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A Monodromy Criterion for the Good Reduction of $K3$ Surfaces

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Abstract – We give a criterion for the good reduction of semistable $K3$ surfaces over p-adic fields. We use neither p-adic Hodge theory nor transcendental methods as in the analogous proofs of criteria for good reduction of curves or $K3$ surfaces. We achieve our goal by realizing the special fiber X_s of a semistable model X of a $K3$ surface over the p-adic field K, as a special fiber of a log-family in characteristic p and use an arithmetic version of the Clemens-Schmid exact sequence in order to obtain a Kulikov-Persson-Pinkham classification theorem in characteristic p.

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1. Introduction

Let $p > 0$ be a prime integer and K a finite extension of $\mathbb{Q}_p$. Consider a smooth, proper and geometrically irreducible scheme X_K over Spec K. The question of whether X_K has good reduction or not can be answered via ℓ-adic or p-adic criteria in some cases. For example, if $X_K = A_K$ is an abelian scheme and G_K the absolute Galois group of K we have that A_K has good reduction if and only if for all $\ell \neq p$ (equivalently, for some $\ell \neq p$), the ℓ-adic G_K-representation $T_\ell(A_K)$ is unramified (see [24, theorem 1]). The p-adic criterion says that A_K has good reduction if and only if the p-adic G_K-representation $T_p(A_K)$ is crystalline (see [6, Theorem II.4.7] and [2, Corollaire 1.6]). Recall that in general for semistable p-adic representations

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this is equivalent to its associated (φ, N)-module having trivial monodromy operator.

For more general varieties the criteria from the preceding paragraph are not valid, but in some cases, different criteria can be obtained. For example if X_K is a proper algebraic curve with semistable reduction, Oda (in [21, Section 3]) obtained an ℓ-adic criterion looking at the Galois action on the étale fundamental group (via an analogous trascendental result) and Andreatta-Iovita-Kim ([1, Theorem 1.6]) obtained its p-adic version studying the monodromy action on the de Rham unipotent fundamental group (via p-adic Hodge Theory). This means that it is not enough to look at the first cohomology group with its Galois/monodromy action but one needs to look at the whole (unipotent) fundamental group (i.e., not only its abelianization).

In this article we obtain a p-adic criterion for $K3$ surfaces. Namely we suppose that $p > 3$ and X_K is a smooth, projective $K3$ surface over Spec K having a minimal semistable model X over the ring of integers O_K of K. We may assume to have combinatorial reduction (see [20, Proposition 3.5]). Then since we are dealing with $K3$ surfaces, the first de Rham cohomology group is trivial, as well as the connected de Rham fundamental group. Therefore we look at the monodromy action N on the second de Rham cohomology group $H^2_{\mathrm{DR}}(X_K)$, which is described in the framework of the theory of log-schemes and log-crystalline cohomology (see [8, Theorem 5.1]). Then our result is the following:

Theorem 1.1. *Under the hypotheses above, the $K3$ surface X_K has good reduction if and only if the monodromy N is zero on $H^2_{DR}(X_K)$.*

In fact we obtain more than that. We know that the operator N is always nilpotent (N^3 is always trivial). In the case the reduction of X_K is not good we can refine the theorem just stated: the type of bad reduction is determined by the order of nilpotency of N. For the complete result, see theorem 6.1.

One can also note that this criterion for good reduction in terms of the monodromy operator on log-crystalline cohomology can also be formulated in "étale terms". Namely, X_K has good reduction if and only if $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline representation. This is a consequence of our criterion and the comparison theorem [27, Theorem 0.2].

In the classical situation (over the complex numbers), given a semistable degeneration of $K3$ surfaces the work of Kulikov [13], Persson-Pinkham [23] and Morrison [18] show how the monodromy action on the generic fiber determines the behavior of the special one. To prove this one uses all the information coming from the structure of the family: the weight-monodromy conjecture and the Clemens-Schmid exact sequence. Our proof has been inspired by these methods.

The monodromy on the de Rham cohomology of X_K is given by the monodromy operator on the log-crystalline cohomology of the special fiber X_s (which is a characteristic p scheme) endowed with the induced log-structure (see for example [8, Theorem 5.1]). Using Nakkajima's results on deformations of $K3$ surfaces ([20]) we may construct a log-smooth deformation of our special fiber over the ring of formal power series $k[[t]]$, where k is the residue field of K. Then using Popescu's version of Artin's approximation theorem we get a deformation of X_s over a smooth scheme Y over $k[t]$ (possibly of dimension larger than 1). Finally by taking a well-chosen curve inside Y we are reduced to the case of a family over a smooth curve, so we can use Chiarellotto-Tsuzuki's results for this setting. In particular for such a family we can use the weight-monodromy conjecture and the existence of a Clemens-Schmid type exact sequence. This gives the elements to rephrase Kulikov-Persson-Pinkham's and Morrison's results in characteristic p, allowing us to get our main theorem (theorem 6.1) which is similar to the one obtained by Pérez Buendía in [22].

This method of proof is different to the one used by Matsumoto in [17], who also obtains results similar to ours, but for a different case, allowing algebraic spaces as models and always working with the generic fiber.

One might have hoped to use our methods to study the case of semistable Enriques surfaces. More precisely, we work with a semistable model, under some hypothesis about the canonical bundle over the DVR. This is compatible with [16], where again they study only the generic fiber. Then, we can follow our techniques along the lines we used for $K3$ surfaces since again by Nakkajima's work [20] we have a classification of the possible special fibers (see the beginning of our section 6 to see the complete classification). But when we apply Morrison's methods we find that the monodromy is zero on the second cohomology group without any connection with the fact that the special fiber is smooth or not. We plan to investigate this problem in another article (see also the parallel work [16]).

