

Groups with few self-centralizing subgroups which are not self-normalizing

MAHMOUD HASSANZADEH (*) – ZOHREH MOSTAGHIM (**)

ABSTRACT – A self-normalizing subgroup is always self-centralizing, but the converse is not necessarily true. Given a finite group G , we denote by $w(G)$ the number of all self-centralizing subgroups of G which are not self-normalizing. We observe that $w(G) = 0$ if and only if G is abelian, and that if G is nonabelian nilpotent then $w(G) \geq 3$. We also prove that if $w(G) \leq 20$ then G is solvable. Finally, we provide structural information in the case when $w(G) \leq 3$.

MATHEMATICS SUBJECT CLASSIFICATION (2010). 20D25, 20D15, 20E07.

KEYWORDS. self-centralizing subgroup, self-normalizing subgroup, minimal simple group, A-group.

1. Introduction

Let G be a group, and H a subgroup of G . We say that H is a *self-normalizing subgroup* of G if $H = N_G(H)$, the normalizer of H in G . Moreover, H is a *self-centralizing subgroup* of G if $H \subseteq C_G(H)$, the centralizer of H in G . This is equivalent to require that $C_G(H) = Z(H)$, the center of H .

Self-normalizing or self-centralizing subgroups are widely used for recognition on groups. An interesting question is to study groups in which all subgroups not having a given property are self-centralizing. In [2] and [6] locally finite groups, in which all noncyclic subgroups are self-centralizing, are classified.

(*) *Indirizzo dell'A.*: School of Mathematics, Iran University of Science & Technology, Tehran, Iran.

E-mail: mahassanzadeh@mathdep.iust.ac.ir

(**) *Indirizzo dell'A.*: School of Mathematics, Iran University of Science & Technology, Tehran, Iran.

E-mail: mostaghim@iust.ac.ir

A still unsolved problem posed by Berkovich [1, p.439, research problem 9] is to classify finite p -groups in which every nonabelian subgroup is self-centralizing. In [3], Delizia et al. studied such groups and provided some partial answer to the above question. Later, Pavel Zalesskii suggested another related problem: to classify finite groups in which every nonabelian subgroup is self-normalizing. This problem has been solved in [4]. Groups in which every nonnilpotent subgroup is self-normalizing have been considered in [5].

Clearly, every self-normalizing subgroup is self-centralizing. The converse is, in general, false. Our purpose is to study groups having only few self-centralizing subgroups which are not self-normalizing. All groups considered in this paper are finite. Given any group G , we denote by $w(G)$ the number of all self-centralizing subgroups of G which are not self-normalizing. In Section 2 we gather some preliminary results. In Section 3 we prove that $w(G) = 0$ if and only if G is abelian. Furthermore, if G is nilpotent and nonabelian then $w(G) \geq 3$. In Section 4 we show that if $w(G) \leq 20$ then G is solvable. In Section 5 we investigate the structure of finite groups G with $w(G) \leq 3$. In particular, if $w(G) \leq 2$ then G is an A-group, i.e. all Sylow subgroups of G are abelian.

2. Preliminaries

We say that a subgroup H of a group G is a *W-subgroup*, if it is a self-centralizing and non-self-normalizing subgroup, i.e. $C_G(H) \leq H \not\leq N_G(H)$. We denote by $w(G)$ the number of all W-subgroups of G . It is obvious from the definition that the trivial subgroups are not W-subgroups, and that $w(G) = 0$ for all abelian groups G .

PROPOSITION 2.1. *If G is a finite group, then for every subgroup H of G , either $\langle H, C_G(H) \rangle$ is a W-subgroup or $N_G(H) = \langle H, C_G(H) \rangle$.*

In particular, for every abelian subgroup $B \leq G$, either $C_G(B)$ is a W-subgroup or $N_G(B) = C_G(B)$.

PROOF. Let $K = \langle H, C_G(H) \rangle$. Suppose K is not a W-subgroup, thus K is non-self-centralizing or self-normalizing. But K is always self-centralizing, because that $C_G(K) \leq C_G(H) \leq K$.

Thus K is self-normalizing, so $N_G(K) = K$ and we have $N_G(H) \leq N_G(K) = K \leq N_G(H)$, since $N_G(H) \leq N_G(C_G(H))$ and $K = \langle H, C_G(H) \rangle \leq N_G(H)$. Therefore $N_G(H) = K = \langle H, C_G(H) \rangle$.

