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On H_σ-permutably embedded subgroups of finite groups*

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Abstract – Let G be a finite group. Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and n an integer. We write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$, $\sigma(G) = \sigma(|G|)$. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member of $\mathcal{H} \setminus \{1\}$ is a Hall σ_i-subgroup of G for some σ_i and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. A subgroup A of G is called: (i) a *σ-Hall subgroup* of G if $\sigma(A) \cap \sigma(|G : A|) = \emptyset$; (ii) *$\sigma$-permutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^x A$ for all $H \in \mathcal{H}$ and all $x \in G$. We say that a subgroup A of G is *H_σ-permutably embedded* in G if A is a σ-Hall subgroup of some σ-permutable subgroup of G. We study finite groups G having an H_σ-permutably embedded subgroup of order $|A|$ for each subgroup A of G. Some known results are generalized.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, n is an integer, $\mathbb{P}$ is the set of all primes, and if $\pi \subseteq \mathbb{P}$, then $\pi' = \mathbb{P} \setminus \pi$. The symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. We use n_π to denote the π-part of n, that is, the largest π-number dividing n; n_p denotes the largest degree of p dividing n.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; Π is a subset of σ and $\Pi' = \sigma \setminus \Pi$.

Let $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$. Then we say that G is *σ-primary* [1] if G is a σ_i-group for some $\sigma_i \in \sigma$.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G [2, 3] if every member of $\mathcal{H} \setminus \{1\}$ is a Hall σ_i-subgroup of G for some σ_i and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. We say that G is *σ-full* if G possesses a complete Hall σ-set. Throughout this paper, G is always supposed to be a σ-full group.

A subgroup A of G is called [1]: (i) a *σ-Hall subgroup* of G if $\sigma(A) \cap \sigma(|G:A|) = \emptyset$; (ii) *$\sigma$-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$; (iii) *σ-quasinormal* or *σ-permutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^x A$ for all $H \in \mathcal{H}$ and all $x \in G$. In particular, A is called *S-quasinormal* or *S-permutable* in G [4, 5] provided $AP = PA$ for all Sylow subgroups P of G.

Definition 1.1. We say that a subgroup A of G is *H_σ-subnormally* (respectively *H_σ-permutably*, *H_σ-normally*) embedded in G if A is a σ-Hall subgroup of some σ-subnormal (respectively σ-permutable, normal) subgroup of G.

In the special case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ the definition of H_σ-normally embedded subgroups is equivalent to the concept of Hall normally embedded subgroups in [6], the definition of H_σ-permutably embedded subgroups is equivalent to the concept of Hall S-quasinormally embedded subgroups in [7] and the definition of H_σ-subnormally embedded subgroups is equivalent to the concept of Hall subnormally embedded subgroups in [8].

Example 1.2. (i) For any σ, all σ-Hall subgroups and all σ-subnormal subgroups of any group S are H_σ-subnormally embedded in S. Now, let $G = (C_7 \rtimes C_3) \times A_5$, where $C_7 \rtimes C_3$ is a non-abelian group of order 21 and A_5 is the alternating group of degree 5, and let $H = (C_7 \rtimes C_3)A$, where A is a Sylow 2-subgroup of A_5. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{7\}$ and $\sigma_2 = \{7\}'$. Then H is σ-subnormal in G and C_3A_5 is a σ-Hall subgroup of G. In view of Lemma 2.1(1)(5) below, the subgroup C_3A is neither σ-subnormal in G nor H_σ-normally embedded in G.

(ii) For any σ, all σ-Hall subgroups and all σ-permutable subgroups of any group S are H_σ-permutably embedded in S. Now, let $p > q > r$ be primes, where

r^2 divides $q-1$. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{q, r\}$ and $\sigma_2 = \{q, r\}'$. Let $H = Q \rtimes R$ be a group of order qr^2, where $C_H(Q) = Q$. Let P be a simple $\mathbb{F}_pH$-module which is faithful for H and $G = P \rtimes H$. Let R_1 be a subgroup of R of order r. Then the subgroup $V = PR_1$ is σ-permutable in G and R_1 is a σ-Hall subgroup of V. Hence R_1 is H_σ-permutably embedded in G. It is also clear that G has no an S-permutable subgroup W such that R_1 is a Hall subgroup of W, so R_1 is neither H_σ-normally embedded nor S-permutably embedded in G.

(iii) For any σ, all σ-Hall subgroups and all normal subgroups of any group S are H_σ-normally embedded in S. Now, let P be a simple $\mathbb{F}_{11}(C_7 \rtimes C_3)$-module which is faithful for $C_7 \rtimes C_3$. Let $G = (P \rtimes (C_7 \rtimes C_3)) \times A_5$. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{5, 7, 11\}$ and $\sigma_2 = \{5, 7, 11\}'$. Then the subgroup $M = (P \rtimes C_7) \times A_5$ is normal in G and a subgroup B of A_5 of order 12 is a σ-Hall subgroup of M, so B is H_σ-normally embedded in G. Finally, if $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{7\}$ and $\sigma_2 = \{7\}'$, then B is not H_σ-normally embedded in G.

