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On W-S-permutable Subgroups of Finite Groups*

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Abstract – A subgroup H of a finite group G is said to be W-S-permutable in G if there is a subgroup K of G such that $G = HK$ and $H \cap K$ is a nearly S-permutable subgroup of G. In this article, we analyse the structure of a finite group G by using the properties of W-S-permutable subgroups and obtain some new characterizations of finite p-nilpotent groups and finite supersolvable groups. Some known results are generalized.

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1. Introduction

All groups considered in this paper are finite.

Recall that two subgroups A and B of a group G are said to permute if $AB = BA$, i.e. AB is a subgroup of G. A subgroup H of G is called π-quasinormal in G if H permutes with every Sylow p-subgroup of G for all $p \in \pi$, where π is a set of primes [11]. A subgroup T of G is said to be S-permutable (S-quasinormal) in G if T is $\pi(G)$-quasinormal in G, where $\pi(G)$ denote a set of primes dividing $|G|$. The relationship between the structure of a group G and its S-permutable subgroups has been extensively studied by many authors (for example, see [4],[5], [12],[17]). On the other hand, a subgroup H of a group G is C-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$,

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where H_G is the core of H in G [6]. A subgroup H of a group G is said to be weakly S-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G [14]. Recently, Khaled A. Al-Sharo introduced the concept of nearly S-permutable subgroups and obtained many interesting results[1]. A subgroup H of a group G is said to be nearly S-permutable in G if the normalizer $N_K(H)$ contains some Sylow p-subgroup of K for every subgroup K of G containing H and for every prime p with $(p,|H|)= 1$. As inspired by the above research, it is good for us to give the following definition:

Definition 1.1. A subgroup H of a group G is said to be W-S-permutable in G if there is a subgroup K of G such that $G = HK$ and $H \cap K$ is a nearly S-permutable subgroup of G.

Remark 1.2. It is clear that C-supplemented subgroups and nearly S-permutable subgroups are W-S-permutable subgroups. However, the converses do not hold in general, for example:

(1) Let $G = S_4$, the symmetric group of degree 4. Take $H = \langle(12)\rangle$. Then it is easy to see H is W-S-permutable in G. But H is not nearly S-permutable in G since $N_{S_3}(\langle(12)\rangle)$ does not contain any Sylow 3-subgroup of S_3.

(2) Let $P = \langle x, y | x^{16} = y^4 = 1, x^y = x^3\rangle$. Then it is clear that $\Phi(P)=\langle x^2\rangle\times\langle y^2\rangle$ and $\langle y^2\rangle$ is S-permutable in G, and so $\langle y^2\rangle$ is W-S-permutable in G. But $\langle y^2\rangle$ is not C-supplemented in G.

In the present paper, we first give some properties of W-S-permutable subgroups, and then we try to investigate the structure of groups. In fact, some new conditions for a group to be p-nilpotent or supersolvable are given by using the assumption that some kinds of subgroups of prime power order are W-S-permutable, and many known results are generalized.

2. Preliminaries

In this section we will list some basic or known results which are useful for us in the paper.

First we recall that a class $\mathfrak{F}$ of groups is a formation if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$, and if $G/N_i \in \mathfrak{F}$, $i = 1, 2$, then $G/N_1 \cap N_2 \in \mathfrak{F}$. Furthermore, a formation $\mathfrak{F}$ is said to be a saturated formation if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$, where $\Phi(G)$ is the Frattini subgroup of G. In this paper, U denotes the class of all supersolvable groups. It is well-known that U is a saturated formation.

Lemma 2.1 (Lemma 2.2, [1]). *Let H be a nearly S-permutable subgroup of a group G and N a normal subgroup of G. Then*

(1) *HN is nearly S-permutable in G;*

(2) *If H is a group of prime power order, then $H \cap N$ is nearly S-permutable in G;*

(3) *If H is a group of prime power order, then HN/N is nearly S-permutable in G/N;*

(4) *If $|H| = p^n$ for some prime p, then $H \leq O_p(G)$.*

Lemma 2.2. *Suppose that V is a W-S-permutable subgroup of a group G and N is a normal subgroup of G. Then*

(1) *V is W-S-permutable in K whenever $V \leq K \leq G$;*

(2) *Suppose that V is a p-group for some prime p. If $N \leq V$, then V/N is W-S-permutable in G/N;*

(3) *Suppose that V is a p-group for some prime p and N is p'-subgroup, then VN/N is W-S-permutable in G/N.*

Proof. By the hypotheses, there is a subgroup T of G such that $G = TV$ and $T \cap V$ is a nearly S-permutable subgroup of G. It follows from that $K = V(K \cap T)$ and $V \cap (K \cap T) = (V \cap T) \leq K$. Obviously, $V \cap T$ is nearly S-permutable in K. Hence, V is W-S-permutable in K and (1) is true.

Also we have $G/N = (V/N)(TN/N)$ and $(V/N) \cap (TN/N) = (V \cap TN)/N = (V \cap T)N/N$. By Lemma 2.1(3), $(V \cap T)N/N$ is nearly S-permutable in G/N. Hence, V/N is W-S-permutable in G/N and (2) is true.

