On M-O.Ore determinants

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Abstract – The existence of certain $\mathbb{F}_q$-spaces of differential forms of the projective line over a field $K$ containing $\mathbb{F}_q$ leads us to prove an identity linking the determinant of the Moore matrix of $n$ indeterminates with the determinant of the Moore matrix of the cofactors of its first row. These same spaces give an interpretation of Elkies pairing in terms of residues of differential forms. This pairing puts in duality the $\mathbb{F}_q$-vector space of the roots of a $\mathbb{F}_q$-linear polynomial and that of the roots of its reversed polynomial.

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To Marco Garuti, 
a friend too soon lost

1. Introduction

Marco Garuti was rapporteur for Guillaume Pagot’s thesis ([7, 10]) and the origin of this note is the following remark ([7] p. 68).

Let $K$ be a field with characteristic $p > 0$, $n \geq 2$ and $W \subset K[X]$, be an $n$ dimensional $\mathbb{F}_p$-subspace in $K[X]$ whose non zero elements have the same degree $d$ and

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let $P$ be a non zero polynomial which is a common multiple of the polynomials in $W$. Let $(P_1, P_2, \cdots, P_n)$, be a $\mathbb{F}_p$-basis of $W$ such that for $1 \leq i \leq n$ each differential form $\omega_i := \frac{P_i}{P} dX$ is a logarithmic differential, then (Proposition 4.1) there is $\gamma \in K^*$ such that

\[(1.1) \quad \Delta_n(P_1, P_2, \cdots, P_n) = \gamma P^{1+p+p^2+\cdots+p^{n-2}}\]

where $\Delta_n(P_1, P_2, \cdots, P_n)$ is the Moore determinant of the polynomials $P_1, P_2, \cdots, P_n$ (Definition 2.1).

In Proposition 3.1, we adapt the method of ([2, 7]) where such a $\mathbb{F}_p$-space $W$ for $n \geq 2$ was built, in order to build some $\mathbb{F}_q$-space of differential forms in $\Omega^1_{K^*}(K(X))$ that we call a $L^q_{\mu+1,n}$-space.

In section 3.2, we give a first application to a construction of Elkies pairing ([1] § 4.35). N. Elkies takes up and extends the results of O. Ore. For a $\mathbb{F}_q$-linear polynomial $P := c_0 X + c_1 X^q + \cdots + c_n X^{q^n}$ with $c_0 c_n \neq 0$, his pairing induces a duality between the $\mathbb{F}_q$-space of roots of $P$ and that of its reversed polynomial. In our construction the role of the $\mathbb{F}_q$-vector space of the roots of the reversed polynomial is played by a $L^q_{\mu+1,n}$-space of differential forms and the pairing is expressed by the residue of these forms evaluated at the roots of the polynomial $P$.

The rest of the note deals with the evaluation of the constant $\gamma$ in (1) when we apply it to $L^q_{\mu+1,n}$-subspaces of $L^q_{\mu+1,n+m}$-spaces constructed in Proposition 3.1.

Thus formula (1) takes the following form (Corollary 4.1)

**Corollary 4.1.** Let $(Y) := (Y_1, Y_2, \cdots, Y_n)$ and $(X) := (X_1, X_2, \cdots, X_m)$, be $n+m$ indeterminates over $\mathbb{F}_q$ with $n \geq 2$, $m \geq 0$ and the convention that $X = \emptyset$ and $\Delta_m(X) = 1$ for $m = 0$. We write $(\hat{Y}_i) := (Y_1, \cdots, Y_{i-1}, Y_{i+1}, \cdots, Y_n)$ for $1 \leq i \leq n$. Then we have the following equality in $\mathbb{F}_q(Y, X)$

\[
\frac{\Delta_n(\Delta_{n-1+m}(\hat{Y}_1, X), \cdots, (-1)^{i+1} \Delta_{n-1+m}(\hat{Y}_i, X), \cdots, (-1)^{n+1} \Delta_{n-1+m}(\hat{Y}_n, X))}{\Delta_m(X)^{q^{n-1}} \Delta_{n+m}(Y, X)^{1+q+\cdots+q^{n-2}}} = \frac{\Delta_n(\Delta_{n-1}(\hat{Y}_1), \cdots, (-1)^{i+1} \Delta_{n-1}(\hat{Y}_i), \cdots, (-1)^{n+1} \Delta_{n-1}(\hat{Y}_n))}{\Delta_n(Y)^{1+q+\cdots+q^{n-2}}} = \gamma
\]

Thanks to the work of Ore and Elkies (cf. Proposition 2.5) we know that $\gamma \in \mathbb{F}_q^*$. We show (Theorem 4.1) that $\gamma = 1$. We give three proofs. The first one shows it for $m = 1$ and by induction on $n$. It is a technical exercise in computing determinants. The second proof is a matrix equality which is in itself original and which translates a relation between a generic Moore matrix and the Moore matrix of the cofactors of its
first row (see theorem below); a relation analogous to the classical relation between a square matrix and its comatrix. The \( m = 0 \) case of Theorem 4.1 is immediately deduced.

**Theorem 4.2.** Let \( Y_1, Y_2, \ldots, Y_n, \) be indeterminates over \( \mathbb{F}_q \) and \( M_n(\Delta_n(\hat{Y}_1), \ldots, (-1)^{i-1}\Delta_n(\hat{Y}_i), \ldots, (-1)^{n-1}\Delta_n(\hat{Y}_n)) \) be the Moore matrix of the cofactors \( (\Delta_n(\hat{Y}_1), \ldots, (-1)^{i-1}\Delta_n(\hat{Y}_i), \ldots, (-1)^{n-1}\Delta_n(\hat{Y}_n)) \) of the first row of \( M_n(Y) \) where for \( 1 \leq i \leq n \). We write \( (\hat{Y}_1, \ldots, Y_i-1, Y_{i+1} \ldots, Y_n) \), then we have

\[
M_n(\Delta_n(\hat{Y}_1), \ldots, (-1)^{i-1}\Delta_n(\hat{Y}_i), \ldots, (-1)^{n-1}\Delta_n(\hat{Y}_n)) t M_n(Y) = \begin{pmatrix}
0 & \cdot & \cdot & \cdots & 0 & (-1)^{n-1}\Delta_n(Y) \\
\Delta_n(Y) & 0 & \cdot & \cdots & 0 & 0 \\
\alpha_1 & \Delta_n(Y)^q & 0 & \cdots & 0 & 0 \\
\alpha_2 & \alpha_1^q & \Delta_n(Y)^{q^2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\alpha_{n-2} & \alpha_1^{q^{n-3}} & \cdots & \alpha_1^{q^{n-2}} & \Delta_n(Y)^{q^{n-2}} & 0 \\
\end{pmatrix}
\]

where \( \alpha_k := \Delta_n(\hat{Y}_1)^{q^{k+i}}Y_1 + \cdots + (-1)^{i-1}\Delta_n(\hat{Y}_i)^{q^{k+i}}Y_i + \cdots + (-1)^{n-1}\Delta_n(\hat{Y}_n)^{q^{k+i}}Y_n \).

The third proof is a generalization of the above theorem which gives a matrix equality (Theorem 4.3) from which we deduce Theorem 4.1 without invoking Corollary 4.1.

In section 5, we offer two illustrations of M-O.O. Ore determinants. In the first one we study the application \((a_1, \ldots, a_n) \in K^n \rightarrow (\Delta_{n-1}(\hat{a}_i))_{1 \leq i \leq n} \in K^n \) and in the second we study a \( K \)-étale algebra defined by \( n \) Artin-Schreier equations. In this context we express a group action in terms of an appropriate Elkies pairing.

## 2. Generalities and motivations

### 2.1 – Notations

In this note all rings are commutative and unitary and \( A \) (resp. \( K \)) denotes a ring (resp. a field) of characteristic \( p > 0 \) containing the field \( \mathbb{F}_q \) where \( q := p^s \). Finally \( F: A \rightarrow A \) with \( F(a) = a^q \), denotes the Frobenius endomorphism. We denote by \( K^{alg} \) a \( K \) algebric closure. We adopt the following notations when the context is not ambiguous.

Let \( n \geq 1, m \geq 0 \) be integers. \( (a) := (a_1, a_2, \ldots, a_n) \), with \( a_i \in A \) and \( n \geq 1 \),
\((X) := (X_1, X_2, \cdots, X_m)\) be indeterminates over \(A\) and let \(m \geq 0\) with the convention \(X = 0\) if \(m = 0\). The integer \(m\) is determined by the context.

\((\hat{a}_i, X) := (a_1, \cdots, \hat{a}_i, \cdots, a_n, X)\), i.e. we omit \(a_i\) and \(X\) may be empty. The integers \(n\) and \(m\) are determined by the context.

**Definition 2.1.** Let \(A\) be a commutative ring containing the finite field \(\mathbb{F}_q\). Let \(m, n \geq 1\) be integers and \(a := (a_1, \cdots, a_n) \in A^n\). We call Moore matrix of size \(m, n\) associated to \(a\), the matrix of \(M_{m,n}(A)\) denoted by \(M_{m,n}(a)\), \((M_n(a)\) if \(n = m\)) where

\[
M_{m,n}(a) := \begin{pmatrix}
a_1 & a_2 & \cdots & a_n \\
a_1^q & a_2^q & \cdots & a_n^q \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{q^{m-1}} & a_2^{q^{m-1}} & \cdots & a_n^{q^{m-1}}
\end{pmatrix}
\]

and Moore determinant associated to \(a\), the determinant of \(M_n(a)\) denoted by \(\Delta_n(a)\).

### 2.2 – Additive polynomials and Moore determinants

**Definition 2.2.** Let \(K\) be a field containing \(\mathbb{F}_q\). We call \(\mathbb{F}_q\)-linear polynomial a polynomial of the form \(c_nX^n + c_{n-1}X^{n-1} + \cdots + c_{1}X^q + \cdots + c_{0}X \in K[X]\).

It is easy to see that a polynomial \(P \in K[X]\) is a \(\mathbb{F}_q\)-linear polynomial if and only if it satisfies the following two conditions.

1. \(P(X + Y) = P(X) + P(Y)\) in the polynomials ring \(K[X, Y]\)
2. \(P(\lambda X) = \lambda P(X)\) for all \(\lambda \in \mathbb{F}_q\).

A polynomial is additive if it satisfies condition 1. An additive polynomial is said to be reduced if it is separable. If it is non-zero, this means that the coefficient of \(X\) is non-zero.

Let \(P \in K[X]\) be a \(\mathbb{F}_q\)-linear polynomial, let \(\text{Ker} P := \{x \in K^{\text{alg}} \mid P(x) = 0\}\) be the set roots of \(P\); it is a \(\mathbb{F}_q\)-subspace of \(K^{\text{alg}}\).

The application \(x \in K \to P(x) \in K\) is a \(\mathbb{F}_q\)-linear endomorphism of \(K\). Thus we can consider the \(\mathbb{F}_q\)-subspace of \(K\) which is the kernel of this endomorphism. It coincides with \(\text{Ker} P\) when \(\text{Ker} P \subset K\).

**Proposition 2.1.** ([4], [6] and [1] prop.1 p.80) Let \(A\) be an integral commutative ring containing \(\mathbb{F}_q\). The \(n\) elements of \(A\), \(a_1, a_2, \cdots, a_n\) are \(\mathbb{F}_q\)-linearly independent if and only if \(\Delta_n(a) \neq 0\). In other words the \(n\) elements \(a_1, a_2, \cdots, a_n\) of \(A\) are \(\mathbb{F}_q\)-linearly independent if and only if the \(n\) vectors \(a, F(a), \cdots, F^{n-1}(a)\) of \(A^n\), are \(\mathbb{F}_q\)-linearly independent.
This proposition is a consequence of Moore’s identity ([1] (3.4) p. 80, (3.6) p. 81) which says that if \(a_1, a_2, \cdots, a_n\) are elements of \(A\), then

\[
\Delta_n(a) = \prod_{1 \leq i \leq n} \prod_{\epsilon_i \in \mathbb{F}_q} (a_i + \epsilon_i a_{i-1} + \cdots + \epsilon_1 a_1)
\]  

(2.1)

**Proposition 2.2.** Let \(K\) be a field containing \(\mathbb{F}_q\), \(W\) a \(\mathbb{F}_q\)-vector subspace of \(K\) with \(\dim_{\mathbb{F}_q} W = n\), then there exists a unique unit polynomial of degree \(q^n\), denoted \(P_W\) with \(W = \{x \in K \mid P_W(x) = 0\}\). Moreover \(P_W\) is a \(\mathbb{F}_q\)-linear polynomial which is reduced and if \(\underline{w} := (w_1, \cdots, w_n) \in K^n\) is a \(\mathbb{F}_q\)-basis of \(W\) then

\[
P_W(X) = \prod_{\underline{w} \in W} (X - w) = \frac{\Delta_{n+1}(\underline{w}, X)}{\Delta_n(\underline{w})} = X^{q^n} + \cdots + (-1)^n \Delta_n(\underline{w})^{q-1} X
\]  

(2.2)

The following proposition is the version adapted to hyperplanes in Proposition 2.2.

