Rend. Sem. Mat. Univ. Padova 1xx (201x) **Rendiconti del Seminario Matematico della Università di Padova**

A brief journey through extensions of rational groups

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Abstract – Let A and B be rational groups, i.e. torsion-free groups of rank-1 and thus subgroups of the rational numbers. This paper gives a short overview of the structure of $\mathrm{Ext}(A, B)$ especially considering some interesting classes of torsion-free pairs.

Mathematics Subject Classification (2010). 20K15, 20K35.

Keywords. Abelian group, extension, rational group, torsion-free.

1. Introduction

Throughout this paper the phrase extension of rational groups means extension of a rational group by a rank-1 group.
For the convenience of the reader, we give a short summary of the concept of types: For any element $a \neq 0$ of a group A the height sequence $(h_p)_{p\in\mathbb{P}}$ is defined by $h_p = n$ if there is a non-negative integer n with $a \in p^n A \setminus p^{n+1} A$ and $h_p = \infty$ if no such n exists. The set of height sequences has a partial ordering given by $\alpha = (\alpha_p) \leq (\beta_p) = \beta$ if $\alpha_p \leq \beta_p$ for each $p \in \mathbb{P}$. It forms a lattice by defining $\sup\{\alpha, \beta\} = (\max\{\alpha_p, \beta_p\})$ and $\inf\{\alpha, \beta\} = (\min\{\alpha_p, \beta_p\})$.
Two height sequences (α_p) and (β_p) are said to be equivalent if they only differ in finitely many entries and if $\alpha_p \neq \beta_p$, both have to be finite. The arising equivalence classes are called types and build a lattice induced by the lattice structure of the height sequences, where $[(\alpha_p)] \leq [(\beta_p)]$ iff $\alpha_p \leq \beta_p$ for all but finitley many primes $p \in \mathbb{P}$ and if $\alpha_p \nleq \beta_p$, then α_p is an integer.
It is easy to see that in a rank-1 group A all elements have equivalent height sequences. Hence the lattice of isomorphism classes of rank-1 groups is isomorphic to the lattice of types, which was shown by Reinhold Baer in 1935. Due to this fact it is obvious to identify a rank-1 group A by its type $tp(A)$. For simplicity, we write $tp(A) = (\alpha_p)$ without explicitly indicating that this is an equivalence class.

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Furthermore we can define an addition of types: if $tp(A) = (\alpha_p)$ and $tp(B) = (\beta_p)$ we put $tp(A)+tp(B) = (\alpha_p+\beta_p)$. In particular, this is the type of the group $A\otimes B$.

Recall the definition of the *nucleus* of a group A, which was originally given by Phil Schultz:

Definition 1.1. For any group A we call

$$\mathrm{Nuc}(A) := \left\langle \frac{1}{p^\omega} \mid p \in \mathbb{P} \textit{ with } (\cdot p) \in \mathrm{Aut}(A) \right\rangle \leq \mathbb{Q}$$

the nucleus of A denoted by A_0.

In other words, A_0 is the largest subring of $\mathbb{Q}$ such that A is still an A_0-module. Thus for any group A we have $tp(A_0) = (\alpha_p)$ with $\alpha_p = \infty$ if A is p-divisible and $\alpha_p = 0$ otherwise. Hence $tp(A_0)$ is an idempotent type. In particular $tp(A_0) \leq tp(A)$ applies for any rational group A.

One of the very valuable properties of the functor Ext in the category of Abelian groups is the fact that given a torsion-free Abelian group A the group $\mathrm{Ext}(A,B)$ is divisible for any Abelian group B. Hence its structure is very much determined and $\mathrm{Ext}(A,B)$ must be of the form $\mathrm{Ext}(A,B) = \bigoplus_{r_0} \mathbb{Q} \oplus \bigoplus_p [\bigoplus_{r_p} \mathbb{Z}_{p^\infty}]$ for some uniquely determinded cardinals r_0 and r_p which are called the *torsion-free rank* and the *p-rank* of $\mathrm{Ext}(A,B)$, respectivley. In [2] it was shown what values for these cardinals are possible in general. We will now apply these results on extensions of rank-1 groups.

2. The structure of Ext by comparing types

At first we consider the case $tp(A) \leq tp(B)$. By [3, Theorem 2.1.4] we know that $\mathrm{Ext}(A,B)$ is torsion-free if and only if the following applies:

$$OT((A \otimes B_0)/D) \leq IT(B),$$

with D being the divisible subgroup of $A \otimes B_0$ for any torsion-free groups A and B of finite rank and $OT(B) \neq tp(\mathbb{Q})$.

