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On the singular spectrum of tempered ultrahyperfunctions corresponding to proper convex cones

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Abstract – We address in this paper the issue of singularities of ultrahyperfunctions. Following the Carmichael's approach for ultrahyperfunctions, we study the relation between the singular spectrum of a class of tempered ultrahyperfunctions corresponding to proper convex cones and their expressions as boundary values of holomorphic functions. In passing, a simple version of the celebrated edge of the wedge theorem for this setting is derived from the integral representation without using cohomology.

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1. Introduction

In 1958 Sebastião e Silva [28, 29] introduced the class of generalized functions called by him tempered ultradistributions, which are the Fourier image of L. Schwartz distributions of exponential type. Later, Hasumi [16] considered the global theory of tempered ultrahyperfunctions in the higher dimensional space, and Morimoto [19, 20, 21] (who coined the name tempered ultrahyperfunctions) localized the theory of tempered ultrahyperfunctions in the imaginary direction.

The interest in ultrahyperfunctions arose simultaneously with the growing interest in various classes of analytic functionals and various attempts to develop a theory of such functionals which would be analogous to the Schwartz theory of distributions. Since then, various developments of the theory have been proposed by many authors, among others we refer the reader to [35, 22, 8, 9, 10, 34, 26, 30, 3, 27, 15, 14, 4, 13, 5, 6, 25, 7, 12]. More recently, the study of ultrahyperfunctions has been stimulated by relativistic quantum field theory with a fundamental length [3, 14, 4, 5].

Using the cohomological approach, Morimoto gave a clear description of the cotangential components of singularities of ultrahyperfuntions in Ref. [22] (the ultrahyperfunctions were called in [22] cohomological ultradistributions), not necessarily assumed to be tempered. Following Nishiwada [23, 24], we present an alternative way of describing the singularities of a class of tempered ultrahyperfunctions corresponding to proper convex cones in terms of generalized boundary values of holomorphic functions; namely, the singular spectrum of a tempered ultrahyperfunction u is characterized by the directions from which the boundary values can be taken in an analytic representation of u.

Our approach for tempered ultrahyperfunctions parallels that of Carmichael addressed in [8, 9, 10]. In this approach, boundary values of holomorphic functions in tube domains with edge an open set X of $\mathbb{R}^n$ are tempered ultrahyperfunctions in X; and conversely a tempered ultrahyperfunction in $\mathbb{R}^n$ is the sum of boundary values of holomorphic functions in tube domains. Thus, an analysis of the arguments of Nishiwada and those of Carmichael shows that the connection between the singular spectrum and the boundary values of holomorphic functions can be expressed naturally in the setting of ultrahyperfunctions. We are therefore in the ideal situation to study the singularities of tempered

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ultrahyperfunctions in a manner which follows hyperfunction theory as closely as possible. An important tool for the analysis leading to this study is the extension of the Paley-Wiener-Schwartz theorem for this class of generalized functions. In addition, we prove the edge of the wedge theorem for tempered ultrahyperfunctions corresponding to a convex cone.

2. Tempered ultrahyperfunctions in a nutshell

To begin with, we shall recall very briefly the basic definition of tempered ultrahyperfunctions. Firstly, we shall consider the function

$$h_K(\xi) = \sup_{x\in K} \bigl|\langle \xi, x\rangle\bigr| \;, \quad \xi \in \mathbb{R}^n \;,$$

the indicator of K, where K is a compact set in $\mathbb{R}^n$. $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = \bigl[-k, k\bigr]^n$, $0 < k < \infty$, the indicator function $h_K(\xi)$ can be easily determined:

$$h_K(\xi) = \sup_{x\in K} \bigl|\langle \xi, x\rangle\bigr| = k|\xi| \;, \quad \xi \in \mathbb{R}^n \;, \quad |\xi| = \sum_{i=1}^{n} |\xi_i| \;.$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_b(\mathbb{R}^n;K)$ (b stands for bounded) defines the space of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)}D^\alpha\varphi(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α. One defines in $H_b(\mathbb{R}^n;K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{\xi\in\mathbb{R}^n;\alpha\leqslant N} \bigl\{e^{h_K(\xi)}|D^\alpha\varphi(\xi)|\bigr\} < \infty \;, \quad N = 0,1,2,\dots \;. \tag{1}$$

The space $H_b(\mathbb{R}^n;K)$ equipped with the topology given by the seminorms (1) is a Fréchet space [16, 19]. If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1}(\xi) \leqslant h_{K_2}(\xi)$, and thus the canonical injection $H_b(\mathbb{R}^n;K_2) \hookrightarrow H_b(\mathbb{R}^n;K_1)$ is continuous. Let O be a convex open set of $\mathbb{R}^n$. To define the topology of $H(\mathbb{R}^n;O)$ it suffices to let K range over an increasing sequence of convex compact subsets $K_1, K_2, \dots$ contained in O such that for each $i = 1, 2, \dots$, $K_i \subset K^\circ_{i+1}$ (K°_{i+1} denotes the interior of K_{i+1}) and $O = \bigcup_{i=1}^\infty K_i$. Then the space $H(\mathbb{R}^n;O)$ is the projective limit of the spaces $H_b(\mathbb{R}^n;K)$ according to restriction mappings above, *i.e.*

$$H(\mathbb{R}^n;O) = \lim_{K\subset O}\operatorname{proj}\; H_b(\mathbb{R}^n;K) \;, \tag{2}$$

where K runs through the convex compact sets contained in O. By $H'(\mathbb{R}^n;O)$ we denote the dual space of $H(\mathbb{R}^n;O)$.

Proposition 2.1 (Hasumi [16, Proposition 3], Morimoto [19, Theorem 5]). *A distribution $V \in H'(\mathbb{R}^n;O)$ may be expressed as a finite order derivative of a continuous function of exponential growth*

$$V = D^\gamma_\xi[e^{h_K(\xi)}g(\xi)] \;,$$

where $g(\xi)$ is a bounded continuous function.

In the space $\mathbb{C}^n$ of n complex variables $z_i = x_i + iy_i$, $1 \leqslant i \leqslant n$, we denote by $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ the tubular set of all points z, such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, *i.e.*, Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$, then $\mathfrak{H}_b(T(K))$ defines the space of all continuous functions φ on $T(K)$ which are holomorphic in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \leqslant \mathbf{M}_{K,N}(\varphi)(1+|z|)^{-N} \tag{3}$$

is valid. The best possible constants in (3) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{K,N} = \inf\Bigl\{\mathbf{M}_{K,N}(\varphi) \mid \sup_{z\in T(K)}(1+|z|)^N|\varphi(z)| < \infty, N = 0,1,2,\dots\Bigr\} \;. \tag{4}$$

If $K_1 \subset K_2$ are two convex compact sets, we have that the canonical injection

$$\mathfrak{H}_b(T(K_2)) \hookrightarrow \mathfrak{H}_b(T(K_1)) \;, \tag{5}$$

is continuous.

Given that the spaces $\mathfrak{H}_b(T(K_i))$ are Fréchet spaces, with topology defined by the seminorms (4), the space $\mathfrak{H}(T(O))$ is characterized as a projective limit of Fréchet spaces:

$$\mathfrak{H}(T(O)) = \lim_{K \subset O} \operatorname{proj} \, \mathfrak{H}_b(T(K)) \ , \tag{6}$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings above.

Let K be a convex compact set in $\mathbb{R}^n$. Then the space $\mathfrak{H}(T(K))$ is characterized as a inductive limit

$$\mathfrak{H}(T(K)) = \lim_{K_1 \supset K} \operatorname{ind} \, \mathfrak{H}_b(T(K_1)) \ , \tag{7}$$

where K_1 runs through the convex compact sets such that K is contained in the interior of K_1 and the inductive limit is taken following the restriction mappings (5).