**Proposition 2.3.** Let \(W\) be a \(\mathbb{F}_q\)-subspace of \(K\) with \(\dim_{\mathbb{F}_q} W = n\). Let \(\underline{w} := (w_1, w_2, \cdots, w_n)\) be a \(\mathbb{F}_q\)-basis of \(W\) and \(\underline{w}^* := (w_1^*, \cdots, w_n^*)\) its dual basis. Let \(\varphi\) be a non-zero \(\mathbb{F}_q\)-linear form on \(W\) and \(\ker \varphi\) the hyperplane of \(W\) kernel of \(\varphi\). Let \(\underline{\alpha} := (\alpha_1, \cdots, \alpha_n) \in \mathbb{F}_q^n - \{(0, \cdots, 0)\}\) such that \(\varphi = \sum_{1 \leq i \leq n} \alpha_i w_i^*\) and

\[
\Delta_{\varphi}(\underline{w}, X) =
\begin{vmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n & 0 \\
w_1 & w_2 & \cdots & w_n & X \\
w_1^q & w_2^q & \cdots & w_n^q & X^q \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
w_1^{q^{n-1}} & w_2^{q^{n-1}} & \cdots & w_n^{q^{n-1}} & X^{q^{n-1}}
\end{vmatrix}
\]

\[
\delta_{\varphi}(\underline{w}) :=
\begin{vmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
w_1 & w_2 & \cdots & w_n \\
w_1^q & w_2^q & \cdots & w_n^q \\
\vdots & \vdots & \ddots & \vdots \\
w_1^{q^{n-2}} & w_2^{q^{n-2}} & \cdots & w_n^{q^{n-2}}
\end{vmatrix}
\]

Then \(\delta_{\varphi}(\underline{w}) \neq 0\) and like in (2.2) we can write

\[
P_{\ker \varphi} = \prod_{\underline{w} \in \ker \varphi} (X - w) = \frac{\Delta_{\varphi}(\underline{w}, X)}{\delta_{\varphi}(\underline{w})} = X^{q^{n-1}} + \cdots + (-1)^{n+1} \delta_{\varphi}(\underline{w})^{q-1} X
\]  

(2.3)
it is a $\mathbb{F}_q$-linear polynomial of degree $q^{n-1}$ which is reduced. Moreover we have for $1 \leq i \leq n$, $\Delta_{w^i}(w, X) = (-1)^{i-1}\Delta_n(\hat{w}_i, X)$, $\Delta_{\varphi}(w, X) = \sum_{1 \leq i \leq n} \alpha_i \Delta_{w^i}(w, X) = \sum_{1 \leq i \leq n} \alpha_i(-1)^{i-1}\Delta_n(\hat{w}_i, X)$ and $\delta_{\varphi}(w) = \sum_{1 \leq i \leq n} (-1)^i \alpha_i \Delta_{n-1}(\hat{w}_i)$.

Proof. First we show that $\Delta_{\varphi}(w, X)$ is not the null polynomial. Let us assume the opposite. Since $w, F(w), \ldots, F^{n-1}(w)$ are $\mathbb{F}_q$-linearly independent and since for $j \in \{0, \ldots, n-1\}$, the coefficient of $X^q$ is zero, we get $\alpha \in \sum_{i \in \{0, \ldots, n-1\}, i \neq j} \mathbb{F}_q F^i(w)$; thus its $(j+1)$-th coordinate in the $\mathbb{F}_q$-basis $w, F(w), \ldots, F^{n-1}(w)$ is zero, which contradicts the non-nullity of $\alpha$.

The polynomial $\Delta_{\varphi}(w, X)$ is thus a $\mathbb{F}_q$-linear polynomial of degree $\leq q^{n-1}$. We check that it is zero on the hyperplane $\varphi$; thus its degree is equal to $q^{n-1}$. Hence the proposition.

Corollary 2.1. Let $K$ be a field containing $\mathbb{F}_q$, $w := (w_1, w_2, \ldots, w_n) \in K^n$ and for $1 \leq i \leq n$, $\Delta_{n-1}(\hat{w}_i) := \Delta_{n-1}(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$, then $\Delta_n(w) \neq 0$ if and only if $\Delta_n(\Delta_{n-1}(\hat{w}_i)) := \Delta_n(\Delta_{n-1}(\hat{w}_1), \Delta_{n-1}(\hat{w}_2), \ldots, \Delta_{n-1}(\hat{w}_n)) \neq 0$.

Proof. Let us assume that $\Delta_n(w) \neq 0$, then by Proposition 2.1 and 2.3, $\delta_{\varphi}(w) = \sum_{1 \leq i \leq n} \alpha_i \Delta_{n-1}(\hat{w}_i) \neq 0$ for all $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F}_q^n - \{(0, \ldots, 0)\}$ where $\varphi = \sum_{1 \leq i \leq n} \alpha_i w^i$. It follows from Proposition 2.1 that $\Delta_n(\Delta_{n-1}(\hat{w}_i)) \neq 0$.

Let us assume that $\Delta_n(w) = 0$, then by Proposition 2.1, there is $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathbb{F}_q^n - \{(0, \ldots, 0)\}$ with $\sum_{1 \leq i \leq n} \epsilon_i w_i = 0$. Let $f$ be the $\mathbb{F}_q$-linear form over $\mathbb{F}_q^n$ such that $f((\alpha_1, \ldots, \alpha_n)) = \sum_{1 \leq i \leq n} \epsilon_i \alpha_i$ and $(\alpha_1, \ldots, \alpha_n) \in \ker f - \{0\}$, then

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
w_1 & w_2 & \cdots & w_n \\
w_1^q & w_2^q & \cdots & w_n^q \\
\vdots & \vdots & \ddots & \vdots \\
w_1^{q^{n-2}} & w_2^{q^{n-2}} & \cdots & w_n^{q^{n-2}}
\end{vmatrix} = 0, \text{ so } \sum_{1 \leq i \leq n} (-1)^{i-1} \alpha_i \Delta_{n-1}(\hat{w}_i) = 0.
\]

Thus $\Delta_n(\Delta_{n-1}(\hat{w}_i)) = 0$.

Definition 2.3. Let $P(X) := c_n X^{q^n} + c_{n-1} X^{q^{n-1}} + \cdots + c_0 X \in K[X]$ be a reduced $\mathbb{F}_q$-linear polynomial (i.e. $c_0 \neq 0$) of degree $q^n$ (i.e. $c_n \neq 0$). With Ore we consider the reversed polynomial $\rho P$ of the polynomial $P$ where

\[
(\rho P)(X) := \sum_{0 \leq m \leq n} c_{n-m}^q X^{q^m}.
\]

It is a reduced $\mathbb{F}_q$-linear polynomial of degree $q^n$.

O. Ore shows the following result (see [1] Theorem 5 p.88).
Proposition 2.4. Let $K$ be a field containing $\mathbb{F}_q$. Let $P = \sum_{0 \leq i \leq n} c_i X^q^i \in K[X]$ be a reduced $\mathbb{F}_q$-linear polynomial of degree $q^n$, $\rho P$ its reversed polynomial (Definition 2.3). We assume that the roots of $P$ are in $K$. Let $W := \text{Ker} P \subset K$ (Definition 2.2) and $w := (w_1, w_2, \cdots, w_n) \in K^n$ be a $\mathbb{F}_q$-basis of $W$. Let $\hat{W} \subset K$, the $\mathbb{F}_q$-subspace of $K$ spanned by the $n$ minors $\Delta_{n-1}(\hat{w}_i), 1 \leq i \leq n$, then if $U := \text{Ker} \rho P$, we have $U = c_n^{-1} (\frac{W}{\Delta_n(\hat{w})})^q$; this is a $\mathbb{F}_q$-subspace of $K$ of dimension $n$.

Remark 2.1. It follows from Proposition 2.4 (see also [9], Corollary 1.7.14 p.18) that if the $n$ elements $w_1, \cdots, w_n$ in $K$ are $\mathbb{F}_q$-independent, so are the $n$ elements $\Delta_{n-1}(\hat{w}_i), 1 \leq i \leq n$. Although not explicitly written, O. Ore ([6]) and N. Elkies ([1]) show the following result

Proposition 2.5. Let $K$ be a field containing $\mathbb{F}_q$, $w := (w_1, w_2, \cdots, w_n) \in K^n$ and for $1 \leq i \leq n$, $\Delta_{n-1}(\hat{w}_i) := \Delta_{n-1}(w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n)$. Let us assume that $\Delta_n(w) \neq 0$. Then

$$ \left( \Delta_n \left( \Delta_{n-1}(\hat{w}_i) \right) \right)^q = \Delta_n(w)^{q^{n-1}}. $$

Thus

$$ \frac{\Delta_n \left( \Delta_{n-1}(\hat{w}_i) \right)}{\Delta_n(w)^{1+q+\cdots+q^{n-2}}} \in \mathbb{F}_q^*. $$

Proof. Let $W \subset K$ be the $\mathbb{F}_q$-vector space $\bigoplus_{1 \leq i \leq n} \mathbb{F}_q w_i$ and $P_W$ the polynomial associated to $W$ by Proposition 2.2. Let $\hat{W} := \bigoplus_{1 \leq i \leq n} \mathbb{F}_q \Delta_{n-1}(\hat{w}_i)$ in $K$. With (2.2) we have $P_W(X) := \frac{\Delta_{n+1}(wX)}{\Delta_n(wX)}$ and if $\Delta[i](w) := \det(w, F(w), \cdots, \hat{F}^i(w), \cdots, F^n(w))$ (i.e. the line $F^i(w)$ is left out), then $P_W(X) = \sum_{0 \leq m \leq n} (-1)^{n-m} \Delta[m](w) X^m$, thus $P_W(X) = \sum_{0 \leq m \leq n} c_m X^m$ with $c_m = (-1)^{n-m} \Delta[m](w) \Delta_n(w)^{\frac{m}{n}}$. Applying the above to the family $\Delta_{n-1}(\hat{w}_i)$ and the polynomial

$$ P_{\hat{W}}(X) := \frac{\Delta_{n+1}(\Delta_{n-1}(\hat{w}_i)X)}{\Delta_n(\Delta_{n-1}(\hat{w}_i))} =: \sum_{0 \leq m \leq n} \hat{c}_m X^m $$

we obtain the following identities for $0 \leq m \leq n$, $\hat{c}_m = (-1)^{n-m} \Delta[m](\Delta_{n-1}(\hat{w}_i)) \Delta_n(\Delta_{n-1}(\hat{w}_i))$. In particular $\hat{c}_0 = (-1)^n (\Delta_n(\Delta_{n-1}(\hat{w}_i)))^{q-1}$. Elkies (cf. [1] 4.28) shows following Ore that

$$ P_{\hat{W}}(X) = X^{q^n} + (-1)^n \left( \sum_{1 \leq m \leq n} \hat{c}_m X^m \right) $$

Thus (2.4) is fulfilled.

In the case where $1 \leq m \leq n - 1$ we obtain the equality

$$ P_{\hat{W}}(X) = X^{q^n} + (-1)^n \left( \sum_{1 \leq m \leq n} \hat{c}_m X^m \right) $$
Conversely, if (2.4) gives dimensional (2.9) \( G \) of (2.7)

\[ (\Delta[m] (\Delta_{n-1}(\hat{w}_l)))^{q-1} = \Delta_n(w)^{q^n-q^{m+1}+q^{m-1}} (\Delta[n-m](w))^{q^{m-1}(q-1)}. \]

Remark 2.2. If we take into account Theorem 4.1, we can specify the equalities (2.5) and (2.6). Thus we have

\[ \frac{\Delta_n(\Delta_{n-1}(\hat{w}_l))}{\Delta_n(w)^{1+q+\ldots+q^{n-2}}} = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \]

where \( \left\lfloor \frac{n}{2} \right\rfloor \) is the lower integer part of \( \frac{n}{2} \), and

\[ \Delta[m] (\Delta_{n-1}(\hat{w}_l)) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \Delta_n(w)^{\frac{q^{n-1}}{q^n-q^{m+1}-q^{m-1}} \Delta[n-m](w)^{q^{m-1}}}. \]

Next proposition takes up results of Ore and Elkies ([1], Proposition 3) which specify the link between composition of two \( \mathbb{F}_q \)-linear polynomials and the geometry of sets of roots.

Proposition 2.6. Let \( K \) be a field which contains \( \mathbb{F}_q \). Let \( W \) be a \( \mathbb{F}_q \)-vector space of \( K \) with \( \dim_{\mathbb{F}_q} W = n \), \( W_1 \) be a \( \mathbb{F}_q \)-subvector space of \( W \) and \( P_W(X) := \prod_{x \in W} (X - x) \) (resp. \( P_{W_1}(X) := \prod_{x \in W_1} (X - x) \)). Let \( P_W(W) := \{ P_{W_1}(x) \mid x \in W \} \), it is a finite dimensional \( \mathbb{F}_q \)-subspace of \( K \) and

\[ P_W(X) = P_{P_{W_1}(W)}(P_{W_1}(X)). \]

Conversely, if \( Q \) is a monic \( \mathbb{F}_q \)-linear polynomial such that \( P_W(X) = Q(P_{W_1}(X)) \), we have \( Q = P_{P_{W_1}(W)} \).

Proof. Let us assume that \( \dim_{\mathbb{F}_q} W_1 = m \) then \( \deg P_{W_1} = q^m \). Since \( x \in W \to P_{W_1}(x) \in K \) is a \( \mathbb{F}_q \)-linear map whose kernel is \( W_1 \), it follows that \( P_{W_1}(W) \) is a \( \mathbb{F}_q \)-subspace of \( K \) of dimension \( n - m \); thus the polynomial \( P_{P_{W_1}(W)}(P_{W_1}(X)) \) is a monic \( \mathbb{F}_q \)-linear polynomial of degree \( q^{n-m} q^m = q^n \). Since it is by construction zero on \( W \), it follows that \( P_W(X) \) divides \( P_{P_{W_1}(W)}(P_{W_1}(X)) \) in \( K[X] \), hence the equality.