Recall that G is *σ-nilpotent* [9] if $G = H_1 \times \cdots \times H_t$ for some σ-primary groups $H_1, \ldots, H_t$. The *σ-nilpotent* residual $G^{\mathfrak{N}_\sigma}$ of G is the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N, $G^{\mathfrak{N}}$ denotes the nilpotent residual of G. It is clear that every subgroup of a σ-nilpotent group G is σ-permutable and σ-subnormal in G.

THEOREM 1.3. *Let $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ be a complete Hall σ-set of G and $D = G^{\mathfrak{N}_\sigma}$. Then any two of the following conditions are equivalent:*

(i) G has an H_σ-permutably embedded subgroup of order $|A|$ for each subgroup A of G.

(ii) D is cyclic of square-free order and $|\sigma_i \cap \pi(G)| = 1$ for all i such that $\sigma_i \cap \pi(D) \neq \emptyset$.

(iii) For each set $\{A_1, \ldots, A_t\}$, where A_i is a subgroup (respectively normal subgroup) of H_i for all $i = 1, \ldots, t$, G has an H_σ-permutably embedded (respectively H_σ-normally embedded) subgroup of order $|A_1| \cdots |A_t|$.

Let $\mathfrak{F}$ be a class of groups. A subgroup H of G is said to be an $\mathfrak{F}$-covering subgroup of G [10, VI, Definition 7.8] if $H \in \mathfrak{F}$ and for every subgroup E of G such that $H \leq E$ and $E/N \in \mathfrak{F}$ it follows that $E = NH$. We say that a subgroup H of G is a *σ-Carter subgroup* of G if H is an $\mathfrak{N}_\sigma$-covering subgroup of G, where $\mathfrak{N}_\sigma$ is the class of all σ-nilpotent groups.

A group G is said to have a *Sylow tower* if G has a normal series $1 = G_0 < G_1 < \cdots < G_{t-1} < G_t = G$, where $|G_i/G_{i-1}|$ is the order of some Sylow subgroup of G for each $i \in \{1, \ldots, t\}$. A chief factor of G is said to be *σ-central* (in G) [1] if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary. Otherwise, H/K is called *σ-eccentric* (in G).

We say that G is an *$H\sigma E$-group* if the following conditions are hold: (i) $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is a σ-Hall subgroup of G and $|\sigma(D)| = |\pi(D)|$; (ii) D has a Sylow tower and every chief factor of G below D is σ-eccentric; (iii) M acts irreducibly on every M-invariant Sylow subgroup of D.

We do not still know the structure of a group G having an H_σ-subnormally embedded subgroup of order $|A|$ for each subgroup A of G. Nevertheless, the following fact is true.

THEOREM 1.4. *Any two of the following conditions are equivalent:*
(i) Every subgroup of G is H_σ-subnormally embedded in G.
(ii) Every σ-subnormal subgroup H of G is an $H\sigma E$-group of the form $H = D \rtimes M$, where $D = H^{\mathfrak{N}_\sigma}$ and M is a σ-Carter subgroup of H.
(iii) Every σ-subnormal subgroup of G is an $H\sigma E$-group.

Now, let us consider some corollaries of Theorems 1.3 and 1.4. First note that since a nilpotent group G possesses a normal subgroup of order n for each integer n dividing $|G|$, in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, Theorem 1.3 covers Theorem 11 in [6], Theorem 2.7 in [8] and Theorems 3.1 and 3.2 in [7].

From Theorem 1.3 we also get the following result.

COROLLARY 1.5. *Suppose that G possesses a complete Hall σ-set $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ such that H_i is nilpotent for all $i = 1, \ldots, t$. Then G has an H_σ-normally embedded subgroup of order $|H|$ for each subgroup H of G if and only if $G^{\mathfrak{N}}$ is cyclic of square-free order and $|\sigma_i \cap \pi(G)| = 1$ for all i such that $\sigma_i \cap \pi(D) \neq \emptyset$.*

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Corollary 1.5 the following known result.

COROLLARY 1.6 (Ballester-Bolinches, Qiao, [11]). *G has a Hall normally embedded subgroup of order $|H|$ for each subgroup H of G if and only if $G^{\mathfrak{N}}$ is cyclic of square-free order.*

On the basis of Theorems 1.3 and 1.4 we prove also the next two theorems.

THEOREM 1.7. *Any two of the following conditions are equivalent:*
(i) Every subgroup of G is H_σ-normally embedded in G.
(ii) $G = D \rtimes M$ is an $H\sigma E$-group, where $D = G^{\mathfrak{N}_\sigma}$ is a cyclic group of square-free order and M is a Dedekind group.
(iii) $G = D \rtimes M$, where D is a σ-Hall cyclic subgroup of G of square-free order with $|\sigma(D)| = |\pi(D)|$ and M is a Dedekind group.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Theorem 1.7 the following known result.