Theorem 2.1. *For any rational groups A and B the following statements are equivalent:*

(1) $\mathrm{Ext}(A,B)$ *is torsion-free*

(2) $tp(A) \leq tp(B)$ *or* $A \otimes B_0 = \mathbb{Q}$

Proof. First let be $tp(A) \leq tp(B)$. Since inner type, outer type and the type of any rational group are all equal, $\mathrm{Ext}(A,B)$ is torsion-free by a result of Pat

Goeters, see [4, Prop. 1.7]. If otherwise $A \otimes B_0 = \mathbb{Q}$ we conclude $\mathrm{Ext}(A, B) \cong \mathrm{Ext}(A \otimes B_0, B) \cong \mathrm{Ext}(\mathbb{Q}, B)$ is torsion-free since $\mathbb{Q}$ is divisible. See [2, Lemma 2.6] for the first isomorphism.
Now let $\mathrm{Ext}(A, B)$ be torsion-free. If $tp(B) = tp(\mathbb{Q})$, then trivially $tp(A) \leq tp(B)$ because $tp(\mathbb{Q})$ is the maximal element in the lattice of types. So assume $tp(B) \neq tp(\mathbb{Q})$ and we have to consider $tp((A \otimes B_0)/D)$. Either $A \otimes B_0 = \mathbb{Q}$ or $A \otimes B_0$ has no divisible subgroup since it is a rank-1 group. Thus $tp(A) \leq tp(A \otimes B_0) = OT((A \otimes B_0)/D) \leq tp(B)$. □

In particular, the group of self-extensions $\mathrm{Ext}(A, A)$ is torsion-free for any rational group A.

One of the main results of [2] says that $r_0(\mathrm{Ext}(A, B)) = 0$ iff $\mathrm{Ext}(A, B) = 0$, or $r_0 = 2^{\aleph_0}$. Thus a not-vanishing torsion-free extension of rational groups is of the form $\mathrm{Ext}(A, B) = \bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

Assuming the stricter condition $tp(A) \leq tp(B_0)$ it is possible to point out when Ext vanishes for rational groups A and B. By [2] this happens if and only if $A \otimes B_0$ is a free B_0-module. In this case we receive:

THEOREM 2.2. *For any rational groups A and B the following are equivalent:*
(1) $\mathrm{Ext}(A, B) = 0$
(2) $tp(A) \leq tp(B_0)$

PROOF. So let be $\mathrm{Ext}(A, B) = 0$. Thus $A \otimes B_0 = B_0$ since it is a free B_0-module of rank-1. Hence $tp(A \otimes B_0) = tp(A) + tp(B_0) = tp(B_0)$ which is equivalent to $tp(A) \leq tp(B_0)$. □

Following pat Goeters we define the *support* of a group A as $supp(A) = \{p \in \mathbb{P} | pA \neq A\}$, that is the set of all primes not dividing A. Trivially, $supp(A) \subseteq supp(B)$ if $tp(A) > tp(B)$ because for a rational group $A = (\alpha_p)$ the support of A is given by $supp(A) = \{p \in \mathbb{P} | \alpha_p \neq \infty\}$

THEOREM 2.3. *For any rational groups A and B the following are equivalent:*
(1) $r_p(\mathrm{Ext}(A, B)) = 1$ *for any* $p \in supp(A) \cap supp(B)$
(2) $tp(A) > tp(B)$ *or the types are incomparable*

PROOF. Assume (2) holds. Due to Warfiled it is well-known that the p-rank of $\mathrm{Ext}(A, B)$ can be calculated by $r_p(\mathrm{Ext}(A, B)) = r_p(A) \cdot r_p(B) - r_p(\mathrm{Hom}(A, B))$ for finite rank Abelian groups A and B, where $r_p(A) = dim_{\mathbb{Z}/p\mathbb{Z}}(A/pA)$ if A is torsion-free. But there are no homomorphisms $\varphi : A \to B$ except the trivial one and hence $\mathrm{Hom}(A, B) = 0$ iff $tp(A) > tp(B)$ or the types are incomparable. Therefore we conclude $r_p(\mathrm{Ext}(A, B)) = r_p(A) \cdot r_p(B)$ and thus $r_p(\mathrm{Ext}(A, B)) = 1$ if both A and B are not p-divisible.
If we assume the negation of (2), $\mathrm{Ext}(A, B)$ is torsion-free by 2.1 and thus $r_p(\mathrm{Ext}(A, B)) = 0$. Hence the assertion holds. □

So any not torsion-free extension of rational groups is of the form
$\mathrm{Ext}(A,B) = \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \bigoplus_{p} \mathbb{Z}_{p^\infty}$ with $p \in supp(A) \cap supp(B)$.