Let us give an outline of this article. In section 2 we establish our notation and setting. This includes the theory of log deformations that

we need.
In section 3, we use Néron-Popescu desingularization (see [25, Theorem 1.1]) to write the ring of formal power series $k[[t]]$ as a limit of smooth $k[t]$-algebras:

$$k[[t]] = \varinjlim_{\alpha} A_\alpha,$$

and this allows, in a similar way to what is done in [9, Section 4], to see our situation as a fiber inside a larger family of varieties $f : X_A \to Y = \text{Spec } A$, where $A = A_\alpha$ for some α.

Then, in section 4 we construct a smooth curve C inside Y in such a way that we can restrict the family $X_A \to Y$, to a smaller family X_C over this curve. In particular this allows us to use the main result in [5], which is a Clemens-Schmid exact sequence in characteristic p:

$$\cdots \to H^m_{rig}(X_s) \to H^m_{log-crys}((X_s, M_s)/W^\times) \otimes K_0 \xrightarrow{N}$$
$$H^m_{log-crys}((X_s, M_s)/W^\times) \otimes K_0(-1) \to H^{m+2}_{X_s,rig}(X_C) \to H^{m+2}_{rig}(X_s) \to \cdots$$

In section 5 we use this Clemens-Schmid exact sequence to get criteria for N to be the zero map on $H^1_{log-crys}$ or $H^2_{log-crys}$, assuming that we are dealing with a semistable family of varieties over a smooth curve over a finite field. For this we use the fact that the monodromy and weight filtrations on the special fiber coincide (as proved in [5], using the fact that we deal with a family of varieties). For a more general situation, i.e., if we do not assume that the special fiber is inside a semistable family of varieties, this is known only for the case of curves and surfaces (see [19, Sections 5 and 6]). The criteria that we get in this section are in terms of the Betti numbers of the dual graph of the special fiber which can be easily described in the case of combinatorial reduction. As we mentioned before we can always restrict ourselves to this case after a finite base extension.

Finally in section 6, we apply the criteria from section 5 to the case of $K3$ surfaces, assuming that the special fiber is combinatorial, i.e., it is one of three possible types. We obtain that the degree of nilpotency determines the type of degeneracy of the special fiber. Our main result will be stated in theorem 6.1: the trivial monodromy action on the second de Rham cohomological group is equivalent to good reduction.

2. Notation and setting

In this article we fix a prime integer $p > 3$. Let K be a finite extension of $\mathbb{Q}_p$ and denote by O_K its ring of integers, π a uniformizer of O_K and k its residue field. For a finite field k, we denote by $W = W(k)$ its ring of Witt vectors and K_0 its field of fractions. We shall denote by the same letter W the formal scheme Spf W with the trivial log structure, and we denote by $W^\times$ the same formal scheme with the log structure given by $1 \mapsto 0$.

We consider a smooth, projective $K3$ surface X_K over K, i.e., a smooth, projective surface with trivial canonical sheaf and irregularity 0. Moreover, we assume that X_K has a semi-stable model $\mathfrak{X} \to \operatorname{Spec} O_K$, i.e., $\mathfrak{X}$ is a proper scheme over O_K, étale locally étale over a scheme of the form

$$\operatorname{Spec}\ (O_K[x_1, ..., x_n]/(x_1 \cdots x_r - \pi)).$$

Let $X_s := \mathfrak{X} \otimes_{O_K} k$ be the special fiber of $\mathfrak{X}$ and assume it is a combinatorial $K3$ surface. In particular, we may assume that we are in one of the following cases:

I) X_s is a smooth $K3$ surface over k

II) $X_s = X_0 \cup X_1 \cup \cdots \cup X_{j+1}$ is a chain of smooth surfaces, with X_0, X_{j+1} rational and the others are elliptic ruled and double curves on each of them are rulings.

III) $X_s = X_0 \cup X_1 \cup \cdots \cup X_{j+1}$ is a chain of smooth surfaces and every X_i is rational, and the double curves on X_i are rational and form a cycle on X_i. The dual graph of X is a triangulation of the sphere S^2.

We shall refer to each of these as surface of type I, II and III, respectively.

Remark 2.1. The definition of a combinatorial $K3$ surface [20, Definition 3.2] requires for the cases II) and III) that the geometric special fiber $X_{\overline{s}}$ has a decomposition of those types and not necessarily X_s, but this implies that there exists a finite extension k' of k such that the base change $X_{k'} = X_s \otimes_k k'$ has such decomposition. Since k' is again a finite field, we may assume that it is X_s the one that admits such decomposition.

Remark 2.2. The condition of having a combinatorial special fiber is achieved for example if the model satisfies $\omega_{\mathfrak{X}/O_K} = 0$. If we take any general semistable model of X_K, one can apply Kawamata's minimal model program (only in the case $p > 3$, see [11],[12]) and we get the same situation, but with a model which is an algebraic space and not necessarily a scheme (even if the fibers are schemes themselves). The details of this can be found in [17].

Remark 2.3. To study surfaces of type II we use the fact that for a smooth, proper, rational surface Y over a field (such as X_0 and X_{j+1}), we have $H^1_{rig}(Y)$ is zero. Indeed, first note that since Y is smooth and proper over a field, then Y is necessarily projective (see [15, Remark 3.5, Ch.9]). Then, we use Castelnuovo-Zariski's criterion in characteristic p as stated in [14, Theorem 4.6] to get that the first ℓ-adic étale cohomology group is trivial. Since Y is smooth and proper, we conclude that the dimension of the first rigid cohomology group is also 0, since rigid cohomology is a Weil cohomology (see [3]).