The Fourier transformation is well defined on the space $H(\mathbb{R}^n;O)$. If $\varphi \in H(\mathbb{R}^n;O)$, the Fourier transform of φ belongs to the space $\mathfrak{H}(T(O))$, for any open convex nonempty set $O \subset \mathbb{R}^n$. By the dual Fourier transform $H'(\mathbb{R}^n;O)$ is topologically isomorphic with the space $\mathfrak{H}'(T(-O))$ [19].

Remark 2.2. We will put $\mathfrak{H} = \mathfrak{H}(\mathbb{C}^n) = \mathfrak{H}(T(\mathbb{R}^n))$ and, as usual, we shall denote the dual space of $\mathfrak{H}$ by $\mathfrak{H}'$.

Definition 2.3. A tempered ultrahyperfunction is by definition a continuous linear functional on $\mathfrak{H}$.

In this article we are interested in the class of tempered ultrahyperfunctions corresponding to proper convex cones. Therefore, we start by recalling some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if C (unless specified otherwise, all cones will have their vertices at zero) is invariant under positive homoteties, *i.e.*, if for all $\lambda > 0$, $\lambda C \subset C$. A cone C is an open connected cone if C is an open connected set. Moreover, C is called convex if $C + C \subset C$ and *proper* if it contains no any straight line (observe that if C is a proper cone, it follows that if $y \in C$ and $y \neq 0$ then $-y \notin C$). A cone C' is called compact in C – we write $C' \Subset C$ – if the projection $\mathsf{pr}\overline{C}' \overset{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathsf{pr}C \overset{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in y-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \bigl\{\xi \in \mathbb{R}^n \mid \langle \xi, y \rangle \geqslant 0, \forall\, y \in C\bigr\}$. The cone C^* is called the *dual cone* of C.

Definition 2.4. Let C be a proper open convex cone, and let $C' \Subset C$. Let $B[0;r]$ denote a **closed** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. We define by

$$T_\delta^{(C' \setminus (C' \cap B[0;r]))} = \Bigl\{ x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in \bigl(C' \setminus \bigl(C' \cap B[0;r]\bigr)\bigr), |y| < \delta < \infty \Bigr\} \ ,$$

where $\delta > 0$ is an arbitrary but fixed number, the **truncated** tube domain.

We are going to introduce a space of holomorphic functions which satisfy certain estimate according to Carmichael [8, 9, 10]. We want to consider the space consisting of holomorphic functions f such that

$$\bigl|f(z)\bigr| \leqslant \mathbf{M}(C')(1+|z|)^N e^{h_{C^*}(y)} \ , \quad z = x + iy \in T_\delta^{(C' \setminus (C' \cap B[0;r]))} \ , \tag{8}$$

where $h_{C^*}(y) = \sup\bigl\{\langle \xi, y \rangle \mid \xi \in C^*\bigr\}$ is the supporting function of C^*, $\mathbf{M}(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. Clearly, as C^* is a convex set, one has $h_{C^*}(y) = h_{\operatorname{ch} C^*}(y)$, where $\operatorname{ch} C^*$ is the convex hull of C^*. The set of all functions f which for every cone $C' \Subset C$ are holomorphic in $T_\delta^{(C' \setminus (C' \cap B[0;r]))}$ and satisfy the estimate (8) will be denoted by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.

Remark 2.5. Here, it should be noted that the truncated cone acts such as to ensure that the supporting function $h_{C^*}(y)$ is finite. In fact, we have $\langle \xi, y \rangle \leqslant |\xi||y| < |\xi|\delta$, and since C^* is a cone, if we take $\alpha\xi$, with $\alpha > 0$, instead of ξ, then our old supremum is multiplied by α to get the new supremum, but this new supremum *must be still finite* if we define $\delta' = \alpha\delta$ in order to obtain a new upper bound for $|y|$.

Remark 2.6. The space of functions $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ constitutes a generalization of the space $\mathfrak{A}_{\omega}^{i}$ of Sebastião e Silva [28] and the space $\mathfrak{A}_{\omega}$ of Hasumi [16] to arbitrary tubular radial domains in $\mathbb{C}^n$ (a tube domain is said to be radial if its base is a connected cone in $\mathbb{R}^n$).

We now shall introduce another space of holomorphic functions whose elements are analytic in a domain which is larger than $T_\delta^{(C'\setminus(C'\cap B[0;r]))}$ and has boundary values in $\mathbb{R}^n$. Let $B(0;r)$ denote an **open** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Let $T_\delta^{(C'\setminus(C'\cap B(0;r)))}$ denote the subset of $\mathbb{C}^n$ defined by $T_\delta^{(C'\setminus(C'\cap B(0;r)))} = \Big\{x+iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in \big(C' \setminus \big(C' \cap B(0;r)\big)\big), |y| < \delta < \infty\Big\}$, where $\delta > 0$ is an arbitrary but fixed number.

Definition 2.7. Let C be a proper open convex cone, and let $C' \Subset C$. Denote by $T_\delta(C')$ the tube domain

$$\Big\{x+iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C', |y| < \delta < \infty\Big\}$$

We say that the function $f = f(z)$ is in the space $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ if it is holomorphic in $T_\delta(C')$ and satisfies the estimate

$$\big|f(z)\big| \leqslant \mathbf{M}(C')(1+|z|)^N e^{h_{C^*}(y)}\;, \quad z = x+iy \in T_\delta^{(C'\setminus(C'\cap B(0;r)))}\;. \tag{9}$$

Note that $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}} \subset \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ for any open convex cone C. Following Hasumi [16], we define the *kernel* of the mapping $f : \mathfrak{H} \to \mathbb{C}$ by $\mathbf{\Pi}$, where $\mathbf{\Pi}$ is the set of all pseudo-polynomials in one of the variables $z_1,\ldots,z_n$. We recall that a pseudo-polynomial is a function the form $\sum_\alpha z_j^\alpha G_\alpha(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n)$, where $G_\alpha(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n)$ are functions in $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ with respect to $(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n)$. Then, $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ belongs to the kernel $\mathbf{\Pi}$ if and only if $\langle f(z), \psi(x)\rangle = 0$, with $\psi \in \mathfrak{H}$ and $x = \operatorname{Re} z$. Put $\mathscr{U}_c = \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}/\mathbf{\Pi}$, that is, $\mathscr{U}_c$ is the quotient space of $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ by set of pseudo-polynomials $\mathbf{\Pi}$.

Definition 2.8. The set $\mathscr{U}_c$ is the subspace of the tempered ultrahyperfunctions generated by $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ corresponding to a proper open convex cone $C \subset \mathbb{R}^n$.

Definition 2.9. We denote by $H'_{C^*}(\mathbb{R}^n;O)$ the subspace of $H'(\mathbb{R}^n;O)$ of distributions of exponential growth with support in the cone C^*:

$$H'_{C^*}(\mathbb{R}^n;O) = \Big\{V \in H'(\mathbb{R}^n;O) \mid \operatorname{supp}(V) \subseteq C^*\Big\}\;. \tag{10}$$

3. Boundary values of holomorphic functions

In this section the boundary values of holomorphic functions are always considered in the distribution sense defined below. We say that $f \in T_\delta(C')$ has a boundary value $U = BV(f)$ in $\mathfrak{H}'$ as $y \to 0$, $y \in C' \Subset C$, if for all $\psi \in \mathfrak{H}$ the limit

$$\langle U,\psi\rangle = \lim_{\substack{y\to 0\\ y\in C'}} \int_{\mathbb{R}^n} f(x+iy)\psi(x)\,dx\;,$$

exists.

