

## On the values of unipotent characters in bad characteristic

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ABSTRACT – Let  $G(q)$  be a Chevalley group over a finite field  $\mathbb{F}_q$ . By Lusztig’s and Shoji’s work, the problem of computing the values of the unipotent characters of  $G(q)$  is solved, in principle, by the theory of character sheaves; one issue in this solution is the determination of certain scalars relating two types of class functions on  $G(q)$ . We show that this issue can be reduced to the case where  $q$  is a prime, which opens the way to use computer algebra methods. Here, and in a sequel to this article, we use this approach to solve a number of cases in groups of exceptional type which seemed hitherto out of reach.

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### 1. Introduction

Let  $G(q)$  be a group of Lie type over a finite field with  $q$  elements. This paper is concerned with the problem of computing the values of the irreducible characters of  $G(q)$ . The work of Lusztig [16], [23] has led to a general program for solving this problem. In this framework, one has to establish certain identities of class functions on  $G(q)$  of the form  $R_x = \xi\chi_A$ , where  $R_x$  denotes an “almost character” (that is, an explicitly known linear combination of irreducible characters) and  $\chi_A$  denotes the characteristic function of a suitable “character sheaf” on the underlying algebraic group  $G$ ; furthermore,  $\xi$  is a scalar of absolute value 1. This program has been successfully carried out in many cases, see, e.g., Bonnafé [2], Lusztig [22], [24], Shoji [36], [40] and Waldspurger [43], but not in complete generality.

In this paper, we will assume that  $G(q)$  is of split type and only consider the above problem as far as unipotent characters of  $G(q)$  are concerned, as defined by Deligne and Lusztig [7]. By Shoji’s work [37], [38], we know that the desired

identities  $R_x = \xi\chi_A$  as above hold, but the scalars  $\xi$  are not yet determined in all cases. And even in those cases where they are known, this often required elaborate computations. In such cases, the scalars then turn out to behave rather uniformly as  $q$  varies (see, e.g., Shoji [36], [40]). The main theoretical result of this paper, Proposition 3.4, provides a partial, *a priori* explanation for this phenomenon; the proof is an elaboration of ideas which are already contained in Lusztig’s and Shoji’s papers. The fact that we can formulate this result without any assumptions on  $q$  essentially relies on Lusztig [26], where the “cleanness” of cuspidal character sheaves is established in complete generality and, consequently, the principal results of [17]–[21] (e.g. [21, Theorems 23.1 and 25.2]) hold unconditionally.

The main observation of this paper is that the statement of Proposition 3.4 can also be exploited in a different way, as follows. For a given type of group, we consider the base case where  $q = p$  is a prime. For a specific value of  $p$ , we can use ad hoc methods and/or computer algebra systems like GAP [10] to perform all kinds of computations within the fixed finite group  $G(p)$ . If we succeed in this way to determine the scalars  $\xi$  for  $G(p)$ , then Proposition 3.4 tells us that the analogous result will hold for any power of  $p$ . This is particularly relevant for “bad” primes  $p = 2, 3, 5$  which, typically, are known to cause additional complications and require separate arguments. We illustrate this procedure with a number of examples. In particular, we determine  $\xi$  in two cases where the character table of  $G(\mathbb{F}_p)$  is explicitly known; namely,  $F_4$ ,  $E_6$  and  $p = 2$ . For type  $F_4$ , our results complete earlier results of Marcelo–Shinoda [31]. See also [13] for the discussion of further examples, where the complete character table of  $G(\mathbb{F}_p)$  is not known.

We assume some familiarity with the character theory of finite groups of Lie type; see, e.g., [5], [12]. The basic reference for the theory of character sheaves are Lusztig’s papers [17]–[21]. In Section 2, we review the classification of unipotent characters of  $G(q)$  and the analogous classification of the unipotent character sheaves on  $G$ . These two classifications are known to be the same for  $G(q)$  of split type (a fact which has only recently found a conceptual explanation; see Lusztig [28]). In Section 3, we can then formulate in precise terms the problem of equating class functions  $R_x = \xi\chi_A$  as above, and establish Proposition 3.4. Finally, Sections 4, 5, 6 contain a number of examples where we show how the scalars  $\xi$  can be determined using standard functions and algorithms in GAP.

1.1. *Notation.* Let  $\ell$  be a prime such that  $\ell \nmid q$ . If  $\Gamma$  is a finite group, we denote by  $\text{CF}(\Gamma)$  the vector space of  $\overline{\mathbb{Q}}_\ell$ -valued functions on  $G^F$  which are constant on the conjugacy classes of  $\Gamma$ . (We take  $\overline{\mathbb{Q}}_\ell$  instead of  $\mathbb{C}$  since, in the framework of [7], [23], class functions on  $\Gamma = G(q)$  are constructed whose values are cyclotomic numbers in  $\overline{\mathbb{Q}}_\ell$ .) Given  $f, f' \in \text{CF}(\Gamma)$ , we denote by  $\langle f, f' \rangle = |\Gamma|^{-1} \sum_{g \in \Gamma} f(g)\overline{f'(g)}$  the standard scalar product of  $f, f'$  where the bar denotes a field automorphism which maps roots of unity to their inverses. Let  $\text{Irr}(\Gamma)$  be the set of irreducible characters of  $\Gamma$  over  $\overline{\mathbb{Q}}_\ell$ ; these form an orthonormal basis of  $\text{CF}(\Gamma)$ .

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## 2. Unipotent character sheaves and almost characters

Let  $p$  be a prime and  $k = \overline{\mathbb{F}}_p$  be an algebraic closure of the field with  $p$  elements. Let  $G$  be a connected reductive algebraic group over  $k$  and assume that  $G$  is defined over the finite subfield  $\mathbb{F}_q \subseteq k$ , where  $q = p^f$  for some  $f \geq 1$ . Let  $F: G \rightarrow G$  be the corresponding Frobenius map. Let  $B \subseteq G$  be an  $F$ -stable Borel subgroup and  $T \subseteq B$  be an  $F$ -stable maximal torus. Let  $W = N_G(T)/T$  be the corresponding Weyl group. We assume that  $F$  acts trivially on  $W$  and that  $F(t) = t^q$  for all  $t \in T$ . Then the group of rational points  $G^F = G(\mathbb{F}_q)$  is a finite group of Lie type of “split type”.

2.1. For each  $w \in W$ , let  $R_w$  be the virtual character of  $G^F$  defined by Deligne–Lusztig [7, §1]. Let  $\mathfrak{U}(G^F)$  be the set of unipotent characters of  $G^F$ , that is,

$$\mathfrak{U}(G^F) = \{\rho \in \text{Irr}(G^F) \mid \langle \rho, R_w \rangle \neq 0 \text{ for some } w \in W\}.$$

Now [16, Main Theorem 4.23] provides a classification of  $\mathfrak{U}(G^F)$  in terms of

- a parameter set  $X(W)$  and a pairing  $\{ , \}: X(W) \times X(W) \rightarrow \overline{\mathbb{Q}}_\ell$  (which only depend on  $W$ ),
- an embedding  $\text{Irr}(W) \hookrightarrow X(W)$ ,  $\epsilon \mapsto x_\epsilon$ .

(See [16, 4.21] for precise definitions; recall that we assume that  $F$  acts trivially on  $W$ ). Indeed, there is a bijection

$$\mathfrak{U}(G^F) \leftrightarrow X(W), \quad \rho \leftrightarrow x_\rho,$$

such that for any  $\rho \in \mathfrak{U}(G^F)$  and any  $\epsilon \in \text{Irr}(W)$ , we have

$$\langle \rho, R_\epsilon \rangle = \delta_\rho \{x_\rho, x_\epsilon\} \quad \text{where} \quad R_\epsilon := \frac{1}{|W|} \sum_{w \in W} \epsilon(w) R_w.$$

Here, for  $\rho \in \mathfrak{U}(G^F)$ , we define a sign  $\delta_\rho = \pm 1$  by the condition that  $\delta_\rho D_G(\rho) \in \mathfrak{U}(G^F)$ , where  $D_G$  denotes the duality operator on the character ring of  $G$ ; see [16, 6.8]. Note that [16, 6.20] identifies  $\delta_\rho$  with the sign  $\Delta(x_\rho)$  appearing in the formulation of [16, 4.23].

REMARK 2.2. Assume that  $G$  is simple modulo its centre. The bijection  $\mathfrak{U}(G^F) \leftrightarrow X(W)$  in 2.1 is not uniquely determined by the properties stated above.

We shall make a definite choice according to [16, 12.6] and the tables in the appendix of [16]. In particular, this means the following. Let us fix a square root of  $q$  in  $\overline{\mathbb{Q}}_\ell$ . Then it is well-known that the irreducible characters of  $G^F$  which occur in the character of the permutation representation of  $G^F$  on the cosets of  $B^F$  are naturally parametrised by the irreducible characters of  $W$ ; see, e.g., [16, 8.7]. If  $\epsilon \in \text{Irr}(W)$ , we denote the corresponding irreducible character of  $G^F$  by  $\rho_\epsilon$ ; clearly,  $\rho_\epsilon \in \mathfrak{U}(G^F)$ . Hence, under a bijection  $\mathfrak{U}(G^F) \leftrightarrow X(W)$  as in 2.1, the character  $\rho_\epsilon$  will correspond to an element  $x_{\rho_\epsilon}$ . By [16, Prop. 12.6], we automatically have  $x_\epsilon = x_{\rho_\epsilon}$  except when  $\epsilon(1) = 512$  and  $G$  is of type  $E_7$ , or when  $\epsilon(1) = 4096$  and  $G$  is of type  $E_8$ . In these exceptional cases,  $\mathfrak{U}(G^F) \leftrightarrow X(W)$  can still be chosen such that  $x_\epsilon = x_{\rho_\epsilon}$ ; see the tables for  $E_7, E_8$  in the appendix of [16].

In order to obtain a full uniqueness statement, one has to take into account Harish-Chandra series and further invariants of the characters in  $\mathfrak{U}(G^F)$ , namely, the ‘‘eigenvalues of Frobenius’’ as determined in [16, 11.2]; see [8, Prop. 6.4], [27, §3], [12, §4].

