If G is a finite group with $OC(G) = OC(U_4(2)) = \{2^6.3^4, 5\}$ and G is not a 2-Frobenius group, then by (Theorem 1, [29]) and Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(|H|) \cup \pi(|G/K|) \subseteq \pi_1(G)$, H is nilpotent and K/H is a nonabelian simple group. Since $|G| = 2^6.3^4.5$, according to [34], $K/H \cong A_5, A_6$ or $U_4(2)$. If $K/H \cong A_5, A_6$, then since $G/H \leq \text{Aut}(K/H)$, we have $3 \mid |H|$. Let $H_3 \in \text{Syl}_3(H)$ and $G_5 \in \text{Syl}_5(G)$. Thus $|H_3| = 3^i$, where $i = 2$ or 3 . Since G does not have an element of order 15, we can conclude that G_5 acts fixed point freely on H_3 and hence, $5 \mid (3^i - 1)$ ($i = 2, 3$), a contradiction. Thus $K/H \cong U_4(2)$ which implies that $G \cong U_4(2)$, as desired. If $OC(G) = OC(U_5(2)) = \{2^{10}.3^5.5, 11\}$, and G is not a 2-Frobenius group, then a similar argument implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(|H|) \cup \pi(|G/K|) \subseteq \pi_1(G)$, H is nilpotent and $K/H \cong L_2(11), M_{11}, M_{12}$ or $U_5(2)$. So, it is enough to replace the roles of 5 and 11 in the previous argument to get $G \cong U_5(2)$. \square

Let p be a prime number. Recall that a character χ in $\text{Irr}(G)$ is said to be of p -defect zero if p does not divide $|G|/\chi(1)$. By a fundamental result of R. Brauer (Theorem 8.17, [23]) if $\chi \in \text{Irr}(G)$ is of p -defect zero then, for every element $g \in G$ such that p divides $o(g)$, we have $\chi(g) = 0$.

LEMMA 2.6 (Proposition 2.1, [14]). *Let S be a nonabelian simple group and p a prime number. If S is of Lie type, or if $p \geq 5$, then there exists $\chi \in \text{Irr}(S)$ of p -defect zero.*

REMARK 2.7. If χ vanishes on a p -element of G , then $\chi(1)$ is divisible by p .

PROOF. According to (Corollary 22.26, [25]) the proof is straightforward. \square

LEMMA 2.8. [33] *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic nonabelian simple groups and $G/K \leq \text{Out}(K/H)$.*

LEMMA 2.9. *Let G be a finite group of even order. Suppose that there exists $p \in \pi(|G|)$ such that p and 2 are nonadjacent in $\text{GK}(G)$. If G is nonsolvable, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group, $|G/K| \mid |\text{Out}(K/H)|$ and $K/H \leq G/H \leq \text{Aut}(K/H)$.*

PROOF. According to Theorem 3 in [11] and the proof of Lemma 1 in [33], the proof is straightforward. \square

LEMMA 2.10 (Theorem 1, [2]). *Let G be a finite nonsolvable simple group whose order g is divisible by $p > g^{\frac{1}{3}}$. Then G is isomorphic either to $L_2(p)$, where $p > 3$ is a prime or $L_2(p-1)$, where $p > 3$ is a Fermat prime.*

3. Main Results

The following general results play a role in the proof of Theorems A and B:

LEMMA 3.1. *Let G be a finite group and let S be a finite simple group with disconnected Gruenberg-Kegel graph such that $S \not\cong A_7$ and there exists $2 \leq i \leq t(S)$ such that for every $p \in \pi_i(S)$, we have $p \parallel |S|$. If $\text{Vo}(G) = \text{Vo}(S)$ and $|G| = |S|$, then $m_i(S) \in \text{OC}(G)$. Particularly, the Gruenberg-Kegel graph of G is disconnected.*

PROOF. According to Lemma 2.4(1) and the fact that $\text{Vo}(G) = \text{Vo}(S)$, we have $\Gamma(G) = \Gamma(S) = \text{GK}(S)$. Since $|G| = |S|$, there exists $2 \leq i \leq t(S)$ such that for every $p \in \pi_i(S)$, we have p divides $|G|$ and p^2 does not divide $|G|$. Suppose the assertion of the lemma is false. Thus there exists $q \in \pi_j(S)$, where $1 \leq j \leq t(S)$ and $i \neq j$, such that p and q are adjacent in $\text{GK}(G)$. Since $p \mid |S|$, according to Lemma 2.6 and the fact that $\text{Vo}(S) = \text{Vo}(G)$, we have $p \in \text{Vo}(G)$. So G has an element g of order p such that $\chi(g) = 0$ for some irreducible complex character χ of G . Now, Remark 2.7 implies that p divides $\chi(1)$. Since $p \parallel |S|$ and $|S| = |G|$, χ is an irreducible character of p -defect zero of G . Thus p and q are adjacent in $\Gamma(G)$, which is a contradiction to the fact that $\Gamma(G) = \Gamma(S) = \text{GK}(S)$. \square

According to the above lemma, we have the following corollary:

COROLLARY 3.2. *Let G be a finite group and S be a finite simple group with disconnected Gruenberg-Kegel graph except A_7 . Assume that for every $p \in \pi_i(S)$, where $2 \leq i \leq t(S)$, we have $p \parallel |S|$. If $\text{Vo}(G) = \text{Vo}(S)$ and $|G| = |S|$, then $\text{OC}(G) = \text{OC}(S)$.*

Proof of Theorem A. The proof of Theorem A falls naturally into three parts:

Part 1. Let M be a sporadic simple group. Then according to [32], the Gruenberg-Kegel graph components of M are shown in Table 1 and hence, M and G satisfy the conditions of Corollary 3.2. Thus according to [6], we have $G \cong M$.