For the reciprocal we remark that \( (Q - P_{P_{W_1}(W)}(P_{W_1}(X))) \) is the null polynomial in \( K[X] \) and that \( P_{W_1}(X) \) is transcendental over \( K \).
Finally, the following corollary specifies Proposition 2.3

**Corollary 2.2.** Let $K$ be a field which contains $\mathbb{F}_q$. Let $W \subset K$ be a $\mathbb{F}_q$-vector space with $\dim_{\mathbb{F}_q} W = n$, $w := (w_1, w_2, \cdots, w_n) \in K^n$ a $\mathbb{F}_q$-basis of $W$ and $(w_1^*, w_2^*, \cdots, w_n^*)$ its dual basis. Let $\varphi$ be a non-zero $\mathbb{F}_q$-linear form on $W$ and $\text{Ker} \varphi$ be the hyperplane of $W$ kernel of $\varphi$. Let $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}_q^n = \{(0, 0, \cdots, 0)\}$ and such that $\varphi = \sum_{1 \leq i \leq n} \alpha_i w_i^*$, $\Delta_\varphi (w, X) = \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \Delta_n (\hat{w}_i, X)$ and $\delta_\varphi (w) = \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \Delta_{n-1} (\hat{w}_i)$. Let $P_W \in K[X]$ (resp. $P_{\text{Ker} \varphi} \in K[X]$) be the monic and reduced polynomial whose set of roots is $W$ (resp. $\text{Ker} \varphi$).

Then (2.9) is satisfied with $W_1 := \text{Ker} \varphi$, and $P_{\text{Ker} \varphi} (X) = \frac{\Delta_\varphi (w, X)}{\delta_\varphi (w)}$, $P_{\text{Ker} \varphi} (W) (X) = Xq - (\Delta_n (w)) q^{-1} X$, thus $\frac{P_{\text{Ker} \varphi} (X)}{P_W (X)} = \frac{\Delta_\varphi (w, X)}{\delta_\varphi (w)} \frac{\Delta_n (w, X)}{\Delta_n (w, X)} = \frac{1}{P_{\text{Ker} \varphi} (X) q^{-1} - (\Delta_n (w)) q^{-1}}$.

**Proof.** Since $\text{Ker} \varphi$ is a hyperplane of $W$, it follows that $P_{\text{Ker} \varphi} (W) (X) = Xq - c_\varphi X \in K[X]$. Thus with (2.9) we have $P_W (X) = P_{W_1} (X) q - c_\varphi P_{W_1} (X)$. Since $\text{coeff}_X P_W = (-1)^n \Delta_n (w)^q - 1$ (cf. (2.2) and (2.3)) $\text{coeff}_X P_{W_1} = (-1)^{n+1} \delta_\varphi (w)^q$, the corollary follows.

3. Vector spaces of differentials and Moore determinants

3.1 – The $\mathbb{F}_q$-spaces $L_{q}^{\mu+1, n}$

**Definition 3.1.** Let $K$ be a field of characteristic $p > 0$. Let $\mu \in \mathbb{N}$ with $\mu \geq 2$ prime to $p$ and $n \geq 1$. We call space $L_{\mu+1, n}$ a $\mathbb{F}_p$-vector space of dimension $n$ of logarithmic differential forms in $\Omega_{K}^{\mu+1} (K(X))$, whose nonzero elements have $(\mu - 1)\infty$ as zero divisor and their poles are in $K$ (such a form has $\mu + 1$ poles and they are simple).

One can show ([7] lemme 6 p. 63) that if such a space exists then $p^{n-1}$ divides $\mu + 1$.

Such $\mathbb{F}_p$-vector spaces have been constructed for $n \geq 1$ in [2] in order in particular to lift in null characteristic certain $(\mathbb{Z}/p\mathbb{Z})^n$-coverings of the projective line $\mathbb{P}_K^1$ into Galoisian coverings of group $(\mathbb{Z}/p\mathbb{Z})^n$. See [5] for a presentation of recent contributions on the subject.

The spaces $L_{\mu+1, n}$ have been defined and studied by Guillaume Pagot in his thesis ([10] p. 19), a part of which is published in [7]. See also [3] for complements.

We will consider a generalization to the case where $\mathbb{F}_p$ is replaced by the field $\mathbb{F}_q$ with $q = p^s$.  

Definition 3.2. Let $K$ be a field which contains $\mathbb{F}_q$. Let $\mu \in \mathbb{N}$ with $\mu \geq 2$ prime to $p$ and $n \geq 1$. Let $L^q_{\mu+1,n}$ be a $\mathbb{F}_q$-vector space of dimension $n$ of differential forms in $\Omega^1_K(K(X))$ whose nonzero elements have $(\mu - 1)\infty$ as zero divisor, such that their poles are simple and in $K$ (so there are $\mu + 1$ poles) and such that their residues are in $\mathbb{F}_q$.

Proposition 3.1. Let $K$ be a field containing $\mathbb{F}_q$, $W$ a $\mathbb{F}_q$-subspace of $K$ with $\dim_{\mathbb{F}_q} W = n$, $w := (w_1, \ldots, w_n) \in K^n$ a $\mathbb{F}_q$-basis of $W$ and $w^*_i : (w_1^*, \ldots, w_n^*)$ its dual basis. For $j \in \{1, \ldots, n\}$, we note

$$\omega_j := \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathbb{F}_q^n} \frac{\epsilon_j}{X - \sum_{i=1}^n \epsilon_i w_i} dX.$$ 

Let $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q^n - \{(0, \ldots, 0)\}$, $\varphi = \sum_{1 \leq i \leq n} \alpha_i w_i^*$ and

$$\omega_\varphi := \sum_{1 \leq j \leq n} \alpha_j \omega_j = \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathbb{F}_q^n} \frac{\sum_{1 \leq j \leq n} \alpha_j \epsilon_j}{X - \sum_{i=1}^n \epsilon_i w_i} dX.$$ 

Then, $\omega_\varphi = -\Delta_n(w)^{q-1} \frac{\Delta_\varphi(w, X)}{\Delta_{n+1}(w, X)} dX$ and $\Delta_\varphi(w, X) | \Delta_{n+1}(w, X)$ (cf. Proposition 2.3).

Thus $\Omega_W := \sum_{1 \leq j \leq n} \mathbb{F}_q \omega_j \subset \Omega^1_K(K(X))$ is a $L^q_{\mu+1,n}$ with $\mu + 1 := q^{n-1}(q - 1)$.

Proof. Let $\omega \in \Omega_W$ and $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F}_q^n - \{(0, 0, \ldots, 0)\}$ with $\omega = \sum_{1 \leq j \leq n} \alpha_j \omega_j$. Then $\omega = \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathbb{F}_q^n} \frac{\sum_{1 \leq j \leq n} \alpha_j \epsilon_j}{X - \sum_{i=1}^n \epsilon_i w_i} dX$. It follows that the poles of $\omega$ are the elements of $W$ deprived of the zeros of the $\mathbb{F}_q$-linear form $\varphi = \sum_{1 \leq i \leq n} \alpha_i w_i^*$ so of cardinal $q^n - q^{n-1} := \mu + 1$ and they are simple. The residues by construction are in $\mathbb{F}_q$. In particular $\omega \neq 0$ and therefore $\Omega_W$ is a $\mathbb{F}_q$-vector space of dimension $n$.

It remains to see that the zero divisor of $\omega$ is $(\mu - 1)\infty$. For that we consider the fraction $F(X) := -\Delta_n(w)^{q-1} \frac{\Delta_\varphi(w, X)}{\Delta_{n+1}(w, X)}$. The poles of $F$ are the elements of $W - \ker \varphi$ and they are simple (Corollary 2.2).

Let $w \in W$ with $\varphi(w) \neq 0$, then $\text{res}_w F(X) = -\Delta_n(w)^{q-1} \frac{\Delta_\varphi(w, w)}{\Delta_{n+1}(w, X)(w)}$. We have $w = \sum_{i=1}^n \epsilon_i (w) w_i$ with $\epsilon_i (w) \in \mathbb{F}_q$; thus (cf. Proposition 2.3)

$$\Delta_\varphi(w, w) = \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \Delta_n(\tilde{w}_i, w) = \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \epsilon_i (w) \Delta_n(\tilde{w}_i, w_i) = \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \epsilon_i (w) (-1)^{n-i} \Delta_n(w) = (-1)^{n-1} \Delta_n(w) \sum_{1 \leq i \leq n} \alpha_i \epsilon_i (w).$$

Finally (cf. Proposition 2.2) $\Delta_{n+1}(w, X)(w) = (-1)^n \Delta_n(w)^q$.

Thus $\text{res}_w F(X) = -\Delta_n(w)^{q-1} (-1)^{n-1} \Delta_n(w) \sum_{1 \leq i \leq n} \alpha_i \epsilon_i (w) = \sum_{1 \leq i \leq n} \alpha_i \epsilon_i (w)$, and so $\omega = \omega_\varphi$. It follows that the zeros of $\omega$ are concentrated at infinity (Corollary 2.2).

Remark 3.1. In ([10] Remark 4 p.29) Pagot remarks that if $K$ is algebraically closed then the pullback by a morphism $\Phi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ with $\Phi(X) = \alpha X + X P(X^p)$
where $\alpha \in K^*$ and $P \in K[X]$, of a space $L_{\mu+1,n}$ is a space $L_{(\mu+1)\deg \Phi,n}$. Similarly an exercise shows that the pullback of a $L_{\mu+1,n}^q$ is a space $L_{(\mu+1)\deg \Phi,n}^q$. We can thus construct new spaces $L_{\mu+1,n}^q$ for example from Proposition 3.1.

3.2 – Spaces $L_{\mu+1,n}^q$ and Elkies pairing

In this section $K$ denotes a field that contains $\mathbb{F}_q$, $W$ is a $\mathbb{F}_q$-vector space of $K$ with $\dim_{\mathbb{F}_q} W = n$, $w := (w_1, \ldots, w_n) \in K^n$, a $\mathbb{F}_q$-basis of $W$ and $w^* := (w_1^*, \ldots, w_n^*)$ its dual basis. Finally let $\hat{W} := \bigoplus_{1 \leq i \leq n} \mathbb{F}_q \Delta_n^{-1}(\hat{\omega}_i)$ and $U := (\frac{\hat{W}}{\Delta_n(w)})^q$. Recall that $U = \text{Ker} \rho P_W$ where $\rho P_W$ is the reversed polynomial of the polynomial $P_W$ (cf. Proposition 2.2, Definition 2.3, Proposition 2.4).

We will recall the construction of Elkies pairing attached to the monic $\mathbb{F}_q$-linear polynomial $P_W$. It puts in duality the two $\mathbb{F}_q$-subvector spaces of $K$ which are $W$ and $U$ and we will give a differential interpretation of it using the spaces $L_{\mu+1,n}^q$ defined in Proposition 3.1.

A. Elkies pairing

Elkies ([1], § 4.35) defines a $\mathbb{F}_q$-perfect pairing $E : W \times U \to \mathbb{F}_q$ as follows. He first observes that if $(w, u) \in W \times U$ then $0 = w((\rho P_W)(u))^{q^n} - u P_W(w) = E(w, u) - E(w, u)^q$ where $E(w, u) := \sum_{1 \leq m \leq n} \sum_{0 \leq i \leq m-1} ((c_m w)^{q^m} w)^i$, $P_W(X) = \sum_{0 \leq m \leq n} c_m X^{q^m}$ and $c_n = 1$. It follows that $E(w, u) \in \mathbb{F}_q$.

B. The pairing $f : W \times U \to \mathbb{F}_q$

We will see that Proposition 3.1 allows to define a $\mathbb{F}_q$-pairing $f : W \times U \to \mathbb{F}_q$ given by the residue of differential forms.

Precisely, if $w \in W$ and $u \in U$ are different from 0, we can write $w = \sum_{i=1}^n \epsilon_i (w, w) w_i$ in the basis $w$ of $W$ where $(\epsilon_1(w, w), \ldots, \epsilon_n(w, w)) \in \mathbb{F}_q^n - \{0\}$ and by definition of $U$ we can write $u = (\frac{\sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \Delta_{n-1}(\hat{\omega}_i)}{\Delta_n(w)})^q$ where $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q^n - \{(0, \ldots, 0)\}$. Let $\varphi := \sum_{1 \leq i \leq n} \alpha_i w_i^*$, then (cf. Proposition 2.3)

$$
(3.1) \quad u = (\frac{\delta_\varphi(w)}{\Delta_n(w)})^q \text{ with } \delta_\varphi(w) = \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \Delta_{n-1}(\hat{\omega}_i).
$$

Let $\Omega^o_W := \sum_{1 \leq j \leq n} \mathbb{F}_q \omega_j \subset \Omega_K^1(K(X))$, be the $\mathbb{F}_q$-space $L_{\mu+1,n}^q$ with $\mu + 1 := q^{n-1}(q - 1)$ as defined in Proposition 3.1 and let

$$
\omega_\varphi := \sum_j \alpha_j \omega_j = \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathbb{F}_q^n} \sum_{1 \leq j \leq n} \alpha_j \epsilon_j \frac{\sum_{1 \leq j \leq n} \alpha_j \epsilon_j dX}{\sum_{1 \leq i \leq n} \epsilon_i w_i} \in \Omega^o_W - \{0\}.
$$
Then we define

\[(3.2) \quad f(w, u) := (-1)^{n-1} \text{res}_w \omega_\varphi = (-1)^{n-1} \sum_{1 \leq j \leq n} \alpha_j \varepsilon_j(w, w) \in \mathbb{F}_q.\]

**Lemma 3.1.** The pairing $f$ is perfect.

**Proof.** Let $u \in U$, let us assume that $f(w, u) = 0$ for all $w \in W$. It follows from (3.1) that $u = \left(\frac{\delta_\varphi(w)}{\Delta_n(w)}\right)^q$ with $\delta_\varphi(w) = \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \Delta_n-1(\tilde{w}_i)$ where $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q^n$ and $\varphi = \sum_{1 \leq i \leq n} \alpha_i w_i^*$.