2.3. The ‘‘Fourier matrix’’  $\Upsilon := (\{x, y\})_{x, y \in X(W)}$  is hermitian, and  $\Upsilon^2$  is the identity matrix (see [16, 4.14]). For each  $x \in X(W)$ , the corresponding unipotent ‘‘almost character’’  $R_x$  is defined by

$$R_x := \sum_{\rho \in \mathfrak{U}(G^F)} \delta_\rho \{x_\rho, x\} \rho; \quad \text{see [16, 4.24.1].}$$

Note that  $R_{x_\epsilon} = R_\epsilon$  for  $\epsilon \in \text{Irr}(W)$ . For any  $x, y \in X(W)$  we have

$$\langle R_x, R_y \rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\Upsilon^2$  is the identity matrix, we obtain

$$\rho = \delta_\rho \sum_{x \in X(W)} \{x, x_\rho\} R_x \quad \text{for any } \rho \in \mathfrak{U}(G^F).$$

Thus, the problem of computing the values of  $\rho \in \mathfrak{U}(G^F)$  is equivalent to the analogous problem for the unipotent almost characters  $R_x, x \in X(W)$ .

2.4. Let  $\hat{G}$  be the set of character sheaves on  $G$  (up to isomorphism). These are certain simple perverse sheaves in the bounded derived category  $\mathcal{D}G$  of constructible  $\overline{\mathbb{Q}}_\ell$ -sheaves on  $G$  (in the sense of Beilinson, Bernstein, Deligne [1]), which are equivariant for the action of  $G$  on itself by conjugation. For  $w \in W$  let  $K_w^{\mathcal{L}_0} \in \mathcal{D}G$  be defined as in [17, 2.4], where  $\mathcal{L}_0 = \overline{\mathbb{Q}}_\ell$  is the constant local system on the maximal torus  $T$ . Let  $\hat{G}^{\text{un}}$  be the set of unipotent character sheaves, that is, those  $A \in \hat{G}$  which are constituents of a perverse cohomology sheaf  ${}^p H^i(K_w^{\mathcal{L}_0})$  for some  $w \in W$  and some  $i \in \mathbb{Z}$  (see [17, Def. 2.10]). Let  $\epsilon \in \text{Irr}(W)$ . In analogy to the above definition of  $R_\epsilon$ , we formally define

$$K_\epsilon^{\mathcal{L}_0} := \frac{1}{|W|} \sum_{w \in W} \epsilon(w) \sum_{i \in \mathbb{Z}} (-1)^{i + \dim G} {}^p H^i(K_w^{\mathcal{L}_0});$$

see [19, 14.10.3]. (We write  $K_\epsilon^{\mathcal{L}_0}$  in order to avoid confusion with  $R_\epsilon$  in 2.1.) As in [19, 14.10.4], we also denote by  $(A : K_\epsilon^{\mathcal{L}_0})$  the multiplicity of  $A \in \hat{G}^{\text{un}}$  in  $K_\epsilon^{\mathcal{L}_0}$  (in the appropriate Grothendieck group).

2.5. Now [21, Theorem 23.1] (see also the comments in [26, 3.10]) provides a classification of  $\hat{G}^{\text{un}}$  in terms of similar ingredients as in 2.1. Indeed, let the parameter set  $X(W)$ , the pairing  $\langle \cdot, \cdot \rangle : X(W) \times X(W) \rightarrow \overline{\mathbb{Q}}_\ell$  and the embedding  $\text{Irr}(W) \hookrightarrow X(W)$  be as above. Then there is a bijection

$$\hat{G}^{\text{un}} \leftrightarrow X(W), \quad A \leftrightarrow x_A,$$

such that  $(A : K_\epsilon^{\mathcal{L}_0}) = \hat{\varepsilon}_A \{x_A, x_\epsilon\}$  for any  $A \in \hat{G}^{\text{un}}$  and  $\epsilon \in \text{Irr}(W)$ . Here, we set

$$\hat{\varepsilon}_K := (-1)^{\dim G - \dim \text{supp}(K)} \quad \text{for any } K \in \mathcal{D}G,$$

where  $\text{supp}(K)$  is the Zariski closure of the set  $\{g \in G \mid \mathcal{H}_g^i(K) \neq \{0\} \text{ for some } i\}$ . (Cf. [19, 15.11].) Here,  $\mathcal{H}_g^i(K)$  are the stalks at  $g \in G$  of the cohomology sheaves of  $K$ , for any  $i \in \mathbb{Z}$ .

Assume that  $G$  is simple modulo its centre. Then, again, the bijection  $\hat{G}^{\text{un}} \leftrightarrow X(W)$  is not uniquely determined by the above properties. But one obtains a full uniqueness statement by an analogous scheme as in Remark 2.2; see [27, §3].

2.6. Consider any object  $A \in \mathcal{D}G$  and suppose that its inverse image  $F^*A$  under the Frobenius map is isomorphic to  $A$  in  $\mathcal{D}G$ . Let  $\phi : F^*A \xrightarrow{\sim} A$  be an isomorphism. Then  $\phi$  induces a linear map  $\phi_{i,g} : \mathcal{H}_g^i(A) \rightarrow \mathcal{H}_g^i(A)$  for each  $i$  and  $g \in G^F$ . This gives rise to a class function  $\chi_{A,\phi} \in \text{CF}(G^F)$ , called ‘‘characteristic function’’ of  $A$ , defined by

$$\chi_{A,\phi}(g) = \sum_i (-1)^i \text{Trace}(\phi_{i,g}, \mathcal{H}_g^i(A)) \quad \text{for } g \in G^F,$$

see [18, 8.4]. Note that  $\phi$  is unique up to a non-zero scalar; hence,  $\chi_{A,\phi}$  is unique up to a non-zero scalar.

Now assume that  $A \in \hat{G}$ . Then one can choose an isomorphism  $\phi_A : F^*A \xrightarrow{\sim} A$  such that the values of  $\chi_{A,\phi_A}$  are cyclotomic integers and  $\langle \chi_{A,\phi_A}, \chi_{A,\phi_A} \rangle = 1$ ; see [21, 25.6, 25.7] (and also the comments in [26, 3.10]). The precise conditions which guarantee these properties are formulated in [19, 13.8], [21, 25.1]; note that these conditions specify  $\phi_A$  up to multiplication by a root of unity. In the following, we will tacitly assume that  $\phi_A$  has been chosen in this way whenever  $A \cong F^*A$ .

**THEOREM 2.7** (Shoji [37, 5.7], [38, 3.2, 4.1]). *Assume that  $Z(G)$  is connected and that  $G/Z(G)$  is simple; also recall that  $F$  is assumed to act trivially on  $W$ . Let  $A \in \hat{G}^{\text{un}}$  and  $x \in X(W)$  be such that  $x = x_A$ . Then  $F^*A \cong A$  and  $R_x$  is equal to  $\chi_{A,\phi_A}$ , up to a non-zero scalar multiple.*

As already mentioned, Shoji’s results also apply to non-split groups and to non-unipotent characters. In [37], [38] it is assumed, however, that  $p$  is ‘‘almost

good”, that is, the following conditions hold. If  $G$  is type  $A_n, B_n, C_n$  or  $D_n$ , no condition. If  $G$  is of type  $E_6$ , then  $p \neq 2$ . If  $G$  is of type  $G_2, F_4$  or  $E_7$ , then  $p \neq 2, 3$ . If  $G$  is of type  $E_8$ , then  $p \neq 2, 3, 5$  (see [21, 23.0.1]). Since Lusztig [26] has established the “cleanness” of cuspidal character sheaves in full generality, the methods in [37], [38] which were used to prove Theorem 2.7 for  $G$  of exceptional type and almost good  $p$  can now also be applied for any  $p$ . Consequently, as Shoji pointed out to the author, Theorem 2.7 holds as stated above, without any condition on  $p$ .

DEFINITION 2.8. In the setting of Theorem 2.7, let  $A \in \hat{G}^{\text{un}}$  and  $x \in X(W)$  be such that  $x = x_A$ . Recall that  $\phi_A: F^*A \xrightarrow{\sim} A$  is assumed to be chosen as in 2.6. Then we define  $0 \neq \zeta_A \in \overline{\mathbb{Q}}_\ell$  by the condition that  $R_x = (-1)^{\dim G} \hat{\varepsilon}_A \zeta_A \chi_{A, \phi_A}$ .

2.9. In the setting of Remark 2.2, let  $\epsilon \in \text{Irr}(W)$  and consider the corresponding character  $\rho_\epsilon$ . We have  $\delta_{\rho_\epsilon} = 1$  and  $x_\epsilon = x_{\rho_\epsilon}$ . So, using the formula in 2.3, we obtain:

$$\begin{aligned} \rho_\epsilon &= \sum_{x \in X(W)} \{x, x_{\rho_\epsilon}\} R_x = \sum_{x \in X(W)} \{x, x_\epsilon\} R_x \\ &= (-1)^{\dim G} \sum_{A \in \hat{G}^{\text{un}}} \hat{\varepsilon}_A \zeta_A \{x_A, x_\epsilon\} \chi_{A, \phi_A} \quad (\text{see Def. 2.8}) \\ &= (-1)^{\dim G} \sum_{A \in \hat{G}^{\text{un}}} \zeta_A (A : K_\epsilon^{\mathcal{L}_0}) \chi_{A, \phi_A} \quad (\text{see 2.5}). \end{aligned}$$

Such an expression for  $\rho_\epsilon$  as a linear combination of characteristic functions first appeared in [19, 14.14]; it is actually an important ingredient in the proof of Theorem 2.7. On the other hand, the argument in [19, 14.14] relies on an alternative interpretation of the coefficients  $\zeta_A$ , which we will consider in more detail in the following section. Note that, for a given  $A \in \hat{G}^{\text{un}}$ , there always exists some  $\epsilon \in \text{Irr}(W)$  such that  $(A : K_\epsilon^{\mathcal{L}_0}) \neq 0$ ; see [19, 14.12]. Furthermore, by [21, 25.2] (and the comments in [26, 3.10]), the functions  $\{\chi_{A, \phi_A} \mid A \in \hat{G}^{\text{un}}\}$  are linearly independent. It follows that the coefficients  $\zeta_A$  are uniquely determined by the above system of equations, where  $\epsilon$  runs over  $\text{Irr}(W)$ .

### 3. The scalars $\zeta_A$

We keep the basic assumptions of the previous section; we also assume that  $Z(G)$  is connected and  $G/Z(G)$  is simple. We fix a square root of  $q$  in  $\overline{\mathbb{Q}}_\ell$ . For any  $A \in \hat{G}^{\text{un}}$ , we know by Theorem 2.7 that  $F^*A \cong A$ ; we assume that an isomorphism  $\phi_A: F^*A \xrightarrow{\sim} A$  has been chosen as in 2.6. Our aim is to get hold of the coefficients  $\zeta_A$  in Definition 2.8.