It is clear that $N \leq T$, $G/N = (VN/N)(T/N)$ and $(VN/N) \cap (T/N) = (VN \cap T)/N = (V \cap T)N/N$. By Lemma 2.1(3), $(V \cap T)N/N$ is nearly S-permutable in G/N. Hence, VN/N is W-S-permutable in G/N and (3) is true. □

Lemma 2.3. *Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent for some prime p. Then*

(1) *G has a normal Sylow p-subgroup P and $G = P \rtimes Q$, where Q is non-normal cyclic Sylow q-subgroup for some prime $q \neq p$;*

(2) *the exponent of P is 2 or 4 if $p = 2$; the exponent of P is p if $p \neq 2$;*

(3) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;*

(4) *$\Phi(P) = Z_\infty(G) \cap P$.*

Proof. For (1)–(3) see [10, III, Satz 5.2 and IV, Satz 5.4].

(4) According to $Z_\infty(G) \cap P \trianglelefteq G$ and (3), we have $P \cap Z_\infty(G) \leq \Phi(P)$. On the other hand, $\Phi(P) \leq Z(G)$. So (4) holds. □

Lemma 2.4 (A, 1.2, [7]). *Let T, V and W are subgroups of a group G. Then the following are equivalent:*

(1) *$T \cap VW = (T \cap V)(T \cap W)$;*

(2) *$VT \cap WT = (V \cap W)T$.*

Lemma 2.5 (Lemma 2.6, [13]). *Let G be a group. Assume that N is a normal subgroup of G ($N \neq 1$) and $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in $F(N)$.*

Lemma 2.6 (Corollary 2, [2]). *Let P be a Sylow 2-subgroup of a group G. If P has no section isomorphic to Q_8 and $\Omega_1(P) \leq Z(G)$, then G is 2-nilpotent, where Q_8 is the quaternion group of order 8.*

Lemma 2.7 (Theorem A, [9]). *Suppose that a group G has a Hall π-subgroup, where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.8 (Lemma 2.16, [14]). *Let $\mathfrak{F}$ be a saturated formation containing U, let G be a group with a normal subgroup H such that $G/H \in \mathfrak{F}$. If H is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.9 (Lemma 2.8, [16]). *Let M be a maximal subgroup of a group G and P be a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is normal in G.*

Lemma 2.10 (Theorem 3.1, [16]). *Let $\mathfrak{F}$ be a saturated formation containing U and G a group with a solvable normal subgroup H such that $G/H \in \mathfrak{F}$. If for every maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathfrak{F}$.*

Lemma 2.11. *Let p be the smallest prime divisor of the order of a group G. If G has no section isomorphic to Q_8 and every subgroup of G with order p is W-S-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. By Lemma 2.2 (1), it is easy to see that G is a minimal non-p-nilpotent group. By Lemma 2.3, G has a normal Sylow p-subgroup G_p such that $G = G_p \rtimes G_q$ for a cyclic Sylow q-subgroup G_q $(q > p)$. Suppose that every subgroup of G_p with order p is normal in G. Then $\Omega_1(G) \leq Z(G)$. If $p \neq 2$, then, by [10, IV,Satz5.5(a)], G is p-nilpotent, a contradiction. If $p = 2$, then G is p-nilpotent by Lemma 2.6, again a contradiction. Therefore there exists some minimal subgroup H of G such that H is not normal in G. So G_p is non-abelian and $H \nsubseteq \Phi(G_p)$. By the hypotheses, there is a subgroup K of G such that $G = HK$ and $H \cap K$ is nearly S-permutable in G. If $H \cap K = H$, then H is nearly S-permutable in G. By Lemma 2.1(3), $H\Phi(G_p)/\Phi(G_p)$ is nearly S-permutable in $G/\Phi(G_p)$. Then there exists some Sylow q-subgroup $Q\Phi(G_p)/\Phi(G_p)$ of $G/\Phi(G_p)$ such that $Q\Phi(G_p)/\Phi(G_p) \leq N_{G/\Phi(G_p)}(H\Phi(G_p)/\Phi(G_p))$. Since $G_p/\Phi(G_p)$ is abelian, $H\Phi(G_p)/\Phi(G_p) \trianglelefteq G/\Phi(G_p)$. By Lemma 2.3(3), $H\Phi(G_p)/\Phi(G_p) = G_p/\Phi(G_p)$ is a cyclic group. Burnside's Theorem[10, IV, Satz 2.6] implies that $G/\Phi(G_p)$ is p-nilpotent and so G is p-nilpotent by [10, VI, Hilfssatz 6.3], a contradiction. If $H \cap K = 1$, then $K \trianglelefteq G$. The choice of G implies that K is p-nilpotent and therefore G is p-nilpotent, a contradiction. The proof of the lemma is complete. □

3. Main Results

THEOREM 3.1. *Let P be a Sylow p-group of a group G, where p is the smallest prime divisor of the order of G. If every cyclic subgroup H of P with prime order or order 4 (P is non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) either is W-S-permutable or has a supersolvable supplement in G, then G is p-nilpotent.*

