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Erdélyi-Kober fractional integral operators on ball Banach function spaces

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Abstract – We establish the boundedness of the Erdélyi-Kober fractional integral operators on ball Banach function spaces. In particular, it gives the boundedness of the Erdélyi-Kober fractional integral operators on amalgam spaces and Morrey spaces.

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Contents

1. Introduction

The main result of this paper is the boundedness of the Erdélyi-Kober fractional integral operators on ball Banach function spaces.

The family of Erdélyi-Kober fractional integral operator is one of the most important operators on fractional calculus. They have a big number

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of applications on several aspects of applied mathematics, just for example we mainly recall physical science and statistics [13, 20].

The ball Banach function spaces are special cases of the ball quasi-Banach function spaces introduced in [19]. This family includes several function spaces used in analysis such as amalgam spaces, Morrey space and Banach function spaces.

In this paper, we extend the mapping properties of the Erdélyi-Kober fractional integral operators to ball Banach function spaces. In particular, our main result gives the boundedness of the Erdélyi-Kober fractional integral operators on amalgam spaces and Morrey spaces.

This paper is organized as follows. Section 2 gives the definitions of the ball Banach function spaces and the Erdélyi-Kober fractional integral operators. The boundedness of the Erdélyi-Kober fractional integral operators on ball Banach function spaces is established in Section 3.

2. Definitions

Let $\mathcal{M}$ and L_{loc} denote the set of Lebesgue measurable functions and the set of locally integrable functions on $\mathbb{R}_+ = [0, \infty)$, respectively.

We use the definition of the Erdélyi-Kober fractional integral operators from [13, (0.7)] and [20]. Let $\delta, \eta > 0$ and $\gamma \in \mathbb{R}$. For any locally integrable function f, the Erdélyi-Kober fractional integral operators are defined as

$$I_\eta^{\gamma,\delta} f(t) = \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t s^{\eta(\gamma+1)-1}(t^\eta - s^\eta)^{\delta-1} f(s)ds, \quad t \geq 0, \tag{1}$$

$$K_\eta^{\gamma,\delta} f(t) = \frac{t^{\eta\gamma}}{\Gamma(\delta)} \int_t^\infty (s^\eta - t^\eta)^{\delta-1} s^{-\eta(\gamma+\delta)+\eta-1} f(s)ds, \quad t \geq 0 \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function.

When $\gamma = 0$ and $\eta = 1$, we have

$$I_1^{0,\delta} f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)ds = t^{-\delta} R_\delta f(t), \quad t \geq 0$$

where $R_\delta f$ is the Riemann-Liouville integral of order δ.

Furthermore, the fractional integral operators (1) and (2) offer generalizations on a number of well known integral operators such as ordinary n-fold integrals, the Weyl integrals, the Uspensky integral transform and the generalized Gelfond-Leontiev integration with respect to the Mittag-Leffler functions, see [13, p.15-17].

We now turn to the studies of ball Banach function spaces.

For any $x \in \mathbb{R}_+$ and $r > 0$, define $B(x,r) = \{y \in \mathbb{R}_+ : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}_+, r > 0\}$.

Definition 2.1. A Banach space $X \subset \mathcal{M}$ is said to be a ball Banach function space (B.f.s.) on $\mathbb{R}_+$ if it satisfies

(1) $\|f\|_X = 0 \Leftrightarrow f = 0$ a.e.,

(2) $|g| \le |f|$ a.e. $\Rightarrow \|g\|_X \le \|f\|_X$,

(3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$,

(4) $B \in \mathbb{B} \Rightarrow \chi_B \in X$,

(5) $B \in \mathbb{B} \Rightarrow \int_B |f(x)|dx < C_B\|f\|_X$, $\forall f \in X$ for some $C_B > 0$.

Whenever $X \subset \mathcal{M}$ satisfies (1)-(3) and

(i) $\chi_E \in \mathcal{M}$ and $|E| < \infty \Rightarrow \chi_E \in X$,

(ii) $\chi_E \in \mathcal{M}$ and $|E| < \infty \Rightarrow \int_E |f(x)|dx < C_E\|f\|_X$, $\forall f \in X$ for some $C_E > 0$,

then X is a Banach function space defined in [1, Chapter 1, Definitions 1.1 and 1.3].