3.1. For the further discussion, it will be convenient to change the notation and label everything by elements of  $X(W)$ . Thus, via the bijection  $\mathfrak{U}(G^F) \leftrightarrow X(W)$

in 2.1 (arranged as in Remark 2.2), we can write

$$\mathfrak{U}(G^F) = \{\rho_x \mid x \in X(W)\} \quad \text{where} \quad \rho_\epsilon = \rho_{x_\epsilon} \text{ for } \epsilon \in \text{Irr}(W).$$

For  $x \in X(W)$ , we write  $\delta_x := \delta_{\rho_x}$ . Then  $R_y = \sum_{x \in X(W)} \delta_x \{x, y\} \rho_x$  for all  $y \in X(W)$ . Next, via the bijection  $\hat{G}^{\text{un}} \leftrightarrow X(W)$  in 2.5, we can write

$$\hat{G}^{\text{un}} = \{A_x \mid x \in X(W)\}.$$

For  $x \in X(W)$ , we denote an isomorphism  $F^*A_x \cong A_x$  as in 2.6 by  $\phi_x$  and the corresponding characteristic function simply by  $\chi_x$ . Then the relation in Definition 2.8 is rephrased as

$$R_x = (-1)^{\dim G} \hat{\epsilon}_x \zeta_x \chi_x \quad \text{where} \quad \hat{\epsilon}_x := \hat{\epsilon}_{A_x} \text{ and } \zeta_x := \zeta_{A_x}.$$

For  $\epsilon \in \text{Irr}(W)$ , the identity in 2.9 now reads:

$$\rho_\epsilon = (-1)^{\dim G} \sum_{x \in X(W)} \zeta_x(A_x : K_\epsilon^{\mathcal{L}_0}) \chi_x.$$

3.2. Let us fix an integer  $m \geq 1$ . Then  $G$  is also defined over  $\mathbb{F}_{q^m}$  and  $F^m : G \rightarrow G$  is the corresponding Frobenius map. Clearly,  $F^m$  acts trivially on  $W$  and we have  $F^m(t) = t^{q^m}$  for all  $t \in T$ . So the whole discussion in Section 2 can be applied to  $F^m$  instead of  $F$ . As in 3.1, we write

$$\mathfrak{U}(G^{F^m}) = \{\rho_x^{(m)} \mid x \in X(W)\}.$$

Again, the unipotent characters of  $G^{F^m}$  which occur in the character of the permutation representation of  $G^{F^m}$  on the cosets of  $B^{F^m}$  are naturally parametrised by  $\text{Irr}(W)$ . If  $\epsilon \in \text{Irr}(W)$ , we denote the corresponding character of  $G^{F^m}$  by  $\rho_\epsilon^{(m)}$ . As in Remark 2.2, the labelling of  $\mathfrak{U}(G^{F^m})$  is arranged such that

$$\rho_\epsilon^{(m)} = \rho_{x_\epsilon}^{(m)} \quad \text{for } \epsilon \in \text{Irr}(W).$$

For  $y \in X(W)$ , the corresponding unipotent almost character of  $G^{F^m}$  is given by

$$R_y^{(m)} = \sum_{x \in X(W)} \delta_x \{x, y\} \rho_x^{(m)}.$$

Note that  $\delta_{\rho_x^{(m)}} = \delta_{\rho_x} = \delta_x$  by [16, 4.23, 6.20].

3.3. Let  $x \in X(W)$ . Then  $\phi_x : F^*A_x \xrightarrow{\sim} A_x$  naturally induces isomorphisms

$$F^*(\phi_x) : (F^*)^2 A_x \xrightarrow{\sim} F^* A_x, \quad (F^*)^2(\phi_x) : (F^*)^3 A_x \xrightarrow{\sim} (F^*)^2 A_x, \quad \dots,$$

which give rise to an isomorphism

$$\tilde{\phi}_x^{(m)} := \phi_x \circ F^*(\phi_x) \circ \dots \circ (F^*)^{m-1}(\phi_x) : (F^*)^m A_x \xrightarrow{\sim} A_x.$$

We also have a canonical isomorphism  $(F^*)^m A_x \cong (F^m)^* A_x$  which, finally, induces an isomorphism

$$\phi_x^{(m)}: (F^m)^* A_x \xrightarrow{\sim} A_x \quad (\text{see [37, 1.1]}).$$

The latter isomorphism again satisfies the conditions in 2.6; we denote the corresponding characteristic function by  $\chi_x^{(m)}: G^{F^m} \rightarrow \overline{\mathbb{Q}}_\ell$ . Note that, if  $g$  is an element in  $G^F$  (and not just in  $G^{F^m}$ ), then

$$\chi_x^{(m)}(g) = \sum_i (-1)^i \text{Trace}((\phi_x)_{i,g}^m, \mathcal{H}_g^i(A_x)).$$

(See again [37, 1.1].) As in Definition 2.8, we define  $0 \neq \zeta_x^{(m)} \in \overline{\mathbb{Q}}_\ell$  by the condition that

$$R_x^{(m)} = (-1)^{\dim G} \hat{\varepsilon}_x \zeta_x^{(m)} \chi_x^{(m)}.$$

(Thus, if  $m = 1$ , then  $\zeta_x^{(1)} = \zeta_x$ .) We can now state the main result of this section.

**PROPOSITION 3.4.** *In the setting of 3.2, 3.3, we have  $\zeta_x^{(m)} = \zeta_x^m$  for  $x \in X(W)$ .*

A result of this kind is implicitly contained in Lusztig [19, §14] and Shoji [37, §2 and 5.19]; the proof will be given in 3.7. First, we need some preparations.

3.5. We recall some constructions from [19, §12, §13]. For any  $w \in W$ , we assume chosen once and for all a representative  $\dot{w} \in N_G(T)$ . There is a corresponding complex  $\bar{K}_{\dot{w}}^{\mathcal{L}_0} \in \mathcal{D}G$  as defined in [19, 12.1]. Then  $\hat{G}^{\text{un}}$  can also be characterized as the set of isomorphism classes of simple perverse sheaves on  $G$  which occur as constituents of a perverse cohomology sheaf  ${}^p H^i(\bar{K}_{\dot{w}}^{\mathcal{L}_0})$  for some  $w \in W$  and some  $i \in \mathbb{Z}$ . Furthermore, for each  $i$ , there is a natural isomorphism

$$\varphi_{i,\dot{w}}: F^*({}^p H^i(\bar{K}_{\dot{w}}^{\mathcal{L}_0})) \xrightarrow{\sim} {}^p H^i(\bar{K}_{\dot{w}}^{\mathcal{L}_0}); \quad \text{see [19, 12.2, 13.8]}.$$

The advantage of using  $\bar{K}_{\dot{w}}^{\mathcal{L}_0}$  instead of  $K_{\dot{w}}^{\mathcal{L}_0}$  (see 2.4) is that  $\bar{K}_{\dot{w}}^{\mathcal{L}_0}$  is semisimple (see [19, 12.8]). Let us now fix  $i, w$  and denote  $K := {}^p H^i(\bar{K}_{\dot{w}}^{\mathcal{L}_0})$ . We also set  $\varphi := \varphi_{i,\dot{w}}: F^* K \xrightarrow{\sim} K$  and recall that, for each  $x \in X(W)$ , we are given  $\phi_x: F^* A_x \xrightarrow{\sim} A_x$ . Now, following Lusztig [19, 13.8.2], there is a canonical isomorphism

$$K \cong \bigoplus_{x \in X(W)} (A_x \otimes V_x)$$

where  $V_x$  are finite-dimensional vector spaces over  $\overline{\mathbb{Q}}_\ell$  endowed with linear maps  $\psi_x: V_x \rightarrow V_x$  such that, under the above direct sum decomposition, the map  $\phi_x \otimes \psi_x$  corresponds to the given  $\varphi: F^* K \xrightarrow{\sim} K$ . More precisely,  $V_x$  and  $\psi_x$  are as follows (cf. [18, 10.4] and [22, 3.5]). We have  $V_x = \text{Hom}(A_x, K)$  and

$$\psi_x(v) = \varphi \circ F^*(v) \circ \phi_x^{-1} \quad \text{for } v \in V_x,$$

where  $F^*(v) \in \text{Hom}(F^* A_x, F^* K)$  is the map induced by  $v: A_x \rightarrow K$ .

THEOREM 3.6 (Lusztig [19, 13.10, 14.14]). *In the setting of 3.5, all eigenvalues of  $\psi_x: V_x \rightarrow V_x$  are equal to  $\zeta_x q^{(i-\dim G)/2}$ .*

More precisely, Lusztig first shows in [19, 13.10] that there is a constant  $0 \neq \xi_x \in \overline{\mathbb{Q}}_\ell$  (which only depends on  $\phi_x$ , the choice of a square root of  $q$  and the choice of the representatives  $\dot{w}$ ) such that, for any  $i$  and  $w$ , all eigenvalues of  $\psi_x: V_x \rightarrow V_x$  are equal to  $\xi_x q^{(i-\dim G)/2}$ . (The ‘‘cleanness’’ assumption in [19, 13.10] holds in general by [26].) It is then shown in [19, 14.14] that

$$\rho_\epsilon = (-1)^{\dim G} \sum_{x \in X(W)} \xi_x(A_x: K_\epsilon^{\mathcal{L}^0}) \chi_x \quad \text{for all } \epsilon \in \text{Irr}(W).$$

(There are also coefficients  $\nu(A_x)$  in the formula in [19, 14.14] but these are all equal to 1 in our situation.) Finally, a comparison with the formula in 3.1 implies that  $\xi_x = \zeta_x$  for all  $x \in X(W)$ .

3.7. *Proof of Proposition 3.4.* Let  $x \in X(W)$ . We also fix  $i, w$  and place ourselves in the setting of 3.5. The whole discussion there can be repeated with  $F$  replaced by  $F^m$ . As in 3.3, we have isomorphisms

$$\tilde{\phi}_x^{(m)}: (F^*)^m A_x \xrightarrow{\sim} A_x \quad \text{and} \quad \phi_x^{(m)}: (F^m)^* A_x \xrightarrow{\sim} A_x.$$

Analogously,  $\varphi: F^* K \xrightarrow{\sim} K$  induces isomorphisms

$$\tilde{\varphi}^{(m)}: (F^*)^m K \xrightarrow{\sim} K \quad \text{and} \quad \varphi^{(m)}: (F^m)^* K \xrightarrow{\sim} K.$$

Let us consider again the canonical isomorphism

$$K \cong \bigoplus_{x \in X(W)} (A_x \otimes V_x) \quad \text{where} \quad V_x = \text{Hom}(A_x, K).$$

Here, as before, each  $V_x$  is endowed with a linear map  $\psi_x^{(m)}: V_x \rightarrow V_x$  such that

$$\psi_x^{(m)}(v) = \varphi^{(m)} \circ (F^m)^*(v) \circ (\phi_x^{(m)})^{-1} \quad \text{for all } v \in V_x.$$

By Theorem 3.6, the scalar  $\zeta_x^{(m)}$  is determined by the eigenvalues of  $\psi_x^{(m)}$ .