Part 2. Let $M = A_n$ be an alternating group. If $\text{GK}(G)$ is not connected, then according to Table 1, one of the numbers n , $n-1$ or $n-2$ is prime. Thus Corollary 3.2 and [1] imply that $G \cong M$. So, to complete the proof, we should consider the case $\text{GK}(G)$ is connected, i.e., n , $n-1$ and $n-2$ are not primes. We will prove the cases $n = 10$ and $n \geq 16$, separately.

- If $n = 10$, then $\text{Vo}(G) = \{2, \dots, 10, 12, 15, 21\}$ and $|G| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$. Since 7 divides $|G|$ and 7^2 does not divide $|G|$, Remark 2.7 implies that G has an irreducible character of 7-defect zero. Thus G does not have any element of order 14. Now we claim that G is nonsolvable. If not, then G has a subgroup K of order 35. We can easily see that K is nilpotent and hence, G has an element of order 35. But

Table 1. The Gruenberg-Kegel graph components of some simple groups

M	Restriction on M	$\pi_1(M)$	m_2	m_3	m_4	m_5	m_6
A_n	$6 < n = p, p + 1, p + 2$ not both $n, n - 2$ prime	$\frac{n!}{2p}$	p				
A_p	$6 < p$ $p, p - 2$ are primes	$\frac{(p)!}{2p(p-2)}$	p	$p - 2$			
M_{12}		{2, 3, 5}	11				
J_2		{2, 3, 5}	7				
Ru		{2, 3, 5, 7, 13}	29				
He		{2, 3, 5, 7}	17				
McL		{2, 3, 5, 7}	11				
Co_1		{2, 3, 5, 7, 11, 13}	23				
Co_3		{2, 3, 5, 7, 11}	23				
Fi_{22}		{2, 3, 5, 7, 11}	13				
HN		{2, 3, 5, 7, 11}	19				
$L_2(q)$	$3 < q \equiv \varepsilon \pmod{4}, \varepsilon = \pm 1$	$\pi(q - \varepsilon)$	$\pi(q)$	$\frac{q+\varepsilon}{2}$			
$L_2(q)$	$3 < q, q$ even	{2}	$q - 1$	$q + 1$			
$L_3(4)$		{2}	3^2	5	7		
$L_3(q)$	$q \neq 2, 4$	$\pi(q(q^2 - 1))$	$\frac{q^3-1}{(q-1)(3,q-1)}$				
$L_4(3)$		{2, 3, 5}	13				
$S_4(q)$		$\pi(q(q^2 - 1))$	$\frac{q^2+1}{(2,q-1)}$				
$S_6(2)$		{2, 3, 5}	7				
$O_8^+(2)$		{2, 3, 5}	7				
$G_2(3)$		{2, 3}	7	13			
$U_3(q)$		$\pi(q(q^2 - 1))$	$\frac{q^3+1}{(q+1)(3,q+1)}$				
$U_4(2)$		{2, 3}	5				
$U_4(3)$		{2, 3}	5	7			
$U_5(2)$		{2, 3, 5}	11				
${}^3D_4(2)$		{2, 3, 7}	13				
${}^2F_4'(2)$		{2, 3, 5}	13				
M_{11}		{2, 3}	5	11			
M_{23}		{2, 3, 5, 7}	11	23			
M_{24}		{2, 3, 5, 7}	11	23			
J_3		{2, 3, 5}	17	19			
HiS		{2, 3, 5}	7	11			
Suz		{2, 3, 5, 7}	11	13			
Co_2		{2, 3, 5, 7}	11	23			
Fi_{23}		{2, 3, 5, 7, 11, 13}	17	23			
F_3		{2, 3, 5, 7, 13}	19	31			
F_2		{2, 3, 5, 7, 11, 13, 17, 19, 23}	31	47			
M_{22}		{2, 3}	5	7	11		
J_1		{2, 3, 5}	7	11	19		
$O'N$		{2, 3, 5, 7}	11	19	31		
LyS		{2, 3, 5, 7, 11}	31	37	67		
Fi_{24}'		{2, 3, 5, 7, 11, 13}	17	23	29		
F_1		{2, 3, 5, 7, 11, 13, 17, 19, 23, $29, 31, 47}$	41	59	71		
J_4		{2, 3, 5, 7, 11}	23	29	31	37	43

this is a contradiction to the fact that G has an irreducible character of 7-defect zero and $35 \notin \text{Vo}(G)$.

Now from Lemma 2.9 we deduce that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group, $|G/K| \mid |\text{Out}(K/H)|$ and $K/H \leq G/H \leq \text{Aut}(K/H)$. According to $|G|$ and [34], K/H is one of the simple groups A_n , where $n \in \{5, 6, 7, 8, 9, 10\}$, $U_4(2)$, $L_3(4)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, J_2 . Moreover, we know that in these cases, $\text{Out}(K/H)$ is a $\{2, 3\}$ -group. So we have the following three characterizable cases:

Case 1. If 7 does not divide $|K/H|$, then K/H is one of the groups A_5 , A_6 or $U_4(2)$. In this case, we can easily see that $|H| = 35k$, where 35 and k are coprimes. Let P be a Sylow 7-subgroup H , then the Frattini argument implies that $G = HN_G(P)$ and hence, $5 \mid |C_G(P)|$. Thus G has an element of order 35. But this is a contradiction to the fact that G has an irreducible character of 7-defect zero and $35 \notin \text{Vo}(G)$.