The reader is referred to [12] for the use of ball Banach function spaces on characterization of BMO. For further generalization of ball Banach function spaces such as ball quasi-Banach function spaces, see [19, 21, 22, 23].

Obviously, L^p, $1 \le p \le \infty$ are ball Banach function spaces. Furthermore, Banach function spaces are ball Banach functions. Therefore, the Lorentz spaces, the Orlicz spaces and the Lebesgue spaces with variable exponents are ball Banach function spaces.

We give more examples on ball Banach function spaces. We first consider the Morrey type spaces.

Definition 2.2. Let X be a Banach function space and $u : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. The Morrey-Banach space M_X^u consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{M_X^u} = \sup_{y \in \mathbb{R}_+, r>0} \frac{1}{u(y,r)} \|\chi_{B(y,r)} f\|_X < \infty. \tag{3}$$

When $X = L^p$, $1 \le p < \infty$ and $u(y,r) = r^{\frac{1}{p}-\frac{1}{q}}$, $p \le q$, then the Morrey-Banach space M_X^u becomes the classical Morrey space M_p^q. Furthermore, the

Morrey-Banach space includes the Morrey-Lorentz spaces [17], the Orlicz-Morrey spaces [16], the Morrey spaces with variable exponents [14]. The reader is referred to [5, 6, 7, 8, 9, 15, 18] and reference therein, for the extrapolation theory, the mapping properties of singular integral operators and weak estimates of singular integral operators on Morrey-Banach spaces.

Under some mild conditions on u and X, M^u_X is a ball Banach function space. Precisely, [9, Proposition 2.3] asserts the following result.

Proposition 2.3. *Let X be a Banach function space and $u(x,r) : (0,\infty)\times(0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If there exists a constant $C > 0$ such that for any $x \in \mathbb{R}_+$ and $r > 0$, u fulfills*

$$C \le u(x,r), \quad \forall x \in \mathbb{R}_+ \quad \text{and} \quad r \ge 1, \tag{4}$$

$$\|\chi_{B(x,r)}\|_X \le Cu(x,r), \quad \forall x \in \mathbb{R}_+ \quad \text{and} \quad r < 1, \tag{5}$$

then for any $B \in \mathbb{B}$, $\chi_B \in M^u_X$.

Therefore, whenever u and X fulfill the above conditions, M^u_X satisfies Item (4) of Definition 2.1.

To verify Item (5) of Definition 2.1, we need the notion of block space [11, Definition 2.4].

Definition 2.4. Let X be a Banach function space and $u(x,r) : (0,\infty)\times(0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. A $b \in \mathcal{M}$ is an (u,X)-block if supp$b \subseteq B(x_0,r)$, $x_0 \in \mathbb{R}$, $r > 0$, and

$$\|b\|_X \le \frac{1}{u(x_0,r)}. \tag{6}$$

Define $\mathfrak{B}_{u,X}$ by

$$\mathfrak{B}_{u,X} = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k \ : \ \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \text{ is an } (u,X)\text{-block} \right\}. \tag{7}$$

The space $\mathfrak{B}_{u,X}$ is endowed with the norm

$$\|f\|_{\mathfrak{B}_{u,X}} = \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty} \lambda_k b_k \text{ a.e.} \right\}. \tag{8}$$

In view of [11, Theorem 3.4] and [1, Chapter 1, Corollaries 4.3 and 4.4], we have the following result.

THEOREM 2.5. *Let X be a Banach function space and $u(x,r):(0,\infty)\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. If X is reflexive, then*

$$(\mathfrak{B}_{u,X'})^*=M_X^u.$$

The reader is referred to [11] for the studies of sublinear operators on $\mathfrak{B}_{u,X}$.

Obviously, $\chi_B\in\mathfrak{B}_{u,X}$ for any $B\in\mathbb{B}$, therefore, the above theorem guarantees that M_X^u fulfills Item (5) of Definition 2.1. Consequently, we identify the conditions that guarantee that M_X^u is a ball Banach function space.

PROPOSITION 2.6. *Let X be a reflexive Banach function space and $u(x,r):(0,\infty)\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. If u satisfies (4) and (5), then M_X^u is a ball Banach function space.*

We now turn to the amalgam spaces.