Now, a simple induction on  $m$  shows that

$$\psi_x^m(v) = \tilde{\varphi}^{(m)} \circ (F^*)^m(v) \circ (\tilde{\phi}_x^{(m)})^{-1} \quad \text{for all } v \in V_x.$$

Next, we use again that we have canonical isomorphisms  $(F^*)^m A_x \cong (F^m)^* A_x$  and  $(F^*)^m K \cong (F^m)^* K$ . Under these isomorphisms, the above map  $(F^*)^m(v)$  corresponds to the map  $(F^m)^*(v): (F^m)^* A_x \rightarrow (F^m)^* K$ . Hence, we also have

$$\psi_x^m(v) = \varphi^{(m)} \circ (F^m)^*(v) \circ (\phi_x^{(m)})^{-1} \quad \text{for all } v \in V_x.$$

Thus, we have  $\psi_x^{(m)} = \psi_x^m$  for all  $x \in X(W)$  and so the eigenvalues of  $\psi_x^{(m)}$  are obtained by raising the eigenvalues of  $\psi_x$  to the  $m$ -th power. It remains to use Theorem 3.6. This completes the proof of Proposition 3.4.  $\square$

REMARK 3.8. We can, and will assume that  $G$  is defined and split over the prime field  $\mathbb{F}_p$  of  $k$ . Let  $F_0: G \rightarrow G$  be the corresponding Frobenius map. If  $q = p^f$  where  $f \geq 1$ , then  $F = F_0^f$ . Hence, Proposition 3.4 means that it will be sufficient to determine the scalars  $\zeta_x$  ( $x \in X(W)$ ) for the group  $G^{F_0} = G(\mathbb{F}_p)$ . For specific values of  $p$  (e.g., bad primes  $p = 2, 3, 5$ ), we may then use ad hoc information which is available, for example, in the Cambridge ATLAS [6], or via computer algebra methods (using GAP [10], CHEVIE [32]). This is the basis for the discussion of the examples below.

#### 4. Cuspidal character sheaves and small rank examples

We keep the notation of the previous section; in particular, we label all objects by the parameter set  $X(W)$  as in 3.1. By [22, 3.5], the computation of the scalar  $\zeta_x$  can be reduced to the case where  $A_x$  is a cuspidal character sheaf (in the sense of [17, Def. 3.10]). So let us look in more detail at this case.

4.1. Assume that  $Z(G)^\circ = \{1\}$ . Let  $x \in X(W)$  be such that  $A_x \in \hat{G}^{\text{un}}$  is cuspidal. Then there exists an  $F$ -stable conjugacy class  $C$  of  $G$  and an irreducible,  $G$ -equivariant  $\overline{\mathbb{Q}}_\ell$ -local system  $\mathcal{E}$  on  $C$  such that  $F^*\mathcal{E} \cong \mathcal{E}$  and  $A = \text{IC}(\overline{C}, \mathcal{E})[\dim C]$ ; see [17, 3.12]. In particular,  $\text{supp}(A_x) = \overline{C}$  and so  $\hat{\varepsilon}_{A_x} = (-1)^{\dim G - \dim C}$ . Let us fix  $g_1 \in C^F$  and set  $A(g_1) := C_G(g_1)/C_G^\circ(g_1)$ . Then  $F$  induces an automorphism  $\gamma: A(g_1) \rightarrow A(g_1)$ . We further assume that:

- (\*) the local system  $\mathcal{E}$  is one-dimensional and, hence, corresponds to a linear character  $\lambda: A(g_1) \rightarrow \overline{\mathbb{Q}}_\ell^\times$  which is invariant under  $\gamma$  (via [25, 19.7]).

(This assumption will be satisfied in all examples that we consider.) We form the semidirect product  $\tilde{A}(g_1) = A(g_1) \rtimes \langle \gamma \rangle$  such that, inside  $\tilde{A}(g_1)$ , we have the identity  $\gamma(a) = \gamma a \gamma^{-1}$  for all  $a \in A(g_1)$ . By (\*), we can canonically extend  $\lambda$  to a linear character

$$\tilde{\lambda}: \tilde{A}(g_1) \rightarrow \overline{\mathbb{Q}}_\ell^\times, \quad a\gamma \mapsto \lambda(a).$$

For each  $a \in A(g_1)$  we have a corresponding element  $g_a \in C^F$ , well-defined up to conjugation within  $G^F$ . (We have  $g_a = h g_1 h^{-1}$  where  $h \in G$  is such that  $h^{-1} F(h) \in C_G(g_1)$  has image  $a \in A(g_1)$ .) We define a class function  $\chi_{g_1, \lambda}: G^F \rightarrow \overline{\mathbb{Q}}_\ell$  by

$$\chi_{g_1, \lambda}(g) = \begin{cases} q^{(\dim G - \dim C)/2} \lambda(a) & \text{if } g = g_a \text{ for some } a \in A(g_1), \\ 0 & \text{if } g \notin C^F. \end{cases}$$

Now, we can choose an isomorphism  $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$  such that the induced map on the stalk  $\mathcal{E}_{g_1}$  is scalar multiplication by  $q^{(\dim G - \dim C)/2}$ . Then this isomorphism canonically induces an isomorphism  $\phi_x: F^*A_x \xrightarrow{\sim} A_x$  which satisfies the requirements in 2.6, and we have  $\chi_x = \chi_{A_x, \phi_x} = \chi_{g_1, \lambda}$ . (This follows from the fact that  $A_x$  is “clean” [26], using the construction in [25, 19.7].) With this choice of  $\phi_x$ , we

also have for all  $m \geq 1$ :

$$\chi_x^{(m)}(g_a^{(m)}) = \begin{cases} q^{m(\dim G - \dim C)/2} \lambda(a) & \text{if } g = g_a^{(m)} \text{ for some } a \in A(g_1), \\ 0 & \text{if } g \notin C^{F^m}. \end{cases}$$

(Here,  $g_a^{(m)} = hg_1h^{-1}$  where now  $h \in G$  is such that  $h^{-1}F^m(h) \in C_G(g_1)$  has image  $a \in A(g_1)$ ; see again [25, 19.7].) The identity in Definition 2.8 now reads:

$$R_x = \sum_{y \in X(W)} \delta_y\{y, x\} \rho_y = (-1)^{\dim C} \zeta_x \chi_{(g_1, \lambda)}.$$

REMARK 4.2. Let  $G$  be of (split) classical type. Then Shoji has shown that we always have  $\zeta_x = 1$  for cuspidal  $A_x \in \hat{G}^{\text{un}}$ ; see [36, Prop. 6.7] for  $p \neq 2$ , and [40, Theorem 6.2] for  $p = 2$ . Note that this involves, in each case, the choice of a particular representative in the conjugacy class supporting  $A_x$ . Since classical groups of low rank appear as Levi subgroups in groups of exceptional type, it will be useful to work out explicitly the relevant identities  $R_x = (-1)^{\dim C} \zeta_x \chi_{(g_1, \lambda)}$  for  $G$  of type  $C_2$ ,  $D_4$  and  $p = 2$ . This also provides a good illustration for: (a) the role of the choice of a class representative as above and (b) the strategy that we will employ when dealing with groups of exceptional type.

EXAMPLE 4.3. Let  $G = \text{Sp}_4(k)$  be the 4-dimensional symplectic group. Then  $G = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in k \rangle$  with root system  $\Phi = \{\pm a, \pm b, \pm(a+b), \pm(2a+b)\}$ . The Weyl group  $W = \langle s_a, s_b \rangle$  is dihedral of order 8 and we have

$$\text{Irr}(W) = \{1_W, \text{sgn}, \text{sgn}_a, \text{sgn}_b, r\}$$

where  $1_W$  is the trivial character,  $\text{sgn}$  is the sign character,  $r$  has degree 2, and  $\text{sgn}_a, \text{sgn}_b$  are linear characters such that  $\text{sgn}_a(s_a) = 1$ ,  $\text{sgn}_a(s_b) = -1$ ,  $\text{sgn}_b(s_a) = -1$ ,  $\text{sgn}_b(s_b) = 1$ . By [5, p. 468], we have

$$X(W) = \{x_1, x_{\text{sgn}}, x_{\text{sgn}_a}, x_{\text{sgn}_b}, x_r, x_0\}$$

where  $\rho_{1_W}(1) = 1$ ,  $\rho_{\text{sgn}}(1) = q^4$ ,  $\rho_{\text{sgn}_a}(1) = \rho_{\text{sgn}_b}(1) = \frac{1}{2}q(q^2 + 1)$ ,  $\rho_r(1) = \frac{1}{2}q(q+1)^2$  and  $\rho_{x_0}(1) = \frac{1}{2}q(q-1)^2$ . By the explicit description of the Fourier matrices in [5, p. 471], we find that

$$R_{x_0} = \frac{1}{2}(\rho_r - \rho_{\text{sgn}_a} - \rho_{\text{sgn}_b} + \rho_{x_0}).$$