The following theorem shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ have distributional boundary values in $\mathfrak{H}'(T(O))$. Further, it shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ satisfy a strong boundedness property in $\mathfrak{H}'(T(O))$.

Theorem 3.1 (Franco [13, Theorem 5.2]). *Let C be a proper open convex cone, and let $C' \Subset C$. Let $V = D_\xi^\gamma[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n;O)$. Then*

(i) $f(z) = (2\pi)^{-n}\big\langle V, e^{i\langle\xi,z\rangle}\big\rangle$ is an element of $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$,

(ii) $\big\{f(z) \mid y = \operatorname{Im} z \in C' \Subset C, |y| \leqslant \delta\big\}$ is a strongly bounded set in $\mathfrak{H}'(T(O))$, where δ is an arbitrarily but fixed positive real number,

(iii) $f(z) \to \mathscr{F}^{-1}[V] \in \mathfrak{H}'(T(O))$ in the strong (and weak) topology of $\mathfrak{H}'(T(O))$ as $y = \operatorname{Im} z \to 0$, $y \in C' \Subset C$.

The functions $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ can be recovered as the (inverse) Fourier-Laplace transform of the constructed distribution $V \in H'_{C^*}(\mathbb{R}^n;O)$. This result is a version of the Paley-Wiener-Schwartz theorem in the tempered ultrahyperfunction set-up.

THEOREM 3.2 (Franco [13, Theorem 5.3]). *Let $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$, where C is a proper open convex cone. Then the distribution $V \in H'_{C^*}(\mathbb{R}^n;O)$ has a uniquely determined inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}\big\langle V, e^{i\langle \xi,z\rangle}\big\rangle$ which is holomorphic in $T_\delta(C')$ and satisfies the estimate (8), with $B[0;r]$ replaced by $B(0;r)$.*

REMARK 3.3. We would like to emphasize that in Theorems 4.1 and 4.2 above, we are considering the inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}\big\langle V, e^{i\langle \xi,z\rangle}\big\rangle$, in contrast to the inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}\big\langle V, e^{-i\langle \xi,z\rangle}\big\rangle$ adopted in Theorems 5.2 and 5.3 in [13]. This is due to the convention of signs in the Fourier transform which is used here.

The same proof as in Carmichael [9, Theorem 1, equation (4)] combined with the proofs of Theorems 3.1 and 3.2 shows that the following proposition is true.

PROPOSITION 3.4. *Let C be a proper open convex cone, and let $C' \Subset C$. Let $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$. Then there exists a unique element $V \in H'_{C^*}(\mathbb{R}^n;O)$ such that*

$$f(z) = \mathscr{F}^{-1}\big[e^{-\langle \xi,y\rangle}V\big]\ , \quad z \in T_\delta^{(C'\setminus(C'\cap B(0;r)))}\ , \tag{11}$$

where (11) holds as an equality in $\mathfrak{H}'(T(O))$.

At this point we must clarify a fact that may not have been clear in Refs. [14, 13]. Let $\mathfrak{H}'_C(T(O))$ denote the subset of $\mathfrak{H}'(T(O))$ defined by $\mathfrak{H}'_C(T(O)) = \big\{U \in \mathfrak{H}'(T(O)) \mid U = \mathscr{F}^{-1}[V], V \in H'_{C^*}(\mathbb{R}^n;O)\big\}$. Then, the boundary value mapping given in *(iii)*, Theorem 3.1, maps $\mathscr{H}_c^{*\,o}$ on $\mathfrak{H}'_C(T(O))$. This follows immediately because of the recovery of the elements $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ as the (inverse) Fourier-Laplace transform of the constructed distribution $V \in H'_{C^*}(\mathbb{R}^n;O)$ and the fact that the Fourier and inverse Fourier transforms from $\mathfrak{H}'_C(T(O))$ to $H'_{C^*}(\mathbb{R}^n;O)$ and $H'_{C^*}(\mathbb{R}^n;O)$ to $\mathfrak{H}'_C(T(O))$, respectively, are isomorphisms! In fact, it is easy to verify the

PROPOSITION 3.5. *$\mathscr{U}_c$ is isomorphic to the space $\mathfrak{H}'_C(T(O))$, with the isomorphism being defined by the mapping BV. If $u \in \mathscr{U}_c$ and $U \in \mathfrak{H}'_C(T(O))$ is the corresponding element under the isomorphism, then*

$$\langle BV_{C'}(f),\psi\rangle = \lim_{\substack{y\to 0\\ y\in C'}} \int_{\mathbb{R}^n} f(x+iy)\psi(x)\,dx = \langle U,\psi\rangle\ , \quad \psi \in \mathfrak{H}\ , \tag{12}$$

where $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ is any representative of u.

REMARK 3.6. We use the notation $BV_{C'}(f)$ in order to emphazise that the boundary value of f is obtained taking the limit $y = \operatorname{Im} z \to 0$ from the directions of the cone C'.

COROLLARY 3.7 (See Sebastião e Silva [28, Proposition 11.1]). *Every element $u \in \mathscr{U}_c$ is represented by a holomorphic function $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ under the form*

$$f(z) = \int_{C^*} V(\xi)e^{i\langle \xi,z\rangle}d\xi\ , \tag{13}$$

where $V \in H'_{C^*}(\mathbb{R}^n;O)$. We will note it by $u = [f]$.

REMARK 3.8. In what follows, we always shall understand the results below as related to the subset $\mathfrak{H}'_C(T(O))$ of $\mathfrak{H}'(T(O))$.

Our next step is to prove that the mapping $f \to BV(f)$ is injective. We do this in a constructive way, beginning by proving the following

THEOREM 3.9. *Let $\Omega \subset \mathbb{C}^n$ be an open set. Then, $\mathfrak{H}(T(\mathbb{R}^n))$ is dense in $\mathfrak{H}(\Omega)$.*

An indication of proof of this theorem was given in [19]. Here, our approach to this problem makes use of a construction of a suitable approximation of the identity modeled on the kernel of the heat equation due to Baouendi-Treves [1, 32]. This construction has been a key tool for many function and distributional spaces [18] (see also [2]).

Proof of Theorem 3.9. To begin with, let $B_0, B_1 \subset \mathbb{R}^n$ be two open balls, both centered at the origin, and $\Phi = (\Phi_1, \ldots, \Phi_n) : B_0 \to B_1$ a C^∞ map. Assume a map $Z : B_0 \to \Omega$, where

$$\Omega = \Big\{ Z(x) = x + iy \mid x \in B_0, y = \Phi(x) \in B_1 \Big\} \subset \mathbb{C}^n \ , \tag{14}$$

is an open set. Furthermore, $\Phi_j(0) = 0$ and $\partial\Phi_j(0)/\partial x_k = 0$, $j = 1, \ldots, n$. Note that

$$\left(\frac{\partial Z_j}{\partial x_k}\right)_{1 \leqslant j,k \leqslant n} = \left(\delta_{jk} + i\frac{\partial \Phi_j}{\partial x_k}\right)_{1 \leqslant j,k \leqslant n} \ , \tag{15}$$

are elements of a nonsingular complex square matrix. We will assume that

$$\big|\Phi(x)\big| < \frac{1}{2}\big|x\big| \ , \quad \forall\, x \in B_0 \ , \tag{16}$$

and we may assume that

$$\left|\frac{\partial \Phi_j}{\partial x_k}\right| < \frac{1}{2} \ , \forall\, x \in \overline{B}_0 \ . \tag{17}$$