COROLLARY 1.8 (Li, Liu, [8]). *Every subgroup of G is a Hall normally embedded subgroup of G if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is a cyclic Hall subgroup of G of square-free order and M is a Degekind group.*

THEOREM 1.9. *Any two of the following conditions are equivalent:*
(i) Every subgroup of G is H_σ-permutably embedded in G.
(ii) $G = D \rtimes M$ is an $H\sigma E$-group, where $D = G^{\mathfrak{N}_\sigma}$ is a cyclic group of square-free order.
(iii) $G = D \rtimes M$, where D is a σ-Hall cyclic subgroup of G of square-free order with $|\sigma(D)| = |\pi(D)|$ and M is σ-nilpotent.

COROLLARY 1.10. *Every subgroup of G is a Hall S-quasinormally embedded subgroup of G if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is a cyclic Hall subgroup of G of square-free order and M is a Carter subgroup of G.*

In conclusion of this section, consider the following example.

EXAMPLE 1.11. Let $5 < p_1 < p_2 < \cdots < p_n$ be a set of primes and p a prime such that either $p > p_n$ or p divides $p_i - 1$ for all $i = 1, \ldots, n$. Let A be a group of order p and P_i a simple $\mathbb{F}_{p_i}A$-module which is faithful for A. Let $L_i = P_i \rtimes A$ and $B = (\ldots((L_1 \curlywedge L_2) \curlywedge L_3) \curlywedge \cdots) \curlywedge L_n$ (see [10, p. 50]). We can assume without loss of generality that $L_i \leq B$ for all $i = 1, \ldots, n$. Let $G = B \times A_5$, where A_5 is the alternating group of degree 5, and let σ be a partition of $\mathbb{P}$ such that for some different indices $i, j, i_1, \ldots, i_n \in I$ we have $p \in \sigma_i$, $\{2, 3, 5\} \subseteq \sigma_j$ and $p_k \in \sigma_{i_k}$ for all $k = 1, \ldots, n$. Then $D = P_1P_2 \cdots P_n = G^{\mathfrak{N}_\sigma}$ is a σ-Hall subgroup of G and $G = D \rtimes (A \times A_5)$.

We show that every subnormal subgroup H of G satisfies Condition (ii) in Theorem 1.4. If $H^{\mathfrak{N}_\sigma} = 1$, it is evident. Hence we can assume without loss of generality $A \leq H$ since every p'-subgroup of G is σ-nilpotent. But then $H = (H \cap D) \rtimes (A \times (H \cap A_5))$ by Lemma 2.1(4) below, where $H \cap D$ is a normal σ-Hall subgroup of H and $M = A \times (H \cap A_5)$ is a σ-nilpotent subgroup of H. Moreover, $H \cap A_5$ induces on every non-identity Sylow subgroup of $H \cap D$ a non-trivial irreducible group of automorphisms. Therefore $H^{\mathfrak{N}_\sigma} = H \cap D$ and $|\sigma(H^{\mathfrak{N}_\sigma})| = |\pi(H^{\mathfrak{N}_\sigma})|$. It is also clear that M is a σ-Carter subgroup of H and every chief factor of H below $H^{\mathfrak{N}_\sigma}$ is σ-eccentric in H. Thus G satisfies Condition (ii) in Theorem 1.4, and so every subgroup H of G is H_σ-subnormally embedded in G. On the other hand, the subgroup DAC_2, where C_2 is a subgroup of order 2 of G, is not Hall subnormally embedded in G since C_2 is not a Sylow subgroup of any subnormal subgroup of G.

Finally, if p divides $p_i - 1$ for all $i = 1, \ldots, n$, then $|P_i| = p_i$ for all $i = 1, \ldots, n$, so G satisfies Condition (ii) in Theorem 1.9 and hence satisfies Condition (ii) in Theorem 1.3.

2. Basic lemmas

An integer n is called a Π-*number* if $\sigma(n) \subseteq \Pi$. A subgroup H of G is called a *Hall* Π-*subgroup* of G [1] if $|H|$ is a Π-number and $|G : H|$ is a Π'-number.

Lemma 2.1 (Lemma 2.6, [1]). *Let A, K and N be subgroups of G, where A is σ-subnormal in G and N is normal in G.*

(1) $A \cap K$ is σ-subnormal in K.

(2) If K is σ-subnormal in G, then $A \cap K$ and $\langle A, K\rangle$ are σ-subnormal in G.

(3) AN/N is σ-subnormal in G/N.

(4) If $H \neq 1$ is a Hall Π-subgroup of G and A is not a Π'-group, then $A \cap H \neq 1$ is a Hall Π-subgroup of A.

(5) If $|G : A|$ is a σ_i-number, then $O^{\sigma_i}(A) = O^{\sigma_i}(G)$.

(6) If V/N is a σ-subnormal subgroup of G/N, then V is σ-subnormal in G.

(7) If K is a σ-subnormal subgroup of A, then K is σ-subnormal in G.

A group G is said to be σ-*soluble* [1] if every chief factor of G is σ-primary.

Lemma 2.2 (Lemmas 2.8 and 3.2 and Theorems B and C, [1]). *Let A, K and N be subgroups of G, where A is σ-permutable in G and N is normal in G.*

(1) AN/N is σ-permutable in G/N.