If  $q$  is odd, then the identification of  $R_{x_0}$  with a characteristic function of a cuspidal character sheaf is explained in the appendix of Srinivasan [42]. Now assume that  $q = 2^f$  where  $f \geq 1$ . Then, by [20, 22.2], there is a unique cuspidal character sheaf  $A_0$  on  $G$ , and it is contained in  $\hat{G}^{\text{un}}$ . By the explicit description in [30, 2.7], we have  $A_0 = A_{x_0}$  and  $A_0 = \text{IC}(\overline{C}, \mathcal{E})[\dim C]$  where  $C$  is the class of regular unipotent elements and  $\mathcal{E} \not\cong \overline{\mathbb{Q}}_\ell$ ; we have  $\dim G = 10$  and  $\dim C = 8$ . Let us fix

$$g_1 = x_a(1)x_b(1) \in C^F.$$

One checks that  $g_1$  has order 4 and that  $g_1$  is conjugate in  $G^F$  to  $g_1^{-1} = x_b(1)x_a(1)$ . Furthermore,  $A(g_1) \cong \mathbb{Z}/2\mathbb{Z}$  is abelian and  $F$  acts trivially on  $A(g_1)$ . Let  $\lambda$  be the non-trivial character of  $A(g_1)$ . Then, as in Example 4.1, we obtain:

$$\chi_{g_1, \lambda}(g) = \begin{cases} q & \text{if } g = g_1, \\ -q & \text{if } g = g_1', \\ 0 & \text{if } g \notin C^F, \end{cases}$$

where  $g_1' \in C^F$  corresponds to the non-trivial element of  $A(g_1)$ . We now have  $R_{x_0} = \zeta_{x_0} \chi_{g_1, \lambda}$ . In order to determine  $\zeta_{x_0}$  it is sufficient, by Remark 3.8, to consider the case where  $q = 2$ . But  $\mathrm{Sp}_4(\mathbb{F}_2)$  is isomorphic to the symmetric group  $\mathfrak{S}_6$ ; an explicit isomorphism is described in [15, 9.21]. One checks that, under this isomorphism,  $g_1 = x_a(1)x_b(1)$  corresponds to an element of cycle type  $(4, 2)$  in  $\mathfrak{S}_6$ . We also need to identify  $\rho_r, \rho_{\mathrm{sgn}_a}, \rho_{\mathrm{sgn}_b}, \rho_{x_0}$  in the character table of  $\mathfrak{S}_6$ . Now,  $B(\mathbb{F}_2)$  is a Sylow 2-subgroup of  $\mathrm{Sp}_4(\mathbb{F}_2)$ . Working out the character of the permutation representation on the cosets of this subgroup, one can identify the 5 characters which are of the form  $\rho_\epsilon$  for some  $\epsilon \in \mathrm{Irr}(W)$ . Looking also at character degrees, we can then identify  $\rho_{\mathrm{sgn}}$  and the sum  $\rho_{\mathrm{sgn}_a} + \rho_{\mathrm{sgn}_b}$ ; finally,  $\rho_{x_0}$  corresponds to the sign character of  $\mathfrak{S}_6$ . By inspection of the table of  $\mathfrak{S}_6$ , we find that  $R_{x_0}(g_1) = 2$  (for  $q = 2$ ) and, hence,  $\zeta_{x_0} = 1$  for any  $q = 2^f$  (using Proposition 3.4). — Of course, this could also be deduced from the explicit knowledge of the “generic” character table of  $G^F = \mathrm{Sp}_4(\mathbb{F}_q)$  for any  $q = 2^f$  (see Enomoto [9]). But the point is that, once  $\zeta_{x_0}$  is known in advance, the task of computing such a generic character table is considerably simplified!

EXAMPLE 4.4. Let  $G = \mathrm{SO}_8(k)$  be the 8-dimensional special orthogonal group. The Weyl group  $W$  has 13 irreducible characters, which are labelled by certain pairs of partitions. By [5, p. 471], we have

$$X(W) = \{x_\epsilon \mid \epsilon \in \mathrm{Irr}(W)\} \cup \{x_0\}.$$

By the explicit description of the Fourier matrices in [5, p. 472], we find that

$$R_{x_0} = \frac{1}{2}(\rho_{(21,1)} - \rho_{(22,\emptyset)} - \rho_{(2,11)} + \rho_{x_0});$$

here,  $(1, 21)$ ,  $(\emptyset, 22)$ ,  $(11, 2)$  indicate irreducible characters  $\epsilon \in \mathrm{Irr}(W)$  (as in [5, p. 449]). If  $q$  is odd, then the values of  $R_{x_0}$  are explicitly computed in [14, Prop. 4.5] (and this provides an identification of  $R_{x_0}$  with a characteristic function of a cuspidal character sheaf). Now assume that  $q = 2^f$  where  $f \geq 1$ . Then, by [20, 22.3], there is a unique cuspidal character sheaf  $A_0$  on  $G$ , and it is contained in  $\tilde{G}^{\mathrm{un}}$ . Again, by the explicit description in [30, 3.3], we have  $A_0 = A_{x_0} = \mathrm{IC}(\overline{C}, \mathcal{E})[\dim C]$  where  $C$  is the class of regular unipotent elements and  $\mathcal{E} \not\cong \overline{\mathbb{Q}}_\ell$ ; we have  $\dim G = 28$  and  $\dim C = 24$ . Let us fix

$$g_1 = x_a(1)x_b(1)x_c(1)x_d(1) \in C^F$$

where  $\{a, b, c, d\}$  is a set of simple roots in the root system of type  $D_4$ . One checks that  $g_1$  has order 8 and that, if  $a', b', c', d'$  is any permutation of  $a, b, c, d$ , then

$g_1$  is conjugate in  $G^F$  to  $x_{a'}(1)x_{b'}(1)x_{c'}(1)x_{d'}(1)$ . Furthermore,  $A(g_1) \cong \mathbb{Z}/2\mathbb{Z}$  is abelian and  $F$  acts trivially on  $A(g_1)$ . Let  $\lambda$  be the non-trivial character of  $A(g_1)$ . As above, we obtain:

$$\chi_{g_1, \lambda}(g) = \begin{cases} q^2 & \text{if } g = g_1, \\ -q^2 & \text{if } g = g'_1, \\ 0 & \text{if } g \notin C^F, \end{cases}$$

where  $g'_1 \in C^F$  corresponds to the non-trivial element of  $A(g_1)$ . Now we have  $R_{x_0} = \zeta_{x_0} \chi_{g_1, \lambda}$ . In order to show that  $\zeta_{x_0} = 1$ , we can use the known character table of  $\mathrm{SO}_8^+(\mathbb{F}_2)$ ; see the ATLAS [6, p. 85]. In fact, using an explicit realization in terms of orthogonal  $8 \times 8$ -matrices, one can create  $\mathrm{SO}_8^+(\mathbb{F}_2)$  as a matrix group in GAP and simply re-calculate that table using the `CharacterTable` function. The advantage of this re-calculation is that GAP also computes a list of representatives of the conjugacy classes of  $\mathrm{SO}_8^+(\mathbb{F}_2)$ . So one can identify the class to which  $g_1$  belongs. Arguing as in the previous example, one can identify the characters  $\rho_{x_0}$ ,  $\rho_{(21,1)}$  and the sum  $\rho_{(22, \emptyset)} + \rho_{(2,11)}$  in the table of  $\mathrm{SO}_8^+(\mathbb{F}_2)$ . (We omit the details.) In this way, one finds that  $R_{x_0}(g_1) = 4$  (for  $q = 2$ ), as required.

REMARK 4.5. Assume that  $Z(G)^\circ = \{1\}$ , as above. Let  $x \in X(W)$  be such that  $A_x$  is cuspidal and let  $C$  be the  $F$ -stable conjugacy class of  $G$  such that  $\mathrm{supp}(A_x) = \overline{C}$ . The above examples highlight the importance of singling out a specific representative  $g_1 \in C^F$  in order to determine a characteristic function  $\chi_x$  of  $A_x$  and the scalar  $\zeta_x$ . This problem is, of course, not a new one. If  $C$  is a unipotent class and  $p$  is a good prime for  $G$ , there is a notion of “split” elements in  $C^F$  which solves this problem in almost all cases; see Shoji’s survey [35, §5]. Despite of much further work (e.g., Shoji [39]), the question of finding general conditions which single out a distinguished representative  $g_1 \in C^F$  appears to be open. In the above examples (and those below), we are able to choose a representative  $g_1 \in C^F$  according to the following principles:

- $g_1$  belongs to  $C^{F_0} = C(\mathbb{F}_p)$  (cf. Remark 3.8) and is conjugate in  $G(\mathbb{F}_p)$  to all powers  $g_1^n$  where  $n \in \mathbb{Z}$  is coprime to the order of  $g_1$ .
- The unipotent part of  $g_1$  has a “short” expression in terms of the Chevalley generators  $x_\alpha(t)$  of  $G$  where  $\alpha$  is a root and  $t \in \mathbb{F}_p$ .

These principles also work in the further examples discussed in [13].

## 5. Type $F_4$ in characteristic 2

Throughout this section, we assume that  $G$  is simple of type  $F_4$  and  $p = 2$ . We have  $G = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in k \rangle$  where  $\Phi$  is the root system of  $G$  with respect to  $T$ . Let  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$  be the set of simple roots with respect to  $B$ , where we choose the notation such that  $\alpha_1, \alpha_2$  are long roots,  $\alpha_3, \alpha_4$  are short roots, and  $\alpha_2, \alpha_3$  are not orthogonal. (This coincides with the conventions in Shinoda [34], where the conjugacy classes of  $G$  are determined.) Also note that, since  $p = 2$ , we do

not have to worry about the precise choice of a Chevalley basis in the underlying Lie algebra. Our aim is to determine the exact relation between characteristic functions of cuspidal unipotent character sheaves and almost characters in this case, using the approach illustrated in the previous section. For this purpose, we essentially rely on the knowledge of the character table of  $F_4(\mathbb{F}_2)$ ; see the ATLAS [6, p. 167]. (The correctness of this table has been verified independently in [4], and it is available in the library of GAP [10].) We shall also rely on a number of explicit computational results obtained through general algorithms and functions concerning matrix groups and character tables in GAP.

5.1. There are seven cuspidal character sheaves  $A_1, \dots, A_7$  on  $G$ , described in detail by Shoji [37, §7] (based on earlier work of Lusztig and Spaltenstein). Firstly, they are all contained in  $\hat{G}^{\text{un}}$ ; see [37, 7.6]. Secondly, as in 4.1, each  $A_j$  corresponds to a pair  $(g_1, \lambda)$  where  $g_1 \in G^F$  and  $\lambda \in \text{Irr}(A(g_1))$  is a non-trivial linear character of  $A(g_1)$ ; see [37, §7.2]. Here, in all seven cases,  $F$  acts trivially on  $A(g_1)$ . The correspondences  $A_j \leftrightarrow (g_1, \lambda)$  are given as follows.