In particular, if B is any abelian subgroup of G , we have $C_G(B) = \langle B, C_G(B) \rangle$. \square

LEMMA 2.2. *Let G be a finite group and H be a subgroup of G , then*

$$w(H) \leq w(G).$$

If K is a W -subgroup of H , then there exists $\hat{K} \leq G$ such that \hat{K} is a W -subgroup in G and $K = H \cap \hat{K}$.

PROOF. Suppose $w(H) = s$ and K_1, \dots, K_s are pairwise distinct W -subgroups of H . Then by Proposition 2.1 for every K W -subgroup of H , we have $N_H(K) = \langle K, C_H(K) \rangle$ or $\langle K, C_H(K) \rangle = K_i$, for some $i \in \{1, \dots, s\}$.

Let $\hat{K} = \langle K, C_G(K) \rangle$, then it is self-centralizing in G , since $C_G(\hat{K}) \leq C_G(K) \leq \hat{K}$.

Notice that \hat{K} is not self-normalizing in G . Indeed, there exists $h \in N_H(K) \setminus K$, so $h \notin C_H(K)$, hence $h \notin \hat{K}$, but $h \in N_G(\hat{K})$. Since $C_H(K) \leq K$, by the modular law we have

$$\hat{K} \cap H = (K \cdot C_G(K)) \cap H = K \cdot (C_G(K) \cap H) = K \cdot C_H(K) = K.$$

Therefore \hat{K} is a W -subgroup in G and $K = H \cap \hat{K}$.

The subgroups $\hat{K}_1, \dots, \hat{K}_s$ are pairwise distinct, since if $\hat{K}_i = \hat{K}_j$ then $\hat{K}_i \cap H = \hat{K}_j \cap H$, thus $K_i = K_j$ and so $i = j$.

Therefore for W -subgroups K_1, \dots, K_s of H , there exist pairwise distinct W -subgroups $\hat{K}_1, \dots, \hat{K}_s$ of G . Hence $s = w(H) \leq w(G)$. \square

LEMMA 2.3. *Let G be a finite group and N be a normal subgroup of G . Then*

$$w(G/N) \leq w(G).$$

If H/N is a W -subgroup of G/N , then H is a W -subgroup of G .

PROOF. If H/N is a self-centralizing subgroup of G/N , then H is a self-centralizing subgroup of G , since

$$C_G(H)N/N \leq C_{G/N}(H/N).$$

Also H/N is a self-normalizing subgroup of G/N if and only if H is a self-normalizing subgroup of G , since

$$N_{G/N}(H/N) = N_G(H)N/N.$$

Therefore, if $w(G/N) = t$ and $H_1/N, \dots, H_t/N$ are pairwise distinct W-subgroups of G/N , then H_1, \dots, H_t are pairwise distinct W-subgroups of G , thus $w(G/N) \leq w(G)$. \square

COROLLARY 2.4. *If G is a finite group and N is a normal subgroup of G , then*

$$w(N) + w(G/N) \leq w(G).$$

PROOF. Let $w(N) = s$ and $w(G/N) = t$. Let K_1, \dots, K_s be pairwise distinct W-subgroups of N and $H_1/N, \dots, H_t/N$ be pairwise distinct W-subgroups of G/N . By Lemma 2.2 we know that, for all $i = 1, \dots, s$, there exist W-subgroups \hat{K}_i of G such that $K_i = N \cap \hat{K}_i$. Moreover, by Lemma 2.3, H_1, \dots, H_t are W-subgroups of G containing N .

If $\hat{K}_i = H_j$, for some i, j , then $N < H_j = K_i$, so $K_i = N \cap \hat{K}_i = N$, a contradiction. \square

PROPOSITION 2.5. *If G and H are finite groups, then*

$$w(G \times H) \geq w(G) \cdot w(H).$$

PROOF. It is easy to see that:

- $A \times B$ is a self-centralizing subgroup of $G \times H$ if and only if A and B are self-centralizing subgroups of G and H , respectively.
- $A \times B$ is a self-normalizing subgroup of $G \times H$ if and only if A and B are self-normalizing subgroups of G and H , respectively.