Recall from [20, Section 2] that the special fiber X_s can be endowed with a log structure M_s in such a way that we have a log smooth morphism $(X_s, M_s) \to (\text{Spec } k, \mathbb{N}^m)$, where m is the number of connected components of the singular locus of X_s and the log structure is defined by $e_i \mapsto 0$, where e_i denotes the ith canonical generator of $\mathbb{N}^m$.

Let us make an explicit description of M_s. In general, suppose that Y is a normal crossing variety and denote by Y_{sing} the singular locus. Denote by $Y_1, ..., Y_m$ the connected components. For each $i = 1, ..., m$, we can endow Spec $(k[x_0, ..., x_n]/(x_0 \cdots x_r))$ with a log structure given by as follows:

$$\mathbb{N}^{m+r} = \mathbb{N}^{i-1} \oplus \mathbb{N}^{r+1} \oplus \mathbb{N}^{m-i} \to k[x_0, ..., x_n]/(x_0 \cdots x_r)$$

$$e_i \mapsto \begin{cases} 0 & \text{if } e_i \in \mathbb{N}^{i-1} \\ x_{i-1} & \text{if } e_i \in \mathbb{N}^{r+1} \\ 0 & \text{if } e_i \in \mathbb{N}^{m-i} \end{cases}$$

Then,

(1) If x is a smooth point of Y, étale locally on a neighbourhood of x, the log structure is the pull-back of the log structure of the log-point (Spec $k, \mathbb{N}^m$)

(2) If $x \in Y_i$, étale locally on a neighbourhood of x, the log structure is the pull-back of the log structure defined above.

Since X_s is in particular a normal crossing variety over k, we can endow X_s with this log structure and we denote it by M_s. Note that this is not the usual log structure defined for example in [10, Example 3.7 (2)], which we denote here by M'_s. As it is stated in [20, p.358], the relationship between them is

$$(X_s, M'_s) = (X_s, M_s) \times_{(\text{Spec } k, \mathbb{N}^m)} (\text{Spec } k, \mathbb{N}),$$

where the morphism of log schemes $(\mathrm{Spec}\ k, \mathbb{N}) \to (\mathrm{Spec}\ k, \mathbb{N}^m)$ is defined by $\mathbb{N}^m \to \mathbb{N}$ the sum of the components. Moreover, the sheaves of relative log differentials $\omega^\bullet_{(X_s,M_s)/(\mathrm{Spec}\ k,\mathbb{N}^m)}$ and $\omega^\bullet_{(X_s,M'_s)/(\mathrm{Spec}\ k,\mathbb{N})}$ coincide, and there is also a canonical isomorphism

$$H^i_{log-crys}((X_s, M_s)/(W, \mathbb{N}^m)) \cong H^i_{log-crys}((X_s, M'_s)/W^\times),$$

as stated and proved in [20].

Assume for the moment that X_s is either of type I), type III) or type II) and such that the double curve is ordinary. Then by [20, Corollary 5.4, Proposition 5.9], there exists a semistable family $X^{\log}$ over $\mathrm{Spec}\ k[[t]]^{\log}$ such that its special fiber is precisely (X_s, M_s). Then we have the following diagram, with cartesian squares:

$$\begin{tikzcd}
(X_s, M'_s) \arrow[r] \arrow[d] & (X_s, M_s) \arrow[r] \arrow[d] & X^{\log} \arrow[d] \\
s^\times \arrow[r] & (\mathrm{Spec}\ k, \mathbb{N}^m) \arrow[r] & \mathrm{Spec}\ k[[t]]^{\log}
\end{tikzcd} \tag{1}$$

where $s^\times$ denotes the log scheme $(\mathrm{Spec}\ k, \mathbb{N})$ with the log structure given by $1 \mapsto 0$.

3. A construction using Néron-Popescu desingularization

In order to get the desired result, we need to study the cohomology of the special fiber X_s of X over $k[[t]]$ and for this we first use the following theorem of Popescu (see [25, Theorem 1.1]):

Theorem 3.1. *Let $f : R \to \Lambda$ be a morphism of rings. Then, f is geometrically regular if and only if Λ is a filtered colimit of smooth R-algebras.*

We prove the following:

Theorem 3.2. *The natural morphism $k[t] \to k[[t]]$ is geometrically regular, as defined in* [25].

Proof. It is clearly flat since it is a completion. Now there are only two prime ideals of $k[[t]]$. Namely, 0 and (t), and their respective counterpart in $k[t]$ are the only couples to consider in the definition.

Case 1 (the ideals generated by t): in this case, we need to check that $k \otimes_{k[t]} k[[t]]_{(t)} \cong k$ is geometrically regular over k, which is trivial.