DEFINITION 2.7. Let $1\le p,q<\infty$. The amalgam (L_p,l_q) consists of those Lebesgue measurable functions f satisfying

$$\|f\|_{(L_p,l_q)}=\left(\sum_{n=0}^{\infty}\left(\int_n^{n+1}|f(x)|^p dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}<\infty.$$

The amalgam space (L_p,l_q) is a Banach space. When $1\le p=q<\infty$, the amalgam space (L_p,l_q) reduces to the Lebesgue space L_p. The reader is referred to [2, 3] for the use of the amalgam spaces in analysis.

LEMMA 2.8. *Let $1\le p,q<\infty$. The amalgam (L_p,l_q) is a ball Banach function space.*

PROOF. It is easy to see that (L_p,l_q) satisfies Items (1)-(2) of Definition 2.1. For Item (3), suppose that $f_k\uparrow f$ and $f_k\ge 0$, $k\in\mathbb{N}$. For any $n\in\mathbb{Z}$, we have $f_k\chi_{[n,n+1]}\uparrow f\chi_{[n,n+1]}$. Consequently,

$$\left(\int_n^{n+1}|f_k(x)|^p dx\right)^{\frac{1}{p}}\uparrow\left(\int_n^{n+1}|f(x)|^p dx\right)^{\frac{1}{p}}$$

since L_p satisfies Item (3) of Definition 2.1. Moreover, l_q also fulfills Item (3) of Definition 2.1, therefore, $\|f_k\|_{(L_p,l_q)}\uparrow\|f\|_{(L_p,l_q)}$. The definition of (L_p,l_q)

also guarantees that (L_p, l_q) fulfills Item (4) of Definition 2.1. For Item (5), we see that for any $B = B(y, r) \in \mathbb{B}$,

$$\int_B |f(x)|dx \leq \sum_{n=L}^{M} \int_n^{n+1} |f(x)|dx \leq \sum_{n=L}^{M} \left(\int_n^{n+1} |f(x)|^p dx \right)^{\frac{1}{p}}$$
$$\leq C\|f\|_{(L_p, l_q)}$$

where $L = \max(0, |y| - r)$, $M = |y| + r$ and C is a constant independent of f. Therefore, (L_p, l_q) is a ball Banach function space. □

Let $1 \leq q < p < \infty$ and

$$E = \bigcup_{k=1}^{\infty} \left[k, k + \frac{1}{k^{\frac{p}{q}}} \right].$$

Obviously, E is Lebesgue measurable and $|E| < \infty$. On the other hand, we see that

$$\|\chi_E\|_{(L_p, l_q)} > \left(\sum_{k=1}^{N} \frac{1}{k} \right)^{\frac{1}{q}}, \quad \forall N \in \mathbb{N}$$

and, hence, $\chi_E \notin (L_p, l_q)$. Therefore, (L_p, l_q) is not a Banach function space since it does not satisfy Item (i) of Definition 2.1 while (L_p, l_q) is a ball Banach function space. This is the main reason why we study ball Banach function spaces instead of Banach function spaces.

Next, we study the dilation properties of ball Banach function spaces. For any $\lambda > 0$ and $f \in L_{loc}$, define $D_\lambda f(x) = f(x/\lambda)$, $x \in \mathbb{R}_+$.

Definition 2.9. Let X be a ball Banach function space on $\mathbb{R}_+$. We write $X \in D(\beta, \theta)$, $0 \leq \beta, \theta < \infty$ if there is a constant $C > 0$ such that for all $f \in X$

$$\|D_\lambda f\|_X \leq C\lambda^\beta \|f\|_X, \quad \lambda \in (0, 1]$$
$$\|D_\lambda f\|_X \leq C\lambda^\theta \|f\|_X, \quad \lambda \in (1, \infty).$$

Obviously, we have $L^p \in D(\frac{1}{p}, \frac{1}{p})$, $1 \leq p \leq \infty$. Some Lorentz spaces and Orlicz spaces also belong to $D(\beta, \theta)$ for some $0 \leq \beta, \theta < \infty$. Instead of presenting the results for Lorentz spaces and Orlicz spaces only, we consider the rearrangement-invariant Banach function spaces which include Lorentz spaces and Orlicz spaces. A Banach function space X is said to be a rearrangement-invariant Banach function space if $\|f\|_X = \|g\|_X$ whenever

$f^* = g^*$ [1, Chapter 2, Definition 4.1] where f^* and g^* are decreasing rearrangements of f and g, respectively [1, Chapter 2, Definitions 1,1 and 1.5].