- (a)  $A_j \leftrightarrow (u, \lambda_j)$  ( $j = 1, 2$ ) where  $u \in G^F$  is regular unipotent,  $\dim C_G(u) = 4$ ; furthermore,  $A(u) \cong \mathbb{Z}/4\mathbb{Z}$  is generated by the image  $\bar{u} \in A(u)$ , and  $\lambda_j$  are the linear characters such that  $\lambda_1(\bar{u}) = i$ ,  $\lambda_2(\bar{u}) = -i$ , where  $i = \sqrt{-1}$  is a fixed fourth root of unity.
- (b)  $A_3 \leftrightarrow (u, \lambda)$  where  $u \in G^F$  is unipotent such that  $\dim C_G(u) = 6$  and  $A(u) \cong \mathbb{Z}/2\mathbb{Z}$ ; furthermore,  $\lambda$  is the non-trivial character of  $A(u)$ .
- (c)  $A_4 \leftrightarrow (u, \lambda)$  where  $u \in G^F$  is unipotent such that  $\dim C_G(u) = 8$  and  $A(u)$  is isomorphic to the dihedral group  $D_8$ ; furthermore,  $\lambda$  corresponds to the sign character of  $D_8$ .
- (d)  $A_5 \leftrightarrow (u, \lambda)$  where  $u \in G^F$  is unipotent such that  $\dim C_G(u) = 12$  and  $A(u)$  is isomorphic to the symmetric group  $\mathfrak{S}_3$ ; furthermore,  $\lambda$  corresponds to the sign character of  $\mathfrak{S}_3$ .
- (e)  $A_j \leftrightarrow (su, \lambda_j)$  ( $j = 6, 7$ ) where  $s \in G^F$  is semisimple, with  $C_G(s)$  isogenous to  $\text{SL}_3(k) \times \text{SL}_3(k)$ , and  $u \in C_G(s)^F$  is regular unipotent; we have  $\dim C_G(su) = 4$ . Furthermore,  $A(su) \cong \mathbb{Z}/3\mathbb{Z}$  is generated by the image  $\overline{su} \in A(su)$ , and  $\lambda_j$  are linear characters such that  $\lambda_6(\overline{su}) = \theta$ ,  $\lambda_7(\overline{su}) = \theta^2$ , where  $\theta$  is a fixed third root of unity.

(In each case (a)–(d), the conditions on  $\dim C_G(u)$  and  $A(u)$  uniquely determine  $g_1 = u$  up to  $G$ -conjugacy; see Shinoda [34, §2]. Furthermore, the class of  $s$  in (e) is also uniquely determined such that  $C_G(s)$  has type  $A_2 \times A_2$ ; see [34, §3].)

5.2. Let  $A_j$  be one of the cuspidal character sheaves in 5.1. In Table 1, we fix a specific element  $g_1 \in G^F$  for the corresponding pair  $(g_1, \lambda)$ . The characteristic function  $\chi_j := \chi_{A_j, \phi_{A_j}}$  in 4.1 will then be defined with respect to this choice of  $g_1$ .

In the table, we use the following notation. If  $\alpha$  is a positive root, written as  $\alpha = \sum_{1 \leq i \leq 4} n_i \alpha_i$ , then we denote the corresponding root element  $x_\alpha(t)$  by  $x_{n_1 n_2 n_3 n_4}(t)$ . Some further comments about the entries of Table 1.

Table 1. Supports of cuspidal character sheaves in type  $F_4$ ,  $p = 2$ 

In 5.1	representative $g_1$	$A(g_1)$	$ C_G(g_1)^F $	GAP
(a)	$u_{31} = x_{1000}(1)x_{0100}(1)x_{0010}(1)x_{0001}(1)$	$\mathbb{Z}/4\mathbb{Z}$	$4q^4$	16a(76)
(b)	$u_{29} = x_{0122}(1)x_{1000}(1)x_{0100}(1)x_{0010}(1)$	$\mathbb{Z}/2\mathbb{Z}$	$2q^6$	8j(47)
(c)	$u_{24} = x_{1100}(1)x_{0120}(1)x_{0001}(1)x_{0011}(1)$	$D_8$	$8q^8$	8a(38)
(d)	$u_{17} = x_{1110}(1)x_{1220}(1)x_{0011}(1)x_{0122}(1)$	$\mathfrak{S}_3$	$6q^{12}$	41(20)
(e)	$su_{17} = u_{17}s$ with $s \in G^F$ of order 3	$\mathbb{Z}/3\mathbb{Z}$	$3q^4$	12o(68)

(Notation for  $u_i$  as in Shinoda [34, §2])

(1) In the cases (a)–(d),  $g_1$  is unipotent; let  $C$  be the conjugacy class of  $g_1$  in  $G$ . Then we choose  $g_1$  to be that representative of  $C^F$  in Shinoda’s list [34, §2], which has an expression as a product of root elements  $x_\alpha(t)$  whose description does not involve the elements  $\eta, \zeta \in k$  defined in [34, 2.2]. All the coefficients in the expression for  $g_1$  are equal to 1, and so  $g_1 \in F_4(\mathbb{F}_2)$ .

(2) Let  $g_1 \in F_4(\mathbb{F}_2)$  be as in (1). Then we will need to know to which class of the GAP character table of  $F_4(\mathbb{F}_2)$  this element belongs. (There are 95 conjugacy classes in  $F_4(\mathbb{F}_2)$ .) This problem is easy for some cases, e.g.,  $g_1 = u_{17}$  since there is a unique class in the GAP table with centraliser order  $6 \cdot 2^{12} = 24576$ . On the other hand, if  $g_1 = u_{31}$ , then the classes with centraliser order  $4 \cdot 2^4 = 64$  are quite difficult to distinguish in the GAP table, especially the two classes 16a, 16b. In such cases, we use the following (computational) argument. Using the explicit 52-dimensional matrix realization of  $F_4(\mathbb{F}_2)$  from Lusztig [29, 2.3] (see also [11, 4.10]), we create  $F_4(\mathbb{F}_2)$  as a matrix group in GAP. Then we consider the subgroup

$$P := \langle x_{\pm 1000}(1), x_{\pm 0100}(1), x_{\pm 0010}(1), x_{0001}(1) \rangle \subseteq F_4(\mathbb{F}_2).$$

The GAP function `CharacterTable` computes the character table of  $P$  (by a general algorithm, without using any specific properties of  $P$ ), together with a list of representatives of the conjugacy classes. There are 214 conjugacy classes of  $P$  and, again by standard algorithms, GAP can find out to which of these 214 classes any given element of  $P$  belongs. Now the function `PossibleClassFusions` determines all possible fusions of the conjugacy classes of  $P$  into the GAP character table of  $F_4(\mathbb{F}_2)$ . (Here, a “possible class fusion” is a map which assigns to each conjugacy class of  $P$  one of the 95 class labels 1a, 2a, ..., 30b of the GAP table of  $F_4(\mathbb{F}_2)$ , such that certain conditions are satisfied which should hold if the map is a true matching of the classes of  $P$  with those of the GAP table; see the help menu of `PossibleClassFusions` for further details.) As might be expected (since there are non-trivial table automorphisms of the character table of  $F_4(\mathbb{F}_2)$ ), the class fusion is not unique; in fact, it turns out that there are 16 possible fusion maps. But, if  $g_1 \in \{u_{17}, u_{24}, u_{29}, u_{31}\}$ , then  $g_1$  is mapped to the same class label of  $F_4(\mathbb{F}_2)$ , under each of the 16 possibilities. Thus, the fusion of  $g_1$  is uniquely determined, and this is the entry in the last column of Table 1.

(3) Let again  $g_1 \in F_4(\mathbb{F}_2)$  be as in (1). Having identified the class of  $g_1$  in the

GAP table, we check by inspection that  $\chi(g_1) \in \mathbb{Z}$  for all  $\chi \in \text{Irr}(F_4(\mathbb{F}_2))$ . Hence,  $g_1$  is conjugate within  $F_4(\mathbb{F}_2)$  to each power  $g_1^n$  where  $n$  is a positive integer coprime to the order of  $g_1$ ; in particular,  $g_1, g_1^{-1}$  are conjugate in  $F_4(\mathbb{F}_2)$ .

(4) Now let  $g_1 = su$  be as in case (e). Let  $C$  be the  $G$ -conjugacy class of  $su$ . Since  $s$  has order 3 and  $u$  has order 4, the element  $g_1 = su$  has order 12; furthermore,  $|C_G(g_1)^F| = 3q^4$ . All this also works for the base case where  $q = 2$  and so we can assume that  $g_1 \in C(\mathbb{F}_2) \subseteq F_4(\mathbb{F}_2)$ . Then  $C(\mathbb{F}_2)$  splits into 3 classes in  $F_4(\mathbb{F}_2)$ ; all elements in these three classes have order 12 and centraliser order  $3 \cdot 2^4 = 48$ . By inspection of the GAP table, we see that these three classes must be 12o, 12p, 12q; here, 12o has the property that all character values on this class are integers. We now choose  $g_1$  to be in 12o; then  $g_1$  is conjugate in  $F_4(\mathbb{F}_2)$  to all powers  $g_1^n$  where  $n$  is a positive integer coprime to the order of  $g_1$ . In particular,  $g_1$  is conjugate in  $G^F$  (even in  $F_4(\mathbb{F}_2)$ ) to  $g_1^5 = s^{-1}u$  and, consequently,  $s, s^{-1}$  are conjugate in  $C_G(u)^F$ . Now the GAP table also shows that  $g_1^3 = u^3 = u^{-1}$  belongs to 41. We have seen in (2), (3) that both  $u_{17}$  and  $u_{17}^{-1}$  belong to 41. Hence, we can assume that  $u = u_{17}$ . It then also follows from the above results (and Sylow's Theorem) that, if  $t \in C_G(u)^F$  is any element of order 3, then  $tu$  is conjugate to  $su$  in  $G^F$ .