Case 2. If 7 divides $|K/H|$ and 5 divides $|H|$, then K/H is one of the simple groups A_n , where $n \in \{7, 8, 9\}$, $L_3(4)$, $L_2(7)$, $L_2(8)$, $U_3(3)$. Let P be a Sylow 5-subgroup H , then the Frattini argument implies that $G = HN_G(P)$. Since $7 \mid |G/H|$, $7 \mid |N_G(P)|$. Now we can see that $7 \mid |C_G(P)|$. Thus G has an element of order 35 and we can get a contradiction similar to Case 1.

Case 3. If 7 divides $|K/H|$ and 5 does not divide $|H|$, then according to $|\text{Out}(K/H)|$, $K/H = J_2, A_{10}$. Let $K/H = A_{10}$. According to $|G|$, we can easily conclude that $G \cong A_{10}$. Let $K/H = J_2$. Since $|G/K| \mid |\text{Out}(K/H)| = 2$ and $|G|/|K/H| = 3$, we conclude that G is a central extension of a group of order 3 by J_2 . Also, according to the order of the Schur Multiplier of J_2 , we have this extension splits. Thus $G = C_3 \times J_2$, where C_3 is the cyclic group of order 3. It is easy to see that in this case $30 \in \text{Vo}(C_3 \times J_2)$, which is a contradiction to the fact that $30 \notin \text{Vo}(G)$.

• Let $n \geq 16$ and r_n be the largest prime not exceeding n . Since Remark 2.7 enables us to follow the proofs in [30] to conclude $G \cong M$, here we just mention the sketch of the proof in the following three steps:

Step 1. In this step, we prove that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group and $t_n(1) \mid |K/H|$. ($t_n(k) = \prod_{i=1}^k (\prod_{\substack{n \\ i+1 < p \leq \frac{n}{i}}} p_i)^i$, where p_j is defined as 1 if there is no prime between $\frac{n}{j+1}$ and $\frac{n}{j}$.)

Let $1 = H_0 < H_1 < \dots < H_m = G$ be a chief series of G . Suppose p is a prime dividing $t_n(1)$. Since $p \parallel |G|$, we can assume $p \mid |H_{i+1}/H_i|$ and $p \nmid |H_i|$. Moreover, we can assume that $p' \nmid |H_i|$, for every $p' \mid t_n(1)$. Put $K := H_{i+1}$ and $H := H_i$. Since K/H is a direct product of isomorphic simple groups and $p \parallel |K/H|$, K/H is a group of order p or a nonabelian simple group. If K/H is cyclic, then $\frac{G/H}{C_{G/H}(K/H)}$ is embedded in the cyclic group of order $p-1$. Since $n \geq 16$, there is a prime q ($q \neq p$) such that $q \mid t_n(1)$. An easy calculation shows that $q \nmid (p-1)$ and $p \nmid (q-1)$. Thus $q \in \pi(|C_{G/H}(K/H)|)$ which implies $pq \in \omega(G)$. Since $p \parallel |G|$, Remark 2.7 yields $pq \in \text{Vo}(G) = \text{Vo}(A_n) = \omega(A_n)$, which is a contradiction to the fact that $p+q > n$. Therefore, K/H is a nonabelian simple group. To complete

the proof of this step, let $p' \mid t_n(1)$ and $p' \nmid |K/H|$. Thus $p' \mid |G/K|$ and by the Frattini argument, we have $G = N_G(P)K$, where $P \in \text{Syl}_{p'}(K)$. This implies that G has a subgroup of order pp' which is a contradiction, because $p' \nmid (p-1)$, $p \nmid (p'-1)$ and $pp' \notin \omega(G)$.

Step 2. Let $16 \leq n \leq 82$ and assume that $n, n-1$ and $n-2$ are not primes. According to step 1 and [34], we can see that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong A_m$, $r_n \leq m \leq n$. Let N be the inverse image of $C_{G/H}(K/H)$ in G . Thus $A_m \leq G/N \leq S_m$. Moreover, by an easy calculation, we can see that $G/N \cong A_n$ or S_n and hence, $G \cong A_n$. For instance, let $n = 27$. We have $A_m \leq G/N \leq S_m$, where $m \in \{23, 24, 25, 26, 27\}$. If $m = 27$, then since $|G| = |A_{27}|$, we can easily conclude that $G \cong M$, as desired. So, it is enough to get a contradiction for the case $m \neq 27$. In this case, we have $|N| \in \{3^3, 2 \cdot 3^3 \cdot 13, 3^3 \cdot 13, 2 \cdot 3^3 \cdot 5^2 \cdot 13, 3^3 \cdot 5^2 \cdot 13, 2^4 \cdot 3^4 \cdot 5^2 \cdot 13, 2^3 \cdot 3^4 \cdot 5^2 \cdot 13\}$. If $|N| = 3^3$, then since $8.17 \in \omega(A_{27}) = \text{Vo}(A_{27})$, $8.17 \in \omega(G)$ and hence, we can easily see that $8.17 \in \omega(A_m)$ or $\omega(S_m)$, where $23 \leq m \leq 26$, a contradiction. Thus $13 \mid |N|$. If $N_{13} \in \text{Syl}_{13}(N)$, then the Frattini argument shows that $19 \mid |N_G(N_{13})|$ and since $|N_G(N_{13})/C_G(N_{13})| \mid 12$, we conclude that $13.19 \in \omega(G)$. Now, Remark 2.7 implies that $13.19 \in \text{Vo}(G) = \text{Vo}(A_{27})$, a contradiction.