The dilation property of rearrangement-invariant Banach function space X is governed by the Boyd's indices $\underline{\alpha}_X, \bar{\alpha}_X$. For simplicity, we refer the reader to [1, Chapter 3, Definitions 5.10 and 5.12] for the definition of the Boyd's indices $\underline{\alpha}_X, \bar{\alpha}_X$. According to the definition of the Boyd's indices, for any rearrangement-invariant Banach function spaces X and $\epsilon > 0$, we have $X \in D(\underline{\alpha}_X - \epsilon, \overline{\alpha}_X + \epsilon)$. The Boyd's indices of Lorentz spaces $L^{p,q}$, $1 < p < \infty$ and $1 \leq q \leq \infty$, are $\frac{1}{p}$ [1, Chapter 4, Theorem 4.6]. For the Boyd's indices of Orlicz spaces, the reader is referred to [1, Chapter 4, Theorem 8.18].

For Morrey type spaces, we have the following result.

Lemma 2.10. *Let $0 \leq \beta, \theta < \infty$, X be a ball Banach function space on $\mathbb{R}_+$ and $u(y,r) : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. Suppose that $X \in D(\beta, \theta)$ and there exist $\theta_u, \beta_u \in \mathbb{R}$ such that u satisfies*

$$u(y/\lambda, r/\lambda) \leq C_0 \lambda^{-\theta_u} u(y,r), \quad \text{for all} \quad 1 < \lambda < \infty \tag{9}$$

$$u(y/\lambda, r/\lambda) \leq C_0 \lambda^{-\beta_u} u(y,r), \quad \text{for all} \quad 0 < \lambda < 1 \tag{10}$$

for some $C_0 > 0$, then there exists $C > 0$ such that

$$\|D_\lambda f\|_{M_X^u} \leq C \lambda^{\theta - \theta_u} \|f\|_{M_X^u}, \quad \text{for all} \quad 1 < \lambda, \tag{11}$$

$$\|D_\lambda f\|_{M_X^u} \leq C \lambda^{\beta - \beta_u} \|f\|_{M_X^u}, \quad \text{for all} \quad 0 < \lambda \leq 1. \tag{12}$$

The reader is referred to [4, Lemma 2.2] for the proof of the preceding lemma. Notice that the proof in [4, Lemma 2.2] is for rearrangement-invariant Banach function space X, it is easy to see that with some simple modifications, it can be extended to ball Banach function space, therefore, for brevity, we leave the details to the reader.

The classical Morrey space M_p^q, $1 \leq p \leq q < \infty$ satisfies (9) and (10) with $\theta_u = \beta_u = \frac{1}{p} - \frac{1}{q}$. As $L^p \in D(\frac{1}{p}, \frac{1}{p})$, Lemma 2.10 guarantees that $M_p^q \in D(\frac{1}{q}, \frac{1}{q})$. Similarly, we can obtain the dilation properties for the Morrey-Lorentz spaces and the Orlicz-Morrey spaces. For brevity, we leave the details to the reader.

For the amalgam space (L_p, l_q), we have the following results from [10, Theorems 2.2 and 2.3].

Lemma 2.11. *Let $1 \leq p, q < \infty$.*

(1) *When* $p < q$, *we have* $(L_p, l_q) \in D(\frac{1}{q}, \frac{1}{p})$.
(2) *When* $q < p$, *we have* $(L_p, l_q) \in D(\frac{1}{p}, \frac{1}{q})$.

The result obtained in [10, Theorems 2.2 and 2.3] is for amalgam spaces defined on $\mathbb{R}$. Obviously, this result also applies to amalgam spaces defined on $\mathbb{R}_+$. The above result has its own independent interest. It is used to study the integral operators on (L_p, l_q), see [10].

At the end of this section, we establish the Minkowski inequality for ball Banach function spaces. To accomplish this result, we state the Fatou's lemma for ball Banach function space.

Lemma 2.12. *Let* X *be a ball Banach function space. If* $f_n \to f$ *a.e. and* $\liminf_{n\to\infty} \|f_n\|_X < \infty$, *then* $f \in X$ *and*

$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

The proof of the Fatou's lemma just relies on Items (2) and (3) of Definition 2.1 and follows from the proof for the case when X is a Banach function spaces [1, Chapter 1, Lemma 1.5 (ii)], therefore, for simplicity, we omit the proof and leave it to the reader.