(2) If G is σ-soluble, then $A \cap K$ is σ-permutable in K.

(3) If $N \leq K$, K/N is σ-permutable in G/N and G is σ-soluble, then K is σ-permutable in G.

(4) A is σ-subnormal in G.

(5) If G is σ-soluble and K is σ-permutable in G, then $K \cap A$ is σ-permutable in G.

Lemma 2.3. *Let H be a normal subgroup of G. If $H/H \cap \Phi(G)$ is a Π-group, then H has a a Hall Π-subgroup, say E, and E is normal in G. Hence, if $H/H \cap \Phi(G)$ is σ-nilpotent, then H is σ-nilpotent.*

Proof. Let $D = O_{\Pi'}(H)$. Then, since $H \cap \Phi(G)$ is nilpotent, D is a Hall Π'-subgroup of H. Hence by the Schur-Zassenhaus theorem, H has a Hall Π-subgroup, say E. It is clear that H is π'-soluble, where $\pi' = \cup_{\sigma_i \in \Pi'} \sigma_i$, so any two Hall Π-subgroups of H are conjugate. By the Frattini argument, $G = HN_G(E) = (E(H \cap \Phi(G)))N_G(E) = N_G(E)$. Therefore E is normal in G. The lemma is proved. □

Lemma 2.4. *If every chief factor of G below $D = G^{\mathfrak{N}_\sigma}$ is cyclic, then D is nilpotent.*

Proof. Assume that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G. Then from the G-isomorphism $D/D \cap R \simeq DR/R = (G/R)^{\mathfrak{N}_\sigma}$ we know that every chief factor of G/R below DR/R is cyclic, so the choice of G implies that $D/D \cap R \simeq DR/R$ is nilpotent. Hence $R \leq D$ and R is the unique minimal normal subgroup of G. In view of Lemma 2.3, $R \nleq \Phi(G)$ and so $R = C_R(R)$ by [12, A, 15.2]. But by hypothesis, $|R|$ is a prime, hence $G/R = G/C_G(R)$ is cyclic, so G is supersoluble and so $G^{\mathfrak{N}_\sigma}$ is nilpotent since $G^{\mathfrak{N}_\sigma} \leq G^{\mathfrak{N}}$. The lemma is proved. □

The following lemma is evident.

LEMMA 2.5. *The class of all σ-soluble groups is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the σ-soluble group by a σ-soluble group is a σ-soluble group as well.*

Let A, B and R be subgroups of G. Then A is said to *R-permute* with B [13] if for some $x \in R$ we have $AB^x = B^x A$.

If G has a complete Hall σ-set $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ such that $H_i H_j = H_j H_i$ for all i, j, then we say that $\{H_1, \ldots, H_t\}$ is a *σ-basis* of G.

LEMMA 2.6 (Theorems A and B, [2]). *Assume that G is σ-soluble.*

(i) G has a σ-basis $\{H_1, \ldots, H_t\}$ such that for each $i \neq j$ every Sylow subgroup of H_i G-permutes with every Sylow subgroup of H_j.

(ii) For any Π, the following hold: G has a Hall Π-subgroup E, every Π-subgroup of G is contained in some conjugate of E and E G-permutes with every Sylow subgroup of G.

LEMMA 2.7. *Let H, E and R be subgroups of G. Suppose that H is H_σ-subnormally embedded in G and R is normal in G.*

(1) If $H \leq E$, then H is H_σ-subnormally embedded in E.

(2) HR/R is H_σ-subnormally embedded in G/R.

(3) If S is a σ-subnormal subgroup of G, then $H \cap S$ is H_σ-subnormally embedded in G.

(4) If $|G : H|$ is σ-primary, then H is either a σ-Hall subgroup of G or σ-subnormal in G.

PROOF. Let V be a σ-subnormal subgroup of G such that H is a σ-Hall subgroup of V.

(1) This assertion is a corollary of Lemma 2.1(1).

(2) In view of Lemma 2.1(3), VR/R is σ-subnormal subgroup of G/R. It is also clear that HR/R is a σ-Hall subgroup of VR/R. Hence HR/R is H_σ-subnormally embedded in G/R.

(3) By Lemma 2.1(1)(2), $V \cap S$ is σ-subnormal both in V and in G and so $H \cap (V \cap S) = H \cap S$ is a σ-Hall subgroup of $V \cap S$ by Lemma 2.1(4), as required.

(4) Assume that H is not σ-subnormal in G. Then $H < V$. By hypothesis, $|G : H|$ is σ-primary, say $|G : H|$ is a σ_i-number. Then $|V : H|$ is a σ_i-number. But H is a σ-Hall subgroup of V. Hence H is a σ-Hall subgroup of G.