PROOF. Suppose that G is not p-nilpotent. Then G has a minimal non-p-nilpotent subgroup L. By Lemma 2.3, $L = L_p \rtimes L_q$, where L_p is a normal Sylow p-subgroup of L and L_q is a cyclic Sylow q-subgroup of L for some prime $p \neq q$. We may assume that $L_p \leq P$. Let $H = \langle x \rangle$, $x \in L_p \backslash \Phi(L_p)$. Then $|H| = p$ or 4 by Lemma 2.3(2). If $H \subseteq Z_\infty(G) \cap L = Z_\infty(L)$, then $\Phi(L_p) \neq L_p \cap Z_\infty(L)$, which contradicts Lemma 2.3(4). Suppose that H is W-S-permutable in G, then H is also W-S-permutable in L by Lemma 2.2(1). Let T be a subgroup of L such that $L = TH$ and $H \cap T$ is nearly S-permutable in L. If $|L : T| = 4$, then $\langle x^2 \rangle T \trianglelefteq L$, and so $L_q \trianglelefteq L$, a contradiction. If $|L : T| = p$, we also get $L_q \trianglelefteq L$, the same contradiction. Therefore $L = T$ and H is nearly S-permutable in L. By Lemma 2.1(3), $H\Phi(L_p)/\Phi(L_p)$ is also nearly S-permutable in $L/\Phi(L_p)$. Then there exists some Sylow q-subgroup Q of L such that $Q\Phi(L_p)/\Phi(L_p) \subseteq N_{L/\Phi(L_p)}(H\Phi(L_p)/\Phi(L_p))$. Hence $H\Phi(L_p)/\Phi(L_p) \trianglelefteq L/\Phi(L_p)$. Since $L_p/\Phi(L_p)$ is a minimal normal subgroup of $L/\Phi(L_p)$, $L_p = H\Phi(L_p) = H$. In view of the hypotheses and Burnside's Theorem [10, IV, Satz 2.6], L is p-nilpotent, a contradiction. If H has a supersolvable supplement K in G, then $G = HK$ and $L = H(L \cap K)$. Since $L_p/\Phi(L_p)$ is abelian, $(L_p \cap K)\Phi(L_p)/\Phi(L_p) \trianglelefteq L/\Phi(L_p)$. By Lemma 2.3(3), $(L_p \cap K)\Phi(L_p)/\Phi(L_p) = L_p/\Phi(L_p)$ or 1. If $(L_p \cap K)\Phi(L_p)/\Phi(L_p) = 1$, then $L_p = H$. Again applying Burnside's Theorem [10, IV, Satz 2.6], then L is p-nilpotent, a contradiction. Thus $L_p \subseteq K$, and so $L \subseteq K$. Since p is the smallest prime divisor of $|K|$, K is p-nilpotent and so L is p-nilpotent, a contradiction. The proof is complete. □

REMARK 3.2. In theorem 3.1, the hypotheses that subgroups of order 4 are W-S-permutable in G if P is non-abelian 2-group and $H \nsubseteq Z_\infty(G)$ could not be removed. For example, let $G = L \rtimes \langle \alpha \rangle$, where $L = Q_8$ is a quaternion group and α is an automorphism of L with order 3. Then G has a unique minimal normal subgroup H of order 2. Evidently, H is W-S-permutable in G. But G is non-p-nilpotent.

THEOREM 3.3. *Let P be a Sylow p-group of a group G, where p is the smallest prime divisor of the order of G. If every maximal subgroup of P is W-S-permutable in G, then G is p-nilpotent.*

PROOF. Suppose that theorem is false and let G be a counterexample of minimal order. Then

(1) G has a unique minimal normal subgroup $1 \neq N$ such that G/N is p-nilpotent. Moreover, $\Phi(G) = 1$.

Let $1 \neq N$ be a minimal normal subgroup of G. Consider the factor group G/N. If $P \subseteq N$, then it is obvious that G/N is p-nilpotent. Suppose that $P \nsubseteq N$. Let L/N be a maximal subgroup of PN/N. Then there exists a maximal subgroup P_1 of P such that $L = NP_1$. By the hypotheses, G has a subgroup T such that $G = TP_1$ and $P_1 \cap T$ is nearly S-permutable in G. We have $G/N = (TN/N)(L/N) = (TN/N)(P_1N/N)$. Since $(|N : N \cap P_1|, |N : N \cap T|) = 1$, $(N \cap P_1)(N \cap T) = N = N \cap TP_1$. By Lemma 2.4, $P_1N \cap TN = (P_1 \cap T)N$. It follows from Lemma 2.1(3) that $(TN/N) \cap (P_1N/N) = (P_1 \cap T)N/N$ is nearly S-permutable in G/N. Therefore, the theorem is true for G/N. The minimality of G implies that G/N is p -nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we may assume that N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

(2) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) > 1$. Then $N \leq O_{p'}(G)$ by (1). Since $G/O_{p'}(G) \simeq (G/N)/(O_{p'}(G)/N)$, it follows that G is p-nilpotent, a contradiction.

(3) $O_p(G) = 1$.