Thus for any $w \in W$, the residue at $w$ of the differential form $\omega_\alpha = \sum \alpha_i \omega_i$ is zero and since the poles of $\omega_\alpha$ are simple, this form is the null form. Thus $\alpha_i = 0$ for $1 \leq i \leq n$, it follows that $\delta_\varphi(w)$ and thus $u$ are zero.

Now let $w \in W$, let us assume that $f(w, u) = 0$ for all $u$ in $\hat{W}$. It follows from Proposition 3.1 that 0 is the only element of $W$ which is not a pole of one of the $\omega_i$ forms, so $w = 0$.

**C. Comparison of the two pairings $E$ and $f$**

**Proposition 3.2.** The two pairings $E$ and $f$ are equal.

**Proof.** Let $w_1, w_2, \ldots, w_n$ be a basis of $W$, $w \in W - \{0\}$ and $u \in U - \{0\}$. It follows from (3.1) that $u = \left(\frac{\delta_\varphi(w)}{\Delta_n(w)}\right)^q$ where $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q^n - \{(0, \cdots, 0)\}$ and $\varphi = \sum_{1 \leq i \leq n} \alpha_i w_i^*$.

As from Proposition 3.1 we have $\omega_\varphi = -\Delta_n(w)q^{-1}\Delta_\varphi(w, X)\Delta_n(w)X dX$, it follows from (2.3) and Corollary 2.2 that

\[f(w, u) = (-1)^{n-1} \text{res}_w \omega_\varphi = (-1)^{n} \frac{\Delta_n(w)}{\Delta_n(w)}q^{-1} \frac{\delta_\varphi(w) P_{\ker \varphi}(w)}{(-1)^{n} \Delta_n(w)q^{-1}} = u^{1/q} P_{\ker \varphi}(w)\]

where $P_{\ker \varphi}(X)$ is a monic polynomial of degree $q^{n-1}$ (cf. (2.3)). On the other hand $E(w, u) = u^{1/q} P_u(w)$ where $P_u(X)$ is a monic $\mathbb{F}_q$-linear polynomial of degree $q^{n-1}$ ([1] lemma p. 92, proof).

It then remains to compare the two polynomials $P_u(X)$ and $P_{\ker \varphi}(X)$.

As $P_u(X)$ and $P_{\ker \varphi}(X)$ divide $P_W(X)$ in $K[X]$ and as ker $P_{\ker \varphi}$ (resp. ker $P_u$) is an hyperplane in $W$ we have (Proposition 2.6) $P_W = Q_\varphi \circ P_{\ker \varphi} = Q_u \circ P_u$ in $K[X]$ where $Q_\varphi(X) = X^q - q_\varphi X$ and $Q_u = X^q - q_u X$ with $q_\varphi, q_u$ non zero elements in $K$. In particular $\text{coeff}_X(P_W) = -q_\varphi \text{coeff}_X(P_{\ker \varphi}) = -q_u \text{coeff}_X(P_u)$. We have $\text{coeff}_X(P_u) = -(-1)^n q_u \frac{q^{-1}}{q^q}$ ([1] lemma p. 92, proof). From Corollary 2.2 we know that $q_\varphi = \left(\frac{\Delta_n(w)}{\delta_\varphi(w)}\right)^{-1}$ and $\text{coeff}_X(P_W) = (-1)^n \frac{q_{\varphi}^{-1}}{q^q}$ (cf. (2.2)) and so $\text{coeff}_X(P_{\ker \varphi}) = \text{coeff}_X(P_u)$; hence $q_\varphi = q_u$.

The equality $Q_\varphi = Q_u$ then follows from the uniqueness of the decomposition in Proposition 2.6.
4. A property of $L^q_{\mu+1, n}$ spaces

4.1 – The property

**Proposition 4.1.** Let $n \geq 2$ and $\Omega$ be a $L^q_{\mu+1, n}$ space (définition 3.2) and $\omega := (\omega_1, \omega_2, \cdots, \omega_n)$ a $\mathbb{F}_q$-basis. Let $\mathcal{P}(\Omega) \subset K$ be the set of poles of the differentials in $\Omega$ and $P(X) := \prod_{x \in \mathcal{P}(\Omega)} (X - x)$. Let $P_i \in K[X]$ with $\omega_i = \frac{P_i}{P} dX$ for $1 \leq i \leq n$. Then there is $\gamma \in K^*$ with

$$(4.1) \quad \Delta_n(P_1, \cdots, P_n) = \gamma P^{1+q+\cdots+q^n-2},$$

**Proof.** Thanks to (2.1) we can write

$$\Delta_n(P_1, \cdots, P_n) = \prod_{1 \leq i \leq n} \prod_{\epsilon_i \in \mathbb{F}_q} (P_i + \epsilon_{i-1} P_{i-1} + \cdots + \epsilon_1 P_1).$$

By hypothesis each factor $P_i + \epsilon_{i-1} P_{i-1} + \cdots + \epsilon_1 P_1$ divides $P$ and factorizes into a product of distinct irreducible polynomials of degree 1. Thus for $x \in \mathcal{P}(\Omega)$, we must show that $x$ is a root of $1 + q + \cdots + q^n-2$ polynomials $P_i + \epsilon_{i-1} P_{i-1} + \cdots + \epsilon_1 P_1$ with $(\epsilon_1, \epsilon_2, \cdots, \epsilon_{i-1}, 1, 0, 0 \cdots, 0) \in \mathbb{F}_q^n$.

Let $x \in \mathcal{P}(\Omega)$, since $x$ is a pole of at least one of the $\omega_i$ forms, the uplet $(\text{res}_x \omega_i)_{1 \leq i \leq n} \in \mathbb{F}_q^n - \{(0, \cdots, 0)\}$ and let $\varphi_x : \Omega \rightarrow \mathbb{F}_q$ be the linear form with $\varphi_x(\omega) = \sum_{1 \leq i \leq n} \alpha_i \text{res}_x \omega_i$ where $\omega = \sum_{1 \leq i \leq n} \alpha_i \omega_j$, then $\varphi_x$ is a $\mathbb{F}_q$-linear non zero form. If $(\alpha_1 : \alpha_2 : \cdots : \alpha_n) \in \mathbb{P}^n(\mathbb{F}_q)$, $\varphi_x(\omega) = \sum_{1 \leq i \leq n} \alpha_i \text{res}_x \omega_i = 0$ if and only if $\sum_{1 \leq i \leq n} \alpha_i P_i(x) = 0$, then as $\{(\epsilon_1, \epsilon_2, \cdots, \epsilon_{i-1}, 1, 0, 0 \cdots, 0), 1 \leq i \leq n\} \in \mathbb{F}_q^n$ is a system of representatives of the elements of $\mathbb{P}^n(\mathbb{F}_q)$, the multiplicity of the zero $x$ in $\Delta_n(P_1, \cdots, P_n)$ is equal to the number of points of the hyperplane of $\mathbb{P}^n(\mathbb{F}_q)$ induced by $\text{Ker}\varphi_x$, so it is equal to $1 + q + \cdots + q^n-2$.

Finally by Proposition 2.1, $\gamma$ is not zero since the $n$ fractions $\frac{P_i}{P}$ are $\mathbb{F}_q$-linearly independent.

**Remark 4.1.** The Proposition 4.1 is a remark in [7] on page 68 in the framework of $L_{\mu+1, n}$-spaces of logarithmic differentials (i.e. $q = p$).

4.2 – An equality between Moore’s determinants I

**Corollary 4.1.** Let $(Y) := (Y_1, Y_2, \cdots, Y_n)$ and $(X) := (X_1, X_2, \cdots, X_m)$, $n + m$ indeterminates over $\mathbb{F}_q$ where $n \geq 2$, $m \geq 0$ and the convention that $X_0 = 0$ and $\Delta_m(X) = 1$ if $m = 0$. For $1 \leq i \leq n$, we write $(\hat{Y}_i) := (Y_1, \cdots, Y_{i-1}, Y_{i+1} \cdots, Y_n)$. Then we have the following equality in $\mathbb{F}_q(Y, X)$
\[
\frac{\Delta_n(\Delta_{n-1+m}(\hat{Y}_1, X), \ldots, (-1)^{i+1}\Delta_{n-1+m}(\hat{Y}_i, X), \ldots, (-1)^{n+1}\Delta_{n-1+m}(\hat{Y}_n, X))}{\Delta_m(X)q^{n-1}\Delta_{n+m}(Y, X)^{1+q\cdots+q^{n-2}}} = \gamma
\]

\[
\frac{\Delta_n(\Delta_{n-1}(\hat{Y}_1), \ldots, (-1)^{i+1}\Delta_{n-1}(\hat{Y}_i), \ldots, (-1)^{n+1}\Delta_{n-1}(\hat{Y}_n))}{\Delta_n(Y)^{1+q\cdots+q^{n-2}}} = \gamma
\]

(4.2)

**Proof.** Let \(A := \mathbb{F}_q[Y, X]\) and \(K\) be its fraction field. For \(j \in \{1, \ldots, n+m\}\), we denote

\[\omega_j := \sum_{(e_1, e_2, \ldots, e_{n+m}) \in \mathbb{F}_q^{n+m}} \frac{\epsilon_jdZ}{Z - \sum_{i=1}^n e_iY_i - \sum_{i=1}^m e_{n+i}X_i},\]

then \(\Omega_{Y, X} := \sum_{1 \leq j \leq n+m} \mathbb{F}_q \omega_j \subset \Omega^1_K(K(Z))\) is a \(L^q_{\mu+1,n+m}\) space where \(\mu + 1 := q^{n-1+m}(q-1)\) and \(\Omega_{Y} := \sum_{1 \leq j \leq n} \mathbb{F}_q \omega_j \subset \Omega^1_K(K(Z))\) is a \(n\)-dimensional \(\mathbb{F}_q\)-subspace \(\Omega\) of \(\Omega_{Y, X}\), hence it is a \(L^q_{\mu+1,n}\) space.

We apply Proposition 3.1 to this last space \(\Omega\).

For \(1 \leq i \leq n\), we have \(\omega_i = -\Delta_{n+m}(Y, X)^{q-1} \frac{\Delta_{n+m}(\hat{Y}_i, X, Z)}{\Delta_{n+m+1}(Y, X, Z)} dZ\). It follows that \(P(\Omega) = \{\sum_{i=1}^n e_iY_i + \sum_{i=1}^m e_{n+i}X_i\} \subset \Omega^1_K(K(Z))\) is the set of poles \(P(\Omega) \subset K\) of the elements of \(\Omega\).

Thus \(P(Z) := \prod_{z \in P(\Omega)} (Z - z) = \frac{\Delta_m(X)}{\Delta_{n+m}(Y, X, Z) - \Delta_{n+m+1}(Y, X, Z)} (\text{cf. (2.2)})\) and \(\omega_i = \frac{P_i}{P} dZ\)

where \(\frac{P_i}{P} = -\Delta_{n+m}(Y, X)^{q-1} \frac{\Delta_{n+m}(\hat{Y}_i, X, Z)}{\Delta_{n+m+1}(Y, X, Z)}\). The equality (4.1) in Proposition 4.1 then gives the following equality

\[
\Delta_n(\Delta_{n+m}(\hat{Y}_1, X, Z), \ldots, (-1)^{i+1}\Delta_{n+m}(\hat{Y}_i, X, Z), \ldots, (-1)^{n+1}\Delta_{n+m}(\hat{Y}_n, X, Z)) \gamma
\]

\[
(4.3) \quad (-1)^{n+1}\Delta_{n+m}(\hat{Y}_n, X, Z) = \gamma\Delta_{m+1}(X, Z)^{q^{n-1}}\Delta_{n+m+1}(Y, X, Z)^{1+q\cdots+q^{n-2}}
\]

where \(\gamma \in \mathbb{F}_q(Y, X)\).

Finally the comparison in the equality (4.3) of the coefficients of higher degree in \(Z\) gives the equality

\[
\Delta_n(\Delta_{n-1+m}(\hat{Y}_1, X), \ldots, (-1)^{i+1}\Delta_{n-1+m}(\hat{Y}_i, X), \ldots, (-1)^{n+1}\Delta_{n-1+m}(\hat{Y}_n, X)) = \gamma\Delta_m(X)^{q^{n-1}}\Delta_{n+m}(Y, X)^{1+q\cdots+q^{n-2}}.
\]

By making \(X_m\) play the role played by \(Z\) in (4.3) we deduce that

\[
\Delta_n(\Delta_{n+m-2}(\hat{Y}_1, X_1, \ldots, X_{m-1}), \ldots, (-1)^{i+1}\Delta_{n-1+m-1}(\hat{Y}_i, X_1, \ldots, X_{m-1}), \ldots,
\]
\[ (-1)^{n+1} \Delta_{n-1+m-1}(\hat{Y}_n, X_1, \cdots, X_{m-1}) = \gamma \Delta_{m-1}(X_1, \cdots, X_{m-1}) q^{m-1} \Delta_{n-1+m-1}(Y, X_1, \cdots, X_{m-1})^{1+q+\cdots+q^{n-2}}. \]

By iterating the process we exhaust \( X \) hence
\[ \Delta_n(\Delta_{n-1}(\hat{Y}_1), \cdots, (-1)^{j+1} \Delta_{n-1}(\hat{Y}_i), \cdots, (-1)^{n-1} \Delta_{n-1}(\hat{Y}_n)) = \gamma \Delta_n(Y)^{1+q+\cdots+q^{n-2}} \]
as announced.

\[ \quad \blacksquare \]

### 4.2.1. An equality between Moore’s determinants II.

We show in two different ways that the constant \( \gamma \) in (4.2) is equal to 1.

Thus we can state the theorem.