The lemma is proved. □

LEMMA 2.8. *Let H be a σ-subnormal subgroup of a σ-soluble group G. If $|G : H|$ is a σ_i-number and B is a σ_i-complement of H, then $G = HN_G(B)$.*

PROOF. Assume that this lemma is false and let G be a counterexample of minimal order. Then $H < G$, so G has a proper subgroup M such that $H \leq M$, $|G : M_G|$

is a σ_i-number and H is σ-subnormal in M. The choice of G implies that $M = HN_M(B)$. On the other hand, clearly that B is a σ_i-complement of M_G. Since G is σ-soluble, Lemma 2.6 and the Frattini argument imply that $G = M_GN_G(B) = MN_G(B) = HN_M(B)N_G(B) = HN_G(B)$. The lemma is proved. □

The following lemma is well-known (see, for example, [14, 3.29] or [15, 1.10.10]).

Lemma 2.9. *Let H/K be an abelian chief factor of G and V a maximal subgroup of G such that $K \leq V$ and $HV = G$. Then $G/V_G \simeq (H/K) \rtimes (G/C_G(H/K))$.*

3. Proofs of the results

Proof of Theorem 1.3. Without loss of generality we may assume that H_i is a σ_i-group for all $i = 1, \dots, t$.

(i), (iii) ⇒ (ii) Assume that this is false. Then $D \neq 1$ and so $t > 1$.

First we prove the following claim.

(*) *If $p \in \sigma_i \cap \pi(G)$, then G has a σ-permutable subgroup E with $|E| = |G|_{\sigma'_i}p$.*

We can assume without loss of generality that $i = 1$. In fact, to prove Claim (*), we consistently build the σ-permutable subgroups $E_2, \dots, E_t$ such that $|H_2| \cdots |H_j|$ divides $|E_j|$ and $|E_j|_{\sigma_1} = p$ for all $j = 2, \dots, t$.

By hypothesis, G has an H_σ-permutably embedded subgroup X of order p. Let V be a σ-permutable subgroup of G such that X is a σ-Hall subgroup of V. Then $|V|_{\sigma_1} = p$ and G has a complete Hall σ-set $\{1, K_1, \dots, K_t\}$, where K_i is a σ_i-group for all $i = 1, \dots, t$, such that $VK_i = K_iV$ for all $i = 1, \dots, t$. Let $W = VK_2$. Then $|W|_{\sigma_1} = p$.

Next we show that there is an H_σ-permutably embedded subgroup Y of G such that $|Y| = |W|$. It is enough to consider the case when Condition (iii) holds. Let A_1 be a subgroup of H_1 of order p, $A_2 = H_2$ and $A_i = H_i \cap V$ for all $i > 2$. Then $|A_2| = |H_2| = |K_2|$. On the other hand, $V \cap K_i$ and $V \cap H_i$ are Hall σ_i-subgroups of V by Lemmas 2.1(4) and 2.2(4) and so $|V \cap K_i| = |V \cap H_i|$. Also, for every $i > 2$ we have $|W : V \cap K_i| = |VK_2 : V \cap K_i| = |V||K_2| : |V \cap K_2||V \cap K_i|$ is a σ'_i-number and hence $V \cap K_i = W \cap K_i$ is a Hall σ_i-subgroup of W. Therefore, $|W| = p|H_2||V \cap H_3| \cdots |V \cap H_t|$ and so G has an H_σ-permutably embedded subgroup Y of order $|W| = |A_1| \cdots |A_t|$ by hypothesis.

Let E_2 be a σ-permutable subgroup of G such that Y is a σ-Hall subgroup of E_2. Then $|H_2| = |K_2|$ divides $|E_2|$ and $|E_2|_{\sigma_1} = p$. Now, arguing by induction, assume that G has a σ-permutable subgroup E_{t-1} such that $|H_2| \cdots |H_{t-1}|$ divides $|E_{t-1}|$ and $|E_{t-1}|_{\sigma_1} = p$. Then for some Hall σ_t-group L we have $E_{t-1}L = LE_{t-1}$, and if $E_t = E_{t-1}L$, then $|E_t| = |G|_{\sigma'_1}p$ and E_t clearly is σ-permutable in G, as required.

Now, let $p \in \sigma_i \cap \pi(D)$ and let P be a Sylow p-subgroup of D. Then, by Claim (*), G possesses a σ-permutable subgroup E such that $|E| = |G|_{\sigma'_i}p$. Lemma 2.2(4) implies that E is σ-subnormal in G, so Lemma 2.1(4) shows that G/E_G is a σ_i-group. Hence $D \leq E_G \leq E$, so $|P| = p$. Therefore G is supersoluble by [10, IV, 2.9] and so every chief factor of G below D is cyclic. Hence D is nilpotent by Lemma 2.4, so D is cyclic of square-free order.

Finally, assume that $|\sigma_i \cap \pi(G)| > 1$ and let $q \in \sigma_i \cap \pi(G) \setminus \{p\}$. Then G possesses a σ-permutable subgroup F such that $|F| = |G|_{\sigma_i'}q$. Then $D \leq F_G \leq F$. Therefore $D \leq E \cap F$ and so p does not divide $|D|$. This contradiction completes the proof of the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) First we show that for every i and for every subgroup (respectively normal subgroup) A_i of H_i, there is an H_σ-permutably embedded (respectively H_σ-normally embedded) subgroup E_i of G such that $|E_i| = |A_i||G|_{\sigma_i'}$. Since G evidently is σ-soluble, it has a σ_i-complement E by Lemma 2.6. Therefore, it is enough to consider the case when $A_i \neq 1$ since every σ-Hall subgroup of G is an H_σ-normally embedded in G.