If $O_p(G) > 1$, then, by (1), $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Since $O_p(G) \cap M$ is normalized by N and M, the uniqueness of N yields $N = O_p(G)$. Pick some maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Then $P = NP_1$. By the hypotheses, there exists a subgroup T of G such that $G = TP_1$ and $P_1 \cap T$ is nearly S-permutable in G. Suppose that $P_1 \cap T \neq 1$. According to the nearly S-permutability of $P_1 \cap T$ and the minimality of N, we have $N \leq (P_1 \cap T)^G = (P_1 \cap T)^{\langle Q_1, Q_2, ..., Q_s \rangle P} = (P_1 \cap T)^P \leq P_1^P = P_1$, where Q_i is some Sylow q_i-subgroup of G contained in $N_G(P_1 \cap T)$ with $p \neq q_i$, $i = 1, 2, ..., s$. Thus $P = P_1N = P_1$, a contradiction. Hence $P_1 \cap T = 1$. This shows that the Sylow p-subgroup of T is cyclic. By Burnside's Theorem [10, IV, Satz 2.6], T is p-nilpotent. Let $T_{p'}$ be the normal complement of T. Then $G = P_1T = P_1N_G(T_{p'})$. By (1), $M \simeq G/N$ is p-nilpotent. Let $M_{p'}$ be the normal complement of M. By (2) and the maximality of M, $N_G(M_{p'}) = M$. By Lemma 2.7, there exists an element $x \in P_1$ such that $T_{p'}^x = M_{p'}$. Then $G = (P_1N_G(T_{p'}))^x = P_1N_G(T_{p'}^x) = P_1N_G(M_{p'})$. Thus $P = P \cap G = P \cap P_1N_G(M_{p'}) = P_1(P \cap N_G(M_{p'})) = P_1(P \cap M) = P_1$, a contradiction and so (3) holds.

The final contradiction.

Let P_1 be a maximal subgroup of P. By the hypotheses, there exists a subgroup T of G such that $G = TP_1$ and $P_1 \cap T$ is nearly S-permutable in G. By (3), $(P_1 \cap T)^G = (P_1 \cap T)^{\langle Q_1, Q_2, ..., Q_s \rangle P} = (P_1 \cap T)^P = 1$, where Q_i is some Sylow q_i-subgroup of G contained in $N_G(P_1 \cap T)$ and $p \neq q_i$, $i = 1, 2, ..., s$. So $P_1 \cap T = 1$. This implies that the Sylow p-subgroup of T is cyclic. By Burnside's Theorem [10, IV,Satz 2.6], T is p-nilpotent. Let $T_{p'}$ be the normal complement of T. Then $G = P_1T = P_1N_G(T_{p'})$. In view of (2), $P \cap N_G(T_{p'})$ is a proper subgroup of P. Consequently, there exists another maximal subgroup P_2 of P such that $P \cap N_G(T_{p'}) \leq P_2$. By the hypotheses, there exists a subgroup H of G such that$G = HP_2$ and $P_2 \cap H$ is nearly S-permutable in G. By the above proof, we can get $P_2 \cap H = 1$ and H is p-nilpotent. Let $H_{p'}$ be the normal p-complement of H. Then $G = P_2 N_G(H_{p'})$. By Lemma 2.7, there exists $g \in P_2$ such that $(H_{p'})^g = T_{p'}$. Now, $G = (P_2 N_G(H_{p'}))^g = (P_2)^g (N_G(H_{p'}))^g = P_2 N_G(T_{p'})$. Then $P = P \cap P_2 N_G(T_{p'}) = P_2(P \cap N_G(T_{p'})) = P_2$, a contradiction. □

Remark 3.4. In Theorem 3.3, the assumption that p is the smallest prime divisor of the order of a group G is essential, for example, let $G = \langle a, b | a^9 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Clearly, every maximal subgroup of Sylow 3-subgroup of G is W-S-permutable in G. But G is not 3-nilpotent.

Corollary 3.5. *Let p be the smallest prime factor of the order of a group G and N a normal subgroup of G such that G/N is p-nilpotent. If N has a Sylow p-subgroup P such that every maximal subgroup of P is W-S-permutable in G, then G is p-nilpotent.*

Proof. By Theorem 3.3 and Lemma 2.2, N is p-nilpotent. Let $N_{p'}$ be the normal p'-complement of N. Then $N_{p'} \trianglelefteq G$. If $N_{p'} \neq 1$, then, by Lemma 2.2, $G/N_{p'}$ satisfies the hypotheses of the corollary. Hence $G/N_{p'}$ is p-nilpotent by the induction on $|G|$, and so G is p-nilpotent. Suppose that $N_{p'} = 1$. Then N is p-group. Let L/N be the normal Hall p'-complement of G/N. By Schur-Zassenhaus Theorem, there is a Hall p'-subgroup $L_{p'}$ of L such that $L = N \rtimes L_{p'}$. Then L is p-nilpotent by Lemma 2.2 and Theorem 3.3. This implies that $L_{p'}$ is normal p'-subgroup of G. Therefore G is p-nilpotent. □

Theorem 3.6. *Let G be a group which has no section isomorphic to A_4 or Q_8 and let P be a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. Suppose that $N_G(P)$ is p-nilpotent and there exists a positive integer m with $1 < p^m < |P|$ such that all subgroups H of P with order p^m are W-S-permutable in G, then G is p-nilpotent.*

Proof. Suppose that theorem is false and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Let $PO_{p'}(G)/O_{p'}(G)$ be a Sylow p-subgroup of $G/O_{p'}(G)$. By Lemma 2.2(3), every subgroup of $PO_{p'}(G)/O_{p'}(G)$ with order p^m is W-S-permutable in $G/O_{p'}(G)$. Clearly, $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent and $G/O_{p'}(G)$ has no section isomorphic to A_4 or Q_8. Hence $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. By the minimality of G, $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

(2) $m > 1$ and $|P| \neq p^{m+1}$.