**Theorem 4.1.** Let \( (Y) := (Y_1, Y_2, \cdots, Y_n) \) and \( (X) := (X_1, X_2, \cdots, X_m) \) be \( n+m \) indeterminates over \( \mathbb{F}_q \) where \( n \geq 2, m \geq 0 \) and the convention that \( X = \emptyset \) and \( \Delta_m(X) = 1 \) for \( m = 0 \). We write \( (\hat{Y}_i) := (Y_1, \cdots, Y_{i-1}, Y_{i+1}, \cdots, Y_n) \) for \( 1 \leq i \leq n \).

Then we have the following polynomial equalities in \( \mathbb{F}_q[X,Y] \),
\[ \tilde{\Delta}_{n,m}(Y, X) := \]
\[ \Delta_n(\Delta_{n-1+m}(\hat{Y}_1, X), \cdots, (-1)^{j+1} \Delta_{n-1+m}(\hat{Y}_i, X), \cdots, (-1)^{n-1} \Delta_{n-1+m}(\hat{Y}_n, X)) = \]
\[ \Delta_m(X)^{q^{m-1}} \Delta_{n+m}(Y, X)^{1+q+\cdots+q^{n-2}} \]
(4.4)

which is also (compare to (4.1))
\[ \Delta_n\left(\frac{\Delta_{n-1+m}(\hat{Y}_1, X)}{\Delta_m(X)}, \cdots, \frac{\Delta_{n-1+m}(\hat{Y}_i, X)}{\Delta_m(X)}, \cdots, \frac{\Delta_{n-1+m}(\hat{Y}_n, X)}{\Delta_m(X)}\right) = \]
\[ (-1)^\left\lfloor \frac{n}{2} \right\rfloor \left(\frac{\Delta_{n+m}(Y, X)}{\Delta_m(X)}\right)^{1+q+\cdots+q^{n-2}}, \]
(4.5)

where \( \left\lfloor \frac{n}{2} \right\rfloor \) is the lower integer part of \( \frac{n}{2} \). We remark that \( \Delta_m(X) \) divides \( \Delta_{n+m}(Y, X) \) thanks to (2.1).

We deduce by specialization of formula (4.4) the corollary

**Corollary 4.2.** Let \( A \) be a commutative ring containing \( \mathbb{F}_q \). Let \( (a) := (a_1, a_2, \cdots, a_n) \in A^n \) and \( (b) := (b_1, b_2, \cdots, b_m) \in A^m \) where \( n \geq 2, m \geq 0 \) and the convention that \( \emptyset = \emptyset \) and \( \Delta_m(\emptyset) = 1 \) for \( m = 0 \). Then
\[ \Delta_n(\Delta_{n-1+m}(\hat{a}_1, b), \cdots, (-1)^{j+1} \Delta_{n-1+m}(\hat{a}_i, b), \cdots, (-1)^{n+1} \Delta_{n-1+m}(\hat{a}_n, b)) = \]
\[ \Delta_m(b)^{q^{m-1}} \Delta_{n+m}(a, b)^{1+q+\cdots+q^{n-2}}. \]
4.3 – First proof of Theorem 4.1. The case $m = 1$ by induction on $n$.

A. We check (4.4) for $(n, m) = (2, 1)$, i.e., $\Delta_2\left(\frac{\Delta_2(Y_2, X)}{X}, -\frac{\Delta_1(Y_1, X)}{X}\right) = \frac{\Delta_1(Y_1, Y_2, X)}{X}$.

This is an equality between polynomials in the variable $X$ of degree $q^2 - 1$. The terms of higher degree are equal as $\Delta_2(\Delta_1(Y_2 X^{q-1}), -\Delta_1(Y_1 X^{q-1})) = \Delta_2(Y_1, Y_2) X^{q^2 - 1}$.

For $Y_\alpha := \alpha_1 Y_1 + \alpha_2 Y_2$ with $\alpha \in \mathbb{F}_q^2$ we have $\Delta_2(\Delta_2(Y_2, Y_\alpha), \Delta_2(Y_1, Y_\alpha)) = 0$, thus the two polynomials have the same zeros (see (2.2)). Hence the equality.

B. We assume that $m = 1$ and we proceed by induction on $n$. Let us assume that (4.4) is satisfied for $m = 1$ and up to rank $n$. We show it for $m = 1$ and $n + 1$.

B1. We show the equality of the coefficients of highest degree in (4.4) for $m = 1$ and $n + 1$; it is also (4.4) for $m = 0$ and $n + 1$.

Let $(\hat{Y}) := (Y_1, Y_2, \ldots, Y_{n+1})$ be $n + 1$ indeterminates over $\mathbb{F}_q$. Let $j$ such that $1 \leq j \leq n + 1$, we apply (4.4) to the $n$ indeterminates $(Y_1, \ldots, Y_{j-1}, \hat{Y}_j, Y_{j+1}, \ldots, Y_{n+1}) := \hat{Y}_j$ over $\mathbb{F}_q$ and we specialize $X$ in $Y_j$. Thus

\[
\Delta_n(\Delta_n(\hat{Y}_1, \ldots, \hat{Y}_j, \ldots, Y_{n+1}, Y_j), \ldots, \Delta_n(Y_1, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_{n+1}, Y_j)), \ldots,
\]

\[
\Delta_n(Y_1, \ldots, \hat{Y}_j, \ldots, Y_n, \hat{Y}_{n+1}, Y_j)) = (-1)^{\frac{n}{2}} \Delta_n(\hat{Y}_j, Y_j) = (c.f. (4.4))
\]

\[
(-1)^{\frac{n}{2}} Y_j^{q^{n-1}} \Delta_{n+1}(Y_1, \ldots, \hat{Y}_j, \ldots, Y_{n+1}, Y_j)^{1+q+\ldots+q^{n-2}} =
\]

\[
(-1)^{\frac{n}{2}} Y_j^{q^{n-1}}((-1)^{n+1-j} \Delta_{n+1}(Y_j))^{1+q+\ldots+q^{n-2}} =
\]

\[
(4.6) \quad (-1)^{\frac{n}{2}} Y_j^{q^{n-1}}((-1)^{(n+1-j)} Y_j^{q^{n-1}} \Delta_{n+1}(Y_j))^{1+q+\ldots+q^{n-2}}.
\]

In what follows we use the following three identities:

For $1 \leq j < i \leq n + 1$, we have $\Delta_n(Y_1, \ldots, \hat{Y}_j, \ldots, Y_i, \ldots, Y_{n+1}) = (-1)^{n+1-i} \Delta_n(Y_1, \ldots, \hat{Y}_j, \ldots, \hat{Y}_i, \ldots, Y_{n+1}, Y_i)$,

for $1 \leq i < j \leq n + 1$, we have $\Delta_n(Y_1, \ldots, Y_i, \ldots, \hat{Y}_j, \ldots, Y_{n+1}) = (-1)^{n-i} \Delta_n(Y_1, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_{n+1}, Y_i)$,

and for $1 \leq i \leq n + 1$, we have $\Delta_{n+1}(Y_1, \ldots, \hat{Y}_i, \ldots, Y_{n+1}, Y_i) = (-1)^{n+1-i} \Delta_{n+1}(Y_i)$.

Then it follows with (4.6) that for

\[
1 \leq i \leq n + 1, \Delta_n(\Delta_n(\hat{Y}_1), \Delta_n(\hat{Y}_2), \ldots, \hat{\Delta}_n(Y_i), \ldots, \Delta_n(\hat{Y}_{n+1})) =
\]
Thus, by developing the determinant
\[
\Delta_n(\Delta(\hat{\gamma}_1, Y_2, \cdots, Y_i, \cdots, Y_{n+1}), \cdots, \Delta_n(Y_1, \cdots, \hat{\gamma}_{i-1}, Y_i, \cdots, Y_{n+1})),
\]
\[
\Delta_n(Y_1, \cdots, Y_{i-1}, Y_i, \hat{\gamma}_{i+1}, \cdots, Y_{n+1}), \cdots, \Delta_n(Y_1, \cdots, Y_i, \cdots, \hat{\gamma}_{n+1})) =
\]
\[
(-1)^{(n+1-i)(i-1)+(n-i)(n+1-i)} (-1)^{i+1} \Delta_{n+1}(\hat{\gamma}_1, Y_i) =
\]
\[
(-1)^{(n+1-i)(i-1)+(n-i)(n+1-i)} (-1)^{i+1} \Delta_{n+1}(Y_i)^{1+q+\cdots+q^{n-2}} =
\]
(4.7)
\[
(-1)^{i+1} \Delta_{n+1}(Y_i)^{1+q+\cdots+q^{n-2}}.
\]
Thus, by developing the determinant \(\Delta_{n+1}(\Delta_n(\hat{\gamma}_1), \cdots, \Delta_n(\hat{\gamma}_i), \cdots, \Delta_n(\hat{\gamma}_{n+1}))\) along the first row, it follows that
\[
\Delta_{n+1}(\Delta_n(\hat{\gamma}_1), \cdots, \Delta_n(\hat{\gamma}_i), \cdots, \Delta_n(\hat{\gamma}_{n+1})) =
\]
\[
\Delta_n(\hat{\gamma}_1) \Delta_n(\hat{\gamma}_1), \cdots, \Delta_n(\hat{\gamma}_i), \cdots, \Delta_n(\hat{\gamma}_{n+1}))^q -
\]
\[
\Delta_n(\hat{\gamma}_2) \Delta_n(\hat{\gamma}_1), \Delta_n(\hat{\gamma}_2), \cdots, \Delta_n(\hat{\gamma}_i), \cdots, \Delta_n(\hat{\gamma}_{n+1}))^q + \cdots +
\]
\[
(-1)^{n+2} \Delta_n(\hat{\gamma}_{n+1}) \Delta_n(\hat{\gamma}_1), \cdots, \Delta_n(\hat{\gamma}_i), \cdots, \Delta_n(\hat{\gamma}_{n+1}))^q =
\]
\[
(-1)^{i+1} \Delta_n(\hat{\gamma}_i)^q \Delta_{n+1}(Y_i)^{q(1+q+\cdots+q^{n-2})} =
\]
\[
(-1)^{i+1} \Delta_{n+1}(Y_i)^{q(1+q+\cdots+q^{n-2})} =
\]
\[
(-1)^{i+1} \Delta_{n+1}(Y_i)^{q(1+q+\cdots+q^{n-1}).
\]
This is (4.4) for \(m = 0\) and \(n + 1\). This also shows the equality of the coefficients of higher degree in (4.4) for \(m = 1\) and \(n + 1\).

**B2.** We compare the zeros with multiplicity in the two members of (4.4) for \(m = 1\) and \(n + 1\).

We write
\[
G := \Delta_{n+1}(\Delta_{n+1}(\hat{\gamma}_1, X), \cdots, (-1)^{i+1} \Delta_{n+1}(\hat{\gamma}_i, X), \cdots, (-1)^{n+2} \Delta_{n+1}(\hat{\gamma}_{n+1}, X))
\]
and
\[
D := X^{q^n} \Delta_{n+1}(\hat{\gamma}_i, X)^{1+q+\cdots+q^{n-1}}.
\]
We are first interested in \(X = 0\), for that we notice that
\[
\frac{G}{X^{1+q+\cdots+q^n}} = \Delta_{n+1}(\Delta_{n+1}(\hat{\gamma}_1, X), \cdots, (-1)^{i+1} \Delta_{n+1}(\hat{\gamma}_i, X), \cdots, (-1)^{n+2} \Delta_{n+1}(\hat{\gamma}_{n+1}, X))
\]
whose constant term is \((-1)^{n(n+1)} \Delta_{n+1}(\Delta_n(\hat{\gamma}_1), \cdots, (-1)^{i+1} \Delta_n(\hat{\gamma}_i), \cdots, (-1)^{n+2} \Delta_n(\hat{\gamma}_{n+1}))^q\).

On the other hand
\[
\frac{D}{X^{1+q+\cdots+q^n}} = \left(\frac{D_{n+1}(\hat{\gamma}_i, X)}{X}\right)^{1+q+\cdots+q^{n-1}}
\]
whose constant term is
\[
(-1)^{(n+1)n} \Delta_{n+1}(\hat{\gamma}_1)^{q(1+q+\cdots+q^{n-1}).
\]
Then we have equality and non nullity of constant terms by B1., which ensures in particular that the multiplicity of \(X = 0\) is \(1 + q + \cdots + q^n\) in \(G\) and in \(D\).
Thanks to (2.2), we can handle the other zeros. Let \( \varepsilon := (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1}) \in \mathbb{F}_q^{n+1} - (0, \cdots, 0) \), and \( x_{\varepsilon} := \sum_{1 \leq j \leq n+1} \varepsilon_j Y_j \). We need to show that \( x_{\varepsilon} \) is a root of \( G \) with multiplicity \( 1 + q + \cdots + q^{n-1} \).

