Finite groups with $H_L$-embedded subgroups

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ABSTRACT – Let $G$ be a finite soluble group and let $\mathcal{F}$ be a class of groups. A chief factor $H/K$ of $G$ is said to be $\mathcal{F}$-central (in $G$) if $(H/K) \rtimes (G/C_G(H/K)) \in \mathcal{F}$; we write $L_{\mathcal{F}}(G)$ to denote the set of all subgroups $A$ of $G$ such that every chief factor $H/K$ of $G$ between $A_G$ and $A_G$ is $\mathcal{F}$-central in $G$. Let $\mathcal{L}$ be a set of subgroups of $G$. We say that a subgroup $A$ of $G$ is $H_L$-embedded in $G$ provided $A$ is a Hall subgroup of some subgroup $E \in \mathcal{L}$. In this paper, we study the structure of $G$ under the condition that every subgroup of $G$ is $H_L$-embedded in $G$, where $\mathcal{L} = L_{\mathcal{F}}(G)$ for some hereditary saturated formation $\mathcal{F}$. Some known results are generalized.


KEYWORDS. finite group, hereditary saturated formation, Hall subgroup, $K$-$\mathcal{F}$-subnormal subgroup, $H_L$-embedded subgroup.

1. Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$; $C_n$ denotes a cyclic

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group of order \( n \).

Let \( \mathfrak{F} \) be a class of groups. If \( 1 \in \mathfrak{F} \), then we write \( G^\mathfrak{F} \) to denote the intersection of all normal subgroups \( N \) of \( G \) with \( G/N \in \mathfrak{F} \). The class \( \mathfrak{F} \) is said to be a formation

if either \( \mathfrak{F} = \emptyset \) or \( 1 \in \mathfrak{F} \) and every homomorphic image of \( G/G^\mathfrak{F} \) belongs to \( \mathfrak{F} \) for every group \( G \); hereditary (Mal’cev [1]) if \( H \in \mathfrak{F} \) whenever \( H \leq G \in \mathfrak{F} \). The formation \( \mathfrak{F} \) is said to be saturated if \( G \in \mathfrak{F} \) whenever \( G^\mathfrak{F} \leq \Phi(G) \).

In what follows, \( \mathfrak{F}, \mathfrak{U} \) and \( \mathfrak{S} \) are the classes of all nilpotent, of all supersoluble and of all soluble groups, respectively, and \( \mathfrak{F} \) is a hereditary saturated formation containing all nilpotent groups. It is well-known that \( \mathfrak{F} \) and \( \mathfrak{U} \) are hereditary saturated formations.

A subgroup \( H \) of \( G \) is said to be an \( \mathfrak{F} \)-projector of \( G \) [2, p. 101] if \( HN/N \) is \( \mathfrak{F} \)-maximal in \( G/N \), that is, \( HN/N = U/N \) for every subgroup \( U/N \) of \( G/N \) such that \( HN/N \leq U/N \in \mathfrak{F} \), for all \( N \leq G \). It is well-known that the \( \mathfrak{F} \)-projectors of \( G \) are exactly the Carter subgroups of a soluble group \( G \). Also, in the theory of saturated formations

the so-called \( \mathfrak{F} \)-normalizers, which have properties analogues to the properties of the system normalizers of soluble groups, were introduced and studied by many authors (see, in particular, [2, Chapter 4] and [3, Chapter IV]).

If \( K \leq H \) and \( C \) are normal subgroups of \( G \) and \( C \leq C_G(H/K) \), then we can form the semidirect product \( (H/K) \ltimes (G/C) \) putting \( (hK)^c = g^{-1}hgK \) for all \( hK \in H/K \) and \( gC \in G/C \). We say that a chief factor \( H/K \) of \( G \) is \( \mathfrak{F} \)-central in \( G \) [4] if \( (H/K) \ltimes (G/C)(H/K)) \in \mathfrak{F} \). Otherwise, it is called \( \mathfrak{F} \)-eccentric.

We write, following [5], \( \mathcal{L}_{c\mathfrak{F}}(G) \) to denote the set of all subgroups \( A \) of \( G \) such that every chief factor \( H/K \) of \( G \) between \( A_G \) and \( A^G \) is \( \mathfrak{F} \)-central in \( G \).

By definition, every normal subgroup of \( G \) belongs to \( \mathcal{L}_{c\mathfrak{F}}(G) \). In the paper [5], it is proved that the set \( \mathcal{L}_{c\mathfrak{F}}(G) \) forms a sublattice of the lattice of all subgroups of \( G \). Let us note in passing, that lattices of this kind have already found applications in the analysis of various questions [5]–[9]. In this paper we continue study the influence of elements in \( \mathcal{L}_{c\mathfrak{F}}(G) \) on the structure of \( G \). The main tool for that is the following

**Definition 1.1.** Let \( \mathcal{L} \) be any set of subgroups of a group \( G \) containing all its normal subgroups. We say that a subgroup \( H \) of \( G \) is \( H_\mathcal{L} \)-embedded in \( G \) if \( H \) is a Hall subgroup of some subgroup \( E \in \mathcal{L} \).