Assume that $m = 1$. Then, by Lemma 2.11, G is p-nilpotent, a contradiction. If $|P| = p^{m+1}$, then G is p-nilpotent by Theorem 3.3, again a contradiction.

(3) H is p-nilpotent for every subgroup H of G such that $P \leq H < G$.

By Lemma 2.2(1), every subgroup of P with order p^m is W-S-permutable in H. Obviously, $N_H(P)$ is p-nilpotent and H has no section isomorphic to A_4 or Q_8. Consequently, H is p-nilpotent by the choice of G.

(4) $O_p(G) \neq 1$, $G/O_p(G)$ is p-nilpotent and G is solvable.

Suppose that $O_p(G) = 1$. Obviously, $N_G(J(P)) < G$ and $C_G(Z(P)) < G$. Since $P \leq N_G(J(P))$ and $P \leq C_G(Z(P))$, $N_G(J(P))$ and $C_G(Z(P))$ are p-nilpotent by (3). It follows from Glauberman-Thompson Theorem [8] that G is p-nilpotent, a contradiction. Hence $O_p(G) \neq 1$. Let $\overline{G} = G/O_p(G), \overline{P} = P/O_p(G)$, $N_{\overline{G}}(J(\overline{P})) = L_1/O_p(G)$, $C_{\overline{G}}(Z(\overline{P})) = L_2/O_p(G)$. Then $P \leq L_1 < G$ and $P \leq L_2 < G$. By (3), L_1 and L_2 are p-nilpotent. Again applying Glauberman-Thompson Theorem [8], $G/O_p(G)$ is p-nilpotent. In view of Feit-Thompson Theorem on groups of odd order, G is solvable.

(5) Let N be a normal p-subgroup of G such that $1 < |N| < p^m$. Then G/N is p-nilpotent.

By Lemma 2.2(2), every subgroup of P/N with order $p^m/|N|$ is W-S-permutable in G/N. Clearly, $N_{G/N}(P/N) = N_G(P)/N$ and G/N has no section isomorphic to A_4 or Q_8. Consequently G/N is p-nilpotent by the choice of G.

(6) If $1 < |\Phi(G)| \neq p^m$, then G is p-nilpotent.

Assume that $1 < |\Phi(G)| < p^m$. Then, by (5), $G/\Phi(G)$ is p-nilpotent and so G is p-nilpotent by [10, VI, Hilfssatz 6.3], a contradiction. If $|\Phi(G)| > p^m$, let H be a subgroup of $\Phi(G)$ of order p^m and $H \trianglelefteq P$. By the hypotheses, there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is nearly S-permutable in G. Since $H \leq \Phi(G)$, $G = HK = K$ and so H is nearly S-permutable in G. Therefore, for every prime divisor q of the order of G with $q \neq p$, there exists some Sylow q-subgroup Q of G such that $Q \leq N_G(H)$. Since $H \trianglelefteq P$, $H \trianglelefteq G$. By (5) and Burnside's Theorem [10, IV,Satz 2.6], H is non-cyclic. Now take a subgroup L of $\Phi(G)$ of order p^{m+1} such that $H \leq L$. Since H is non-cyclic, so is L. Hence L contains a subgroup H_2 of order p^m such that $H \neq H_2$. As above, H_2 is nearly S-permutable in G. Hence, for every prime factor q of the order of G with $q \neq p$, there exists some Sylow q-subgroup Q_1 of G such that $Q_1 \leq N_G(H_2)$. Then $LQ_1 = \langle H, H_2\rangle Q_1 = Q_1\langle H, H_2\rangle = Q_1L$ is a subgroup of G. By Theorem 3.3, LQ_1 is p-nilpotent and so $Q_1 \leq N_G(L)$. Since $H \trianglelefteq G$, $Q_1 \leq C_G(H)$. Then $|G/C_G(H)| = p^\alpha$ for some integer α. It follows from [18, Appendix C, Theorem 6.3] that $H \leq Z_\infty(G)$, and so G contains a cyclic normal subgroup T of order p. By (2), (5) and Burnside's Theorem [10, IV,Satz 2.6], we have G is p-nilpotent, a contradiction.

(7) $O_p(G)$ is a maximal subgroup of P.