Therefore $A \times B$ is a W-subgroup of $G \times H$ if and only if A and B are self-centralizing subgroups of G and H , respectively, and one of them is W-subgroup. So that $G \times H$ has at least $w(G) \cdot w(H)$ W-subgroups. \square

COROLLARY 2.6. *If G and A are finite groups and A is abelian, then*

$$w(G \times A) = w(G).$$

PROOF. Let H be any W-subgroup of G . Then $H \times A$ is a W-subgroup of $G \times A$, as in the proof of Proposition 2.5. So $w(G) \leq w(G \times A)$. Now let K be any W-subgroup of $G \times A$. Then $C_{G \times A}(K) \leq K$. Since A is abelian, we get $A \leq Z(G \times A) \leq C_{G \times A}(K) \leq K$. Write $H = K \cap G$. Then $HA = (K \cap G)A = K$, so $K = H \times A$. Since $H \times A$ is a W-subgroup of $G \times A$, arguing as in the proof of Proposition 2.5 we have that H is a W-subgroup of G , so $w(G \times A) \leq w(G)$. Therefore $w(G \times A) = w(G)$. \square

3. W-subgroups of nilpotent groups

We start by showing that if in a group G , all self-centralizing subgroups of G are self-normalizing i.e. G has no W-subgroups, then it is abelian.

THEOREM 3.1. (*Zassenhaus*) *If, in a finite group G , the normalizer of every abelian subgroup coincides with the centralizer of that subgroup, then the group G is abelian.*

PROOF. See [15, Th. 7]. □

THEOREM 3.2. *Let G be a finite group. Then $w(G) = 0$ if and only if G is abelian.*

PROOF. It is a consequence of Proposition 2.1 and Theorem 3.1. We also give a direct proof.

Let G be a minimal counterexample. By Lemma 2.2, all proper subgroups of G are abelian, hence G is a minimal nonabelian group.

We have two cases: $G' < G$ or $G' = G$.

If $G' < G$, then there exists a maximal subgroup M of G , such that $G' \leq M$, hence M is normal. By minimality of G , M is a maximal abelian subgroup of G , so it is self-centralizing. Thus M is W-subgroup, so $w(G) > 0$, a contradiction.

Assume now that $G' = G$. We show that G is simple. If N is a nontrivial normal subgroup of G , then by Lemma 2.3, $w(G/N) = 0$, so by minimality of G , we have G/N is abelian, hence $G' \leq N$, a contradiction.

Therefore G is simple and minimal nonabelian. By [9], minimal nonabelian groups are non-simple, a contradiction. □

PROPOSITION 3.3. *Let G be a finite nilpotent group, and assume that G is not abelian. Then $w(G) \geq 3$.*

PROOF. Since G is nilpotent, every proper subgroup of G is properly contained in its normalizer in G . Hence every self-centralizing subgroup of G is a W-subgroup of G . Each element of G is contained in a maximal abelian subgroup of G , and G is the union of them. As G is nonabelian, G has at least three maximal abelian subgroups. Clearly, the latter are self-centralizing subgroups of G . Therefore $w(G) \geq 3$. □

COROLLARY 3.4. *Let G be a finite group with $w(G) < 3$. Then all nilpotent subgroups of G are abelian.*

PROOF. Let H be any nilpotent subgroup of G . By Lemma 2.2 we have $w(H) \leq w(G)$. Since H is nilpotent, by Proposition 3.3 we obtain $w(H) = 0$. So H is abelian by Theorem 3.2. \square

4. Groups with at most 20 W-subgroups

In this section we show that finite groups G with $w(G) \leq 20$, are solvable. One can see, for example with GAP [13] that $w(A_5) = 21$, so our bound is sharp.

If G is a nonsolvable group of minimum order with respect to $w(G) \leq 20$, then it is a minimal simple group. A group is called a *minimal simple* group if it is a nonabelian simple group and every proper subgroup of it is solvable. Thompson in 1968 classified these groups:

THEOREM 4.1. [14, Cor.1] *All the finite minimal simple groups (up to isomorphism) are:*

- *The projective special linear group $PSL(2, 2^p)$, where p is a prime number.*
- *The projective special linear group $PSL(2, 3^p)$, where p is an odd prime.*
- *The projective special linear group $PSL(2, p)$, where $p > 3$ is a prime such that $5|p^2 + 1$.*
- *The Suzuki group $Sz(2^p) = {}^2B_2(2^p)$, where p is an odd prime.*
- *The projective special linear group $PSL(3, 3)$.*

Thus we only have to deal with projective special linear groups and Suzuki groups.