Case 2 (the ideals 0): in this case, we need to check that $k(t) \otimes_{k[t]} k((t))$ is geometrically regular over $k(t)$. Take a nontrivial finite extension k' of $k(t)$ such that $(k')^p \subset k(t)$. Note that this is necessarily $k(t^{1/p})$. Indeed, it is a finite extension of degree p (hence it does not have any subextension) and $(k')^p = k(t)$. Then, we only need to check that $k(t^{1/p}) \otimes_{k(t)} (k(t) \otimes_{k[t]} k((t)))$ is a regular local ring, but

$$k(t^{1/p}) \otimes_{k(t)} (k(t) \otimes_{k[t]} k((t))) \cong k((t^{1/p})),$$

which is clearly regular. □

In particular,

$$k[[t]] = \varinjlim_{\alpha} A_\alpha, \tag{2}$$

where the A_α's are smooth $k[t]$-algebras. One can say even more:

Proposition 3.3. *Let $\overline{k}$ be an algebraic closure of k and A a smooth $\overline{k}[t]$-algebra. Then, there exists a finite extension k' of k and a smooth $k'[t]$-algebra A' such that $A' \otimes_{k'} \overline{k} \cong A$.*

Proof. Let us take a presentation of A of the type $\overline{k}[t][x_1, ..., x_n]/(f_1, ..., f_c)$. Since $f_1, ..., f_c$ are a finite number of polynomials, one needs a finite number of elements of $\overline{k}$ to define them in the variables $t, x_1, ..., x_n$. Let k' be a finite extension of k containing all of those coefficients and define $A' := k'[t][x_1, ..., x_n]/(f_1, ..., f_c)$. We only need to assure that A' is smooth over $k'[t]$. This is a direct consequence of corollary 17.7.3, part ii), in EGAIV. □

Since X is proper over $k[[t]]$, by (2) there exist a smooth $k[t]$-algebra A, a scheme X_A, proper over Spec A, étale locally étale over

$$\operatorname{Spec} A[x_1, ..., x_n]/(x_1 \cdots x_r - t),$$

and such that the following diagram is cartesian:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "F"'] & X_A \arrow[d, "f"] \\ \operatorname{Spec} k[[t]] \arrow[r, "v"'] & \operatorname{Spec} A \end{tikzcd} \tag{3}$$

Moreover, we may assume that $f : X_A \to \mathrm{Spec}\ A$ is flat.

Note that the divisor of $Y = \mathrm{Spec}\ A$, defined by $Y_0 = (t = 0)$, is an NCD; and the fiber product $X_{A,t=0} = Y_0 \times_Y X_A$ is an NCD in X_A. Then, we can naturally endow X_A and Y with fine log structures M_A and N, respectively. The map f is then a morphism of log schemes, and moreover, we have the following:

Lemma 3.4. *The morphism $f : (X_A, M_A) \to (Y, N)$ is log-smooth.*

Proof. We use [10, Theorem 3.5]. First note that f has (étale locally on X_A) a chart $(P_{X_A} \to M_A, Q_Y \to N, Q \to P)$ given by $Q = \mathbb{N}$, $P = \mathbb{N}^r$, and the diagonal map $Q \to P$.

We can easily see also that the kernel and the torsion part of the cokernel of $Q^{gp} \to P^{gp}$ (which is just the diagonal map $\mathbb{Z} \to \mathbb{Z}^r$) are both trivial.

It remains to prove that the induced morphism

$$X_A \to Y \times_{\mathrm{Spec}\ \mathbb{Z}[Q]} \mathrm{Spec}\ \mathbb{Z}[P]$$

is smooth. Recall that X_A is locally étale over

$$V = \mathrm{Spec}\ (A[x_1, ..., x_n]/(x_1 \cdots x_r - t)),$$

and note that

$$\begin{aligned} \mathrm{Spec}\ A \times_{\mathrm{Spec}\ \mathbb{Z}[Q]} \mathrm{Spec}\ \mathbb{Z}[P] &\cong \mathrm{Spec}\ A \times_{\mathrm{Spec}\ \mathbb{Z}[u]} \mathrm{Spec}\ \mathbb{Z}[u_1, ..., u_r] \\ &\cong \mathrm{Spec}\ (A[u_1, ..., u_r]/(u_1 \cdots u_r - t)) =: U. \end{aligned}$$

Now note that there are natural closed immersions $j_V : V \hookrightarrow \mathbb{A}^n_A$, and $j_U : U \hookrightarrow \mathbb{A}^r_A$. Moreover, the following diagram is cartesian:

$$\begin{tikzcd} V \arrow[r, "j_V"] \arrow[d, "h"'] & \mathbb{A}^n_A \arrow[d, "p"] \\ U \arrow[r, "j_U"'] & \mathbb{A}^r_A \end{tikzcd}$$

where h is defined by sending each u_i to x_i for $i = 1, ..., r$, and p is the natural projection from the first r components. Since p is smooth, we get that h (and consequently the map $X_A \to U$) is smooth (in the classical sense). $\square$

Thus, we have the following diagram of log schemes:

$$\begin{tikzcd}
(X_s, M'_s) \arrow[r] \arrow[d, "f_s"'] & X^{\log} \arrow[r] \arrow[d] & (X_A, M_A) \arrow[d, "f"] \\
s^{\times} \arrow[r] & \mathrm{Spec}\ k[[t]]^{\log} \arrow[r] & (Y, N)
\end{tikzcd}$$

Since s is a closed point inside Y, then (X_s, M'_s) is a fiber of the log smooth family $(X_A, M_A) \to (Y, N)$. In the next section we shall see that (X_s, M'_s) can be seen as a fiber of a family defined over a smooth curve.