With (2.1) we get

\[
G = \prod_{1 \leq i \leq n} \prod_{\alpha_{i-1} \in \mathbb{F}_q} (\Delta[i](Y, X) + \alpha_{i-1} \Delta[i - 1](Y, X) + \cdots + \alpha_1 \Delta[1](Y, X))
\]

where \( \Delta[i](Y, X) := (-1)^{i+1} \Delta_{n+1}(\hat{Y}_i, X) \) and so with Proposition 2.3

\[
G = \prod_{1 \leq i \leq n} \prod_{\alpha_{i-1} \in \mathbb{F}_q} \Delta(\alpha_1, \cdots, \alpha_{i-1}, 1, 0, \cdots, 0)(Y, X)
\]

where \( \Delta(\alpha_1, \cdots, \alpha_{i-1}, 1, 0, \cdots, 0) = \Delta_\varphi_i \) with \( \varphi_i = \alpha_1 Y_1^* + \cdots + \alpha_{i-1} Y_i^* + Y_i^* \) and \( (Y_i^*)_{1 \leq i \leq n} \) is the dual basis of \( (Y_i)_{1 \leq i \leq n} \). The roots of \( \Delta(\alpha_1, \cdots, \alpha_{i-1}, 1, 0, \cdots, 0)(Y, X) \) seen as a polynomial in \( X \) and coefficients in \( \mathbb{F}_q(Y) \) are simple (Proposition 2.3) and \( \Delta(\alpha_1, \cdots, \alpha_{i-1}, 1, 0, \cdots, 0)(x_{\varepsilon}) = 0 \) if and only if \( \varepsilon_i + \alpha_{i-1} \varepsilon_{i-1} + \cdots + \alpha_1 \varepsilon_1 = 0 \). Thus the multiplicity of \( x_{\varepsilon} \) in \( G \) is equal to the cardinality of the \( (\alpha_1 : \alpha_2 : \cdots : \alpha_{n+1}) \in \mathbb{F}^n(\mathbb{F}_q) \) which belong to the hyperplane \( \sum_{1 \leq i \leq n+1} \varepsilon_i \alpha_i = 0 \) i.e. \( 1 + q + \cdots + q^{n-1} \). Hence we get (4.4) for \( m = 1 \) and \( n + 1 \).

4.4 – Second proof of Theorem 4.1 by a matrix interpretation in the case \( m = 0 \)

The following theorem is of interest independently of the rest. It gives indeed a relation between a generic Moore matrix and the Moore matrix of the cofactors of its first row; relation analogous to the classical relation between a square matrix and its comatrix. The \( m = 0 \) case of Theorem 4.1 is then an immediate corollary by taking the determinants.

**Theorem 4.2.** Let \( Y_1, Y_2, \cdots, Y_n \), be \( n \) indeterminates over \( \mathbb{F}_q \) and \( M_n(\Delta_n(\hat{Y}_1), \cdots, (-1)^{i-1} \Delta_n(\hat{Y}_i), \cdots, (1)^{n-1} \Delta_n(\hat{Y}_n)) \), the Moore matrix of the cofactors \( (\Delta_n(\hat{Y}_1), \cdots, (-1)^{i-1} \Delta_n(\hat{Y}_i), \cdots, (1)^{n-1} \Delta_n(\hat{Y}_n)) \) of the first row of \( M_n(Y) \). Then one gets

\[
M_n(\Delta_n(\hat{Y}_1), \cdots, (-1)^{i-1} \Delta_n(\hat{Y}_i), \cdots, (1)^{n-1} \Delta_n(\hat{Y}_n)) \cdot M_n(Y) = \\

eq \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\Delta_n(Y) & 0 & \cdots & 0 \\
\alpha_1 & \Delta_n(Y)^q & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{n-2} & \alpha_{n-3}^q & \cdots & \alpha_1^{q^{n-3}} \Delta_n(Y)^{q^{n-2}} & 0
\end{pmatrix}
\]
where \( \alpha_k := \Delta_n(\hat{Y}_1)^{q^{k+1}}Y_1 + \cdots + (-1)^{i-1}\Delta_n(\hat{Y}_i)^{q^{k+1}}Y_i + \cdots + (-1)^{n-1}\Delta_n(\hat{Y}_n)^{q^{k+1}}Y_n \).

**Proof.** We write
\[
\mathcal{M}_n(\Delta_n(\hat{Y}_1), \ldots, (-1)^{i-1}\Delta_n(\hat{Y}_i), \ldots, (-1)^{n-1}\Delta_n(\hat{Y}_n)) \ {}^t \mathcal{M}_n(\hat{Y}_i) := \{m_{i,j}\}_{1 \leq i, j \leq n}.
\]

Since \((-1)^{i-1}\Delta_n(\hat{Y}_i)^q\) is the cofactor of \(Y_i\) in the Moore matrix \(\mathcal{M}_n(\hat{Y}_i)\), we get the following formulas
\[
\Delta_n(\hat{Y}_1)^qY_1 + \cdots + (-1)^{i-1}\Delta_n(\hat{Y}_i)^qY_i + \cdots + (-1)^{n-1}\Delta_n(\hat{Y}_n)^qY_n = \Delta_n(Y)
\]
and for \(1 \leq k \leq n - 1,\)
\[
\Delta_n(\hat{Y}_1)^qY_1^{q^k} + \cdots + (-1)^{i-1}\Delta_n(\hat{Y}_i)^qY_i^{q^k} + \cdots + (-1)^{n-1}\Delta_n(\hat{Y}_n)^qY_n^{q^k} = 0.
\]

Since \((-1)^{i-1}\Delta_n(\hat{Y}_i)^q\) is the cofactor of \(Y_i^{q^{i-1}}\), we get the following formulas
\[
\Delta_n(\hat{Y}_1)Y_1^{q^{i-1}} + \cdots + (-1)^{i-1}\Delta_n(\hat{Y}_i)Y_i^{q^{i-1}} + \cdots + (-1)^{n-1}\Delta_n(\hat{Y}_n)Y_n^{q^{i-1}} = (-1)^{n-1}\Delta_n(Y)
\]
and for \(0 \leq k \leq n - 2,\)
\[
\Delta_n(\hat{Y}_1)Y_1^{q^k} + \cdots + (-1)^{i-1}\Delta_n(\hat{Y}_i)Y_i^{q^k} + \cdots + (-1)^{n-1}\Delta_n(\hat{Y}_n)Y_n^{q^k} = 0.
\]

It follows from the relations (4.11) and (4.12) that \(m_{1,j} = 0\) for \(1 \leq j \leq n_1\) and that \(m_{1,n} = (-1)^{n-1}\Delta_n(Y)\).

Let now \(2 \leq i \leq n\). Raising (4.9) and (4.10) to the power \(q^{i-1}\), it follows that \(m_{i,i-1} = \Delta_n(Y)^{q^{i-1}}\) and \(m_{i,j} = 0\) for \(i \leq j \leq n\).

In conclusion the matrix \([m_{i,j}]_{1 \leq i, j \leq n}\) satisfies (4.8).

By taking the determinant of the matrices in (4.8) we obtain that \(\gamma = 1\) in the case \(m = 0\) of Corollary 4.1, Theorem 4.1 follows.

4.5—A matrix interpretation of the general case \((n, m)\).

The following theorem is a generalization of Theorem 4.2 adapted to a matrix interpretation of the general case of Theorem 4.1. Theorem 4.2 corresponds to the case \(m = 0\) i.e. \(X = 0\).

**Theorem 4.3.** For \(n \geq 2, m \geq 1\) let \(Y_1, Y_2, \ldots, Y_n, X_1, X_2, \ldots, X_m\) be \(n + m\) indeterminates over \(\mathbb{F}_q\), \(\delta_i := (-1)^{i-1}\Delta_{n+m-1}(\hat{Y}_i, (X))\) for \(1 \leq i \leq n,\)
\[
\delta_i := (-1)^{i-1}\Delta_{n+m-1}(\hat{Y}_i, (X)) \text{ for } n + 1 \leq i \leq n + m
\]

Let \(A := [a_{i,j}]_{1 \leq i, j \leq n+m}\), where \(a_{i,j} = (\delta_j)^q^{i-1}\) for \(1 \leq i \leq n, 1 \leq j \leq n + m\) and \(a_{i,i-n} = 1\) for \(n + 1 \leq i \leq n + m\) and \(a_{i,j} = 0\) for \(n + 1 \leq i \leq n + m\) and \(j \neq n - i\).
Hence

\[
A = \begin{pmatrix}
M_n(\delta_1, \delta_2, \cdots, \delta_n) & M_{n,m}(\delta_{1+n}, \delta_{2+n}, \cdots, \delta_{n+m}) \\
0 & \text{Id}_m
\end{pmatrix},
\]

Then

\[
A^t M_{n+m}(Y, X) = [m_{i,j}]_{1 \leq i, j \leq n} =: M, \text{ with } m_{1,j} = 0 \text{ for } 1 \leq j \leq n + m - 1, \quad m_{1,n+m} = (-1)^{n+m-1} \Delta_{n+m}(Y, X), \quad m_{2,1} = \Delta_{n+m}(Y, X),
\]

\[
m_{2,j} = 0 \text{ for } 2 \leq j \leq n + m \text{ and } si 3 \leq i \leq n, \quad 1 \leq j \leq i - 2,
\]

\[
m_{i, i-1} = \Delta_{n+m}(Y, X) q^i \quad \text{and} \quad m_{i,j} = 0 \text{ for } i \leq j \leq n + m.
\]

In matrix writing we have

\[
M = \begin{pmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{pmatrix},
\]

where

\[
M_1 := \begin{pmatrix}
0 & \ldots & 0 \\
\beta_0 & \beta_1 & \ldots & 0 \\
\alpha_1 & \alpha_1 & \ldots & 0 \\
\alpha_2 & \alpha_1 & \beta_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{n-2} & \alpha_1 & \alpha_{n-3} & \ldots & \alpha_1 & \beta_{n-2} & 0
\end{pmatrix}
\]

with \(\beta_i := \Delta_{n+m}(Y, X) q^i\),

\[
M_2 := \begin{pmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\ldots & \ldots & \ldots \\
0 & \ldots & 0
\end{pmatrix},
\]

\[
M_3 = t M_{n,m}(X_1, X_2, \cdots, X_m),
\]

\[
M_4 = t M_m(X_1^{q^n}, X_2^{q^n}, \cdots, X_m^{q^n}).
\]

Proof. We can consider \(\delta^q_i\) as the cofactor of \(Y_i\) or \(X_i\), in the Moore matrix \(M_n(Y, X)\), so we have the following formulas

\[
\delta^q_1 Y_1 + \delta^q_2 Y_2 + \cdots + \delta^q_n Y_n + \delta^q_{n+1} X_1 + \delta^q_{n+2} X_2 + \cdots + \delta^q_{n+m} X_m = \Delta_{n+m}(Y, X)
\]

\[
\delta^q_1 Y_1^k + \delta^q_2 Y_2^k + \cdots + \delta^q_n Y_n^k + \delta^q_{n+1} X_1^k + \delta^q_{n+2} X_2^k + \cdots + \delta^q_{n+m} X_m^k = 0
\]

for \(1 \leq k \leq n + m - 1\).

We can also consider \(\delta_i\) as the cofactor of \(Y_i^{q^{n+m-1}}\) or of \(X_i^{q^{n+m-1}}\) in the Moore matrix \(M_n(Y, X)\). We thus have the following formulas
\[ \delta_1 Y_1^{q_{n+m-1}} + \delta_2 Y_2^{q_{n+m-1}} + \cdots + \delta_n Y_n^{q_{n+m-1}} + \delta_{n+1} X_1^{q_{n+m-1}} + \cdots + \delta_{n+m} X_m^{q_{n+m-1}} = \]
\[ = (-1)^{n+m-1} \Delta_{n+m}(Y, X) \]
(4.18)

\[ \delta_1 Y_1^k + \delta_2 Y_2^k + \cdots + \delta_n Y_n^k + \delta_{n+1} X_1^k + \delta_{n+2} X_2^k + \cdots + \delta_{n+m} X_m^k = 0 \]
for \(0 \leq k \leq n + m - 2\).

It follows from the relations (4.13) and (4.14) that the first line of \( A^t M_n(Y, X) \) is the same as the first line of \( M = [m_{i,j}]_{1 \leq i, j \leq n} \).

Then to show the equality between the lines of index \( i \) with \( 2 \leq i \leq n \), it is enough to raise the relations (4.16) and (4.17) to the power \( q^{i-1} \) and to use the definition of \( \alpha_k \) for \( 1 \leq k \leq n - 2 \).

The equality between the lines of index \( i \) with \( n + 1 \leq i \leq n + m \) is immediate. All this shows the relation (4.14).

\[ \textbf{Corollary 4.3.} \text{ Theorem 4.1 is a consequence of the matrix equality in Theorem 4.3.} \]

\[ \textbf{Proof.} \text{ Expanding the determinant of } M \text{ according to the first line, we have } \]
\[ \det M = \Delta_{n+m}(Y, X) \det N \text{ with } N = \left( \begin{array}{cc}
N_1 & N_2 \\
N_3 & N_4
\end{array} \right) \]
\[ N_1 = \left( \begin{array}{cccc}
\beta_1 & 0 & \cdots & 0 \\
\alpha_0 & \beta_1 & 0 & \cdots \\
\alpha_2 & \alpha_1^q & \beta_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\alpha_{n-2} & \alpha_{n-3}^q & \cdots & \alpha_1^{q_{n-3}} & \beta_{n-2}
\end{array} \right), \text{ with } \beta_i := \Delta_{n+m}(Y, X)^{q^i}, \]
\[ N_2 \text{ is the zero matrix in } M_{n-1,m-1}([\mathbb{F}_q [Y, X]]), \quad N_3 = ' M_{n-1,m-1}(X_1, X_2, \cdots, X_m), \]
\[ N_4 = ' M_m(X_1^{q_{n-1}}, X_2^{q_{n-1}}, \cdots, X_m^{q_{n-1}}). \]

It is then clear that
\[ \det N = \Delta_{n+m}(Y, X)^{1+q+\cdots+q^{n-2}} \Delta_m(X)^{q^{n-1}}. \]

Thus with (4.20) and (4.21), one gets
\[ \det M = \Delta_{n+m}(Y, X) \Delta_{n+m}(Y, X)^{1+q+\cdots+q^{n-2}} \Delta_m(X)^{q^{n-1}}. \]
It follows from (4.14), that \( \det M = \Delta_{n+m}(Y, X) \det A \) and from (2.6) that \( \det A = \Delta_n(\delta_1, \delta_2, \ldots, \delta_n) \), thus

\[
(4.23) \quad \det M = \Delta_{n+m}(Y, X) \Delta_n(\delta_1, \delta_2, \ldots, \delta_n).
\]

Since (cf. Proposition 2.1), \( \Delta_{n+m}(Y, X) \neq 0 \), and that \( \mathbb{F}_q[(Y, X)] \) is an integral ring, the equality (4.4) in Theorem 4.1 for \( m \geq 1 \) follows from (4.22) and (4.23).