Assume that $O_p(G)$ is not maximal in P. Since G is solvable by (4), G contains a normal maximal subgroup of M such that $|G/M| = r$, where r is a prime divisor of the order of G. Pick a Sylow p-subgroup L of M such that $L \leq P$. If $r = p$, then we have $P \leq N_G(L) < G$. By (3), $N_G(L)$ is p-nilpotent and so $N_M(L)$ is p-nilpotent. Then, by Lemma 2.2(1), every subgroup of L with order p^m is W-S-permutable in M. So we have M is p-nilpotent by the minimality of G. Hence $O_{p'}(M) \leq O_{p'}(G)$, which is impossible by (1). If $r \neq p$, then $P \leq M$, and so M is p-nilpotent by (3). The same contradiction is obtained. So we have (7).

(8) If $|\Phi(G)| = p^m$, then G is p-nilpotent.

Suppose that $\Phi(G)$ is cyclic. Then $\Phi(G)$ contains a normal subgroup L of G with order p. By (2), (5) and Burnside's Theorem [10, IV,Satz 2.6], G is p-

nilpotent, which contradicts the choice of G. Hence $\Phi(G)$ is non-cyclic. Now let $T/\Phi(G)$ be any subgroup of $O_p(G)/\Phi(G)$ with order p. Since $\Phi(G)$ is non-cyclic, T is non-cyclic. So T has a maximal subgroup F with $F \neq \Phi(G)$. Then we have $T = F\Phi(G)$. By the hypotheses, there exists a subgroup K of G such that $G = KF$ and $K \cap F$ is nearly S-permutable in G. If $K = G$, then F is nearly S-permutable in G. By Lemma 2.1(3), $F\Phi(G)/\Phi(G)$ is nearly S-permutable in $G/\Phi(G)$. If $K \neq G$, then $G/\Phi(G) = (K\Phi(G)/\Phi(G))(F\Phi(G)/\Phi(G)) = (K\Phi(G)/\Phi(G))(T/\Phi(G))$. Obviously, $(K\Phi(G)/\Phi(G)) \cap (T/\Phi(G)) = 1$ and so $T/\Phi(G)$ is W-S-permutable in $G/\Phi(G)$. Thus we get that any subgroup $T/\Phi(G)$ of $O_p(G)/\Phi(G)$ with order p is W-S-permutable in $G/\Phi(G)$. By Lemma 2.2(1), $T/\Phi(G)$ is W-S-permutable in $O_p(G)K/\Phi(G)$, where K is a Hall p'-subgroup of G. By Theorem 2.11, $O_p(G)K/\Phi(G)$ is p-nilpotent and so $O_p(G)K$ is p-nilpotent by [10, VI, Hilfssatz 6.3]. By (7), $O_p(G)K \trianglelefteq G$ and so G is p-nilpotent, a contradiction.

(9) If N is a minimal normal p-subgroup of G, then $|N| \leq p^m$.

Suppose that $|N| > p^m$. Take a subgroup H of N such that $|H| = p^m$ and $H \trianglelefteq P$. By the hypotheses, there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is nearly S-permutable in G. Since N is abelian, $N \cap K \trianglelefteq G$. By the minimality of N, we have $N \cap K = 1$ or $N \cap K = N$. If $N \cap K = 1$, then $N = N \cap G = H(N \cap K) = H$, a contradiction. Thus $N \cap K = N$. It follows from $G = HK$ and $H \leq N \leq K$ that $K = G$. This implies that H is nearly S-permutable in G. Then, for every prime factor q of $|G|$ with $q \neq p$, there exists some Sylow q-subgroup Q of G such that $Q \leq N_G(H)$. Since $H \trianglelefteq P$, $H \trianglelefteq G$, which contradicts the minimality of N.

(10) If H_1 and H_2 are two distinct minimal normal p-subgroups of G, then $|H_1| < p^m$ or $|H_2| < p^m$.

If $|H_1| \geq p^m$ and $|H_2| \geq p^m$, then, by (9), $|H_1| = p^m$ and $|H_2| = p^m$. In view of (6) and (8), we have $\Phi(G) = 1$. Thus G contains a maximal subgroup M such that $G = MH_1$ and $M \cap H_1 = 1$. Obviously, $P \cap M$ is a Sylow p-subgroup of M and let $\overline{G} = G/H_1$, $\overline{P} = P/H_1$. Then $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ is p-nilpotent and so $N_M(P \cap M)$ is p-nilpotent. By Lemma 2.2(1), every subgroup of $P \cap M$ with order p^m is W-S-permutable in M. Furthermore, M has no section isomorphic to A_4 or Q_8. It follows that M is p-nilpotent by the minimality of G, and so $G/H_1 \simeq M$ is p-nilpotent. As above, we also have G/H_2 is p-nilpotent. Since $G = G/H_1 \cap H_2$ is isomorphic to a subgroup of $G/H_1 \times G/H_2$, G is p-nilpotent, a contradiction.

(11) $O_p(G)$ is a minimal normal subgroup of G.