Table 2. Unipotent characters in the family  $\mathcal{F}_0$  for type  $F_4, q = 2$

$\rho_x$	$\rho_x(1)$	GAP	$\rho_x$	$\rho_x(1)$	GAP
$F_4^{II}[1]$	1326	X.5	$\phi''_{6,6}$	249900	X.24
$F_4^I[1]$	21658	X.7	$B_2 : (1.1)$	269892	X.25
$F_4[-1]$	63700	X.13	$\phi_{4,8}$	322218	X.27
$\phi''_{1,12}$	99450	X.14	$\phi''_{4,7}$	358020	X.30
$\phi'_{1,12}$	99450	X.15	$\phi'_{4,7}$	358020	X.31
$\{F_4[i], F_4[-i]\}$	142884	$\{X.16, X.17\}$	$\phi'_{6,6}$	519792	X.32
$\{F_4[\theta], F_4[\theta^2]\}$	183600	$\{X.20, X.21\}$	$\phi''_{9,6}$	541450	X.33
$B_2 : (\emptyset.2)$	216580	X.22	$\phi'_{9,6}$	541450	X.34
$B_2 : (11.\emptyset)$	216580	X.23	$\phi_{12,4}$	584766	X.37
(Notation for $\rho_x$ as in Carter [5, p. 479])			$\phi_{16,5}$	947700	X.44

5.3. Let us write  $\mathfrak{U}(G^F) = \{\rho_x \mid x \in X(W)\}$  and  $\hat{G}^{\text{un}} = \{A_x \mid x \in X(W)\}$  as in 3.1. Here, the parameter set  $X(W)$  has 37 elements. It is partitioned into 11 “families”, which correspond to the special characters of  $W$ , as defined in [16, 4.1, 4.2]. The corresponding Fourier matrix has a block diagonal shape, with one diagonal block for each family. The 7 elements of  $X(W)$  which label cuspidal character sheaves are all contained in one family, which we denote by  $\mathcal{F}_0$  and which consists of 21 elements in total. The elements of  $\mathcal{F}_0$  are given by the 21 pairs  $(u, \sigma)$  where  $u \in \mathfrak{S}_4$  (up to  $\mathfrak{S}_4$ -conjugacy) and  $\sigma \in \text{Irr}(C_{\mathfrak{S}_4}(u))$ . The diagonal block of the Fourier matrix corresponding to  $\mathcal{F}_0$  is then described in terms of  $\mathfrak{S}_4$  and the pairs  $(u, \sigma)$ ; see [5, §13.6] for further details. The identification of the 21

unipotent characters  $\{\rho_x \mid x \in \mathcal{F}_0\}$  with characters in the GAP table of  $F_4(\mathbb{F}_2)$  is given in Table 2. (This information is available through the GAP function `DeligneLusztigNames`; note that this depends on fixing a labelling of the long and short roots.) The results of Shoji [37, Theorem 7.5] show that the cuspidal character sheaves  $A_j$  in 5.1 are labelled as follows by pairs in  $\mathcal{F}_0$ :

$$\begin{aligned} A_1 &= A_{(g_4, i)}, & A_2 &= A_{(g_4, -i)}, & A_3 &= A_{(g_2, \varepsilon)}, \\ \{A_4, A_5\} &= \{A_{(1, \lambda^3)}, A_{(g'_2, \varepsilon)}\}, & A_6 &= A_{(g_3, \theta)}, & A_7 &= A_{(g_3, \theta^2)}. \end{aligned}$$

Here, we follow the notation in [5, p. 455] for pairs in  $\mathcal{F}_0$ . (Note that, a priori, the statement of Theorem 2.7 alone does not tell us anything about the supporting set of a cuspidal character sheaf  $A_x$ !) Let us consider in detail  $A_6, A_7$ . (The remaining cases have been dealt with already by Marcelo–Shinoda [31]; see Remark 5.5 below.) Using the  $21 \times 21$ -times Fourier matrix printed in [5, p. 456] and the labelling of the unipotent characters  $\{\rho_x \mid x \in \mathcal{F}_0\}$  in terms of pairs  $(u, \sigma)$  as above (see [5, p. 479]), we obtain the following formulae for the unipotent almost characters labelled by  $(g_3, \theta)$  and  $(g_3, \theta^2)$ :

$$\begin{aligned} R_{(g_3, \theta)} &= \frac{1}{3}(\rho_{(12,4)} + F_4^{II}[1] - \rho_{(6,6)'} - \rho_{(6,6)''} + 2F_4[\theta] - F_4[\theta^2]), \\ R_{(g_3, \theta^2)} &= \frac{1}{3}(\rho_{(12,4)} + F_4^{II}[1] - \rho_{(6,6)'} - \rho_{(6,6)''} - F_4[\theta] + 2F_4[\theta^2]). \end{aligned}$$

By 5.2 and Theorem 2.7, we have  $R_{(g_3, \theta)} = \zeta_{(g_3, \theta)}\chi_6$  and  $R_{(g_3, \theta^2)} = \zeta_{(g_3, \theta^2)}\chi_7$ .

**PROPOSITION 5.4.** *With notation as in 5.1–5.3, we have  $\zeta_{(g_3, \theta)} = \zeta_{(g_3, \theta^2)} = 1$ ; hence,  $R_{(g_3, \theta)} = \chi_6$  and  $R_{(g_3, \theta^2)} = \chi_7$ .*

**PROOF.** Let  $C$  be the conjugacy class of  $g_1 = su_{17}$  in  $G$  (cf. Table 1). Recall from 5.2 that the values of  $\chi_6$  and  $\chi_7$  on  $C^F$  are given as follows:

$$\chi_6(g) = \begin{cases} q^4 & (\text{if } g = g_1) \\ q^4\theta & (\text{if } g = g_{\bar{g}_1}) \\ q^4\theta^2 & (\text{if } g = g_{\bar{g}_1^2}) \end{cases} \quad \text{and} \quad \chi_7(g) = \begin{cases} q^4 & (\text{if } g = g_1) \\ q^4\theta^2 & (\text{if } g = g_{\bar{g}_1}) \\ q^4\theta & (\text{if } g = g_{\bar{g}_1^2}) \end{cases}$$

In order to prove that  $\zeta_{(g_3, \theta)} = \zeta_{(g_3, \theta^2)} = 1$  it is sufficient, by Proposition 3.4, to consider the base case where  $q = 2$ . So, in the GAP table of  $F_4(\mathbb{F}_2)$ , we form the above linear combinations of the irreducible characters, giving  $R_{(g_3, \theta)}$  and  $R_{(g_3, \theta^2)}$  for the group  $F_4(\mathbb{F}_2)$ . We find that these linear combinations have value 16 on the class 12o, which contains  $g_1 = su_{17} \in C^F$ . (Note that this does not depend on how we match  $F_4[\theta], F_4[\theta^2]$  with X.20, X.21 in the GAP table, since X.20 and X.21 are algebraically conjugate and all values on 12o are integers.) Comparing with the above formulae we see that  $\zeta_{(g_3, \theta)} = \zeta_{(g_3, \theta^2)} = 1$ , as desired.  $\square$

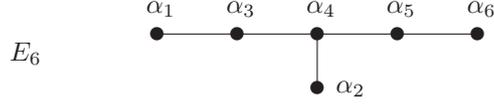
**REMARK 5.5.** The remaining  $A_1, \dots, A_5$  have already been dealt with by Marcelo–Shinoda [31, Theorem 4.1], by completely different (and, in our opinion, computationally somewhat more complicated) methods. They showed that

$$A_4 = A_{(g'_3, \varepsilon)}, \quad A_5 = A_{(1, \lambda^3)} \quad \text{and} \quad \zeta_{A_1} = \zeta_{A_2} = \zeta_{A_3} = \zeta_{A_4} = \zeta_{A_5} = 1.$$

In our setting, this can also be shown by exactly the same kind of argument as in the proof of Proposition 5.4.

## 6. Type $E_6$ in characteristic 2

Throughout this section, we assume that  $G$  is simple of adjoint type  $E_6$  and  $p = 2$ . We have  $G = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in k \rangle$  where  $\Phi$  is the root system of  $G$  with respect to  $T$ . Let  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$  be the set of simple roots with respect to  $B$ , where the labelling is chosen as in the following diagram:



Again, since  $p = 2$ , we do not have to worry about the precise choice of a Chevalley basis in the underlying Lie algebra. Further note that  $G(\mathbb{F}_2)$  is a simple group; it will just be denoted by  $E_6(\mathbb{F}_2)$ . (For  $q$  an even power of 2, the group  $G(\mathbb{F}_q)$  has a normal subgroup of index 3). Now we essentially rely on the knowledge of the character table of  $E_6(\mathbb{F}_2)$ , which has been determined by B. Fischer and is contained in the GAP library [3]. (We have  $|\text{Irr}(E_6(\mathbb{F}_2))| = 180$ .) Although the group is bigger, the further discussion will actually be simpler than that for  $F_4$  in the previous section, since there are fewer cuspidal unipotent character sheaves.

6.1. By Lusztig [20, 20.3] and Shoji [38, 4.6], there are two cuspidal character sheaves  $A_1, A_2 \in \hat{G}^{\text{un}}$ . Both  $A_1, A_2$  have the same support, namely, the closure of the  $G$ -conjugacy class  $C$  of an element  $g_1 = su \in G^F$  such that  $s \in G^F$  is semisimple, with  $C_G(s)^\circ$  isogeous to  $\text{SL}_3(k) \times \text{SL}_3(k) \times \text{SL}_3(k)$ , and  $u \in C_G^\circ(s)^F$  is regular unipotent. Furthermore,  $\dim C_G(g_1) = 6$  and  $A(g_1) \cong C_3 \times C'_3$  where  $C_3, C'_3$  are cyclic of order 3 and  $C_3$  is generated by the image  $\bar{g}_1 \in A(g_1)$ . Finally,  $A_j \leftrightarrow (g_1, \lambda_j)$  where  $\lambda_j$  are linear characters which are trivial on  $C'_3$  and such that  $\lambda_1(\bar{g}_1) = \theta, \lambda_2(\bar{g}_1) = \theta^2$  (where, again,  $\theta$  is a fixed third root of unity). Once we fix a specific element  $g_1 \in C^F$ , we obtain well-defined characteristic functions

$$\chi_j := \chi_{g_1, \lambda_j} = \chi_{A_j, \phi_{A_j}} \quad \text{for } j = 1, 2 \text{ (cf. 4.1).}$$

Table 3. Unipotent characters in the family  $\mathcal{F}_0$  for type  $E_6, q = 2$

$\rho_x$	$\rho_x(1)$	GAP
$\{E_6[\theta], E_6[\theta^2]\}$	45532800	$\{X.14, X.15\}$
$D_4, r$	120645056	X.18
(20, 10)	184660800	X.20
(10, 9)	192047232	X.22
(60, 8)	800196800	X.29
(80, 7)	864212544	X.30
(90, 8)	902358912	X.31

(Notation for  $\rho_x$  as in Carter [5, p. 480])