First suppose that $D \leq E$. Then E/D is normal in G since G/D is σ-nilpotent. Therefore $(E/D) \times (A_iD/D) = EA_i/D$ is σ-permutable (respectively normal) in $G/D = (E/D) \times (H_iD/D)$. Hence $E_i = EA_i$ is σ-permutable (respectively normal) in G by Lemma 2.2(3) and $|E_i| = |A_i||G|_{\sigma_i'}$.

Now suppose that $D \nleq E$. Then $D \cap H_i \neq 1$, so H_i is a p-group for some prime p since for each $\sigma_i \in \sigma(D)$ we have $|\sigma_i \cap \pi(G)| = 1$ by hypothesis. Hence H_2 has a normal subgroup A such that $D_p \leq A$ and $|A| = |A_i|$, where D_p is a Sylow p-subgroup of D. Then $D \leq AE$. Moreover, $AE/D = (DA/D) \times (ED/D)$ since ED/D is a Hall σ_i'-subgroup of G/D. Therefore $E_i = AE$ is σ-permutable (respectively normal) in G by Lemma 2.2(3) and $|E_i| = |A_i||G|_{\sigma_i'}$.

Let $E = E_1 \cap \cdots \cap E_t$. Then $|E| = |A_1| \cdots |A_t|$ since $(|G : E_i|, |G : E_j|) = 1$ for all $i \neq j$. Note that E_i is either a σ-Hall subgroup of G or σ-permutable (respectively normal) in G. Indeed, let V be a σ-permutable (respectively normal) subgroup of G such that E_i is a σ-Hall subgroup of V. Assume that E_i is not σ-permutable (respectively not normal) in G. Then $E_i < V$. Since $|G : E_i|$ is a σ_i-number, $|V : E_i|$ is a σ_i-number. But E_i is a σ-Hall subgroup of V. Hence $E_i = V$ is a σ-Hall subgroup of G.

Assume that $E_1, \ldots, E_r$ are σ-permutable (respectively normal) in G and E_i is a σ-Hall subgroup of G for all $i > r$. Then $E^0 = E_1 \cap \cdots \cap E_r$ is σ-permutable (respectively normal) in G by Lemma 2.2(5) and $E^* = E_{r+1} \cap \cdots \cap E_t$ is a σ-Hall subgroup of G. Now, $E = E^0 \cap E^*$ is a σ-Hall subgroup of E^0 by Lemmas 2.1(4) and 2.2(4), so E is H_σ-permutably (respectively H_σ-normally) embedded in G. Hence (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i) Since G is σ-soluble, H is σ-soluble. Hence H has a σ-basis $\{L_1, \ldots, L_r\}$ such that $L_i \leq H_i$ for all $i = 1, \ldots, r$ by Lemma 2.6. Therefore from the implication (ii) $\Rightarrow$ (iii) we get that G has an H_σ-permutably embedded subgroup of order $|L_1| \cdots |L_r| = |H|$.

The theorem is proved.

Proof of Theorem 1.4. (i) $\Rightarrow$ (ii) Assume that this is false and let G be a counterexample of minimal order. Then some σ-subnormal subgroup V of G is not an $H\sigma E$-group. Moreover, $D = G^{\mathfrak{N}_\sigma} \neq 1$, so $|\sigma(G)| > 1$.

(1) *Condition (ii) is true on every proper section H/K of G, that is, $K \neq 1$ or $H \neq G$. Hence $V = G$* (This directly follows from Lemma 2.7(1)(2) and the

choice of G).

(2) *G is σ-soluble.*

In view of Claim (1) and Lemma 2.5, it is enough to show that G is not simple. Assume that this is false. Then 1 is the only proper σ-subnormal subgroup of G since $|\sigma(G)| > 1$. Hence every subgroup of G is a σ-Hall subgroup of G. Therefore for a Sylow p-subgroup P of G, where p is the smallest prime divisor of $|G|$, we have $|P| = p$ and so $|G| = p$ by [10, IV, 2.8]. This contradiction shows that we have (2).

(3) *If $|G : H|$ is a σ_i-number and H is not a σ-Hall subgroup of G, then H is σ-subnormal in G and a σ_i-complement E of H is normal in G* (This follows from Lemmas 2.7(4) and 2.8).

(4) *D is a σ-Hall subgroup of G. Hence D has a complement M in G.*

Suppose that this is false. Then for some $i \in I$ and for some Hall σ_i-subgroups U and H_i of D and G, respectively, we have $1 < U < H_i$. Let R be a minimal normal subgroup of G contained in D. Claim (2) implies that R is a σ_k-group for some k. Moreover, $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a σ-Hall subgroup of G/R by Claim (1). Hence UR/R is a σ-Hall subgroup of G/R. Suppose that $UR/R \neq 1$, then UR/R is a Hall σ_i-subgroup of G/R.