PROPOSITION 4.2. *Let $G = PSL(2, q)$ with $q \geq 41$. Then $w(G) \geq 21$.*

PROOF. Let $H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in F_q \right\}$. By an easy calculation, one can see that H is an abelian subgroup of G and $C_G(H) = H$, so H is self-centralizing.

Similarly we have

$$N_G(H) = \left\{ \begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} \mid a, x \in F_q, a \neq 0 \right\}.$$

Thus $N_G(H) \neq H$, hence H is not self-normalizing, therefore H is a W-subgroup.

Moreover $|N_G(H)| = q(q-1)$. But it is well-known that

$$|G| = |PSL(2, q)| = \frac{q(q^2 - 1)}{\gcd(2, q - 1)}.$$

Hence

$$|G : N_G(H)| = \frac{q + 1}{\gcd(2, q - 1)} \geq \frac{q + 1}{2} \geq \frac{41 + 1}{2} = 21.$$

Thus H has at least 21 conjugate subgroups in G , and they are W-subgroups. Therefore $w(G) \geq 21$. \square

PROPOSITION 4.3. *Let $G = Sz(2^{2m+1})$ with $m \geq 1$. Then $w(G) \geq 21$.*

PROOF. By Suzuki's work in 1960 [12, p.3], the Suzuki group $Sz(q)$ has a cyclic subgroup A of order $q - 2r + 1$, where $q = 2^{2m+1}$ and $r = 2^m$, such that A is the centralizer of its non-identity elements. So that A is self-centralizing. Moreover, A is of index 4 in its normalizer, hence A is not self-normalizing. Therefore A is a W-subgroup.

But the order of G is $q^2(q-1)(q^2+1)$ and $q = 2^{2m+1} \geq 8$, so that

$$\begin{aligned} [G : N_G(A)] &= \frac{|G|}{|N_G(A)|} = \frac{|G|}{4|A|} \\ &= \frac{1}{4}q^2(q-1)\frac{q^2+1}{q-2r+1} = \frac{1}{4}q^2(q-1)(q+2r+1) \\ &\geq \frac{1}{4}q^2(q-1) \geq \frac{1}{4}8^2(8-1) \geq 21. \end{aligned}$$

Therefore $[G : N_G(A)] \geq 21$, thus H has at least 21 conjugate subgroups in G , and they are W-subgroups. Therefore G has at least 21 W-subgroups, i.e. $w(G) \geq 21$. \square

THEOREM 4.4. *Every finite group G with $w(G) \leq 20$, is solvable.*

PROOF. Let G be a minimal counterexample. Thus G is nonsolvable and $w(G) \leq 20$. By Lemma 2.2, for any proper subgroup H of G , $w(H) \leq w(G) \leq 20$. Thus, minimality of G implies that H is solvable.

Similarly by Lemma 2.3, we obtain that G/N is solvable, for any proper normal subgroup N of G . Now if G is not simple, then there exists a normal subgroup $1 \neq N < G$, hence N and G/N are solvable, therefore G is solvable, a contradiction.

So that G is a (nonabelian) simple group such that all proper subgroups are solvable, i.e. G is a minimal simple group. Thus, by the Thompson's result [14, Theorem 4.1], we can assume that G is either a projective special linear group or a Suzuki group.

By Proposition 4.3, G cannot be a Suzuki group. So assume that G is a projective special linear group. If $G = PSL(2, q)$, then by Proposition 4.2 we have $q \leq 40$. On the other side, for $4 \leq q \leq 40$ we can easily check, using GAP [13], that $w(G) \geq 21$, a contradiction. Similarly if $G = PSL(3, 3)$ then $w(G) \geq 21$. Therefore G is solvable. \square

5. Groups with few W-subgroups

In this section we describe groups with at most three W-subgroups. From now on, the Fitting subgroup of a group G will be denoted by $F(G)$. First we consider groups G with $w(G) = 1$.