Theorem 2.13. *Let* X *be a ball Banach function space. If* F *is a Lebesgue measurable function on* $(0,\infty) \times (0,\infty)$ *satisfying*

$$\int_0^\infty \|F(\cdot, y)\|_X dy < \infty,$$

then

$$\left\| \int_0^\infty F(\cdot, y) dy \right\|_X \leq \int_0^\infty \|F(\cdot, y)\|_X dy.$$

Proof. For any $n \in \mathbb{N}$, we define $X_N = \{g\chi_{B(0,N)} : g \in X\}$ and endow X_N with the norm $\|g\|_{X_N} = \|g\chi_{B(0,N)}\|_X$. In view of Definition 2.1, X_N is a Banach function space associated with Lebesgue measure restricted on $B(0, N)$.

We first establish the Minkowski inequality for X_N. As X_N is a Banach function space, the Lorentz-Luxemburg theorem [1, Chapter 1, Theorem 2.7] assure that

$$\|g\|_{X_N} = \sup_{\substack{h \in X'_N \\ \|g\|_{X'_N} \leq 1}} \int_{B(0,N)} |g(x)h(x)| dx \tag{13}$$

where X'_N is the associate space of X_N [1, Chapter 1, Section 2].

Write $F_N(x) = \chi_{B(0,N)}(x)\int_0^\infty F(x,y)dy$. According to Item (5) of Definition 2.1, we find that

$$\int_0^\infty \int_{B(0,N)} |F(x,y)|dxdy \le C\int_0^\infty \|F(\cdot,y)\|_X dy < \infty.$$

The Fubini theorem guarantees that F_N is a well defined Lebesgue measurable function.

For any $h \in X'_N$ with $\|h\|_{X'_N} \le 1$, we have

$$\begin{aligned}\int_{B(0,N)}\int_0^\infty |F_N(x)h(x)|dx &= \int_{B(0,N)}\int_0^\infty |F(x,y)h(x)|dxdy\\ &\le \int_0^\infty \|F(\cdot,y)\|_{X_N}\|h\|_{X'_N}dy\\ &\le \int_0^\infty \|F(\cdot,y)\|_{X_N}dy < \infty.\end{aligned}$$

Therefore, Item (2) of Definition 2.1 and (13) assure that

$$\begin{aligned}\left\|\chi_{B(0,N)}(\cdot)\int_0^\infty F(\cdot,y)dy\right\|_X &\le \int_0^\infty \|\chi_{B(0,N)}(\cdot)F(\cdot,y)\|_X dy\\ &\le \int_0^\infty \|F(\cdot,y)\|_X dy.\end{aligned}$$

The above inequality shows that for any $x \in B(0,N)$, $\int_0^\infty F(x,y)dy$ is well defined. Since N is arbitrary, $\int_0^\infty F(x,y)dy$ is a well defined Lebesgue measurable function.

Furthermore, since $\chi_{B(0,N)}(x)\int_0^\infty F(x,y)dy \to \int_0^\infty F(x,y)dy$ as $N \to \infty$, by applying the Fatou's lemma, we obtain

$$\left\|\int_0^\infty F(\cdot,y)dy\right\|_X \le \int_0^\infty \|F(\cdot,y)\|_X dy. \qquad \square$$

3. Main results

The mapping properties for the Erdélyi-Kober fractional integral operators on ball Banach function spaces are proved in this section. Here is our main result.

THEOREM 3.1. *Let $\delta,\eta > 0$, $\gamma \in \mathbb{R}$, $0 < \beta,\theta < \infty$. Suppose that X is a ball Banach function space with $X \in D(\beta,\theta)$.*

(1) *If* β *satisfies* $\eta(\gamma+1) > \theta$, *then there is a constant* $C > 0$ *such that for any* $f \in X$, *we have*
$$\|I_\eta^{\gamma,\delta} f\|_X \leq C\|f\|_X.$$
(2) *If* θ *satisfies* $-\eta\gamma < \beta$, *then there is a constant* $C > 0$ *such that for any* $f \in X$, *we have*
$$\|K_\eta^{\gamma,\delta} f\|_X \leq C\|f\|_X.$$