Take, for $\tau > 0$, the following Gaussian function in $\mathbb{C}^n \times \overline{B}_0$:

$$\boldsymbol{\mathcal{E}}_\tau(z;x) = (\tau/\pi)^{n/2}\, e^{-\tau\langle z - Z(x)\rangle^2} \ ,$$

where the notation $\langle z - Z(x)\rangle^2$ means $\sum_{j=1}^n \big(z_j - Z_j(x)\big)^2$. Then, given any holomorphic function $\varphi \in \mathfrak{H}(\Omega)$, we define the basic operator in the Baouendi-Treves approximation formula:

$$\varphi_\tau(z) \stackrel{\text{def}}{=} [\boldsymbol{\mathcal{K}}_\tau \varphi](z) = \int_{\overline{B}_0} \boldsymbol{\mathcal{E}}_\tau(z;x)\varphi(Z(x)) \det\left(\frac{\partial Z_j}{\partial x_k}\right)\, dx \ .$$

It is clear that φ_τ is an entire function of z.

Lemma 3.10. *$\varphi_\tau \in \mathfrak{H}(T(\mathbb{R}^n))$.*

Proof of Lemma 3.10. It is convenient to make the hypothesis that $Z_j \in C^\infty(\overline{B}_0)$, $j = 1, \ldots, n$. Let $Q_k = \big\{ z \in \mathbb{C}^n \mid |\mathscr{I}m\, z| \leqslant k; j = 1, \ldots, n \big\}$. Thus,

$$\begin{aligned} \sup_{z \in Q_k} (1+|z|)^N |\varphi_\tau(z)| &\leqslant \sup_{z \in Q_k} \int_{\overline{B}_0} (1+|z|)^N |\boldsymbol{\mathcal{E}}_\tau(z;x)\varphi(Z(x))| \left|\det\left(\frac{\partial Z_j}{\partial x_k}\right)\right| dx \\ &\leqslant \mathsf{M} \sup_{z \in Q_k} \int_{\overline{B}_0} (1+|z|)^N \frac{|\boldsymbol{\mathcal{E}}_\tau(z;x)|}{(1+|x|)^M} \left|\det\left(\frac{\partial Z_j}{\partial x_k}\right)\right| dx \ . \end{aligned}$$

Taking $k \gg \operatorname{diam} B_0$ (possibly after decreasing the radius of B_0), we obtain that

$$|\boldsymbol{\mathcal{E}}_\tau(z;x)| \leqslant (\tau/\pi)^{n/2} e^{-\tau|\mathscr{R}e\, z - x|^2} e^{\tau k^2} \ .$$

We also observe that $\left|\det\left(\frac{\partial Z_j}{\partial x_k}\right)\right|$ is bounded because of (16). Hence,

$$\sup_{z \in Q_k} (1+|z|)^N |\varphi_\tau(z)| \leqslant \mathsf{M}'(\tau/\pi)^{n/2} e^{\tau k^2} \sup_{\mathscr{R}e\, z \in \mathbb{R}^n} \int_{\overline{B}_0} \frac{(1+k+|\mathscr{R}e\, z|)^N}{(1+|x|)^M} e^{-\tau|\mathscr{R}e\, z - x|^2}\, dx \ .$$

Finally, note that

$$\frac{(1+k+|\mathscr{R}e\, z|)^N}{(1+|x|)^M} e^{-\tau|\mathscr{R}e\, z - x|^2} \ ,$$

is a bounded function of $\mathscr{R}e\, z$ and x for $M > N$. In turn, this estimate can be continued by

$$\sup_{z \in Q_k} (1+|z|)^N |\varphi_\tau(z)| \leqslant \mathsf{M}''(\tau/\pi)^{n/2} e^{\tau k^2} \int_{\mathbb{R}^n} e^{-\tau|\mathscr{R}e\, z - x|^2}\, dx < \infty \ .$$

This proves that $\varphi_\tau(z) \in \mathfrak{H}(T(\mathbb{R}^n))$. $\square$

The following lemma can easily be established using standard techniques.

LEMMA 3.11. *Let φ_τ be a bounded sequence in $\mathfrak{H}(\Omega)$ and $\varphi \in \mathfrak{H}(\Omega)$. If the sequence of functions φ_τ converges to φ uniformly, as $\tau \to \infty$, then φ_τ converges to φ in the topology of $\mathfrak{H}(\Omega)$ defined by semi-norms $\|\varphi\|_{\Omega,N} = \sup_{z\in\Omega}(1+|z|)^N|\varphi(z)|$.*

From now on we fix our attention on the convergence of $\varphi_\tau \to \varphi$. At this point, we use an argument used in [2, page 55]: modifying the functions Φ_j off neighborhood of $\overline{B}_0$ may assume without loss of generality that the functions $\Phi_j(x)$, $j = 1, \ldots, n$ are defined throughout $\mathbb{R}^n$, have compact support and satisfy (17) everywhere, that is

$$\left|\frac{\partial \Phi_j}{\partial x_k}\right| < \frac{1}{2} \;, \forall\, x \in \mathbb{R}^n \;, \tag{18}$$

and

$$\varphi_\tau(z) = \int_{\mathbb{R}^n} \boldsymbol{\mathcal{E}}_\tau(z;x)\varphi(Z(x)) \det\left(\frac{\partial Z_j}{\partial x_k}\right)\, dx \;.$$

Now, consider the following *restriction* of the entire functions φ_τ to Ω:

$$\varphi_\tau(Z(x')) = (\tau/\pi)^{n/2}\int_{\mathbb{R}^n} e^{-\tau\langle Z(x')-Z(x)\rangle^2}\varphi(Z(x)) \det\left(\frac{\partial Z_j}{\partial x_k}\right)\, dx \;, \tag{19}$$

Introducing the change of variables $x \to x' + \tau^{-1/2}x$ in the integral (19), we get

$$\varphi_\tau(Z(x')) = \pi^{-n/2}\int_{\mathbb{R}^n} e^{-\tau\langle Z(x')-Z(x'+\tau^{-1/2}x)\rangle^2}\Gamma(Z(x'+\tau^{-1/2}x))\, dx \;, \tag{20}$$

where we define

$$\Gamma(Z(x'+\tau^{-1/2}x)) \stackrel{\text{def.}}{=} \varphi(Z(x'+\tau^{-1/2}x)) \det\left(\frac{\partial Z_j}{\partial x'_k}\right)(x'+\tau^{-1/2}x) \;.$$

It is worth emphasizing that the differentiability of Z at x' allows us to write

$$Z(x'+\tau^{-1/2}x) = Z(x') + Z_{x'}(x')\cdot\tau^{-1/2}x + \mathbf{R}(\tau^{-1/2}x) \;,$$

where $Z_{x'}$ denotes the $n\times n$ matrix (15) and $\mathbf{R}(\tau^{-1/2}x)$ is the remainder term. We also note that

$$\tfrac{|\mathbf{R}(\tau^{-1/2}x)|}{|\tau^{-1/2}x|} \xrightarrow{\tau\to+\infty} 0 \;,$$

uniformly for $x \in \overline{B}_0$. We have then that the exponential in the integral (20) rewrites as

$$e^{-\langle Z_{x'}(x')x\rangle^2}\cdot e^{-\langle(\tau^{1/2}\mathbf{R}(\tau^{-1/2}x))\cdot\boldsymbol{\psi}(\tau,x',x)\rangle} \;,$$

with

$$\boldsymbol{\psi}(\tau,x',x) \stackrel{\text{def.}}{=} 2Z_{x'}(x')x + \tau^{1/2}\mathbf{R}(\tau^{-1/2}x) \;.$$

Note that $\boldsymbol{\psi}(\tau,x',x)$ is uniformly bounded for $\tau > 0$ and $x', x \in \overline{B}_0$.