4. Reduction to the case of a family over a curve

By [7, Theorem 7, Section 4], there exists a $W[t]$-algebra $\widetilde{A}$ which is smooth over W and such that $\widetilde{A}/p\widetilde{A} = A$. Let $\hat{A}$ be the p-adic completion of $\widetilde{A}$, and $\mathcal{Y} = \mathrm{Spf}\ \hat{A}$. We can define a log structure $\mathcal{N}$ on $\mathcal{Y}$ by $1 \mapsto t$, and then we have the following diagram:

$$\begin{tikzcd}
(X_s, M'_s) \arrow[r] \arrow[d, "f_s"'] & (X_A, M_A) \arrow[d, "f"] & \\
s^{\times} \arrow[r] & (Y, N) \arrow[r] & (\mathcal{Y}, \mathcal{N})
\end{tikzcd} \tag{4}$$

where the lower row consists of two exact closed immersions. This allows us to use all the machinery of log crystalline, log convergent and log analytic cohomology from [26]. More particularly, we have relative cohomology sheaves defined for the family over Y, which we shall restrict to a smaller family. Namely a family over a curve, in order to be in the same situation as in [5].

Let us first construct the curve that we shall use. As stated at the beginning of the preceding section, $\widetilde{A}$ is a smooth W-algebra. Let $\widetilde{Y} = \mathrm{Spec}\ \widetilde{A}$ and $S = \mathrm{Spec}\ W$. Since $Y \to \widetilde{Y}$ is a closed immersion, the image $\hat{s}$ of s inside $\widetilde{Y}$ is a closed point. Since the natural morphism $\widetilde{Y} \to S$ is smooth, there exists an affine open neighborhood $\widetilde{U}$ of $\hat{s}$ and an étale morphism $\sigma : \widetilde{U} \to \mathbb{A}^d_W$ such that $\widetilde{U} \to S$ factors in the following way:

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\sigma"] \arrow[d] & \mathbb{A}^d_W \arrow[dl] \\ S & \end{tikzcd}$$

Let us recall this construction. There exists an open affine subset $\widetilde{U} = \operatorname{Spec} \widetilde{A}_g$ of $\widetilde{Y}$ such that the restriction of $\widetilde{Y} \to S$ is standard smooth. Moreover, we may assume (using the fact that the reduction modulo p is smooth over $k[t]$) that we can write

$$\widetilde{A}_g = W[x_1, ..., x_r, t]/(f_1, ..., f_c),$$

where the polynomial

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_c}{\partial x_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_1}{\partial x_c} & \cdots & \frac{\partial f_c}{\partial x_c} \end{bmatrix}$$

is invertible in $\widetilde{A}_g$. Then, the morphism $W[x_{c+1}, ..., x_r, t] \to \widetilde{A}_g$ is étale, and with $d = r + 1 - c$ we get the desired factorization.

Using this description it is clear how to construct a smooth curve C_W inside $\widetilde{U}$, transversal to $(t = 0)$ and passing through the point $\hat{s}$: by pulling back a curve with these properties inside $\mathbb{A}^d_W$. In particular its reduction C modulo p is a smooth curve inside Y, transversal to $(t = 0)$ and passing throught the point s.

Let N_C be the log structure on C defined to make the closed immersion $(C, N_C) \to (Y, N)$ exact, and then we have a sequence of exact closed immersions

$$s^\times \to (C, N_C) \to (Y, N).$$

Let $(X_C, M_C) = (X_A, M_A) \times_{(Y,N)} (C, N_C)$. Then, we have the following diagram, where all the squares are cartesian:

$$\begin{tikzcd} (X_s, M'_s) \arrow[r] \arrow[d] & (X_C, M_C) \arrow[r] \arrow[d] & (X_A, M_A) \arrow[d] \\ s^\times \arrow[r] & (C, N_C) \arrow[r] & (Y, N) \end{tikzcd} \tag{5}$$

In particular, we are now in the situation where we can use the Clemens-Schmid exact sequence in characteristic p from [5].

5. Monodromy Criteria

As an application of the p-adic version of the Clemens-Schmid exact sequence, we prove a p-adic version of the Monodromy Criteria [18, p.112]. We start with a general situation in which he have an exact sequence of Clemens-Schmid type, as in [5]. Namely, given a finite field k and a smooth curve C over k, we consider a proper and flat morphism

$$g : X_C \to C,$$

where X_C is a smooth variety of dimension $n+1$ over k. Moreover, we assume that there exists a k-rational point $s \in C$ such that the fiber of g at s, which we denote by X_s is an NCD. This defines a log structure M on X_C. We denote by (X_s, M_s') the log scheme with the induced log structure.

Then, the main result of [5] states that there is a long exact sequence:

$$\cdots \to H^m_{rig}(X_s) \to H^m_{log-crys}((X_s, M_s')/W^\times) \otimes K_0 \to$$
$$H^m_{log-crys}((X_s, M_s')/W^\times) \otimes K_0(-1) \to H^{m+2}_{X_s, rig}(X) \to H^{m+2}_{rig}(X_s) \to \cdots$$

We can consider the maps as morphisms of filtered vector spaces, where we give the weight filtration to each of them. Moreover, we know by the results in [5, p.24] that the weight filtration on the log-crystalline cohomology terms coincides with the monodromy one.

Now let us make a description of the filtration on $H^m_{rig}(X_s)$: denote by $X_1, ..., X_r$ the irreducible components of X_s and assume they are proper and smooth. Define the *codimension* p *stratum of* X_s as

$$X^{[p]} := \coprod_{i_0 < \cdots < i_p} X_{i_0} \cap \cdots \cap X_{i_p}.$$

For each $a = 0, ..., p+1$, denote by $\delta_a : X^{[p+1]} \to X^{[p]}$ the natural map that restricted to each component is the inclusion

$$X_{i_0} \cap \cdots \cap X_{i_{p+1}} \hookrightarrow X_{i_0} \cap \cdots \cap X_{i_{a-1}} \cap X_{i_{a+1}} \cap \cdots \cap X_{i_{p+1}}$$

and define

$$\rho_p := (-1)^p \sum_{a=0}^{p} (-1)^a \delta_a^*, \tag{6}$$

where δ_a^* is the morphism of de Rham-Witt complexes $W_n\Omega^\bullet_{X^{[p]}} \to W_n\Omega^\bullet_{X^{[p+1]}}$ induced by δ_a, where we identify $W_n\Omega^\bullet_{X^{[p]}}$ with its direct image in the étale site of X_s.