Step 3. Let $n \geq 83$ and $n, n-1$ and $n-2$ are not primes. According to step 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group. Also, by Remark 2.7, we can easily follow (Lemma 2.1, [30]) to prove that $t_n(6) \mid |K/H|$. This is the main key to show that there exists a normal subgroup N of G such that $G/N \cong A_m$ or S_m , $r_n \leq m \leq n$ in (Lemma 2.4, [30]). Now, it is enough to show that $n = m$. If $m \neq n$, then we derive a contradiction. Let q be the largest prime factor of $n!/m!$. In (Theorem 2.1, [30]), the following results are obtained:

- (1) $q \geq 17$ and $q \geq n - m + 3$.
- (2) If A_m contains the elements of order t , where $\text{gcd}(t, q) = 1$, then $tq \in \omega(G)$.

Since the proof of the above statements relies on the fact that $G/N \cong A_m$ or S_m , we have the same statements here. Put $p_1 := r_m$. If $m - p_1 > 2$, then we take $p_2 = r_{m-p_1}$. Also, if $m - (p_1 + p_2) > 2$, then take $p_3 = r_{m-(p_1+p_2)}$, and so on. Thus there exist certainly some odd primes $p_1 > p_2 > \dots > p_k$ such that $m - 2 \leq p_1 + p_2 + \dots + p_k \leq m$. If $q \neq p_i$, $1 \leq i \leq k$, then from the fact that A_m has an element of order $p_1 p_2 \dots p_k$, we see that G has elements of order $q p_1 p_2 \dots p_k$ from (2). According to (1), we have $p_1 + p_2 + \dots + p_k + q \geq (m-2) + (n-m+3) > n$ which implies that $q p_1 p_2 \dots p_k \notin \omega(A_n)$. But $q \parallel |G|$ and hence, $q p_1 p_2 \dots p_k \notin \omega(G)$, a contradiction. Therefore, there exists $1 \leq i \leq k$ such that $q = p_i$. Put $l = p_1 + p_2 + \dots + p_{i-1}$. Thus $q = r_{m-l}$ and hence, $17 \leq q = p_i \leq m - l \leq 2p_i$. We know that there exists another prime p'_i , $\frac{1}{2}(m-l) < p'_i < m-l$ and $p'_i < p_i$. If $p_1 + p_2 + \dots + p_{i-1} + p'_i \geq m-2$, then we can similarly get a contradiction. Thus $p_1 + p_2 + \dots + p_{i-1} + p'_i < m-2$ and we can assume that $m' = m - (p_1 + p_2 + \dots + p_{i-1} + p'_i) < \frac{1}{2}(m-l)$. We take again $q_1 = r_{m'}$, $q_2 = r_{m'-q_1}, \dots, q_s = r_{m'-(q_1+q_2+\dots+q_{s-1})}$ such that $m' - 2 \leq q_1 + q_2 + \dots + q_{s-1} \leq m'$. Thus $p_1 > p_2 > \dots > p_{i-1} > p'_i > q_1 > q_2 > \dots > q_s$

and $m - 2 \leq p_1 + p_2 + \cdots + p_{i-1} + p'_i + q_1 + q_2 + \cdots + q_s \leq m$. Moreover, $q_i \neq q$, $i = 1, 2, \dots, s$, and hence, we can get a contradiction as above.

Part 3. Let $M = L_2(p)$, where p is an odd prime. Since $\text{Vo}(G) = \text{Vo}(L_2(p))$, according to Lemma 2.4(1) and Table 1, we have $\Gamma(G) = GK(L_2(p))$ and G is a nonsolvable group. Thus Lemma 2.9 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group and $|G/K| \mid |\text{Out}(K/H)|$. According to $|G|$, we can conclude that $p \mid |G/K|$, $p \mid |H|$ or $p \mid |K/H|$.

If $p \mid |G/K|$, then as in the proof of Step 2 in [28], we can get a contradiction. If $p \mid |H|$, then the Frattini argument implies that $G = N_G(P)H$, where P is a Sylow p -subgroup of H . Also, since for every $k > 1$, pk is not an element of $\text{Vo}(G)$, we have $C_G(P) = P$. Thus G/H is isomorphic to a homomorphic image of $N_G(P)/P$. But $N_G(P)/P$ is embedded in the cyclic group $\text{Aut}(P)$. Thus G/H is cyclic, which is a contradiction to the fact that G/H is not solvable. Therefore, $p \mid |K/H|$ and according to $|G|$ and Lemma 2.10, we have $G \cong L_2(p)$, as desired.