LEMMA 3.12 (Berhanu-Cordaro-Hounie [2, Lemma II.1.3]). *Let B be an $n\times n$ matrix with real coefficients and norm $|B| < 1$ and $A = I + i\,B$, where I is the identity matrix. Then*

$$(\det\, A)\pi^{-n/2}\int_{\mathbb{R}^n} e^{-\langle Ax\rangle^2}\, dx = 1 \;.$$

Let x' be an arbitrary but fixed point of B_0. Since x' is fixed, the matrix (15) satisfies the hypotheses of Lemma 3.12 in view of (18). Hence, for a holomorphic function $\varphi \in \mathfrak{H}(\Omega)$, we may write

$$\varphi(Z(x')) = \pi^{-n/2}\int_{\mathbb{R}^n} e^{-\langle Z_{x'}(x')x\rangle^2}\Gamma(Z(x'))\, dx \;,$$

where (again) $Z_{x'}$ in the exponential denotes the $n \times n$ matrix (15) and

$$\Gamma(Z(x')) \stackrel{\text{def.}}{=} \varphi(Z(x')) \det \left(\frac{\partial Z_j}{\partial x'_k} \right) (x') \ .$$

Then,

$$\varphi_\tau(Z(x')) - \varphi(Z(x')) = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-\langle Z_{x'}(x')x \rangle^2} \Big\{ e^{-\langle (\tau^{1/2}\mathbf{R}(\tau^{-1/2}x)) \cdot \boldsymbol{\psi}(\tau,x',x) \rangle}$$

$$\Gamma(Z(x' + \tau^{-1/2}x)) - \Gamma(Z(x')) \Big\} \ dx \ .$$

In order to estimate this difference, we first note that the differentiability of $\varphi(Z(x' + \tau^{-1/2}x))$ and $\det\left(\partial Z_j/\partial x'_k\right)(x' + \tau^{-1/2}x)$ implies that

$$\Gamma(Z(x' + \tau^{-1/2}x)) = \Gamma(Z(x')) + \tau^{-1/2}\Gamma'(Z(x')) \cdot Z_{x'}(x')x + \mathbf{S}(\tau^{-1/2}, x', x) \ ,$$

with

$$\tfrac{|\mathbf{S}(\tau^{-1/2},x',x)|}{|\tau^{-1/2}|} \xrightarrow{\tau \to +\infty} 0 \ ,$$

uniformly for $x', x \in \overline{B}_0$. Hence,

$$\varphi_\tau(Z(x')) - \varphi(Z(x')) = I_1 + I_2 \ ,$$

where

$$I_1 = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-\langle Z_{x'}(x')x \rangle^2} \Big\{ e^{-\langle (\tau^{1/2}\mathbf{R}(\tau^{-1/2}x)) \cdot \boldsymbol{\psi}(\tau,x',x) \rangle} - 1 \Big\} \Gamma(Z(x')) \ dx \ ,$$

and

$$I_2 = \pi^{-n/2}\tau^{-1/2} \int_{\mathbb{R}^n} e^{-\langle Z_{x'}(x')x \rangle^2} \Big\{ e^{-\langle (\tau^{1/2}\mathbf{R}(\tau^{-1/2}x)) \cdot \boldsymbol{\psi}(\tau,x',x) \rangle} \Gamma'(Z(x')) Z_{x'}(x')x \Big\} \ dx \ .$$

To estimate I_2 we observe that $\left| e^{-\langle Z_{x'}(x')x \rangle^2} \right| = e^{-|x|^2 + |\Phi_{x'}(x')x|^2} \leqslant e^{-(3/4)|x|^2}$, in view of (18). Moreover, it follows from (3) that φ is bounded for $x' \in \mathbb{R}^n$, while $\det\left(\partial Z_j/\partial x'_k\right)(x')$ is bounded in $\overline{B}_0$. Thus, I_2 admits the following estimate uniform for $x', x \in \overline{B}_0$:

$$|I_2| \leqslant \mathbf{M}_1 \tau^{-1/2} \int_{\mathbb{R}^n} e^{-(3/4)|x|^2} |x| \ dx \leqslant \mathbf{M}_2 \tau^{-1/2} \ ,$$

where $\mathbf{M}_2$ is a positive constant. In turn, to estimate I_1 we first observe that

$$e^{-\langle (\tau^{1/2}\mathbf{R}(\tau^{-1/2}x)) \cdot \boldsymbol{\psi}(\tau,x',x) \rangle} - 1 = \left[-(\tau^{1/2}\mathbf{R}(\tau^{-1/2}x)) \cdot \boldsymbol{\psi}(\tau,x',x) \right] \cdot \boldsymbol{\Theta}(\tau,x',x) \ ,$$

where $|\boldsymbol{\Theta}(\tau,x',x)|$ is uniformly bounded for $\tau > 0$ and $x', x \in \overline{B}_0$. Hence, I_1 admits the following estimate

$$|I_1| \leqslant \mathbf{M}_3 \tau^{1/2} |\mathbf{R}(\tau^{-1/2}x)| \ .$$

Recalling that $\mathbf{R}(\tau^{-1/2}x))$ is the rest of differentiability, we have that $\tau^{1/2}|\mathbf{R}(\tau^{-1/2}x)| < \varepsilon$, once $\tau >> 0$, with $\varepsilon > 0$ arbitrarily fixed. In summary, for each $\varepsilon > 0$, arbitrarily fixed, corresponds τ_o such that

$$|\varphi_\tau(Z(x')) - \varphi(Z(x'))| \leqslant \mathbf{M}_3 \cdot \varepsilon + \mathbf{M}_2 \cdot \tau^{-1/2} \ , \qquad \forall \, x' \in \overline{B}_0 \ ,$$

if $\tau \geqslant \tau_o$ (where the constants $\mathbf{M}_3, \mathbf{M}_2$ depend on ε and τ_o). Therefore, it follows from Lemma 3.11 that $\varphi_\tau(Z(x'))$ converges to $\varphi(Z(x'))$ uniformly on Ω, as $\tau \to \infty$, w.r.t. the semi-norm $\|\cdot\|_{\Omega,N}$. This completes the proof of Theorem 3.9. □

Remark 3.13. It should be mentioned that another proof of this theorem essentially along these lines is contained in [3, Theorem 2.13]. There, as for the proof of the theorem above, the authors have used a construction inspired by an argument of Hörmander [17, Proposition 9.1.2].

Finally, we take advantage of the approximation scheme described above in order to prove that the mapping $f \to BV(f)$ is injective.

THEOREM 3.14. *If the boundary value $BV(f)$ of $f \in \mathscr{H}^{*\,o}_{\boldsymbol{c}}$ vanishes identically, then the function f itself vanishes.*

PROOF. The following proof is similar to the proof of Theorem II.1.2 of Ref. [11]. It is convenient to assume that the cone C is connected (then we may assume that C' is also connected). Consider an open set $U \subset \mathbb{R}^n$ (with ∂U smooth). Define, for any $\varphi \in \mathfrak{H}$

$$\langle \mu^{\gamma}_{U,f}, \varphi \rangle = \int_{U+i\gamma} f(z)\varphi(z)\,dz\;, \quad \forall\, \varphi \in \mathfrak{H}\;,$$

where the vector $\gamma \in C'$, with $|\gamma| < \delta$. The above formula defines a functional carried by $\overline{U} + i\gamma$.