This gives a double complex

$$0 \longrightarrow W_n\Omega^\bullet_{X^{[0]}} \xrightarrow{\rho_0} W_n\Omega^\bullet_{X^{[1]}} \xrightarrow{\rho_1} W_n\Omega^\bullet_{X^{[2]}} \xrightarrow{\rho_2} \cdots \tag{7}$$

and by taking projective limit, we get the double complex

$$0 \longrightarrow W\Omega^\bullet_{X^{[0]}} \xrightarrow{\rho_0} W\Omega^\bullet_{X^{[1]}} \xrightarrow{\rho_1} W\Omega^\bullet_{X^{[2]}} \xrightarrow{\rho_2} \cdots \tag{8}$$

This allows to define a spectral sequence with

$$E_1^{p,q} = H^q_{rig}(X^{[p]}) \tag{9}$$

with $d_1^{p,q}$ induced by ρ_p.

Theorem 5.1. *The spectral sequence (9) degenerates at E_2 and converges to $H^*_{rig}(X_s)$.*

Proof. Since $X^{[p]}$ is smooth and proper, then $H^q_{rig}(X^{[p]})$ is pure of weight of q. Since $E_2^{p,q}$ is a sub quotient of this, we have that

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

has to be the zero morphism, which proves the degeneracy. To prove that it converges to $H^*_{rig}(X_s)$ it is enough to notice that the simple complex associated to (8) gives this cohomology. This is given by [4, Proposition 1.8 and Theorem 3.6]. □

The weight filtration on rigid cohomology (given by the Frobenius operator) is induced by the spectral sequence (9). Now we list some properties of this filtration, denoted by $W_\bullet$, and its respective graded modules $Gr_\bullet$ on $H^m_{log-crys} := H^m_{log-crys}((X_s, M'_s)/W^\times) \otimes K_0$ and $H^m_{rig} := H^m_{rig}(X_s)$, which are just a consequence of the previous remarks and theorem 5.1.

Proposition 5.2. *(i) N^k induces an isomorphism of vector spaces*

$$Gr_{m+k}H^m_{log-crys} \xrightarrow{\sim} Gr_{m-k}H^m_{log-crys}$$

for all $k \geq 0$.

(ii) *For $k \leq m$, we have a decomposition*

$$Gr_k(H^m_{log-crys}) = \bigoplus_{a=0}^{[k/2]} Gr_{k-2a}(\mathcal{K}_m),$$

where $\mathcal{K}_m = \ker N \subset H^m_{log-crys}$ is the kernel of N acting on $H^m_{log-crys}$, and the filtration on $\mathcal{K}_m$ is induced by the one on $H^m_{log-crys}$.

(iii) $Gr_0(H^m_{rig}) = H^m(|\Gamma|)$, *where Γ is the dual graph associated to X_s.*

(iv) $Gr_k(H^m_{rig}) = E_2^{m-k,k}$.

We also need the following, which is an immediate corollary of the Clemens-Schmid exact sequence.

Proposition 5.3. *For all $k < m$, $W_k(H^m_{rig}) \cong W_k(\mathcal{K}_m)$.*

Proof. It is enough to note that since $H^m_{X_s,rig}(X)$ has weights $> m-1$, when restricting the Clemens-Schmid sequence to the W_k-parts we get an exact sequence:

$$0 \to W_k(H^m_{rig}) \to W_k(\mathcal{K}_m) \to 0$$

for $k < m$. □

With the two previous propositions in hand, we can prove the following monodromy criteria. Note that the statement and proof of this theorem are analogous to those found in [18].

Theorem 5.4. *Denote $H^i_{log-crys} := H^i_{log-crys}((X_s, M'_s)/W^\times) \otimes K_0$. Let $h^k(|\Gamma|) = \dim H^k(|\Gamma|)$, $b_k(X_s) = \dim H^k_{log-crys}$, $h^k(X^{[j]}) = \dim H^k_{rig}(X^{[j]})$ and $\Phi = \dim Gr_1 H^1_{rig}$. Then, we have the following:*

(i) $N = 0$ *on* $H^1_{log-crys}$ *if and only if* $h^1(|\Gamma|) = 0$ *if and only if* $b_1(X_s) = \Phi$.

(ii) $N^2 = 0$ *on* $H^2_{log-crys}$ *if and only if* $h^2(|\Gamma|) = 0$.