Proof of Theorem B. We have divided the proof of Theorem B into a sequence of cases:

Case 1. Let $M = S_6(2)$. According to Table 1 and Corollary 3.2, we can see that $OC(G) = OC(S_6(2))$. Thus Lemmas 2.1, 2.5, 2.9 imply that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group, $|G/K| \mid |\text{Out}(K/H)|$ and $K/H \leq G/H \leq \text{Aut}(K/H)$. According to [34], K/H is isomorphic to one of the following simple groups

$$A_5, A_6, A_7, A_8, A_9, U_4(2), L_2(7), L_2(8), U_3(3), L_3(4), S_6(2).$$

If $K/H \cong A_5, A_6, U_4(2)$, then 7 does not divide $|G/H|$. Since $5 \in \text{Vo}(G/H)$, $\text{Van}(G/H)$ contains an element xH of order 5. Without loss of generality we can assume that $o(x) = 5$. Thus xH is a subset of $\text{Van}(G)$. Fix $L = \langle x \rangle H$. If $R \in \text{Syl}_7(H)$, then Frattini argument implies that $L = N_L(R)H$. Since $5 \mid [L : H] = [N_L(R) : N_H(R)]$, we deduce that $5 \mid |N_L(R)|$. Thus there exist $h \in H$ and $1 \leq i \leq 4$ such that $x^i h \in N_L(R)$ has order 5. Since G does not contain any element of order 35, $\langle x^i h \rangle$ acts fixed point freely on R and hence, $5 \mid 7 - 1$, a contradiction. If $K/H \cong L_2(7), L_2(8), U_3(3)$, then 5 does not divide $|G/H|$ and $7 \mid |G/H|$. Thus replacing the roles of 5 and 7 in the previous argument leads us to get a contradiction. If $K/H \cong A_7, A_8, L_3(4)$, then replacing 7 with 3 and 5 with 7 in the argument given in the above leads us to get a contradiction. Let $K/H \cong A_9$. If $G/H \cong S_9$, then $|H| = 2$ and if $G/H \cong A_9$, then $|H| = 8$. Now applying the previous argument for 2 and 7 shows that $7 \mid (|H| - 1)$ and hence, $G/H \cong A_9$ and $|H| = 8$. If H_1 is a normal minimal subgroup of G such that $H_1 \leq H$, then applying the above argument shows that $7 \mid (|H_1| - 1)$ and hence, $|H_1| = 8$. Thus H is a normal minimal subgroup of G and hence, $H \cong Z_2 \times Z_2 \times Z_2$. Therefore, $G/C_G(H) \leq \text{Aut}(H) \cong GL_3(2)$. Therefore, $2^6 \cdot 3^3 \cdot 5 \mid |C_G(H)|$ and $|C_G(H)| \mid |G|/7$. Also, $C_G(H)/H$ is a normal subgroup of $G/H = K/H$ and hence, simplicity of K/H forces $C_G(H)/H = K/H$ or $C_G(H)/H = 1$, which is a contradiction. Therefore $K/H \cong S_6(2)$ which implies that $G \cong S_6(2)$.

Case 2. Let $M = U_5(2)$. According to Table 1 and Corollary 3.2, we have $OC(M) = OC(G)$. It follows from Lemma 2.5 that G is a 2-Frobenius group or $G \cong M$. We claim that G is not a 2-Frobenius group. Conversely, suppose that G is a 2-Frobenius group with normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$. Since $OC(M) = OC(G)$, according to Table 1 and Lemma 2.3, we have $\pi(|K/H|) = \pi_2(G) = \{11\}$, $|K/H| = 11$ and $|G/K| \mid 10$. Thus by $|U_5(2)| = 2^{10}.3^5.5.11$, $|H| \in \{2^9.3^5.5, 2^{10}.3^5, 2^9.3^5\}$.

Let $Q \in \text{Syl}_{11}(K)$. Since Q acts fixed point freely on H , Thompson's nilpotency criterion shows that H is nilpotent. Thus if $P \in \text{Syl}_p(H)$, where $p \mid |H|$, then $P \trianglelefteq K$ and hence, $11 \mid (|P| - 1)$. This forces $|H| = 2^{10}.3^5$ which implies that $|G/K| = 5$. According to [12], $4, 8 \in \omega(G)$. Thus if $P_2 \in \text{Syl}_2(H)$, then P_2 is not an elementary abelian 2-group. Now, assume that N is a normal minimal subgroup of G such that $N \leq P_2$. Since G is solvable, we conclude that N is an elementary abelian 2-group of order 2^t , where $t > 0$. Thus our assumption on P_2 implies that $1 < 2^t < 2^{10}$. But K/H acts fixed point freely on N and hence, $11 \mid (2^t - 1)$, which is impossible by checking the different values of t . This shows that G is not 2-Frobenius and hence, $G \cong M$.

The proof for $M = U_4(2)$ is similar and we omit the details for the sake of convenience.

Case 3. Let $M = S_4(7)$. Note that $|S_4(7)| = |G| = 2^8.3^2.5^2.7^4$ and the components of $\Gamma(G)$ are $\{2, 3, 7\}$ and $\{5\}$. Let G be solvable and let $F(G)$ be the Fitting subgroup of G . According to Lemma 2.4(2-3), it is easy to see that G is a nearly 2-Frobenius group. If $5 \in \pi(|F(G)|)$, then since $25 \in \text{Vo}(G)$, we deduce that $25 \in \omega(G)$ and hence, $P \in \text{Syl}_5(F(G))$ is a cyclic normal subgroup of G . Therefore, $G/C_G(P)$ is a cyclic group which its order divides 4. Thus considering the components of $\Gamma(G)$, shows that 5 is an isolated point in $\Gamma(G)$, and Lemma 2.4(4) implies that $G/F(G)$ is a 2-group. Since G is nearly 2-Frobenius, $F(G)/F_2 \leq F(G/F_2)$ and hence, $G/F_2/F(G/F_2)$ is a 2-group. Thus $(G/F_2)/F(G/F_2)$ is not a Frobenius group and hence, G/F_2 is not a 2-Frobenius group, which contradicts to the fact that G is nearly 2-Frobenius. Thus $5 \notin \pi(|F(G)|)$. If there exists an element $x \in G$ such that $o(x) = 5r$, where $r \in \{2, 3, 7\}$, then since $5r \notin \text{Vo}(G)$, x is a non-vanishing element. Lemma 2.4(4) now implies that $o(xF(G)) \mid 2^i$ and hence, $5 \in \pi(|F(G)|)$, which is a contradiction. This shows that $GK(G) = \Gamma(G) = \Gamma(S_4(7)) = GK(S_4(7))$. Therefore, $OC(G) = OC(S_4(7))$. Now according to [27], we have $G \cong S_4(7)$, this contradicts the fact that G is solvable. So G is not solvable and by Lemma 2.8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \leq |\text{Out}(K/H)|$. Considering the orders of $S_4(7)$ and the finite simple K_3 -groups and K_4 -groups show that $K/H \cong A_5, A_5 \times A_5, A_6, L_2(7), L_2(7) \times L_2(7), L_2(8), A_7, A_8, L_2(49), L_3(4)$ or $S_4(7)$.