Finally, the equality of the coefficients of highest degree in \( X_i \) in formula (4.4) in Theorem 4.1 for \( m = 1 \) gives, as it is noticed in the first proof, formula (4.4) in Theorem 4.1 for \( m = 0 \).

\[
\square
\]

5. Two illustrations of M-O.Ore determinants

5.1 – The map \((a_1, \ldots, a_n) \in K^n \rightarrow (\Delta_{n-1}(\hat{a}_i))_{1 \leq i \leq n} \in K^n\)

**Proposition 5.1.** Let \( K \) be an algebraically closed field with characteristic \( p > 0 \). Let us denote by \( V(\Delta_n) := \{(a_1, a_2, \ldots, a_n) := a \in K^n \mid \Delta_n(a) = 0\} \). The map \( \varphi : a := (a_1, a_2, \ldots, a_n) \in K^n \rightarrow (\Delta_{n-1}(\hat{a}_i))_{1 \leq i \leq n} \in K^n \) induces an onto map from \( K^n - V(\Delta_n) \) to itself. Moreover for \( a \) and \( a' \) in \( K^n - V(\Delta_n) \), one has \( \varphi(a) = \varphi(a') \) if and only if \( a' = \lambda a \) where \( \lambda^{1+q^1+\cdots+q^{n-2}} = 1 \).

**Proof.** Let \((a_1, a_2, \ldots, a_n) \in K^n - V(\Delta_n) \) and \( b_i := \Delta_{n-1}(\hat{a}_i) \) for \( 1 \leq i \leq n \). Since \( \Delta_n(b_i) = (-1)^{\frac{n+1}{2}} \Delta_{n}(a_1^{1+q^n-1}q^{n-2}) \) (we recognize (4.4) for \( m = 0 \)), it follows that \( \varphi(K^n - V(\Delta_n)) \subset K^n - V(\Delta_n) \). Then,

\[
\varphi^2(a) = (\Delta_{n-1}(\hat{b}_i))_{1 \leq i \leq n} =
\]

\[
(\Delta_{n-1}(\Delta_{n-1}(\hat{a}_1)), \ldots, \Delta_{n-1}(\hat{a}_{i-1}), \Delta_{n-1}(\hat{a}_{i+1}), \ldots, \Delta_{n-1}(\hat{a}_n))_{1 \leq i \leq n}.
\]

We recognize there the equality (4.7) at rank \( n \) obtained in the first proof of Theorem 4.1. It follows that

\[
\varphi^2(a) = (-1)^{\frac{n+1}{2}} \Delta_n(a)^{1+q^1+\cdots+q^{n-2}}(a)^{q^{n-2}}.
\]

Let \( \lambda \in K - \{0\} \), then

\[
\varphi^2(\lambda a) = \lambda^{(1+q^1+\cdots+q^{n-1})(1+q^2+\cdots+q^{n-2})+q^{n-2}} \varphi^2(a).
\]

Note that we have the equality \( (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-3}) + q^{n-2} = (1 + q + \cdots + q^{n-2})^2 \) according to the fact that \( \varphi(\lambda a) = \lambda^{1+q^1+\cdots+q^{n-2}} \varphi(a) \). Thus by taking
\( \lambda \) with \( \lambda^{(1+q+\cdots+q^{n-1})^2} = \Delta_n(a)^{-(1+q+\cdots+q^{n-1})} \), one gets \( \varphi^2(\lambda a) = (a)^{q^{n-2}} \). Hence the surjectivity of \( \varphi^2 \) and therefore of \( \varphi \).

Let us now examine the injectivity defect of the map \( \varphi \).

Let \( a \) and \( a' \) be in \( K^n - V(\Delta_n) \) such that \( \varphi(a) = \varphi(a') \), then \( \varphi^2(a) = \varphi^2(a') \) and so \( \Delta_n(a)^{1+q+\cdots+q^{n-1}} = a_i^{q^{n-2}} \Delta_n(a')^{1+q+\cdots+q^{n-1}} a'_i^{q^{n-2}} \). Thus there is \( \lambda \in K \) such that \( a' = \lambda a \), hence \( \lambda^{1+q+\cdots+q^{n-2}} \varphi(a) = \varphi(a) \) and so \( \lambda^{1+q+\cdots+q^{n-2}} = 1 \). The converse is immediate.

**Remark 5.1.** Proposition 5.1 works the same if we replace the map \( \varphi \) by the map \( \varphi_1 : a := (a_1, a_2, \cdots, a_n) \in K^n \rightarrow ((-1)^{i-1} \Delta_n(a_i))_{1 \leq i \leq n} \in K^n \) as \( \varphi^2 = \varphi_1^2 \).

### 5.2 – On \( K \)-étale algebras and Elkies pairing

In this paragraph, unless expressly mentioned, \( K \) is a field of characteristic \( p > 0 \), \( K^{alg} \) is an algebraic closure of \( K \) and \( F \) is the Frobenius automorphism defined by \( F(x) = x^p \) for \( x \in K^{alg} \).

Let \( f : = (f_1, f_2, \cdots, f_n) \in K^n \) with \( \Delta_n(f) \neq 0 \), i.e. \( f_1, f_2, \cdots, f_n \) are \( \mathbb{F}_p \) free. We intend to study the \( K \)-algebra \( A := \frac{K[\mathbb{Z}/p\mathbb{Z}, 1 \leq i \leq n]}{(F - \text{Id})_i \mid 1 \leq i \leq n} \), in particular its group of \( K \)-automorphisms \( \text{Aut}_K A \), and to exhibit a special generator of the \( K \)-algebra \( A \) and a subgroup \( (\mathbb{Z}/p\mathbb{Z})^n \subset \text{Aut}_K A \) whose action on \( A \) is dictated by an associated Elkies pairing (Section 3.2.A.).

**Proposition 5.2.** Let \( n \geq 1 \) and \( f : = (f_1, f_2, \cdots, f_n) \in K^n \) where \( \Delta_n(f) \neq 0 \).

Let \( V \) be the \( \mathbb{F}_p \)-vector space \( \frac{(\Sigma_{1 \leq i \leq n} \mathbb{F}_p f_i) + (F - \text{Id})_i \mid 1 \leq i \leq n}}{(F - \text{Id})_i \mid 1 \leq i \leq n} \) of dimension \( r \leq n \) and \( I \subset \mathbb{N} \) be a partition of \( \{1, 2, \cdots, n\} \) such that \( f_i \mid i \in I \) induces an \( \mathbb{F}_p \)-basis of the vector space \( V \). Let \( A \) be the \( K \)-algebra \( \frac{K[\mathbb{Z}/p\mathbb{Z}, 1 \leq k \leq n]}{(P_k)_{1 \leq k \leq n}} \), where \( P_k := W_k - W_k - f_k \), then \( A \) is an étale \( K \)-algebra isomorphic to \( L^n \), the cartesian product of \( p^{n-r} \), \( \mathbb{Z}/p\mathbb{Z} \) copies of \( L \), where \( L \subset \mathbb{F}_{p^{n-1}} \) is a field which is a Galois extension of \( K \) of group \( (\mathbb{Z}/p\mathbb{Z})^r \) and \( L \cong \frac{K[\mathbb{Z}/p\mathbb{Z}, k \in I]}{(P_k)_{k \in I}} \).

The group of \( K \)-automorphisms \( \text{Aut}_K A \) is then isomorphic to a semidirect product of the groups \( S_{p^{n-r}} \) and \((L/p^nL)^{p^{n-r}}\).

Moreover if \( w_i \) denotes the canonical image of \( W_i \) in \( A \) and if

\[
(5.1) \quad w := \delta_w(f) := \begin{vmatrix}
w_1 & w_2 & \cdots & w_n \\
f_1 & f_2 & \cdots & f_n \\
f_1^p & f_2^p & \cdots & f_n^p \\
\vdots & \vdots & \cdots & \vdots \\
f_1^{p^{n-2}} & f_2^{p^{n-2}} & \cdots & f_n^{p^{n-2}}
\end{vmatrix} \in K[w_1, w_2, ..., w_n],
\]
then

\[
w_i = \frac{\Delta_n(\Delta_{n-1}(f_1), \cdots, (1)^{-2}\Delta_{n-1}(f_{i-1}), w_i, (1)^i\Delta_{n-1}(f_{i+1}), \cdots, (1)^{n-1}\Delta_{n-1}(f_n))}{\Delta_n(\Delta_{n-1}(f_1), \cdots, (1)^i\Delta_{n-1}(f_{i+1}), \cdots, (1)^{n-1}\Delta_{n-1}(f_n))} - (f_i + f_i^p + \cdots + f_i^{p-2}) \in K[w],
\]

where \( \Delta_n(\hat{f}_1) = \Delta_{n-1}(f_1, f_2, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n) \) and

\[
A = K[w_1, w_2, \cdots, w_n] = K[w] \cong \frac{K[W]}{(Q(W))}
\]

where

\[
Q(W) = \frac{\Delta_{n+1}(\Delta_{n-1}(\hat{f}_1), \cdots, \Delta_{n-1}(\hat{f}_n), W)}{\Delta_n(\Delta_{n-1}(\hat{f}_1), \cdots, \Delta_{n-1}(\hat{f}_n))} - \Delta_n(f)^{p^n-1} = W^{p^n} + (\sum_{1 \leq i \leq n-1} (-1)^{n-i}\Delta_n(f)^{p^{n-1}}(\Delta[n - i](f)^{p^{i-1}}W^{p^{i-1}})) + (-1)^{n-1}\Delta_n(f)^{p^{n-1}}W - \Delta_n(f)^{p^{n-1}},
\]

\( \Delta[i](f) := \det(f, F(f), \cdots, \hat{f}^i(f), \cdots, F^n(f)) \) and \( Q(w) = 0 \).

**Proof.** i) Let us show that \( A \) is isomorphic to the \( K \)-algebra \( L^{p^{n-r}} \).

By definition of \( I \), we have \( V = \frac{(\sum_{i \in I} \mathbb{F}_p f_i)}{(F - Id)(K)} \). The Artin-Schreier theory ([8] chap.5 p.88 §.11 Theorem 5) says that \( (F - Id)^{-1}(\sum_{i \in I} \mathbb{F}_p f_i) \subset K^{alg} \) is a Galois extension \( L/K \) of group \( Hom(V, \mathbb{F}_p) \cong (\mathbb{Z}/p\mathbb{Z})^r \) and that

\[
L = \bigoplus_{0 \leq \alpha_i < p, \ i \in I} K \prod_{i \in I} x_i^{\alpha_i} \text{ where } x_i \in K^{alg} \text{ and } P_i(x_i) = 0.
\]

Let \( \pi \) be the \( K \)-algebra homomorphism of \( K[W_i, i \in I] \) onto \( L \) mapping \( \pi(W_i) \) to \( x_i \).

It follows from (5.3) that \( \pi \) induces a \( K \)-algebra homomorphism

\[
\pi' : \frac{K[W_i, i \in I]}{(P_i, i \in I)} = K[w_i, i \in I] \to L \text{ which is surjective and as }
\]

\[
K[w_i, i \in I] = \sum_{0 \leq \alpha_i < p, \ i \in I} K \prod_{i \in I} w_i^{\alpha_i}, \text{ we get a } K \text{-algebra isomorphism }
\]

\[
\frac{K[W_i, i \in I]}{(P_i, i \in I)} \cong L.
\]

On the other hand we have for \( j \in J, f_j = \sum_{i \in I} \lambda_{j,i}f_i + g_j \) with \( \lambda_{j,i} \in \mathbb{F}_p \)

and \( g_j \in K \). Thus for \( j \in J \), and if \( W'_j := W_j + \sum_{i \in I} \lambda_{j,i}W_i \), one gets \( K[W_k, 1 \leq k \leq n] = K[W_i, i \in I, W'_j, j \in J] \) and for \( j \in J \) one has \( W'_j - W_j - (g_j^p - g_j) = W_j + f_j + \sum_{i \in I} \lambda_{j,i}(W_i - f_i) \) and so if \( P'_j(W_j) := W_j - W_j - (g_j^p - g_j) \)

we have \( A \Rightarrow \frac{K[W_i, i \in I, W'_j, j \in J]}{(P_i, i \in I, P'_j, j \in J)} \).

Now we can apply the following general lemma.
Let $K$ be any field (no condition on the characteristic) and $K^{\text{alg}}$ an algebraic closure.

Let $n \geq 1$ and for $1 \leq k \leq n$, $P_k \in K[W_k]$ be a non constant polynomial. Let $A$ be the $K$-algebra $K[\{W_k, 1 \leq k \leq n\}] = K[w_k, 1 \leq k \leq n]$ where $w_k$ is the canonical image of $W_k$.

Let $I \sqcup J$ be a partition of $\{1, 2, \cdots, n\}$ and $B$ be the $K$-algebra $B := \frac{K[\{W_k, k \in I\}]}{(P_k)_{k \in I}}$. Let $u : K[W_k, k \in I] \to A$ be the $K$-homomorphism with $u(W_k) = w_k$ for $k \in I$ then $\ker u = \sum_{i \in I} P_i K[W_k, k \in I]$ and $u$ induces on one side an isomorphism between the two $K$-algebras $B$ and $K[\{W_k, k \in I\}] \subset A$ and on the other side an isomorphism between the two $K$-algebras $\frac{B[\{W_k, k \in J\}]}{(P_k)_{k \in J}}$ and $A$.