Proof. We first consider $I_\eta^{\gamma,\delta}$. For any $f \in X$, by using the substitution $s = ut$, we have

$$\begin{aligned}
I_\eta^{\gamma,\delta} f(t) &= \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t s^{\eta(\gamma+1)-1}(t^\eta - s^\eta)^{\delta-1} f(s)ds \\
&= \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^1 (ut)^{\eta(\gamma+1)-1}(t^\eta - (ut)^\eta)^{\delta-1} f(ut)tdu \\
&= \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^1 u^{\eta(\gamma+1)-1}(1-u^\eta)^{\delta-1} D_{\frac{1}{u}} f(t) t^{\eta(\gamma+1)-1+\eta(\delta-1)+1} du \\
&= \frac{1}{\Gamma(\delta)} \int_0^1 u^{\eta(\gamma+1)-1}(1-u^\eta)^{\delta-1} D_{\frac{1}{u}} f(t)du.
\end{aligned}$$

Theorem 2.13 gives

$$\begin{aligned}
\|I_\eta^{\gamma,\delta} f\|_X &\leq \frac{1}{\Gamma(\delta)} \left\| \int_0^1 u^{\eta(\gamma+1)-1}(1-u^\eta)^{\delta-1} D_{\frac{1}{u}} f(\cdot)du \right\|_X \\
&\frac{1}{\Gamma(\delta)} \int_0^1 u^{\eta(\gamma+1)-1}(1-u^\eta)^{\delta-1} \|D_{\frac{1}{u}} f(\cdot)\|_X du.
\end{aligned}$$

Since $X \in D(\beta, \theta)$, we have

$$\begin{aligned}
\|I_\eta^{\gamma,\delta} f\|_X &\leq C\left(\int_0^{\frac{1}{2}} u^{\eta(\gamma+1)-1-\theta} du + \int_{\frac{1}{2}}^1 (1-u^\eta)^{\delta-1} du \right) \|f\|_X \\
&\leq C\|f\|_X
\end{aligned}$$

for some $C > 0$ because $\eta(\gamma+1) > \theta$ and $\delta > 0$.

We now consider $K_\eta^{\gamma,\delta}$. By using the substitution $s = ut$, we obtain

$$\begin{aligned}
K_\eta^{\gamma,\delta} f(t) &= \frac{t^{\eta\gamma}}{\Gamma(\delta)} \int_t^\infty (s^\eta - t^\eta)^{\delta-1} s^{-\eta(\gamma+\delta)+\eta-1} f(s)ds \\
&= \frac{t^{\eta\gamma}}{\Gamma(\delta)} \int_1^\infty ((ut)^\eta - t^\eta)^{\delta-1} (ut)^{-\eta(\gamma+\delta)+\eta-1} f(ut)tdu \\
&= \frac{t^{\eta\gamma}}{\Gamma(\delta)} t^{\eta(\delta-1)-\eta(\gamma+\delta)+\eta} \int_1^\infty (u^\eta - 1)^{\delta-1} u^{-\eta(\gamma+\delta)+\eta-1} D_{\frac{1}{u}} f(t)du.
\end{aligned}$$

Since $X \in D(\alpha, \beta)$, the Minkowski inequalities yield

$$\begin{aligned}\|K_\eta^{\gamma,\delta} f\|_X &\leq \frac{1}{\Gamma(\delta)} \left\| \int_1^\infty (u^\eta - 1)^{\delta-1} u^{-\eta(\gamma+\delta)+\eta-1} D_{\frac{1}{u}} f(\cdot) du \right\|_X \\ &\leq \frac{1}{\Gamma(\delta)} \int_1^\infty (u^\eta - 1)^{\delta-1} u^{-\eta(\gamma+\delta)+\eta-1} \|D_{\frac{1}{u}} f(\cdot)\|_X du \\ &\leq \frac{1}{\Gamma(\delta)} \int_1^\infty (u^\eta - 1)^{\delta-1} u^{-\eta(\gamma+\delta)+\eta-1-\beta} \|f\|_X du.\end{aligned}$$

Consequently,

$$\begin{aligned}\|K_\eta^{\gamma,\delta} f\|_X &\leq C \left(\int_1^2 (u^\eta - 1)^{\delta-1} du + \int_2^\infty u^{\eta(\delta-1)-\eta(\gamma+\delta)+\eta-1-\beta} du \right) \|f\|_X \\ &\leq C \left(\int_1^2 (u-1)^{\delta-1} du + \int_2^\infty u^{-\eta\gamma-1-\beta} \right) \|f\|_X \\ &\leq C\|f\|_X\end{aligned}$$

for some $C > 0$ because $\delta > 0$ and $-\eta\gamma < \beta$. □

Theorem 3.1 gives the following boundedness results of the Erdélyi-Kober fractional integral operators on (L_p, l_q).