Suppose now that $BV(f) \equiv 0$; this means that $BV(f)|_U = 0$ and hence $\mu^{\gamma}_{U,f}$ is carried by ∂U. Thus, given an open neighborhood $\mathscr{N}$ of ∂U in $\mathbb{C}^n$ we can choose an ε, with $0 < \varepsilon < \delta$, and a constant $\mathbf{M}_\gamma(C') > 0$ so that

$$\left| \int_{U+i\gamma} f(z)\varphi(z)\,dz \right| \leqslant \mathbf{M}_\gamma(C') \sup_{z \in \mathscr{N}} (1+|z|)^N |\varphi(z)|\;. \tag{21}$$

for all $\gamma \in C'$, $|\gamma| < \varepsilon$. Take, for $\tau > 0$, $\varphi(z) = (\tau/\pi)^{n/2} e^{-\tau\langle z-x-i\gamma\rangle^2}$, where $\langle z-x-i\gamma\rangle^2 = \sum_{j=1}^{n}\bigl(z_j - x_j - i\gamma_j\bigr)^2$, with $x \in U$ away from $\mathscr{N}$. Then, on the basis of approximation scheme described in Theorem 3.9, we conclude that the left-hand side of Eq.(21) converges to $|f(x+i\gamma)|$. On the other hand, the right-hand side is dominated by

$$\mathbf{M}_\gamma(C')(\tau/\pi)^{n/2} \sup_{z \in \mathscr{N}} (1+\varepsilon+|\mathscr{R}e\, z|)^N e^{-\tau|z-x|^2} e^{\tau\varepsilon^2}\;,$$

Then, if we select $\mathscr{N}$ sufficiently "thin" around ∂U to ensure that there is an open subset of $\mathbb{R}^n$, $U_1 \subset U$, such that $\varepsilon << \operatorname{dist}(x,\mathscr{N})$, $\forall\, x \in U_1, z \in \mathscr{N}$, the right-hand side will converge to zero, implying that $f = 0$ in an open of the type $U_1 + iC'$. It now follows from connectedness of $T_\delta(C')$ that $f \equiv 0$, which is what we wanted to prove. □

4. Carrier of analytic functionals in $\mathfrak{H}'(T(O))$ and ultrahyperfunctional singular spectrum

We address in this section the issue of singularities of ultrahyperfunctions. Following the Carmichael's approach for ultrahyperfunctions, we study the relation between the singular spectrum of the tempered ultrahyperfunctions corresponding to proper convex cones and their expressions as boundary values of holomorphic functions. Of course, it will be important to specify in which sense the boundary values are considered. This issue will be addressed in Lemma 4.3.

Let Ω be a closed set in $T(O)$. Let Ω_m be a closed neighborhood of Ω defined by

$$\Omega_m = \bigl\{ z \in \mathbb{C}^n \mid \operatorname{dist}(z,\Omega) \leqslant 1/m \bigr\}\;.$$

For a closed set Ω_m of $\mathbb{C}^n$, $\mathfrak{H}_b(\Omega_m)$ is the space of all continuous functions ψ on Ω_m which are holomorphic in the interior of Ω_m and satisfy

$$\|\psi\|_{\Omega_m,N} = \sup_{\substack{z \in \Omega_m \\ N \in \mathbb{N}}} \bigl\{(1+|z|)^N |\psi(z)|\bigr\}\;.$$

$\mathfrak{H}_b(\Omega_m)$ is a Fréchet space with the seminorms $\|\psi\|_{\Omega_m,N}$. If $m' < m$, $\Omega_m \subset \Omega_{m'}$, then we have the canonical injections

$$\mathfrak{H}_b(\Omega_{m'}) \hookrightarrow \mathfrak{H}_b(\Omega_m)\;. \tag{22}$$

We define the space $\mathfrak{H}(\Omega)$

$$\mathfrak{H}(\Omega) = \liminf_{m\to\infty} \mathfrak{H}_b(\Omega_m)\;, \tag{23}$$

where the inductive limit is taken following the restriction mappings (22).

Given that Ω is a closed set in $T(O)$, $\Lambda_\Omega^{T(O)} : \mathfrak{H}(T(O)) \to \mathfrak{H}(\Omega)$ defines the restriction map. In general the transpose of $\Lambda_\Omega^{T(O)}$, ${}^t\Lambda_\Omega^{T(O)} : \mathfrak{H}'(\Omega) \to \mathfrak{H}'(T(O))$, need not be injective. It follows from Hahn-Banach theorem that the injectivity of ${}^t\Lambda_\Omega^{T(O)} : \mathfrak{H}'(\Omega) \to \mathfrak{H}'(T(O))$ is equivalent to the density of $\Lambda_\Omega^{T(O)}(\mathfrak{H}(T(O)))$ in $\mathfrak{H}(\Omega)$.

Definition 4.1. An analytic functional $U \in \mathfrak{H}'(T(O))$ is said to be carried by the closed set $\Omega \subset \mathbb{C}^n$, if for every closed neighborhood Ω_m of Ω the functional U belongs to the range of ${}^t\Lambda_{\Omega_m}^{T(O)}$.

Proposition 4.2 below shows that an analytic functional $U \in \mathfrak{H}'(T(O))$ is carried by a closed set $\Omega \subset \mathbb{C}^n$ if, and only if, there is a decreasing sequence $\{\Omega_m\}_{m=1}^\infty$ of closed neighborhoods of Ω such that, for every m, the functional U is already a functional on the space $\mathfrak{H}(T(O))|_{\Omega_m}$ of restrictions to Ω_m of functions in $\mathfrak{H}(T(O))$, where $\mathfrak{H}(T(O))|_{\Omega_m}$ carries the topology induced by $\mathfrak{H}(T(O))$.

Proposition 4.2. *For $U \in \mathfrak{H}'(T(O))$ to be carried by the closed set Ω of $T(O)$ it is necessary and sufficient that there be a fundamental sequence of closed neighborhood Ω_m of Ω and a constant $C > 0$ such that, for all $\psi \in \mathfrak{H}(T(O))$*

$$|\langle U, \psi\rangle| \leqslant C \sup_{\substack{z\in\Omega_m \\ N\in\mathbb{N}}} \left\{(1+|z|)^N |\psi(z)|\right\} . \tag{24}$$

Proof. Suppose that there is $\widetilde{U} \in \mathfrak{H}'(\Omega)$ such that $U = {}^t\Lambda_\Omega^{T(O)}\widetilde{U}$. Then, given any $\psi \in \mathfrak{H}(T(O))$, $\langle U, \psi\rangle = \langle \widetilde{U}, \Lambda_\Omega^{T(O)}\psi\rangle$. There is a fundamental sequence of closed neighborhood Ω_m of Ω and a constant C such that $|\langle \widetilde{U}, \psi\rangle| \leqslant C\|\psi\|_{\Omega_m,N}$, $\forall \varphi \in \mathfrak{H}_b(\Omega_m)$, for every m, whence (24). Conversely, suppose that (24) holds. First we note that $\Lambda_\Omega^{T(O)}\psi \to \langle U, \psi\rangle$ is a well defined linear functional on the subspace $\Lambda_\Omega^{T(O)}(\mathfrak{H}(T(O)))$ of $\mathfrak{H}(\Omega)$: by (24), $\Lambda_\Omega^{T(O)}\psi = 0 \Rightarrow \langle U, \psi\rangle = 0$. The right hand side of (24) defines a continuous seminorm not only on $\Lambda_\Omega^{T(O)}(\mathfrak{H}(T(O)))$ but on $\mathfrak{H}(\Omega)$ too. By the Hahn-Banach theorem, the linear functional $\Lambda_\Omega^{T(O)}\psi \to \langle U, \psi\rangle$ extends from $\Lambda_\Omega^{T(O)}(\mathfrak{H}(T(O)))$ to $\mathfrak{H}(\Omega)$ as a linear functional $\widetilde{U}$ that satisfies (24). Clearly $U = {}^t\Lambda_\Omega^{T(O)}\widetilde{U}$. □