**Proof.** We have $\ker u := \{P \in K[W_k, k \in I] \mid P = \sum_{1 \leq k \leq n} Q_k P_k\}$ where $Q_k \in K[W_k, 1 \leq k \leq n]$. Let $z_k \in K^{\text{alg}}$ with $P_k(z_k) = 0$. Let $\sigma : K[W_k, 1 \leq k \leq n] \to K^{\text{alg}}[W_k, k \in I]$ such that $\sigma(a) = a$ for $a \in K$, $\sigma(W_k) = W_k$ for $k \in I$ and $\sigma(W_k) = z_k$ for $k \in J$, then

\begin{equation}
(5.5) \quad P = \sigma(P) = \sum_{k \in I} \sigma(Q_k) P_k, \quad \text{and} \quad \sigma(Q_k) \in K^{\text{alg}}[W_k, k \in I].
\end{equation}

It follows that there is a finite field extension $L/K$ inside $K^{\text{alg}}$ with $\sigma(Q_k) \in L[W_k, k \in I]$. Let $\{e_0 = 1, e_1, \cdots, e_m\}$ a basis for $L/K$, then $L[W_k, k \in I] = \bigoplus_{0 \leq s \leq m} K[W_k, k \in I] e_s$. It follows from (5.5) there is $R_k \in K[W_k, k \in I]$ with $P = \sum_{k \in I} R_k P_k$, thus $\ker u = \sum_{i \in I} P_i K[W_k, k \in I]$.

Let $\pi : K[W_k, k \in I] \to B$, be the canonical $K$-homomorphism and let $v : B \to A$ be the unique $K$-homomorphism with $u = v \circ \pi$. Then $v$ induces an isomorphism from $B$ to $K[\{w_k, k \in I\}] \subset A$.

So we have the following commutative diagram

\[
\begin{array}{ccc}
K[W_k, k \in I] & \xrightarrow{\pi} & B \\
\downarrow & & \downarrow^v \\
A & \xrightarrow{u} & A
\end{array}
\]

it extends in the following commutative diagram

\[
\begin{array}{ccc}
K[W_k, k \in I][W_k, k \in J] & \xrightarrow{\tilde{\pi}} & A \\
\downarrow & & \downarrow^\tilde{\pi} \\
B[W_k, k \in J] & \xrightarrow{\tilde{u}} & A
\end{array}
\]

where $\tilde{u}(W_k) = w_k$, $\tilde{\pi}(W_k) = W_k$, $\tilde{v}(W_k) = w_k$ for $k \in J$.

We claim that $\ker \tilde{v} = \sum_{k \in J} P_k B[W_k, k \in J]$. 
Let $Q \in \ker \tilde{v}$ and $\tilde{Q} \in K[W_k, k \in I; W_k, k \in J]$ such that $Q = \tilde{\pi}(\tilde{Q})$. Then $\tilde{u}(\tilde{Q}) = \tilde{\nu}(\tilde{Q}) = 0$ and so $\tilde{Q} \in \sum_{1 \leq k \leq n} K[W_t, 1 \leq t \leq n]P_k$ and $Q \in \pi(\sum_{1 \leq k \leq n} P_k K[W_t, 1 \leq t \leq n]) = \sum_{k \in J} P_k B[W_k, k \in J]$. 

As $A \simeq \frac{K[W_i, i \in I, W'_j, j \in J]}{(P_i, i \in I, P'_j, j \in J)}$, it follows from Lemma 5.1 that $A = \frac{K[w_i, i \in I, W'_j, j \in J]}{(P'_j, j \in J)}$, where $K[w_i, i \in I] = \frac{K[W_i, i \in I]}{(P_i, i \in I)}$. Now with (5.4) we deduce that $A \simeq \frac{L[W'_j, j \in J]}{(P_j)} \simeq L^{p^{n-r}}$. Moreover $A$ is a $K$-étale algebra since $L/K$ is separable.

ii) We show that the group $\text{Aut}_K A$ is a semidirect product of the groups $\mathfrak{S}_{p^{n-r}}$ and $(\mathbb{Z}/p\mathbb{Z})^{p^{n-r}}$.

This follows from i) and the following lemma.

**Lemma 5.2.** Let $K$ be a commutative field (no condition on the characteristic) and $L/K$ be a finite Galois extension of group $G$. Let $t \geq 1$ and $A := L^t$ and $\text{Aut}_K A$ be the group of $K$-automorphisms of $A$. Let $\rho : \mathfrak{S}_t \to \text{Aut}_K A$, where $\rho(\sigma)(x_1, x_2, \ldots, x_t) := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(t)})$ and $\varphi : G^t \to \text{Aut}_K A$ such that

$$
\varphi(g_1, g_2, \ldots, g_t)(x_1, x_2, \ldots, x_t) := (g_1(x_1), g_2(x_2), \ldots, g_t(x_t)),
$$

then $\rho$ and $\varphi$ are two injective homomorphisms of groups with

$$
\rho(\sigma)\varphi(g_1, g_2, \ldots, g_t)\rho(\sigma)^{-1} = \varphi(g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(t)})
$$

and $\text{Aut}_K A$ is the internal semidirect product of the groups $\rho(\mathfrak{S}_t) \simeq \mathfrak{S}_t$ and $\varphi(G^t) \simeq G^t$.

**Proof.** We can assume that $t \geq 2$ and we show the last assertion.

Let $\mathfrak{M}_i := \{(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_t)\}$ with $x_j \in L$ for $j \neq i$, then $\mathfrak{M}_i$ is a maximal ideal of $A$ and $A/\mathfrak{M}_i \simeq L$. Then $\{\mathfrak{M}_i, 1 \leq i \leq t\}$ is the set of maximal ideals $\text{Spm}(A)$ of $A$.

Now if $\Phi \in \text{Aut}_K A$, $\Phi$ induces a bijection of $\text{Spm}(A)$, so there is $\sigma \in \mathfrak{S}_t$ with $\Phi(\mathfrak{M}_i) = \mathfrak{M}_{\sigma^{-1}(i)}$ for $1 \leq i \leq t$ hence $\rho(\sigma^{-1})\Phi(\mathfrak{M}_i) = \mathfrak{M}_i$ for $1 \leq i \leq t$.

Let $\Psi := \rho(\sigma^{-1})\Phi$, we have $\Psi(\cap_{j \neq i} \mathfrak{M}_j) = \cap_{j \neq i} \mathfrak{M}_j = (0, 0, \ldots, L, 0, \ldots, 0)$ for $1 \leq i \leq t$ where only the $i$-th component is not zero. Thus $\Psi$ induces a $K$-automorphism $g_i$ of $L$. Thus we have $\Psi = \varphi(g_1, g_2, \ldots, g_t)$ and so $\Phi = \rho(\sigma)\varphi(g_1, g_2, \ldots, g_t)$.

iii) We show (5.2).

Let $v_i := w_i + (f_i + f_i^P + \ldots + f_i^{p^{n-2}})$, then $w = \delta_{v_i}(f) = \delta_{v_i}(f)$ and $u^P = v_i + f_i^{p^{n-1}}$. It follows that $w^{p^j} = \delta_{v_i}(f^{p^j})$ for $0 \leq j \leq n - 1$ and $w^{p^n} = \delta_{v_i}(f^{p^n}) + \Delta_n(f^{p^{n-1}})$. 
whose first term is the monic additive polynomial whose roots are the 
which is zero. We get (5.2) where

\[
\begin{vmatrix}
\Delta_{n-1}(\hat{f}_1) & \cdots & \Delta_{n-1}(\hat{f}_n) & w \\
\Delta_{n-1}(\hat{f}_1)^p & \cdots & \Delta_{n-1}(\hat{f}_n)^p & w^p \\
\vdots & \cdots & \vdots & \vdots \\
\Delta_{n-1}(\hat{f}_1)^{p^{n-1}} & \cdots & \Delta_{n-1}(\hat{f}_n)^{p^{n-1}} & w^{p^{n-1}} \\
\end{vmatrix}
\]

which is zero. We get (5.2) where

\[
Q(W) := \frac{\Delta_{n+1}(\Delta_{n-1}(\hat{f}_1), \cdots, \Delta_{n-1}(\hat{f}_n), W)}{\Delta_n(\Delta_{n-1}(\hat{f}_1), \cdots, \Delta_{n-1}(\hat{f}_n))} - \Delta_n(f)^{p^{n-1}}
\]

whose first term is the monic additive polynomial whose roots are the \(\mathbb{F}_p\)-space \(\oplus_{1 \leq i \leq n} \mathbb{F}_p \Delta_{n-1}(\hat{f}_i)\); then the equality

\[
Q(W) = W^{p^n} + \left( \sum_{1 \leq i \leq n-1} (-1)^{n-i} \Delta_n(f)^{p^{n-1} - p^{i-1} - p^i} (\Delta[n-i](f))^{p^{i-1}} W^{p^{n-i}} \right) + (-1)^n \Delta_n(f)^{p^{n-1} - 1} W - \Delta_n(f)^{p^{n-1}},
\]

follows from Elkies ([1] 4.28) and the proof of Proposition 2.5.

**Remark 5.2.** i) Since \(\Delta_f(f), \, i \in I \neq 0\), Proposition 5.2 applied to the \(K\)-algebra
\[
L = \frac{K[W, k \in I]}{(P_k)_{k \in I}}
\]
gives a generator of the extension \(L/K\).

ii) One may consult ([3]) for an application in the case where \(K = \mathbb{k}((t))\) is a field of formal power series.

**Corollary 5.1.** We keep the notations of the proposition.

Let \(F := \oplus_{1 \leq i \leq n} \mathbb{F}_p \hat{f}_i\), \(Z := \oplus_{1 \leq i \leq n} \mathbb{F}_p \Delta_{n-1}(\hat{f}_i)\) be two \(\mathbb{F}_p\)-subspaces of \(K\) associated to \(f\). Let \(z \in Z\) and \(\sigma_z\) be the \(K\)-algebra automorphism of \(K[W]\) such that \(\sigma_z(W) := W + z\); then \(\sigma_z\) induces a \(K\)-algebra automorphism of \(A\) that we still denote by \(\sigma_z\). The map \(z \in Z \rightarrow \sigma_z \in \text{Aut}_K A\) is an injective group homomorphism and its image is a subgroup \(G\) of \(\text{Aut}_K A\) which is isomorphic to \((\mathbb{Z}/p\mathbb{Z})^n\). Let \(U := (\Delta_n(T))^p \subset K\) be the \(\mathbb{F}_p\)-space of roots of the reversed polynomial of \(P_F(X) := \prod_{f \in F} (X - f) = \frac{\Delta_{n+1}(f, X)}{\Delta_n(f)} = X^{p^n} + \cdots + (-1)^n \Delta_n(f)^{p-1} X\) (cf. (2.2) and Section 3.2).
Let $z \in \mathbb{Z}$, we can write $z = \Delta_n(f)u^{1/p}$ with $u \in U$ and for $\varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{F}_p^n - (0, 0, \cdots, 0)$ let $w_\varepsilon := \sum_{1 \leq i \leq n} \varepsilon_i w_i \in A$ (resp. $f_\varepsilon := \sum_{1 \leq i \leq n} \varepsilon_i f_i \in K$) then $w_\varepsilon^p - w_\varepsilon = f_\varepsilon$ and $K[w_\varepsilon] \subset A$ is isomorphic to the $K$-algebra $K[w_\varepsilon]/(w_\varepsilon^p - f_\varepsilon)$ and is a $K$-subalgebra of dimension $p$. Moreover $\sigma_\varepsilon(w_\varepsilon) = w_\varepsilon + (-1)^n E(f_\varepsilon, u)$ where $E : F \times U \rightarrow \mathbb{F}_p$ is the Elkies pairing (see Section 3.2.A.).

In particular when $r = n$ i.e. $A$ is a field and the group $G$ is the full group $\text{Aut}_K A$, then the set $\{K[w_\varepsilon] \mid \varepsilon \in E\}$ where $E$ is a set of representatives of $\mathbb{F}^{n-1}(\mathbb{F}_p)$, is equal to the $\frac{p^{n-1}}{p-1}$, $p$-cyclic extensions of $K$ inside $A$.

Proof. i) We show the equality $\sigma_\varepsilon(w_\varepsilon) = w_\varepsilon + (-1)^n E(f_\varepsilon, u)$.

We have $z := \sum_{1 \leq i \leq n} \alpha_i (-1)^{i-1} \Delta_{n-1}(\hat{f}_i)$ with $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_p^n$. Thus

\[
\sigma_\varepsilon(w_\varepsilon) = \sum_{1 \leq i \leq n} \varepsilon_i (w_i + \Delta_{n-1}(\hat{f}_i), \ldots, (-1)^{i-1} \Delta_{n-1}(\hat{f}_i)) = w_\varepsilon + \sum_{1 \leq i \leq n} \varepsilon_i \alpha_i = w_\varepsilon + (-1)^n E(f_\varepsilon, u)
\]

where $E : F \times U \rightarrow \mathbb{F}_p$ is the Elkies pairing (see Proposition 3.2 and 3.2.B).

ii) We show that $K[w_\varepsilon] \subset A$ is a $K$-subalgebra of dimension $p$.

As the $w_i$, $0 \leq i \leq n$, are $\mathbb{F}_p$-linear independant, after a $\mathbb{F}_p$-linear change of variables we can assume that $\varepsilon = (0, 0, \cdots, n - 1, 1)$, then the results follows from Lemma 5.1.

The case $n = r$ in Corollary 5.1 then follows from Galois theory.

\[\blacksquare\]

References