If $k \neq i$, then U is a Hall σ_i-subgroup of G by order considerations. This contradicts that $U < H_i$. If $k = i$, then $R \leq U$ and so U/R is a Hall σ_i-subgroup of G/R. It follows that U is a Hall σ_i-subgroup of G, which contradicts that $U < H_i$. Therefore $UR/R = 1$. Consequently, $U \leq R$ and $U = R$. But, clearly, $H_i \nleq UR \leq D$. Thus $R = U = H_i \cap D$ is a Hall σ_i-subgroup of D. Therefore R is the unique minimal normal subgroup of G contained in D.

Now we show that $R \nleq \Phi(G)$. Indeed, assume that $R \leq \Phi(G)$. Then $D \neq R$ by Lemma 2.3 since $D = G^{\mathfrak{N}_\sigma}$. On the other hand, D/R is a $\sigma_i^{'}$-group since $R = U$ is a Hall σ_i-subgroup of D. Hence $O_{\sigma_i^{'}}(D) \neq 1$ by Lemma 2.3. But $O_{\sigma_i^{'}}(D)$ is characteristic in D and so it is normal G. Therefore G has a minimal normal subgroup L such that $L \neq R$ and $L \leq D$. This contradiction shows that $R \nleq \Phi(G)$.

Let S be a maximal subgroup of G such that $RS = G$. Then $|G : S|$ is a σ_i-number. It is also clear that S is not a σ-Hall subgroup of G. Hence S is σ-subnormal in G by Claim (3) and so G/S_G is a σ_i-group, which implies that $R \leq D \leq S_G \leq S$ and so $G = RS = S$. This contradiction completes the proof of (4).

(5) *If $M \leq E < G$, then E is not σ-subnormal in G and so E a σ-Hall subgroup of G.*

Assume that E is σ-subnormal in G. Then G has a proper subgroup V such that $E \leq V$ and G/V_G is σ-primary, so $D \leq V_G$. Hence $V = M(D \cap V) = MD = G$, a contradiction. Hence E is not σ-subnormal in G. By hypothesis, G has a σ-subnormal subgroup W such that E is a σ-Hall subgroup of W. But then $W = G$, so E is a σ-Hall subgroup of G.

(6) *D is soluble, $|\sigma(D)| = |\pi(D)|$ and M acts irreducibly on every M-invariant Sylow subgroup of D.*

Let $p \in \sigma_i \in \sigma(D)$. Lemma 2.6 and Claims (2) and (4) imply that for some Sylow p-subgroup P of G we have $PM = MP$. Moreover, MP is a σ-Hall subgroup of G by Claim (5). Hence $|\sigma_i \cap \pi(G)| = 1$ for all i such that $\sigma_i \cap \pi(D) \neq \emptyset$ and so $|\sigma(D)| = |\pi(D)|$. Therefore D is soluble since G is σ-soluble by Claim (2) and hence M acts irreducibly on every M-invariant Sylow subgroup of D by Claim (5).

(7) *M is a σ-Carter subgroup of G.*

Let R be a minimal normal subgroup of G contained in D and E a subgroup of G containing M. We need to show that $E = E^{\mathfrak{N}_\sigma}M$. Claim (1) implies that RM/R is a σ-Carter subgroup of G/R, so $ER/R = (ER/R)^{\mathfrak{N}_\sigma}(RM/R)$. Hence $ER = E^{\mathfrak{N}_\sigma}MR$ since $(ER/R)^{\mathfrak{N}_\sigma} = E^{\mathfrak{N}_\sigma}R/R$. Claim (6) implies that R is a p-group for some prime p. Claims (4), (5) and (6) imply that R, E and $E^{\mathfrak{N}_\sigma}M$ are σ-Hall subgroups of G. Therefore, if $R \nleq E$, then E and $E^{\mathfrak{N}_\sigma}M$ are Hall p'-subgroups of $ER = E^{\mathfrak{N}_\sigma}MR$, so $E = E^{\mathfrak{N}_\sigma}M$. Finally, assume that $R \leq E$ but $R \nleq E^{\mathfrak{N}_\sigma}M$. Then $R \cap E^{\mathfrak{N}_\sigma} = 1$. On the other hand, since $DE/D \simeq E/D \cap E$ is σ-nilpotent, $E^{\mathfrak{N}_\sigma} \leq D$ and so $M \cap E^{\mathfrak{N}_\sigma} = 1$. Therefore

$$E^{\mathfrak{N}_\sigma} \cap RM = (E^{\mathfrak{N}_\sigma} \cap R)(E^{\mathfrak{N}_\sigma} \cap M) = 1.$$

Then $E/E^{\mathfrak{N}_\sigma} = E^{\mathfrak{N}_\sigma}MR/E^{\mathfrak{N}_\sigma} \simeq MR$ is σ-nilpotent. Hence $M \leq C_G(R)$. Suppose that $C_G(R) < G$ and let $C_G(R) \leq W < G$, where G/W is a chief factor of G. Claim (2) implies that G/W is σ-primary, so $D \leq W$. But then $G = DM \leq W < G$, a contradiction. Therefore $C_G(R) = G$, that is, $R \leq Z(G)$. Let V be a complement to R in D. Then V is a Hall normal subgroup of D, so it is characteristic in D. Hence V is normal in G and $G/V \simeq RM$ is σ-nilpotent, so $D \leq V < D$. This contradiction completes the proof of (7).