In particular, when Ω is contained in $\mathbb{R}^n = \{z \in \mathbb{C}^n \mid z = x + iy, x \in \mathbb{R}^n, y = 0\}$, every function $f \in \mathscr{H}_{\boldsymbol{c}}^{*\,o}$, which for each $y \in C'$ as a function of $x = \operatorname{Re} z$ belongs to $\mathfrak{H}'(T(O))$, is a continuous linear functional on the space of *restrictions* to $\mathbb{R}^n$ of functions in $\mathfrak{H}(T(O))$. Then, according to Theorem 3.1(*iii*), $U = BV(f)$ the distributional boundary value of f is an element of $\mathfrak{H}'(T(O))$ carried by $\mathbb{R}^n$.

Let C be an open cone of the form $C = \bigcup_{j=1}^m C_j$, $m < \infty$, where each C_j is an proper open convex cone. If we write $C' \Subset C$, we mean $C' = \bigcup_{j=1}^m C'_j$ with $C'_j \Subset C_j$. Furthermore, we define by $C_j^* = \{\xi \in \mathbb{R}^n \mid \langle \xi, x\rangle \geqslant 0, \forall x \in C_j\}$ the dual cones of C_j, such that the dual cones C_j^*, $j = 1, \ldots, m$, have the properties

$$\mathbb{R}^n \setminus \bigcup_{j=1}^m C_j^* \ , \tag{25}$$

and

$$C_j^* \cap C_k^* \ , j \neq k \ , j, k = 1, \ldots, m \ , \tag{26}$$

are sets of Lebesgue measure zero. Assume that $V \in H'_{C^*}(\mathbb{R}^n; O)$ can be written as $V = \sum_{j=1}^m V_j$, where we define

$$V_j = D_\xi^\gamma[e^{h_K(\xi)}\lambda_j(\xi)g(\xi)] \ , \tag{27}$$

with $\lambda_j(\xi)$ denoting the characteristic function of C_j^*, $j = 1, \ldots, m$, $g(\xi)$ being a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = [-k, k]^n$.

Lemma 4.3. *Let C be an open cone such that $C = \bigcup_{j=1}^{m} C_j$, where the C_j are open convex cones such that (25) and (26) are satisfied and $V \in H'_{C^*}(\mathbb{R}^n; O)$ be represented as $V = \sum_{j=1}^{m} V_j$ where*

$$V_j = D_\xi^\gamma [e^{h_K(\xi)} \lambda_j(\xi) g(\xi)] ,$$

with $\lambda_j(\xi)$ denoting the characteristic function of C_j^*, $j = 1, \ldots, m$, $g(\xi)$ being a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k, k\big]^n$. Assume that $\mathscr{F}^{-1}[V] \in \mathfrak{H}'(T(O))$ is carried by $\mathbb{R}^n$. Then, $U = \mathscr{F}^{-1}[V]$ is the sum of distributional boundary values in $\mathfrak{H}'(T(O))$ of functions $f_j(z) \in \mathscr{H}^{*\,o}_{c_j}$, $j = 1, \ldots, m$.

Proof. Consider

$$h_y(\xi) = \int_{\mathbb{R}^n} \frac{f(z)}{P(iz)}\, e^{-i\langle \xi, z\rangle}\, dx\,, \quad z \in T_\delta^{(C' \setminus (C' \cap B(0;r)))}\,, \tag{28}$$

with $h_y(\xi) = e^{k|\xi|} g_y(\xi)$, where $g_y(\xi)$ is a bounded continuous function on $\mathbb{R}^n$, and $P(iz) = (-i)^{|\gamma|} z^\gamma$. By hypothesis $f \in \mathscr{H}^{*\,o}_{c}$ and satisfies (8), with $B[0;r]$ replaced by $B(0;r)$. For this reason, for an n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ of non-negative integers conveniently chosen, we obtain

$$\left|\frac{f(z)}{P(iz)}\right| \leqslant \mathbf{M}(C')(1 + |z|)^{-n-\varepsilon} e^{h_{C^*}(y)}\,, \tag{29}$$

where n is the dimension and ε is any fixed positive real number. This implies that the function $h_y(\xi)$ exists and is a continuous function of ξ. Further, by using arguments paralleling the analysis in [33, p.225] and the Cauchy-Poincaré Theorem [33, p.198], we can show that the function $h_y(\xi)$ is independent of $y = \operatorname{Im} z$. Therefore, we denote the function $h_y(\xi)$ by $h(\xi)$.

From (29) we have that $f(z)/P(iz) \in L_2$ as a function of $x = \operatorname{Re} z \in \mathbb{R}^n$, $y \in C' \setminus \big(C' \cap B(0;r)\big)$. Hence, from (28) and the Plancherel theorem we have that $e^{-\langle \xi, y\rangle} h(\xi) \in L_2$ as a function of $\xi \in \mathbb{R}^n$, and

$$\frac{f(z)}{P(iz)} = \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} h(\xi)\big](x)\,, \quad z \in T_\delta^{(C' \setminus (C' \cap B(0;r)))}\,, \tag{30}$$

where the inverse Fourier transform is in the L_2 sense.

From Parseval-type relation for the inverse Fourier transform of a distribution V of exponential growth, we have that

$$\begin{aligned}
\langle \mathscr{F}^{-1}[V], \psi\rangle &= \langle V, \mathscr{F}^{-1}[\psi]\rangle \\
&= \sum_{j=1}^{m} \Big\langle D_\xi^\gamma \big(\lambda_j(\xi) h(\xi)\big), \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(x) e^{i\langle \xi, x\rangle}\, dx \Big\rangle \\
&= \sum_{j=1}^{m} \Big\langle \lambda_j(\xi) h(\xi), \frac{(-1)^{|\gamma|}}{(2\pi)^n} \int_{\mathbb{R}^n} D_\xi^\gamma \Big(\psi(x) e^{i\langle \xi, x\rangle}\Big)\, dx \Big\rangle \\
&= \sum_{j=1}^{m} \Big\langle \lambda_j(\xi) h(\xi), \frac{(-i)^{|\gamma|}}{(2\pi)^n} \int_{\mathbb{R}^n} x^\gamma \psi(x) e^{i\langle \xi, x\rangle}\, dx \Big\rangle \\
&= \sum_{j=1}^{m} \lim_{C' \ni y \to 0} \Big\langle \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} \lambda_j(\xi) h(\xi)\big], (-i)^{|\gamma|} (x + iy)^\gamma \psi(x) \Big\rangle\,,
\end{aligned} \tag{31}$$

where we have used the fact that the differentiation under the integral sign is valid. We note that $\psi(x) \in \mathfrak{H}(T(O))$ implies $(z^\gamma \psi(x)) \in \mathfrak{H}(T(O))$ as a function of $x = \operatorname{Re} z \in \mathbb{R}^n$.