6.2. Let us write  $\mathfrak{U}(G^F) = \{\rho_x \mid x \in X(W)\}$  and  $\hat{G}^{\text{un}} = \{A_x \mid x \in X(W)\}$  as in 3.1. Here, the parameter set  $X(W)$  has 30 elements, and it is partitioned into 17 families. The 2 elements of  $X(W)$  which label cuspidal character sheaves are all contained in one family, which we denote by  $\mathcal{F}_0$  and which consists of 8 elements in total. The elements of  $\mathcal{F}_0$  are given by the 8 pairs  $(u, \sigma)$  where  $u \in \mathfrak{S}_3$  (up to  $\mathfrak{S}_3$ -conjugacy) and  $\sigma \in \text{Irr}(C_{\mathfrak{S}_3}(u))$ . The diagonal block of the Fourier matrix corresponding to  $\mathcal{F}_0$  is then described in terms of  $\mathfrak{S}_3$  and the pairs  $(u, \sigma)$ ; see [5, §13.6] for further details. The identification of the 8 unipotent characters  $\{\rho_x \mid x \in \mathcal{F}_0\}$  with characters in the GAP table of  $E_6(\mathbb{F}_2)$  is given in Table 3. (These characters are uniquely determined by their degrees, except for the two cuspidal unipotent characters  $E_6[\theta]$ ,  $E_6[\theta^2]$  which have the same degree.) The results of Shoji [38, 4.6, 5.2] show that the cuspidal character sheaves  $A_j$  in 6.1 are labelled as follows by pairs in  $\mathcal{F}_0$ :

$$A_1 = A_{(g_3, \theta)} \quad \text{and} \quad A_2 = A_{(g_3, \theta^2)},$$

where we follow again the notation in [5, p. 455] for pairs in  $\mathcal{F}_0$ . Using the  $8 \times 8$ -times Fourier matrix printed in [5, p. 457] and the labelling of the unipotent characters  $\{\rho_x \mid x \in \mathcal{F}_0\}$  in terms of pairs  $(u, \sigma)$  as above (see [5, p. 480]), we obtain the unipotent almost characters labelled by  $(g_3, \theta)$  and  $(g_3, \theta^2)$ :

$$\begin{aligned} R_{(g_3, \theta)} &= \frac{1}{3}(\rho_{(80,7)} + \rho_{(20,10)} - \rho_{(10,9)} - \rho_{(90,8)} + 2E_6[\theta] - E_6[\theta^2]), \\ R_{(g_3, \theta^2)} &= \frac{1}{3}(\rho_{(80,7)} + \rho_{(20,10)} - \rho_{(10,9)} - \rho_{(90,8)} - E_6[\theta] + 2E_6[\theta^2]). \end{aligned}$$

6.3. Let  $C$  be the  $F$ -stable conjugacy class of  $G$  such that  $\text{supp}(A_j) = \overline{C}$  for  $j = 1, 2$  (see 6.1). We now fix a specific representative  $g_1 \in C^F$ , as follows. Using the GAP character table of  $E_6(\mathbb{F}_2)$  and the information in Table 3, we find (for  $q = 2$ ) that  $\{R_{(g_3, \theta)}, R_{(g_3, \theta^2)}\} = \{f_1, f_2\}$  where  $f_1, f_2: E_6(\mathbb{F}_2) \rightarrow \overline{\mathbb{Q}}_\ell$  are class functions with non-zero values as follows:

GAP :	12n (no. 60)	12o (no. 61)	12p (no. 62)
$f_1$	8	$8\theta$	$8\theta^2$
$f_2$	8	$8\theta^2$	$8\theta$

Since  $\chi_j$  only has non-zero values on elements in  $C^F$ , we conclude that  $C$  must contain the elements of  $E_6(\mathbb{F}_2)$  which belong to the three classes 12n, 12o, 12p in the GAP table; these three classes have centraliser order  $192 = 3 \cdot 2^6$ . By inspection of the GAP table, all character values on 12n are integers, while the character values on 12o are complex conjugates of those on 12p. Thus, we choose  $g_1 \in C(\mathbb{F}_2) \subseteq C^F$  to be from the class 12n in  $E_6(\mathbb{F}_2)$ . Then  $g_1$  is conjugate in  $E_6(\mathbb{F}_2)$  to all powers  $g_1^n$  where  $n$  is a positive integer coprime to the order of  $g_1$ . Once  $g_1$  is fixed, we obtain characteristic functions  $\chi_1, \chi_2$  as in 6.1. Then Theorem 2.7 shows that  $R_{(g_3, \theta)} = \zeta_{(g_3, \theta)} \chi_1$  and  $R_{(g_3, \theta^2)} = \zeta_{(g_3, \theta^2)} \chi_2$ .

6.4. Let  $g_1 \in C(\mathbb{F}_2)$  be as in 6.3. We write  $g_1 = su = us$  where  $s \in E_6(\mathbb{F}_2)$  has order 3 and  $u \in E_6(\mathbb{F}_2)$  is unipotent. As in 5.2(4) one sees that all such elements  $s$

of order 3 are conjugate in  $E_6(\mathbb{F}_2)$ . We now identify the unipotent part  $u$  of  $g_1$ , where we use the results and the notation of Mizuno [33]. Let

$$(a) \quad u_{15} := x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)x_{\alpha_2+\alpha_4+\alpha_5}(1) \in E_6(\mathbb{F}_2),$$

as in [33, Lemma 4.3]; in particular, we have

$$(b) \quad A(u_{15}) \cong \mathfrak{S}_3 \quad \text{and} \quad |C_G(u_{15})^F| = 6q^{18}(q-1)^2.$$

We claim that the unipotent part  $u$  of  $g_1$  is conjugate in  $E_6(\mathbb{F}_2)$  to  $u_{15}$ . To see this, we follow a similar approach as in 5.2. Using the explicit 78-dimensional matrix realization of  $E_6(\mathbb{F}_2)$  from Lusztig [29, 2.3] (see also [11, 4.10]), we create  $E_6(\mathbb{F}_2)$  as a matrix group in GAP. We consider the subgroup

$$L := \langle x_{\pm\alpha_2}(1), x_{\pm\alpha_3}(1), x_{\pm\alpha_4}(1), x_{\pm\alpha_5}(1) \rangle \subseteq E_6(\mathbb{F}_2);$$

note that  $u_{15} \in L$ . The group  $L$  is simple of order 174182400. (In fact,  $L \cong \text{SO}_8^+(\mathbb{F}_2)$ .) As in Example 4.4, the GAP function `CharacterTable` computes the character table of  $L$  (by a general algorithm), together with a list of representatives of the conjugacy classes; there are 53 conjugacy classes of  $L$ . One notices that there is a unique class of  $L$  in which the elements have centraliser order 64, and one checks using GAP that our element  $u_{15}$  belongs to this class. Now, as in 5.2, the function `PossibleClassFusions` determines all possible fusions of the conjugacy classes of  $L$  into the GAP table of  $E_6(\mathbb{F}_2)$ . It turns out that there are 2 possible fusion maps. But under each of these two possibilities, the class containing our element  $u_{15}$  is mapped to the class `4k` (no. 18) in the GAP table of  $E_6(\mathbb{F}_2)$ . Thus,  $u_{15}$  belongs to the class `4k` in GAP. According to the GAP table, the element  $g_1^3 = u^{-1}$  also belongs to the class `4k`; furthermore, all character values on `4k` are integers. Hence,  $u, u^{-1}, u_{15}$  are all conjugate in  $E_6(\mathbb{F}_2)$ , as claimed.

To summarize the above discussion, we can assume that the chosen representative  $g_1 \in C^F$  in 6.3 is as follows. We have  $g_1 = us = su \in E_6(\mathbb{F}_2)$  where  $u = u_{15}$  (see 6.4(a)) and  $s \in E_6(\mathbb{F}_2)$  is any element of order 3 such that  $su = us$ .

**PROPOSITION 6.5.** *Let  $g_1 = su_{15} \in E_6(\mathbb{F}_2) \subseteq G^F$  be as above. For  $j = 1, 2$ , define  $\chi_j = \chi_{(g_1, \lambda_j)}: G^F \rightarrow \overline{\mathbb{Q}}_\ell$  as in Example 4.1, with  $\lambda_j$  as in 6.1. Then  $\zeta_{(g_3, \theta)} = \zeta_{(g_3, \theta^2)} = 1$ ; hence,  $R_{(g_3, \theta)} = \chi_1$  and  $R_{(g_3, \theta^2)} = \chi_2$ .*

**PROOF.** Recall from 6.2 that  $R_{(g_3, \theta)} = \zeta_{(g_3, \theta)}\chi_1$  and  $R_{(g_3, \theta^2)} = \zeta_{(g_3, \theta^2)}\chi_2$ . Now it will be sufficient to show that the scalars  $\zeta_{(g_3, \theta)}$  and  $\zeta_{(g_3, \theta^2)}$  are equal to 1 for the special case where  $q = 2$ ; see Proposition 3.4. But this has been observed in 6.3 above; note that, for  $q = 2$ , we have  $\{f_1, f_2\} = \{\chi_1, \chi_2\}$  and  $g_1$  belongs to the class `12n` in the GAP table. Regardless of how we match the functions in these two pairs, the values on  $g_1 \in 12n$  are equal to 8 in both cases. Hence, we must have  $\zeta_{(g_3, \theta)} = \zeta_{(g_3, \theta^2)} = 1$ .  $\square$

REMARK 6.6. We briefly sketch an alternative proof of Proposition 6.5, following the line of argument in Lusztig [22]. Let us fix  $g_1 = su_{15} = u_{15}s$  as above. Arguing analogously to 6.2, we can express the unipotent characters  $\{\rho_x \mid x \in \mathcal{F}_0\}$  in terms of the almost characters  $\{R_x \mid x \in \mathcal{F}_0\}$ . In particular, we obtain:

$$E_6[\theta] = \frac{1}{3}(R_{(80,7)} + R_{(20,10)} - R_{(10,9)} - R_{(90,8)} + 2R_{(g_3,\theta)} - R_{(g_3,\theta^2)}).$$

Now evaluate on  $g_1$ . The values of the almost characters  $R_\epsilon$  ( $\epsilon \in \text{Irr}(W)$ ) occurring in the above expression can be computed using the character formula in [5, 7.2.8] (which expresses the values of the Deligne–Lusztig virtual characters  $R_w$  in terms of the Green functions of  $C_G(s)^\circ$ ) and the fact that Green functions take value 1 on regular unipotent elements (see [7, 9.16]). Furthermore, since all character values on  $g_1$  are integers, we must have  $\zeta_{(g_3,\theta)} = \zeta_{(g_3,\theta^2)} = \pm 1$ , and so  $R_{(g_3,\theta)}(g_1) = R_{(g_3,\theta^2)}(g_1) = \pm q^3$ . This yields the condition that

$$E_6[\theta](g_1) = \frac{1}{3}(\pm q^3 + \text{known value}) \in \mathbb{Z},$$

which uniquely determines the sign of  $\zeta_{(g_3,\theta)}$ . Note that this argument does not require the knowledge of the complete character table of  $E_6(\mathbb{F}_2)$ , but we still need to know some information about the representative  $g_1$ , e.g., the fact that it is conjugate to all powers  $g_1^n$  where  $n$  is coprime to the order of  $g_1$ . (A similar argument will actually work for  $E_6(\mathbb{F}_q)$ , where  $q$  is a power of any prime  $\neq 3$ , once the existence of a representative  $g_1$  as above is guaranteed; for  $q$  a power of 3, a different argument is needed. See [13] for further details.)

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