(iii) $N = 0$ *on* $H^2_{log-crys}$ *if and only if* $h^2(|\Gamma|) = 0$ *and* $\Phi = h^1(X^{[0]}) - h^1(X^{[1]})$

Proof. (i) By the final remark in [4], we have an exact sequence

$$0 \to H^1_{rig} \to H^1_{log-crys} \xrightarrow{N} H^1_{log-crys}.$$

In particular, $\ker N \cong H^1_{rig}$. Then, by part (i) of proposition 5.2, we have $Gr_2 H^1_{log-crys} \cong Gr_0 H^1_{log-crys}$, and by part (ii), we have

$Gr_0 H^1_{log-crys} \cong Gr_0(\mathcal{K}_1) = Gr_0(H^1_{rig})$, and by part (iii), we conclude that $Gr_2 H^1_{log-crys} \cong H^1(|\Gamma|)$.
Similarly, by part (ii) of proposition 5.2, we have

$$Gr_1 H^1_{log-crys} \cong Gr_1(\mathcal{K}_1) = Gr_1 H^1_{rig}.$$

First suppose $N = 0$. Then, $Gr_2 H^1_{log-crys} = 0 = Gr_0 H^1_{log-crys}$, since the first isomorphism is induced by N. Then it follows that $h^1(|\Gamma|) = 0$ and $b_1(X_s) = \Phi$.
Now suppose that $h^1(|\Gamma|) = 0$. Then, $Gr_0 H^1_{log-crys} \cong Gr_0(H^1_{rig}) = 0$. This implies that $Gr_1 H^1_{log-crys} = H^1_{log-crys}$, but $Gr_1 H^1_{log-crys} = Gr_1 \mathcal{K}_1$, hence $Gr_1 \mathcal{K}_1 = H^1_{log-crys}$. By part (ii) of proposition 5.2, we also have $Gr_0 \mathcal{K}_1 \cong Gr_0 H^1_{log-crys} = 0$ and

$$Gr_2 \mathcal{K}_1 \oplus Gr_0 \mathcal{K}_1 = Gr_2 \mathcal{K}_1 \cong Gr_2 H^1_{log-crys} = 0,$$

hence $\mathcal{K}_1 = Gr_1 \mathcal{K}_1 = H^1_{log-crys}$, which proves that $N = 0$.
Finally, note that if $b_1(X_s) = \Phi$, then $Gr_1 H^1_{log-crys} = H^1_{log-crys}$, and this implies that $h^1(|\Gamma|) = 0$.

(ii) For the proof of this and next part, we note that the Clemens-Schmid sequence for even indices can be seen as two exact sequences (since $N = 0$ on $H^0_{log-crys}$):

$$0 \to H^0_{rig} \to H^0_{log-crys} \to 0$$

$$0 \to H^0_{log-crys} \to H^2_{X_s,rig}(X) \to H^2_{rig} \to H^2_{log-crys} \xrightarrow{N} H^2_{log-crys} \to \cdots$$

By part (ii) of proposition 5.2, we have that $Gr_0 H^2_{log-crys} \cong Gr_0 \mathcal{K}_2$, and by proposition 5.3, this is isomorphic to $Gr_0 H^2_{rig} \cong H^2(|\Gamma|)$.
Suppose that $N^2 = 0$ on $H^2_{log-crys}$. Then, by part (i) of proposition 5.2, we have $Gr_4 H^2_{log-crys} \cong Gr_0 H^2_{log-crys} = 0$, and this gives that $h^2(|\Gamma|) = 0$.
Conversely, suppose that $h^2(|\Gamma|) = 0$. Then, $\dim Gr_0 H^2_{rig} = 0$. Note that N^2 takes W_0 to $W_{-4} = 0$, W_1 to $W_{-3} = 0$, W_2 to $W_{-2} = 0$, W_3 to $W_{-1} = 0$ and W_4 to $W_0 = Gr_0 H^2_{rig} = 0$. Thus, $N^2 = 0$.

(iii) By part (ii) of proposition 5.2, we have that $Gr_1 H^2_{log-crys} \cong Gr_1 \mathcal{K}_2$, and by proposition 5.3, this is isomorphic to $Gr_1 H^2_{rig} = E_2^{1,1} = \ker d_1^{1,1} / \operatorname{Im} d_1^{0,1}$.
Note that $d_1^{1,1} : H^1_{rig}(X^{[1]}) \to H^2_{rig}(X^{[1]})$ is the zero map (since $H^2_{rig}(X^{[1]})$ is trivial). Then, we conclude that

$$\dim Gr_1 H^2_{log-crys} = h^1(X^{[1]}) - \dim \operatorname{Im} \left(H^1_{rig}(X^{[0]}) \to H^1_{rig}(X^{[1]})\right)$$

$$= h^1(X^{[1]}) - (h^1(X^{[0]}) - \dim\ker(H^1_{rig}(X^{[0]}) \to H^1_{rig}(X^{[1]})))$$
$$= \Phi - h^1(X^{[0]}) + h^1(X^{[1]}).$$

Now suppose that $N = 0$. Then, $N^2 = 0$ and by the preceding part, we have $h^2(|\Gamma|) = 0$. Moreover, N induces an isomorphism

$$Gr_3(H^2_{log-crys}) \xrightarrow{\sim} Gr_1(H^2_{log-crys}),$$

hence $Gr_1 H^2_{log-crys} = 0$ and $\Phi - h^1(X^{[0]}) + h^1(X^{[1]}) = 0$.
Conversely, suppose that $h^2(|\Gamma|) = 0$ and $\Phi - h^1(X^{[0]}) + h^1(X^{[1]}) = \dim Gr_1 H^2_{log-crys} = 0$ and let us prove that $\mathcal{K}_2 = H^2_{log-crys}$ (hence $N = 0$). First note that $Gr_0 H^2_{log-crys} = 0$, since $h^2(|\Gamma|) = 0$, i.e., $W_0 = 0$. But since $Gr_1 H^2_{log-crys} = 0$, then $W_1 = 0$. By part (i) of proposition 5.2, we have $Gr_3 H^2_{log-crys} = 0$, hence $W_3 = W_2$. By the same argument, $Gr_0 H^2_{log-crys} \cong Gr_4 H^2_{log-crys}$, hence $W_4 = W_3 = W_2 = H^2_{log-crys}$. This gives $Gr_2 H^2_{log-crys} = H^2_{log-crys}$. By part (ii) of proposition 5.2, we get

$$Gr_2 H^2_{log-crys} = Gr_2\mathcal{K}_2 \oplus Gr_0\mathcal{K}_2 = Gr_2\mathcal{K}_2 = \mathcal{K}_2,$$

which concludes the proof.