PROPOSITION 5.1. *Let G be a finite group, and assume that $w(G) = 1$. Then $F(G)$ has prime index in G . Moreover $F(G)$ is the unique proper normal self-centralizing subgroup of G , and the unique W-subgroup of G .*

PROOF. Let H_0 be the unique W-subgroup of G . Then all conjugate subgroups of H_0 in G are W-subgroups of G . Since $w(G) = 1$, H_0 is normal. By Lemma 2.3, $w(G/H_0) \leq w(G) = 1$. Thus either G/H_0 is abelian or $w(G/H_0) = 1$.

If $w(G/H_0) = 1$, with W-subgroup H/H_0 , then by Lemma 2.3, H is a W-subgroup of G , so $H = H_0$. But this is a contradiction, because the trivial subgroup $H/H_0 = H_0/H_0$ is not self-centralizing.

Hence G/H_0 is abelian and $G' \leq H_0$. Thus every proper subgroup K of G containing H_0 is normal and self-centralizing (and W-subgroup), since $G' \leq H_0 \leq K$ and upward-closedness property of self-centralizing subgroups.

Therefore H_0 has prime index in G . Moreover H_0 is the unique proper normal self-centralizing subgroup of G . By Theorem 4.4 and Proposition 3.3, G is solvable and not nilpotent. Then $F(G)$ is a proper normal self-centralizing subgroup of G (see, for instance, [11, 7.4.7]). Therefore $F(G) = H_0$, as required. \square

THEOREM 5.2. *Let G be a finite group with $w(G) = 1$, and let $p = |G : F(G)|$. Then $F(G)$ is abelian, and $G = P \rtimes A$, where P is any Sylow p -subgroup and A is an abelian normal p' -subgroup of G . Moreover all Sylow subgroups of G are abelian.*

PROOF. By Proposition 5.1, $|G : F(G)| = p$, a prime number. Write $|G| = p^t p_1^{t_1} \dots p_k^{t_k}$, where $t \geq 1$, $k \geq 1$, p_i is a prime and $t_i \geq 0$ for all $i = 1, \dots, k$. Then $|F(G)| = p^{t-1} p_1^{t_1} \dots p_k^{t_k}$. Let P be a Sylow p -subgroup of G . For all $i = 1, \dots, k$, let P_i be a Sylow p_i -subgroup of $F(G)$. Since $F(G)$ is nilpotent, each P_i is characteristic in $F(G)$ and so it is normal in G . Hence $A = P_1 \times \dots \times P_k$ is a normal p' -subgroup of G . Therefore $G = P \rtimes A$.

By Corollary 3.4, $F(G)$ and all Sylow subgroups of G are abelian. In particular, A is abelian, as required. \square

Now we consider groups G with $w(G) = 2$.

LEMMA 5.3. *If G is a finite group with $w(G) = 2$, then $F(G)$ is W-subgroup and maximal abelian in G .*

PROOF. By Theorem 4.4, G is solvable. So $F(G)$ is a W-subgroup of G and, by Corollary 3.4, $F(G)$ is abelian. Assume there exists an abelian subgroup A of G with $F(G) \leq A$. Then from $F(G) < A$ would follow that $F(G)$ is a W-subgroup of A , a contradiction since $w(A) = 0$. Thus $F(G) = A$, as required. \square

THEOREM 5.4. *Let G be a finite group with $w(G) = 2$, and let $H_1 = F(G)$ and H_2 be the two W-subgroups of G . Then H_2 is a normal subgroup of G having prime index q in G , and one of the following holds:*

- (1) H_1 has prime index p in G , and $G = P \rtimes A$, where P is a Sylow p -subgroup of G and A is an abelian normal p' -subgroup of G ;
- (2) $H_1 < H_2$, H_1 has prime index $p \neq q$ in H_2 , and $G = (P \times Q) \rtimes A$, where P is a Sylow p -subgroup of G , Q is a Sylow q -subgroup of G , and A is an abelian normal $\{p, q\}'$ -subgroup of G .

PROOF. By Theorem 4.4, G is solvable. Any conjugate subgroup of H_2 in G is a W-subgroup of G . Since H_1 is normal in G and $w(G) = 2$, it follows that H_2 is also normal in G . Moreover H_1 is abelian by Corollary 3.4.

Note that the case $H_2 < H_1 = F(G)$ can not occur, since $F(G)$ is a maximal abelian subgroup of G by Lemma 5.3, so it is minimal self-centralizing [8, Proposition 2]. Hence, by Lemma 2.3, $w(G/H) = 0$, and thus G/H_2 is abelian by Theorem 3.2. Moreover, since H_2 is self-centralizing, every proper subgroup of G containing H_2 is a W-subgroup of G . It follows that G/H_2 is simple, so it has prime order, say q .