If $K/H \not\cong A_5, A_5 \times A_5, A_6, L_2(7), L_2(7) \times L_2(7), L_2(8)$ and $S_4(7)$, then G/H contains an element xH of order 5. Also, for $P \in \text{Syl}_7(H)$, considering the order of G/H forces $1 < |P| \leq 7^3$. Since $G = N_G(P)H$, without loss of generality, we can assume that $x \in N_G(P)$ and x is a 5-element. Also, since G/H does not contain any normal 5-subgroup, we can assume by (Theorem C, [15]) that $xH \in \text{Van}(G/H)$ and hence, Lemma 2.4(iv) shows that $xH \subseteq \text{Van}(G)$. Thus $xP \subseteq \text{Van}(G)$. On the

other hand, 5 is an isolated point in $\Gamma(G)$, so $\langle x \rangle$ acts fixed point freely on P . Thus $5 \mid |P| - 1$, which is impossible. If $K/H \cong L_2(7)$, $L_2(7) \times L_2(7)$ or $L_2(8)$, then replacing the roles of 5 and 7 in the previous argument and if $K/H \cong A_5$ and A_6 , then replacing 5 with 3 and 7 with 5 and the relative subgroups in the previous argument lead us to get a contradiction. Also, since $25 \in \text{Vo}(G)$, $25 \in \omega(G)$, so $K/H \not\cong A_5 \times A_5$. This shows that $K/H \cong S_4(7)$ and hence, $G \cong S_4(7)$, as claimed.

Case 4. Let $M = L_2(49)$. According to Table 1 and Lemma 2.4(1), we obtain that G is nonsolvable. Since $|L_2(49)| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$, Remark 2.7 implies that G has an irreducible character of 3-defect zero. Thus by Lemma 2.8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong S_1 \times \cdots \times S_t$, where S_i , $1 \leq i \leq t$, is a simple K_3 -group or a simple K_4 -group and for every $1 \leq i, j \leq t$, we have $S_i \cong S_j$. Since $3 \mid |S_i|$, where $1 \leq i \leq t$, and $3 \parallel |G|$, we conclude that $K/H \cong S$, where S is a simple K_3 -group or a simple K_4 -group and $K/H \leq G/H \leq \text{Aut}(K/H)$.

Subcase 1. Let K/H be a simple K_3 -group. Since $|K/H| \mid |G|$ and $3 \parallel |G|$, checking the orders of simple K_3 -groups shows that $K/H \in \{A_5, L_2(7)\}$. If $K/H \cong A_5$, then $A_5 \leq G/H \leq S_5$. It follows that $2^2 \cdot 3 \cdot 5 \mid |G/H|$ and $|G/H| \mid 2^3 \cdot 3 \cdot 5$. Thus $|H| = 7^2 \cdot 2 \cdot 5$ or $|H| = 7^2 \cdot 2^2 \cdot 5$. Let $P \in \text{Syl}_5(H)$. By Fattini's argument, we have $G = N_G(P)H$. Thus $G/H \cong N_G(P)/N_H(P)$ and $3 \mid |N_G(P)|$. Put $Q \in \text{Syl}_3(N_G(P))$. Since G has an irreducible character of 3-defect zero and $15 \notin \text{Vo}(M) = \text{Vo}(G)$, we deduce that $15 \notin \omega(G)$. Thus Q acts fixed point freely on P and hence, $3 = |Q| \mid (|P| - 1) = 5 - 1$, which is impossible. If $K/H \cong L_2(7)$, then we conclude that $G/H \leq \text{Aut}(L_2(7))$. Thus $2^3 \cdot 3 \cdot 7 \mid |G/H|$ and $|G/H| \mid 2^4 \cdot 3 \cdot 7$. Therefore, $|H| = 5^2 \cdot 2 \cdot 7$ or $|H| = 5^2 \cdot 7$, which implies that $n_5(H) = 1$. If $P \in \text{Syl}_5(H)$, then $P \trianglelefteq G$ and we have $P \cong \mathbb{Z}_{25}$, because $25 \in \text{Vo}(M) = \text{Vo}(G)$ and $P \in \text{Syl}_5(G)$. Since $P \leq C_G(P)$, $\frac{|N_G(P)|}{|C_G(P)|} \mid 4$. Thus $\frac{|G|}{|C_G(P)|} \mid 4$ and hence, $\frac{|G|}{4} \mid |C_G(P)|$, which implies that $3 \cdot 25 \in \omega(G)$. But G has an irreducible character of 3-defect zero and hence, $3 \cdot 25 \in \text{Vo}(G) = \text{Vo}(M)$, a contradiction.