□

6. Application to $K3$ surfaces

In this section we apply the results from the previous one to the left part of diagram (5). Then, we have the following:

THEOREM 6.1. *(a) X_s is of type I if and only if $N = 0$ on $H^2_{log-crys}$.*
(b) X_s is of type II if and only if $N \neq 0$ and $N^2 = 0$ on $H^2_{log-crys}$.
(c) X_s is of type III if and only if $N^2 \neq 0$ on $H^2_{log-crys}$.

PROOF. We shall prove that if $N = 0$ on $H^2_{log-crys}$, then X_s is neccesarily of type I; if $N \neq 0$ and $N^2 = 0$, then X_s is necessarily of type II; and if $N^2 \neq 0$, then X_s is necessarily of type III. This shall prove the equivalence, since we know that we can be only in one of these three cases.

First assume that X_s is of type I. Then, $X^{[0]} = X_s$, $X^{[1]} = \varnothing$ and the dual graph Γ is only one point. In this case, the spectral sequence has the form

$$E_\infty^{p,q} = E_1^{p,q} = H^q_{rig}(X^{[p]}) = \begin{cases} 0 & \text{if } p \geq 1 \\ H^q_{rig}(X_s) & \text{if } p = 0 \end{cases}$$

and this gives immediately that $\Phi = \dim Gr_1 H^1_{rig} = \dim E_2^{0,1} = 0$. Since $H^1_{rig}(X_s) = H^1_{rig}(X^{[1]}) = 0$, and $h^2(|\Gamma|) = 0$, we conclude that $N = 0$, by theorem 5.4 (iii).

Now assume that X_s is of type II (use the same notation as in the beginning of the section). In this case, it is clear that the dual graph is homeomorphic to $[0,1]$. In particular, $h^2(|\Gamma|) = 0$ and $N^2 = 0$ by theorem 5.4 (ii). By definition of the type II, $X^{[1]}$ is the disjoint union of $j+1$ elliptic curves, hence $h^1(X^{[1]}) = 2j+2$. Since X_0 and X_{j+1} are rational surfaces, by remark 2.3, we have

$$h^1(X^{[0]}) = \sum_{i=1}^{j} h^1(X_i),$$

but the X_i's are ruled, with the double curves rulings. Then, $h^1(X^{[0]}) = 2j$ and we get $h^1(X^{[0]}) - h^1(X^{[1]}) = -2$, but Φ cannot be negative, hence

$$\Phi \neq h^1(X^{[0]}) - h^1(X^{[1]})$$

and $N \neq 0$. Finally, assume that X_s is of type III. In this case, $h^2(|\Gamma|) = h^2(S^2) = 1 \neq 0$, hence $N^2 \neq 0$.

The only remaining case is when X_s is of type II such that the double curve is not ordinary, i.e., supersingular. In this case, by [20, Corollary 6.9], the geometric special fiber $X_{\overline{s}}$ is the special fiber of a projective semistable family $\widetilde{X}$ over Spec $\overline{k}[[t]]$. Now we use the same approximation argument from section 3, and we get the following cartesian diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X_A \arrow[d] \\ \mathrm{Spec}\ \overline{k}[[t]] \arrow[r] & \mathrm{Spec}\ A \end{tikzcd}$$

where A is a smooth $\overline{k}[t]$-algebra. By properness of X_A over A and proposition 3.3, there exists a finite extension k' of k and a $k'[t]$-algebra A' over which we can define $X_{A'}$ to have a cartesian diagram

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & X_{A'} \arrow[d, "f'"] \\ \text{Spec } A \arrow[r] & \text{Spec } A' \end{tikzcd}$$

The composition $A' \to A \to \overline{k}[[t]] \to \overline{k}$ defines a closed point x in Spec A'. Then, the fiber of f' at x, denoted by X_x, satisfies

$$X_x \otimes_{k'} \overline{k} \cong X_s \otimes_k \overline{k} = X_{\overline{s}}$$

Then, there exists a finite extension k'' of k' such that

$$X_x \otimes_{k'} k'' \cong X_s \otimes_k k'' =: X_s''.$$

Since $(X_s \otimes_k k'') \otimes_{k''} \overline{k} = X_s'' \otimes_{k''} \overline{k}$, we get that X_s is of the same type (I, II or III) as X_s''. Moreover, if we denote by K''/K the extension corresponding to k''/k, then the degree of nilpotency on $H^2_{log-crys}$ and $H^2_{log-crys} \otimes_K K''$ is preserved, since any extension of fields is faithfully flat. This completes the proof for all the cases. □

A consequence of this theorem is the following, which is the desired good reduction criterion:

Corollary 6.2. *Let $p > 3$ and K a finite extension of $\mathbb{Q}_p$. Let X_K be a smooth, projective $K3$ surface over K, that admits a semistable model over O_K. Then, X_K has good reduction if and only if the monodromy operator N on $H^2_{DR}(X_K)$ is zero.*

Proof. By the preceding theorem, X_s is smooth if and only the monodromy operator on its log-crystalline cohomology is zero. But since X_K has a semistable model, this is equivalent to have trivial monodromy on $H^2_{DR}(X_K)$ ([8]), which is the desired result. □

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