(8) *D possesses a Sylow tower.*

Let R be a minimal normal subgroup of G contained in D. Then R is a p-group for some prime p by Claim (6). Moreover, the Frattini argument implies that for some Sylow p-subgroup P of D we have $M \leq N_G(P)$ and so $R = P$ since M acts irreducible on P by Claim (6). On the other hand, by Claim (1), D/R possesses a Sylow tower. Hence we have (8).

(9) *Every chief factor of G below D is σ-eccentric.*

Let H/K be a chief factor of G below D. Then H/K is a p-group for some prime p since D is soluble by Claim (6). By the Frattini argument, there exist a Sylow p-subgroup P and a p-complement E of D such that $M \leq N_G(P)$ and $M \leq N_G(E)$. Then $M \leq N_G(P \cap K)$ and $M \leq N_G(P \cap H)$. Hence $P \cap K = 1$ and $P \cap H = P$ by Claim (6), so $H = K \rtimes P$. Let $V = EM$. Then $K \leq V$ and $HV = G$, so V is a maximal subgroup of G. Hence $G/V_G \simeq (H/K) \rtimes (G/C_G(H/K))$ by Lemma 2.9. Therefore, if H/K is σ-central in G, then $D \leq V_G$, which is impossible since evidently p does not divide $|V|$. Thus we have (9).

From Claims (4)–(9) it follows that G is a $H\sigma E$-group, contrary to our assumption on $G = V$. Hence (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) By hypothesis, $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is a σ-Hall subgroup of G, $|\sigma(D)| = |\pi(D)|$ and M acts irreducibly on every M-invariant Sylow subgroup of D.

(*) *Every subgroup A of G containing M is a σ-Hall subgroup of G.*

Let $D_0 = D \cap A$. Then $A = D_0 \rtimes M$ and $D_0 \neq 1$. Let $p \in \pi(D_0)$. The Frattini argument and Lemma 2.6 imply that for some Sylow p-subgroup P_0 of D_0 and some Sylow p-subgroup P of D we have $M \leq N_G(P_0)$, $M \leq N_G(P)$ and $P_0M \leq PM$. Hence, since M acts irreducibly on every M-invariant Sylow subgroup of D, $P_0 = P$. Therefore every Sylow subgroup of A is a Sylow subgroup of G. Hence A is a σ-Hall subgroup of G since $|\sigma(D)| = |\pi(D)|$ and M is a σ-Hall subgroup of G.

Now, let A be a subgroup of G. First assume that $DA < G$. By Lemma 2.1(6), DA is σ-subnormal in G. Therefore every σ-subnormal subgroup of DA is also σ-subnormal in G. Hence Condition (iii) holds for DA, so A is H_σ-subnormally embedded in DA by induction. But then A is H_σ-subnormally embedded in G by Lemma 2.1(7).

Finally, suppose that $DA = G$. Then, since G is σ-soluble, for some x we have $M \leq A^x$ by Lemma 2.6. Hence A^x is a σ-Hall subgroup of G by Claim (*), so A^x is an H_σ-subnormally embedded subgroup of G. But then A is an H_σ-subnormally embedded subgroup of G. Therefore the implication (iii) $\Rightarrow$ (i) is true.

The theorem is proved.

Proof of Theorem 1.9. (i) $\Rightarrow$ (ii) This follows from Lemma 2.2(4) and Theorems 1.3 and 1.4.

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) Let A be any subgroup of G. Then DA is σ-permutable in G by Lemma 2.2(3) since G is σ-soluble. On the other hand, since $|\sigma(D)| = |\pi(D)|$ and D is a cyclic σ-Hall subgroup of G of square-free order, A is a σ-Hall subgroup of DA. Hence A is H_σ-permutably embedded in G. Therefore the implication (iii) $\Rightarrow$ (i) is true.

The theorem is proved.

Proof of Theorem 1.7. (i) $\Rightarrow$ (ii) In view of Theorem 1.9, it is enough to show that if $D \leq L \leq G$ and L is a σ-Hall subgroup of some normal subgroup V of G, then L is normal in G. But since G/D is σ-nilpotent, L/D is σ-subnormal in G/D, so L is σ-subnormal in G by Lemma 2.1(6). Hence L is σ-subnormal in V by Lemma 2.1(1). But then L is a normal in V by Lemma 2.1(4) and so L is a characteristic subgroup of V. It follows that L is normal in G.

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) See the proof of the implication (iii) $\Rightarrow$ (i) in Theorem 1.9.

The theorem is proved.

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