THEOREM 3.2. *Let $\delta, \eta > 0$, $\gamma \in \mathbb{R}$, $1 \leq p, q < \infty$.*

(1) *If p, q satisfy $\eta(\gamma + 1) > \frac{1}{\min(p,q)}$, then there is a constant $C > 0$ such that for any $f \in (L_p, l_q)$, we have*

$$\|I_\eta^{\gamma,\delta} f\|_{(L_p,l_q)} \leq C\|f\|_{(L_p,l_q)}.$$

(2) *If p, q satisfy $-\eta\gamma < \frac{1}{\max(p,q)}$, then there is a constant $C > 0$ such that for any $f \in (L_p, l_q)$, we have*

$$\|K_\eta^{\gamma,\delta} f\|_{(L_p,l_q)} \leq C\|f\|_{(L_p,l_q)}.$$

Similarly, we have the mapping properties for the Erdélyi-Kober fractional integral operators on Morrey type spaces.

THEOREM 3.3. *Let $\delta, \eta > 0$, $\gamma \in \mathbb{R}$ and $0 \leq \beta, \theta < \infty$. Let X be a reflexive Banach function space and $u(y, r) : (0, \infty) \times (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function satisfying (4), (5), (9) and (10). Suppose that $X \in D(\beta, \theta)$ and there exist $\theta_u, \beta_u \in \mathbb{R}$ such that u satisfies (9) and (10).*

(1) *If* θ, θ_u *satisfy* $\eta(\gamma+1)+\theta_u > \theta$*, then there is a constant* $C > 0$ *such that for any* $f \in M_X^u$*, we have*

$$\|I_\eta^{\gamma,\delta} f\|_{M_X^u} \leq C\|f\|_{M_X^u}.$$

(2) *If* β, β_u *satisfy* $-\eta\gamma + \beta_u < \beta$*, then there is a constant* $C > 0$ *such that for any* $f \in M_X^u$*, we have*

$$\|K_\eta^{\gamma,\delta} f\|_{M_X^u} \leq C\|f\|_{M_X^u}.$$

Let X be a reflexive rearrangement-invariant Banach function space. If $\eta(\gamma+1)+\theta_u > \overline{\alpha}_X$, then there is a $\epsilon > 0$ such that $\eta(\gamma+1)+\theta_u > \overline{\alpha}_X + \epsilon$ and $X \in D(\underline{\alpha}_X - \epsilon, \overline{\alpha}_X + \epsilon)$. Consequently, whenever u satisfies (4), (5), (9) and (10) with $\beta = \underline{\alpha}_X - \epsilon$ and $\theta = \overline{\alpha}_X + \epsilon$, Theorem 3.3 yields a constant $C > 0$ such that

$$\|I_\eta^{\gamma,\delta} f\|_{M_X^u} \leq C\|f\|_{M_X^u}.$$

Similarly, if $-\eta\gamma + \beta_u < \underline{\alpha}_X$, we have a constant $C > 0$ such that

$$\|K_\eta^{\gamma,\delta} f\|_{M_X^u} \leq C\|f\|_{M_X^u}.$$

Particularly, we have the boundedness of Erdélyi-Kober fractional integral operators on the classical Morrey spaces M_p^q, $1 < p \leq q < \infty$. If $\eta(\gamma+1) > \frac{1}{q}$, the above theorem yields a constant $C > 0$ such that

$$\|I_\eta^{\gamma,\delta} f\|_{M_p^q} \leq C\|f\|_{M_p^q}.$$

Similarly, if $-\eta\gamma < \frac{1}{q}$, then the above theorem yields a constant $C > 0$ such that

$$\|K_\eta^{\gamma,\delta} f\|_{M_p^q} \leq C\|f\|_{M_p^q}.$$

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