Since H_1 is also self-centralizing, every proper normal subgroup of G containing H_1 is a W-subgroup of G . Hence either G/H_1 is simple or $H_1 < H_2$.

In the former case $|G : H_1| = p$, a prime number. Thus, arguing as in the proof of Theorem 5.2, we obtain $G = P \times A$, where P is a Sylow p -subgroup of G and A is an abelian normal p' -subgroup of G . Therefore (1) holds.

In the latter case, by Lemma 2.2, $w(H_2) = 1$ and H_1 is the unique W-subgroup of H_2 . It follows, by Theorem 5.2, that $H_1 = F(H_2)$ and $|H_2 : H_1|$ is a prime number, say p . Hence $|G : H_1| = pq$. Since H_2/H_1 is a W-subgroup of G/H_1 , by Theorem 3.2 G/H_1 is not abelian. It follows that $p \neq q$. Write $|G| = p^\alpha q^\beta r_1^{\gamma_1} \dots r_k^{\gamma_k}$, where α and β are positive integers, r_i is a prime number different from p and q , and γ_i is a non-negative integer, for all $i = 1, \dots, k$. Thus $|H_2| = p^\alpha q^{\beta-1} r_1^{\gamma_1} \dots r_k^{\gamma_k}$ and $|H_1| = p^{\alpha-1} q^{\beta-1} r_1^{\gamma_1} \dots r_k^{\gamma_k}$. For all $i = 1, \dots, k$, let R_i denote a Sylow r_i -subgroup of H_1 . Since H_1 is abelian, each R_i is normal in G . Hence $A = R_1 \times \dots \times R_k$ is an abelian normal $\{p, q\}'$ -subgroup of G . As $|G : A| = p^\alpha q^\beta$, by Schur-Zassenhaus Theorem (see, for instance, [11, 9.3.6]) G has a subgroup D of order $p^\alpha q^\beta$. Thus $G = D \times A$. Let P and Q be a Sylow p -subgroup and a Sylow q -subgroup of D , respectively. Since

$$|PQ| = \frac{|P| \cdot |Q|}{|P \cap Q|}$$

we have $PQ = D$. Hence P and Q are permutable. Therefore $D = P \times Q$, and (2) holds. \square

Finally, we consider groups G having exactly three W-subgroups. Again, G is solvable by Theorem 4.4. Our first result shows that if G is a p -group with $w(G) = 3$, then $p = 2$.

PROPOSITION 5.5. *Let G be a finite p -group. The following conditions are equivalent:*

- (1) $w(G) = 3$;
- (2) $G/Z(G) \cong C_2 \times C_2$;

(3) $|G : Z(G)| = 4$.

In particular, $p = 2$.

PROOF. Arguing as in the proof of Proposition 3.3 we obtain that $w(G) = 3$ if and only if G has exactly three maximal abelian subgroups, and of course it is the union of them. Thus our statement follows from well-known results due to Scorza [10] (see also [7]). \square

PROPOSITION 5.6. *If G is a finite group with $w(G) = 3$, then all Sylow subgroups of odd order of G are abelian and G has a Sylow 2-subgroup P such that P is abelian or $|P : Z(P)| = 4$.*

PROOF. If P is a Sylow p -subgroup of G , then by Lemma 2.2, $w(P) \leq w(G) = 3$. But P is nilpotent and by Proposition 3.3, $w(P) = 0$ or 3 . Therefore by Theorem 3.2 and Proposition 5.5, P is abelian or $|P : Z(P)| = 4$. \square

PROPOSITION 5.7. *Let G be a finite nilpotent group. Then $w(G) = 3$ if and only if $G = P \times A$, where P is a 2-group with $w(P) = 3$ and A is an abelian group of odd order.*

PROOF. Let $w(G) = 3$. Then G is not abelian, so by Proposition 5.6 the Sylow 2-subgroup P of G is not abelian. Hence G has the required structure. Conversely, let $G = P \times A$, where P is a 2-group with $w(P) = 3$ and A is an abelian group of odd order. Then $w(G) = 3$ by Corollary 2.6. \square

Similar to Theorems 5.2 and 5.4 we have:

THEOREM 5.8. *Let G be a finite group with exactly three W -subgroups. Then they are normal in G .*

Moreover, let G be nonnilpotent. Then the Fitting subgroup $F(G)$ is W -subgroup and maximal abelian in G .