Subcase 2. Assume that K/H is a simple K_4 -group. If K/H is isomorphic to one of the groups listed in Lemma 1.1 (2), then comparing the orders of these groups and K/H forces $K/H \cong L_2(49)$ and hence, $H = 1$ and $K = G \cong L_2(49)$, as desired. If $K/H \cong L_2(r)$, then $r \in \{2, 3, 5, 7\}$, which is impossible. If $K/H \cong L_2(2^m)$, where $m \geq 5$, $2^m - 1 = u$ and $(2^m + 1)/3 = t$ are primes, then since $u, t \in \pi(|G|) = \{2, 3, 5, 7\}$, we get a contradiction. Finally, assume that $K/H = L_2(3^m)$, where m and $(3^m + 1)/4 = t$ are odd primes. But $t \in \pi(|G|) = \{2, 3, 5, 7\}$, which is a contradiction.

Case 5. Let $M = L_2(2^m)$, where $2^m + 1/3 = t$ and $2^m - 1 = u$, are primes greater than 3. Then according to Table 1 and Lemma 2.4(1), we obtain that G is nonsolvable. Thus Lemma 2.3 implies that G is not a 2-Frobenius group. Also, if G is a Frobenius group with kernel F and complement H , then according to Lemma 2.2, we have $OC(G) = \{|F|, |H|\}$. Since $u \parallel |G|$ and $u \in OC(M)$, we obtain $u \in OC(G)$, by Lemma 3.1. If $u = |F|$, then $|H| \mid (u - 1)$. Thus $2^m(2^m + 1) \mid (2^m - 2)$, which is impossible. If $u = |H|$ and $P \in \text{Syl}_t(F)$, then since F is nilpotent, we see that $P \trianglelefteq G$ and hence, H acts fixed point freely on P . Thus,

$(2^m - 1) = |H| \mid (|P| - 1) = 2(2^{m-1} - 1)/3$, which is impossible. Thus according to Lemma 2.1, G has a normal series $1 \trianglelefteq K \trianglelefteq H \trianglelefteq G$ and K/H is a nonabelian simple group such that $u \mid |K/H|$.

Subcase 1. If K/H is a simple K_3 -group, then $K/H \in \{A_5, L_2(7)\}$, because $3 \parallel |G|$ and hence, $3 \parallel |K/H|$. We have $u \in \pi(|K/H|)$ and hence, $u = 5$ or $u = 7$. Since $u = 2^m - 1$, we deduce that $u \neq 5$ and hence $K/H \cong A_5$. If $u = 7$, then $m = 3$, which is a contradiction.

Subcase 2. If K/H is a simple K_4 -group, then since $3 \parallel |G|$, we deduce that $3 \parallel |K/H|$ and hence, $K/H \in \{L_2(16), L_2(25), L_2(49), L_3(5), U_3(7), L_2(2^{m'})\}$, under conditions of Lemma 1.1(2). If $K/H \cong L_2(16)$ or $L_2(25)$, then $2^m - 1 \in \{5, 13, 17, 31, 43\}$, which is impossible. If $K/H \cong L_2(49)$, then $u = 3$, which is impossible. Now, if $K/H \cong L_2(r)$, then $r \in \{u, t\}$. If $r = u = 2^m - 1$, then $|L_2(r)| = r(r^2 - 1)/2 = (2^m - 1)2^m(2^{m-1} - 1) \mid (2^m - 1)2^m(2^m + 1)$, and hence $(2^{m-1} - 1) \mid (2^m + 1)$. It follows that $m = 2$ or $m = 3$, which is impossible. If $r = t$, then $r = t = 2^m + 1/3$. Since $u \mid |L_2(r)| = r(r - 1)(r + 1)/2$, we have $2^m - 1 = u \mid (t - 1)/2 = (2^{m-1} - 1)/3$ or $2^m - 1 = u \mid (t + 1)/2 = 2(2^{m-2} + 1)/3$, which is impossible. Finally, if $K/H \cong L_2(2^{m'})$, then $2^{m'} - 1$ is a prime number. Thus $2^{m'} - 1 = u$ or $2^{m'} - 1 = t$. But $t \mid (2^m + 1)$, and $2^{m'} - 1 = u$. From this, we have $2^{m'} - 1 = u = 2^m - 1$, and hence $m' = m$. It shows that $G \cong L_2(2^m)$, as claimed.

Case 6. Let $M = L_2(25)$. According to Table 1, we obtain that $\Gamma(L_2(25))$ has three components. Thus Lemmas 2.4 and 3.1 show that G is a nonsolvable group and $13 \in OC(G)$. Since G is nonsolvable, G is not 2-Frobenius. Also, Lemma 2.2 and checking the orders imply that G is not a Frobenius group. Thus according to Lemma 2.1, G has a normal series $1 \trianglelefteq K \trianglelefteq H \trianglelefteq G$ such that $13 \in \pi(|K/H|)$. Furthermore, $|K/H| \in \{13 \cdot p^\alpha \cdot q^\beta, 13 \cdot 2^\alpha \cdot 3^\beta \cdot 5^\gamma\}$, where $p, q \in \{2, 3, 5\}$ and $\alpha, \beta, \gamma \in \mathbb{N}$. If $|K/H| = 13 \cdot p^\alpha \cdot q^\beta$, then by checking the orders of simple K_3 -groups in Lemma 1.1(1), we can easily get a contradiction. Thus $|K/H| = 13 \cdot 2^\alpha \cdot 3^\beta \cdot 5^\gamma$ and K/H is one of the groups listed in Lemma 1.1(2). If K/H is a group listed in Lemma 1.1(2-a), then checking the orders of the groups shows that $K/H \cong L_2(25)$. Thus $|G| = |K/H|$ which implies that $G = K \cong L_2(25)$, as desired. If K/H is a group listed in Lemma 1.1(2-b,d), then we can see that $7 \in \pi(|K/H|)$, a contradiction. Also, if K/H is a group listed in Lemma 1.1(2-c), then $K/H \cong L_2(2^m)$, where $m \geq 5$ and $2^m - 1 = u$ is prime. Thus $u \notin \{3, 5, 13\}$, which is a contradiction.