Suppose that $O_p(G)$ is not a minimal normal subgroup of G. In view of (6) and (8), we have $\Phi(G) = 1$. By Lemma 2.5, we may assume that N_1 and N_2 are two distinct minimal normal subgroups of G contained in $O_p(G)$. By (10), if $|N_1| < p^m$ and $|N_2| < p^m$, then G/N_1 and G/N_2 are p-nilpotent by (5), and so G is p-nilpotent, a contradiction. Therefore we may assume that $|N_1| < p^m$ and $|N_2| \geq p^m$. Since G/N_1 is p-nilpotent by (5), $G/N_1 = (P/N_1)(T/N_1)$, where T/N_1 is normal Hall p'-subgroup of G/N_1. Then $P \cap T = N_1$, and therefore $N_2 \cap (P \cap T) = N_2 \cap T = 1$ and $T \leq C_G(N_2)$. Then, by [18, Appenix C, Theorem 6.3], $N_2 \leq Z_\infty(G)$. This implies that G contains a normal subgroup L of order p. By (5), G/L is p-nilpotent and so G is p-nilpotent, a contradiction.

The final contradiction.

By (11), $O_p(G)$ is a minimal normal subgroup of G. Then $|O_p(G)| \leq p^m$ by (9). In view of (7), $|P| \leq p^{m+1}$, which contradicts (2). The proof is complete. □

Remark 3.7. In Theorem 3.6, the assumption that G has no section isomorphic to A_4 or Q_8 is necessary, for example, let $x=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $y=\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$, $z=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ be three generators of $G = GL(2,3)$ and x, y, z satisfies the following relation:

$$x^8 = y^2 = z^3 = 1,\ y^{-1}xy = x^3,\ z^{-1}x^2z = xy,\ z^{-1}xyz = xyx^2,\ y^{-1}zy = z^2.$$

Then $P = \langle x, y\rangle$ is a Sylow 2-subgroup of G and $G'' = \langle x^2, xy\rangle$ is a quaternion group of order 8. We see that $P = N_G(P)$ and $SL(2,3)/Z(G'') \simeq A_4$. Obviously, all subgroups of P with order 2 are W-S-permutable in G. However, G is not 2-nilpotent.

Theorem 3.8. *Let $\mathfrak{F}$ be a saturated formation containing U and let E be a normal subgroup of a group G such that $G/E \in \mathfrak{F}$. Suppose that all maximal subgroups of Sylow subgroups of E are W-S-permutable in G, then $G \in \mathfrak{F}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.2 and Theorem 3.3, E is a Sylow tower group of supersolvable type. Let Q be a Sylow q-subgroup of E, where q is the largest prime divisor of the order of E. Then we have $Q \trianglelefteq G$. Let N be a minimal normal subgroup of G contained in Q. In view of Lemma 2.2, the theorem holds for G/N. By the choice of G, $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, $N \nsubseteq \Phi(G)$ and $N = Q$ is the unique minimal normal subgroup of G. Hence there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Let M_q be a Sylow q-subgroup of M. Then $G_q = NM_q$. Pick a maximal subgroup L of G_q such that $M_q \subseteq L$. Then $Q \cap L = Q_1$ is a maximal subgroup of Q. By the hypotheses, there exists a subgroup T of G such that $G = TQ_1$ and $T \cap Q_1$ is nearly S-permutable in G. On the other hand, $N = N \cap G = N \cap TQ_1 = (N \cap T)Q_1$. We have $N \cap T \trianglelefteq G$ since N is abelian. In view of the minimality of N, $N \cap T = T$ and $N \subseteq T$. Consequently, $G = T$ and $T \cap Q_1 = Q_1$ is W-S-permutable in G. Then, for every prime p of the order of G with $q \neq p$, the normalizer $N_G(Q_1)$ contains some Sylow p-subgroup P of G. Consequently, $Q_1 \trianglelefteq G$. We have $Q_1 = 1$ by the minimality of N and so N is cyclic. By Lemma 2.8, $G \in \mathfrak{F}$, which contradicts the choice of G. The proof of the theorem is complete. □

Theorem 3.9. *Let $\mathfrak{F}$ be a saturated formation containing U and let E be a solvable normal subgroup of a group G such that $G/E \in \mathfrak{F}$. Suppose that all maximal subgroups of Sylow subgroups of $F(E)$ are W-S-permutable in G, then $G \in \mathfrak{F}$.*