PROOF. If G is a p -group then by Proposition 5.5 the W -subgroups of G have index $p = 2$ and thus they are normal.

If G is a nilpotent group, similar to the p -group case and by Proposition 5.7 the W -subgroups of G have index 2.

Let G be a nonnilpotent group. Then $F(G)$ is a normal and self-centralizing subgroup of G , thus it is a W -subgroup of G . Assume that H_1, H_2 and $F(G)$ are distinct W -subgroups of G .

If H_1 is not normal in G , then the conjugate subgroups of H_1 are W-subgroups of G . Since $w(G) = 3$ and by normality of $F(G)$, the number of conjugate subgroups of H_1 equals 2. Hence $|G : N_G(H_1)| = 2$ and so $N_G(H_1)$ is normal in G . Therefore, $N_G(H_1)$ is a W-subgroup of G , since it is a proper, normal and self-centralizing subgroup in G .

But $w(G) = 3$ so $N_G(H_1) = F(G)$ a contradiction, because that by the proof of Theorem 5.4, $F(G)$ is a maximal abelian subgroup of G , so it is minimal self-centralizing.

Therefore H_1 is normal in G . Similarly H_2 is normal in G . □

Acknowledgement

The authors are grateful to the anonymous referee for careful reading of the manuscript and for helpful suggestions.

REFERENCES

- [1] Y. BERKOVICH, Groups of prime power order, Vol. 1, Walter de Gruyter, Berlin, 2008.
- [2] C. DELIZIA – U. JEZERNIK – P. MORAVEC – C. NICOTERA, *Groups in which every noncyclic subgroup contains its centralizer*, J. Algebra Appl. **13**, no. 5 (2014), 1350154 (11p).
- [3] C. DELIZIA – H. DIETRICH – P. MORAVEC – C. NICOTERA, *Groups in which every non-abelian subgroup is self-centralizing*, J. Algebra **462** (2016), pp. 23-36.
- [4] C. DELIZIA – U. JEZERNIK – P. MORAVEC – C. NICOTERA, *Groups in which every non-abelian subgroup is self-normalizing*, Monatsh Math, **185** no. 4 (2018), pp. 591-600.
- [5] C. DELIZIA – U. JEZERNIK – P. MORAVEC – C. NICOTERA, *Groups in which every non-nilpotent subgroups is self-normalizing*, Ars Math. Contemp. **15**, no. 1 (2018), pp. 39-51.
- [6] C. DELIZIA – U. JEZERNIK – P. MORAVEC – C. NICOTERA – C. PARKER, *Locally finite groups in which every non-cyclic subgroup is self-centralizing*, J. Pure Appl. Algebra **221** no. 2 (2017), pp. 401-410.
- [7] S. HABER – A. ROSENFELD, *Groups as Unions of Proper Subgroups*, Amer. Math. Monthly **66** no. 6 (1959), pp. 491 - 494.
- [8] M. Hassanzadeh – Z. Mostaghim, *On maximal non-self-centralizing subgroups*, Preprint 2017 [Quasigroups and Related Systems].
- [9] G. A. MILLER – H. C. MORENO, *Non-abelian groups in which every subgroup is abelian*, Trans. Amer. Math. Soc. **4** (1903) pp. 398-404.

- [10] G. SCORZA, *I gruppi che possono pensarsi come somma di tre loro sottogruppi*, Boll. Un. Mat. It. **5** (1926), pp. 216-218.
- [11] W. R. SCOTT, *Group theory*, Dover Publications, 1964.
- [12] M. SUZUKI, *A new type of simple groups of finite order*, Proc. Natl. Acad. Sci. USA **46** (1960), pp. 868 - 870.
- [13] The GAP Group, *GAP-Groups, Algorithms, and Programming*, Version 4.7.9; 2015 (<http://www.gap-system.org>).
- [14] J. G. THOMPSON, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. **74** (1968), pp. 383 - 437.
- [15] H. J. ZASSENHAUS, *A group-theoretic proof of a theorem of MacLagan-Wedderburn*, Proc. Glasgow Math. Assoc. **2**, no. 1(1952), pp. 53-63.