If $M = L_2(81)$, then replacing 13 with 41 in the argument given for $L_2(25)$ leads us to see that $G \cong L_2(81)$.

Case 7. Let $M = L_2(3^m)$, under conditions of Lemma 1.1(2-d). According to Table 1 and Lemma 2.4(1), we obtain that G is nonsolvable. Thus by Lemma 2.8, G has a normal series $1 \trianglelefteq K \trianglelefteq H \trianglelefteq G$ such that $K/H \cong S_1 \times \cdots \times S_l$, where S_i , $1 \leq i \leq l$, is a simple K_3 -group or a simple K_4 -group and for every $1 \leq i, j \leq t$, we have $S_i \cong S_j$. Since $|L_2(3^m)| = 3^m \cdot (3^m - 1) \cdot (3^m + 1)/2$, conditions of Lemma 1.1(2-d) show that $4 \parallel |G|$ and G has an irreducible character of u -defect zero, where $t = (3^m + 1)/4$. Since $4 \parallel |G|$ and $4 \mid |S_i|$, we deduce that $l = 1$ and

$4 \parallel |K/H|$. Therefore, K/H is a simple K_3 -group or a simple K_4 -group. Let $u \mid (3^m - 1)/2$, under conditions of Lemma 1.1(2-d).

Subcase 1. Let K/H be a simple K_3 -group. Since $4 \parallel |K/H|$, we deduce that $K/H \cong A_5$, by checking the orders of simple K_3 -groups. Thus $5 \in \pi(|G|) = \{2, 3, u, t\}$. Therefore, $5 \mid (3^m - 1)$ or $5 \mid (3^m + 1)$. This shows that $2 \mid m$, which is a contradiction with conditions of Lemma 1.1(2-d).

Subcase 2. Assume that K/H is a simple K_4 -group. Since $4 \parallel |K/H|$, we deduce that $K/H \cong L_2(3^e)$ or $L_2(r)$ satisfying conditions of Lemma 1.1(2-b,d). First let $K/H \cong L_2(3^e)$. Since $\pi(|K/H|) = \pi(|G|)$ and $|K/H| \mid |G|$, we deduce that $e \leq m$ and $u, t \in \pi(|K/H|)$. If $u \mid (3^m - 1)/2$ and $u \mid (3^e - 1)/2$, then $e = m$ and hence, $K/H \cong M$. Since $|G| = |M| = |K/H|$, we deduce that $H = 1$ and $K = G$ and hence, $G \cong M$, as desired. Also, if $t \mid (3^m + 1)/4$ and $t \mid (3^e - 1)/2$, then $t \mid \gcd((3^m + 1)/4, (3^e - 1)/2)$ and hence, $2m \mid e$. This forces e is even, which is a contradiction.

If $K/H \cong L_2(r)$, then we can see at once that $r \in \{t, u\}$. If $r = u$, then since $|K/H| = u(u^2 - 1)/2$ and either $3^m - 1 = 2u$ or 2.11^2 , we deduce that $|K/H| = 3(3^m - 1)(3^m + 1)(3^{m-1} - 1)/8 \mid |G|$ or $|K/H| = 2^2.3.5.11$. Thus either $(3^{m-1} - 1)/4 \mid 3^{m-1}$ or $t = (3^5 + 1)/4 \mid 2^2.3.5$, which is impossible. If $r = t$, then since $|K/H| = t(t^2 - 1)/2$, we deduce that $u \mid (t - 1)$ or $(t + 1)$, which is a contradiction, because $3^m + 1 = 4t$.

If $M \in \{L_3(4), L_2(8), Sz(8), Sz(32)\}$, then according to [17], we have $G \cong M$. Thus it remains to consider the case in which M is one of the groups $L_2(16)$, $L_3(q)$, where $q \in \{3, 5, 7, 8, 17\}$, $U_3(q)$, where $q \in \{3, 4, 5, 7, 8, 9\}$, $S_4(q)$, where $q \in \{4, 5, 9\}$, $L_4(3)$, $U_4(3)$, $D_4(2)$, $G_2(3)$, ${}^3D_4(2)$, ${}^2F_4(2)'$. Thus M satisfies the conditions of Corollary 3.2 and hence, we have $OC(G) = OC(M)$. If $M \in \{L_3(3), U_3(3), U_3(4), U_3(5), {}^2F_4(2)'\}$, then similar argument for the group $U_4(2)$ in Lemma 2.5 shows that $G \cong M$. Moreover, according to [7], [9], [10], [13], [19], [20], [21], [22], [26], [35] the remaining groups are characterizable by their order components and hence the proof of Theorem B is complete.

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