Proof. Let M be a maximal subgroup of G not containing $F(E)$. We only prove that $M \cap F(E)$ is a maximal subgroup of $F(E)$ by Lemma 2.10. Since $F(E) \nsubseteq M$, there exists a Sylow p- subgroup $O_p(E)$ of $F(E)$ such that $O_p(E) \nleq M$ and $G = MO_p(E)$. Let G_p be a Sylow p-subgroup of G and M_p a Sylow p-subgroup of M. Then $G_p = M_pO_p(E)$. Pick a maximal subgroup L of G_p such that $M_p \leq L$. Then $R = L \cap O_p(E)$ is a maximal subgroup of $O_p(E)$, and $R \cap M = (L \cap O_p(E)) \cap M = O_p(E) \cap M_p = O_p(E) \cap M$. By Lemma 2.9, $O_p(E) \cap M \trianglelefteq G$ and so $O_p(E) \cap M \leq R_G$. Furthermore, $O_p(E) \cap M = R_G$. By the hypotheses, there is a subgroup T of G such that $G = TR$ and $T \cap R$ is nearly S-permutable in G. For some Sylow q_i-subgroup Q_i of G with $q_i \neq p, i = 1, 2, ..., n$, we have $Q_i \leq N_G(T \cap R)$. We may assume that $\Phi(O_p(E)) = 1$. Thus $O_p(E) \cap T \trianglelefteq G$. On the other hand, $T \cap R = T \cap O_p(E) \cap L \trianglelefteq G_p$. Consequently $T \cap R \trianglelefteq G$ and $T \cap R \leq R_G$. Let $H = TR_G$. Then $H \cap R = R_G$. Since M is a maximal subgroup of G, $(O_p(E) \cap H)M = M$ or $(O_p(E) \cap H)M = G$. If $(O_p(E) \cap H)M = M$, then $O_p(E) \cap H \leq M \cap O_p(E) = R_G = H \cap R$. This implies that $R = O_p(E)$, a contradiction. Thus we have $(O_p(E) \cap H)M = G$. In view of $O_p(E) \cap M = O_p(E) \cap H \cap M = R_G$, $O_p(E) \leq H$. Thus $R = R \cap H = R_G = M \cap O_p(E)$ and so $|F(E) : F(E) \cap M| = |O_p(E) : O_p(E) \cap M| = p$. This completes the proof. □

4. Some applications

In the literature one can find the following special case of our main theorems.

Corollary 4.1 (Theorem 3.3, [15]). *Let G be a group and E a normal subgroup of G such that G/E is supersolvable. If all maximal subgroups of each Sylow subgroup of E are C-supplemented in G, then G is supersolvable.*

Corollary 4.2 (Theorem 4.1, [16]). *Let $\mathfrak{F}$ be a saturated formation containing U and G a group and E a solvable normal subgroup such that $G/E \in \mathfrak{F}$. If all maximal subgroups of each Sylow subgroup of $F(E)$ are C-supplemented in G, then $G \in \mathfrak{F}$.*

Corollary 4.3 (Theorem 1.3, [3]). *Let G be a group and E a normal subgroup of G with supersolvable quotient G/E. Suppose that all maximal subgroups of any Sylow subgroup of E are S-permutable in G, then G is supersolvable.*

Corollary 4.4 (Theorem 1.4, [3]). *Let $\mathfrak{F}$ be a saturated formation containing U. Suppose that G is a solvable group with a normal subgroup H such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are S-quasinormal in G, then $G \in \mathfrak{F}$.*

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References

[1] A. Al-Sharo, *On nearly S-permutable subgroups of finite groups*, Comm. Algebra, **40** (2012), pp. 315–326.

[2] M. Asaad, *On p-nilpotence of finite groups*, J. Algebra, **277** (2004), pp. 157–164.

[3] M. Asaad, *On maximal subgroups of finite group*, Comm. Algebra, **26** (1998), pp. 3647–3652.

[4] M. Asaad, M. Ramadan, A. Shaalan, *Influence of π-quasinormality on maximal subgroups of Sylow subgroups of Fitting subgroups of a finite group*, Arch. Math. (Basel), **56** (1991), pp. 521–527.

[5] A. Ballester-Bolinches, and M. C. Pedraza-Aguilera, *Sufficient conditions for supersolubility of finite groups*, J. Pure Appl. Algebra, **127** (1998), pp. 113–118.

[6] A. Ballester-Bolinches, Y. Wang and X. Guo, *C-supplemented subgroups of finite groups*, Glasgow Math. J, **42** (2000), pp. 383–389.

[7] K. Doer and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin, New York, 1992.

[8] G. Glauberman, *Subgroups of finite groups*, Bull. Amer. Math. Soc., **73** (1967), pp. 1–12.

[9] F. Gross, *Conjugacy of odd order Hall subgroups*, Bull. London Math. Soc., **19** (1987), pp. 311–319.

[10] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.

[11] H. O. Kegel, *Sylow-Gruppen and Subnoramlteiler endlieher Gruppen*, Math. Z., **78** (1962), pp. 205–221.

[12] Y. M. Li and K. T. Peng, *π-quasinormally embedded and c-supplemented subgroup of finite group*, Front. Math. China, **3** (2008), pp. 511–521.

[13] Y. M. Li, Y. M. Wang and H. Q. Wei, *The influence of π-quasinormality of some subgroups of a finite group*, Arch. Math. (Basel), **81** (2003), pp. 245–252.

[14] A. N. Skiba, *On weakly S-permutable subgroups of finite groups*, J. Algebra, **315** (2007), pp. 192–209.

[15] Y. Wang, *Finite Groups with some subgroups of Sylow subgroups c-supplemented*, J. Algebra, **224** (2000), pp. 467–478.

[16] Y. Wang, H. Wei and Y. Li, *A generalization of a theorem of Kramer and its applications*, Bull. Aust. Math. Soc., **65** (2002), pp. 467–475.

[17] X. B. Wei, X. Y. Guo, *On finite groups with prime-power order S-quasinormally embedded subgroups*, Monatsh. Math, **162** (2011), pp. 329–339.

[18] M. Weinstein, *Between nilpotent and solvable*, Polygonal Publishing House, NJ, USA, 1982.

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