3. Torsion-free pairs

In analogy to Luigi Salces cotorsion pairs we call a pair $(\mathcal{A}, \mathcal{B})$ of classes of groups a *torsion-free pair* if $\mathrm{Ext}(A,B)$ is torsion-free for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and the classes $\mathcal{A}$ and $\mathcal{B}$ are closed with respect to this property. This means X has to be an element of $\mathcal{B}$ if $\mathrm{Ext}(A,X)$ is torsion-free for all $A \in \mathcal{A}$ as well as $X \in \mathcal{A}$ if $\mathrm{Ext}(X,B)$ is torsion-free for all $B \in \mathcal{B}$. Like in [5] we can define a partial order on the class of torsion-free pairs by putting $(\mathcal{A},\mathcal{B}) \leq (\mathcal{A}',\mathcal{B}')$ if $\mathcal{B} \subseteq \mathcal{B}'$ or, equivalently $\mathcal{A}' \subseteq \mathcal{A}$. Then the torsion-free pairs becomes a complete lattice by setting $\bigwedge_{i\in I}(\mathcal{A}_i,\mathcal{B}_i) = ({}^*(\bigcap_{i\in I} \mathcal{B}_i), \bigcap_{i\in I} \mathcal{B}_i)$ and $\bigvee_{i\in I}(\mathcal{A}_i,\mathcal{B}_i) = (\bigcap_{i\in I} \mathcal{A}_i, (\bigcap_{i\in I} \mathcal{A}_i)^*)$ for a family $\{(\mathcal{A}_i,\mathcal{B}_i)\}_{i\in I}$ of torsion-free pairs. We define

(1) $\mathcal{A}^* := \{X \mid \mathrm{Ext}(A,X)$ *is* torsion-free for all $A \in \mathcal{A}\}$

(2) ${}^*\mathcal{B} := \{X \mid \mathrm{Ext}(X,B)$ *is* torsion-free for all $B \in \mathcal{B}\}$

and call $({}^*(\mathcal{A}^*), \mathcal{A}^*)$ the *torsion-free pair co-generated by* $\mathcal{A}$ and $({}^*\mathcal{B}, ({}^*\mathcal{B})^*)$ the *torsion-free pair generated by* $\mathcal{B}$.

One of the main results of [3] is:

Theorem 3.1. *The lattice of types is anti-isomorphic to the lattice of all rational generated* $(\mathfrak{Tffr}, \mathfrak{Tffr})$*-torsion-free pairs, which mean torsion-free pairs restricted on torsion-free groups of finite rank.*

For the proof and more general results we recommend to have a look at [3].

Since our main purpose in this section is to shed some light on the extensions of rational groups, we replace the restriction on torsion-free groups of finite rank by rational groups, the so-called $(\mathfrak{R},\mathfrak{R})$-torsion-free pairs. Unfortunately, 3.1 does not hold for these rational torsion-free pairs:

Theorem 3.2. *There exist rational groups* A *and* B *such that* $tp(A) < tp(B)$ *but* ${}^*A = {}^*B$.

Proof. Take $B = \mathbb{Q}$. Then $\mathrm{Ext}(A,\mathbb{Q}) = 0$ for any group A and thus ${}^*\mathbb{Q} \cap \mathfrak{R} = \mathfrak{R}$. Now consider the group $\mathbb{Q}_p$ of all rational numbers with denominator prime to p. There is only one group which has a type greater than $tp(\mathbb{Q}_p)$, namely $\mathbb{Q}$. Furthermore, any group of uncomparable type has to be p-divisible. So if X is an arbitrary rank-1 group, either $tp(X) \leq tp(\mathbb{Q}_p)$ or $X \otimes \mathbb{Q}_p = \mathbb{Q}$ which implies that also ${}^*\mathbb{Q}_p \cap \mathfrak{R} = \mathfrak{R}$. □

It turns out that 3.1 holds if we restrict on rational groups $\neq \mathbb{Q}$:

Theorem 3.3. *The lattice of types is anti-isomorphic to the lattice of all rational generated* $(\mathfrak{R} \setminus \{\mathbb{Q}\}, \mathfrak{R} \setminus \{\mathbb{Q}\})$*-torsion-free pairs.*

Proof. Let be $tp(A) \leq tp(B)$. If $X \in^* A$ we know by 2.1 that $tp(X) \leq tp(A)$ or $X \otimes A_0 = \mathbb{Q}$. But then also $tp(X) \leq tp(B)$ or $X \otimes B_0 = \mathbb{Q}$ which implies that $\mathrm{Ext}(X, B)$ is also torsion-free and thus ${}^*A \subseteq {}^*B$.
Now consider the strict inequality $tp(A) < tp(B)$ which implies that $A \otimes B_0 = \mathbb{Q}$ is only possible if $B = \mathbb{Q}$. Since this is excluded, $A \otimes B_0$ cannot be divisible, so $B \otimes A_0 \neq \mathbb{Q}$ as well. Hence there has to be a prime p such that A and B are not p-divisible and thus $\mathrm{Ext}(B, A)$ is not torsion-free. Indeed, $\mathrm{Ext}(B, B)$ is torsion-free. So we conclude ${}^*A \subsetneqq {}^*B$. □

Putting 3.1 and 3.3 together we obtain:

Theorem 3.4. *The lattices of all rational generated* $(\mathfrak{Tffr}, \mathfrak{Tffr})$*-torsion-free pairs and*
$(\mathfrak{R} \setminus \{\mathbb{Q}\}, \mathfrak{R} \setminus \{\mathbb{Q}\})$*-torsion-free pairs are isomorphic.*

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Received 06.03.2017; revised 29.08.2017