From (30), we have for $z \in T_\delta^{(C' \setminus (C' \cap B(0;r)))}$

$$\begin{aligned}
\Big\langle i^{|\gamma|} (x + iy)^{-\gamma} f(x + iy), \psi(x) \Big\rangle &= \Big\langle \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} h(\xi)\big], \psi(x) \Big\rangle\,, \\
&= \sum_{j=1}^{m} \Big\langle \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} \lambda_j(\xi) h(\xi)\big], \psi(x) \Big\rangle\,.
\end{aligned} \tag{32}$$

Combining (31) and (32), we obtain

$$\langle \mathscr{F}^{-1}[V], \psi \rangle = \sum_{j=1}^{m} \lim_{C' \ni y \to 0} \langle f_j(x+iy), \psi(x) \rangle = \sum_{j=1}^{m} \langle U_j, \psi \rangle = \langle U, \psi \rangle \ .$$

Thus the inverse Fourier transform $\mathscr{F}^{-1}[V]$ is a distributional boundary value of $\sum_{j=1}^{m} f_j(z)$ in the sense of weak convergence. But from [31, Corollary 1, p.358] the latter implies strong convergence since $\mathfrak{H}(T(O))$ is Montel. □

We will now describe the singularities of a tempered ultrahyperfunction. We start by observing that the relationship between the distributional boundary value and the (analytic) singular spectrum of a tempered ultrahyperfunction u, which will be denoted by $\mathbf{S.S.}(u)$, is that if the family of functions $f_j(x+iy)$ is holomorphic in $T_\delta^{(C' \setminus (C' \cap B(0;r)))}$, then the singular spectrum of the distributional boundary value $U = \sum_{j=1}^{m} \lim_{C' \ni y \to 0} f_j(x+iy)$ is contained in $X \times C^*$, where X is an open set in $\mathbb{R}^n$ and C^* is the dual cone of $C = \bigcup_{j=1}^{m} C_j$. Here we recall that, according to Proposition 3.5, the tempered ultrahyperfunction u is related to U as follows: $u = \sum_{j=1}^{m} BV_{C'}(f_j)$. In short, by considering the Paley-Wiener-Schwartz Theorem 3.2, the singular spectrum of a tempered ultrahyperfunction u is characterized by the directions from which the boundary values can be taken in an analytic representation of u, as shown in the following

Theorem 4.4. *Let X be an open set in $\mathbb{R}^n$ and let $u \in \mathscr{U}_c(X)$, where $\mathscr{U}_c(X)$ is the subspace of the tempered ultrahyperfunctions generated by $\mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ corresponding to a proper open convex cone $C \subset \mathbb{R}^n$. If $V \in H'_{C^*}(\mathbb{R}^n; O)$ (with $O \subseteq \mathbb{R}^n$), then $\mathbf{S.S.}(u) \subset X \times C^*$.*

Proof. Let $\{C_j^*\}_{j \in L}$ be a finite covering of closed properly convex cones of C^*. Decompose $V \in H'_{C^*}(\mathbb{R}^n; O)$ as follows [8, Theorem 4]:

$$V = \sum_{j=1}^{m} V_j \ , \quad \text{such that } V_j \in H'_{C_j^*}(\mathbb{R}^n; O) = \Big\{ V_j \in H'(\mathbb{R}^n; O) \mid \operatorname{supp}(V_j) \subseteq C_j^* \Big\} \ . \tag{33}$$

Next apply the Theorem 3.2 for each V_j. Then, in view of the Lemma 4.3, the decomposition (33) will induce a representation of u in the form of a sum of boundary values of functions $f_j \in \mathscr{H}_{\boldsymbol{c_j}}^{*\,\boldsymbol{o}}$, such that $f_j \to \mathscr{F}^{-1}[V_j] \in \mathfrak{H}'$ in the strong topology of $\mathfrak{H}'$ as $y = \operatorname{Im} z \to 0$, $y \in C'_j \Subset C_j$. According to Theorem 3.2, the family of functions f_j satisfy the estimate

$$\big| f_j(z) \big| \leqslant \mathbf{M}(C')(1+|z|)^N e^{h_{C_j^*}(y)} \ , \quad z = x+iy \in T_\delta^{(C'_j \setminus (C'_j \cap B(0;r)))} \ .$$

unless $\langle \xi, Y \rangle < 0$ for $\xi \in C_j^*$ and $Y \in -C'_j$, with $|Y| < \delta$. Then the cones of “bad” directions responsible for the singularities of these boundary values are contained in the dual cones of the base cones. So, we have the inclusion

$$\mathbf{S.S.}(u) \subset X \times \bigcup_j C_j^* \ . \tag{34}$$

Then, by making a refinement of the covering and shrinking it to C^*, we obtain the desired result. □

Corollary 4.5. *Let condition $\mathbf{S.S.}(u) \subset X \times C^*$ of Theorem 4.4 be satisfied. Then, $u = \sum_{j=1}^{m} BV_{C'}(f_j)$, where $f_j \in \mathscr{H}_{\boldsymbol{c_j}}^{*\,\boldsymbol{o}}$.*

From Theorem 3.2, Proposition 3.5, Corollary 3.7 and Lemma 4.3, we can draw the following

Corollary 4.6. *Let C be an open convex cone and let C' be an arbitrary compact cone contained in C. Let C^* be the dual cone of C. If an element $U \in \mathfrak{H}'$ is the boundary value in the distributional sense of a function f which is analytic in $T_\delta(C')$ and which satisfy the estimate (8), then $\mathbf{S.S.}(U) \subset X \times C^*$.*

In what follows, we give a version of the celebrated edge of the wedge theorem for the space of the tempered ultrahyperfunctions. It is derived from the integral representation without using cohomology (see [22, 34] for a proof using the cohomological approach). We note that in the literature one finds some global versions of the edge of the wedge theorem for tempered ultrahyperfunctions, *e.g.*, Refs. [21, 30, 13]. In turn, in Ref. [7] the authors propose a local version of the same theorem for tempered ultrahyperfunctions.

Theorem 4.7. *Let C_j, $j = 1, 2$ be proper open convex cones. Assume that the distributional boundary values of two holomorphic functions $f_j \in \mathscr{H}^{*\,o}_{\boldsymbol{c}_j}$ $(j = 1, 2)$ agree, that is, $U = BV(f_1) = BV(f_2)$, where $U \in \mathfrak{H}'$ in accordance with the Theorem 3.1. Then f_1 and f_2 can be glued together to a common element $\boldsymbol{f}$ which is holomorphic on $T(\boldsymbol{ch}(C)) = \mathbb{R}^n + i\,\boldsymbol{ch}(C)$, where $\boldsymbol{ch}(C)$ is the convex hull of the cone $C = C_1 \cup C_2$.*

Proof. Let $U = BV(f_1) = BV(f_2)$. It is not difficult to show that $C_1^* \cap C_2^* = (C_1 \cup C_2)^*$. Then, from Theorem 4.4 and Corollaries 4.5 and 4.6 we have $\mathbf{S.S.}(U) \subset X \times \big(C_1^* \cap C_2^*\big)$. If $C_1^* \cap C_2^* = \varnothing$, it follows that $\mathbf{S.S.}(U) = \varnothing$, which means that U is a real analytic function. On the other hand, if $C_1^* \cap C_2^* \neq \varnothing$, then there exists a holomorphic function $\boldsymbol{f}$ such that $U = BV(\boldsymbol{f})$. Since $U = BV(\boldsymbol{f}) = BV(f_1) = BV(f_2)$, it follows that the differences $\boldsymbol{f} - f_j$, $(j = 1, 2)$, vanish on the domain of definition of each f_j by Theorem 3.14. This proves the theorem. □

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