Note that all subgroups in \( \mathcal{L} \) and all Hall subgroups of \( G \) are \( H_\mathcal{L} \)-embedded in \( G \).

Groups with given systems of \( H_\mathcal{L} \)-embedded subgroups for some sublattices \( \mathcal{L} \) of the lattice of all subgroups of \( G \) were studied by many authors (see, for example, the recent papers [10]–[16]).

Our first goal here is to prove the following result in this line researches.

**Theorem 1.2.** Let \( D = G^\mathfrak{F} \) and \( \mathcal{L} = \mathcal{L}_{c\mathfrak{F}}(G) \).

1. If \( G \) possesses an \( H_\mathcal{L} \)-embedded subgroup of order \( |G : A| \) for each subgroup \( A \) of \( G \) of prime power order, then \( D \) is cyclic of odd square free order.

2. If for each integer \( r \) dividing \( |G/D| \) there exists a subgroup of \( G/D \) of order \( r \) and \( D \) is cyclic of square free order, then \( G \) possesses an \( H_\mathcal{L} \)-embedded subgroup of order \( d \) for each integer \( d \) dividing \( |G| \).

3. Every subgroup of \( G \) is \( H_\mathcal{L} \)-embedded in \( G \) if and only if \( D \) is a Hall cyclic subgroup of odd square free order.
**Corollary 1.3** (See Theorem 7 in [11]). If for any subgroup $A$ of $G$ there is a subgroup $B$ such that $|B| = |G : A|$ and $B$ is a Hall subgroup of some normal subgroup of $G$, then $G$ is supersoluble.

From Theorem 1.2 we get also the following characterization of supersolubility.

**Corollary 1.4.** Let $\mathcal{L} = \mathcal{L}_{all}(G)$. Then $G$ is supersoluble if and only if every subgroup of $G$ is $H_{\mathcal{L}}$-embedded in $G$.

Recall that a subgroup $A$ of $G$ is said to be $\mathfrak{F}$-subnormal in $G$ in the sense of Kegel [17] or $K$-$\mathfrak{F}$-subnormal in $G$ [2, 6.1.4] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \lhd A_i$ or $A_i/(A_{i-1})A_i \in \mathfrak{F}$ for all $i = 1, \ldots, n$.

Note that any subnormal subgroup of $G$ is also a $K$-$\mathfrak{F}$-subnormal subgroup of $G$.

**Remark 1.5.** (1) If $G \in \mathfrak{F}$, then for every subgroup $A$ of $G$ the chain $A \leq G$ satisfies that $G/A \in \mathfrak{F}$. That is to say that every subgroup of $G$ is $K$-$\mathfrak{F}$-subnormal in $G$.

(2) Let $G$ be a simple group such that $G \notin \mathfrak{F}$. Clearly the chain $G = A_0 \leq A_1 = G$ satisfies that $A_0$ is normal in $A_1$ and $A_1/(A_0)A_1 = G/G_0 = 1 \in \mathfrak{F}$. So, $G$ is $K$-$\mathfrak{F}$-subnormal in $G$. If $A$ is a proper $K$-$\mathfrak{F}$-subnormal subgroup of $G$, then there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \lhd A_i$ or $A_i/(A_{i-1})A_i \in \mathfrak{F}$ for all $i = 1, \ldots, n$. If $A \neq 1$, then simplicity of $G$ implies that $G/(A_{n-1})G = \mathfrak{F}$ and this is not true. Therefore $\{1, G\}$ is the set of all $K$-$\mathfrak{F}$-subnormal subgroups of $G$.

(3) Now let $\mathcal{L}$ be the set of all $K$-$\mathfrak{F}$-subnormal subgroups of $G$. If $G \in \mathfrak{F}$, then every subgroup of $G$ is $H_{\mathcal{L}}$-embedded in $G$ by Part (1). On the other hand, if $G$ is simple and $G \notin \mathfrak{F}$, then $A$ is $H_{\mathcal{L}}$-embedded in $G$ if and only if $A$ is a Hall subgroup of $G$ by Part (2).

**Definition 1.6.** We say that $G$ has the structure $S_\mathfrak{F}$ if $G$ is the semidirect product $G = D \rtimes M$ for some subgroup $M$ of $G$ and $D = G^\mathfrak{F}$, and the following holds:

1. $D$ is a Hall subgroup of $G$,
2. $D$ possesses a Sylow tower,
3. $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$, and
4. every chief factor of $G$ below $D$ is $\mathfrak{F}$-eccentric in $G$.

**Remark 1.7.** Let $G$ have the structure $S_\mathfrak{F}$ and $D = G^\mathfrak{F}$ with $\pi = \pi(D)$.

(1) If $G \in \mathfrak{F}$, then $G$ has a trivial $S_\mathfrak{F}$-structure since $G^\mathfrak{F} = 1$ and $M = G$.

(2) If $N \trianglelefteq G$, then $G/N$ has the structure $S_{\mathfrak{F}}$.

Indeed, $(G/N)^\mathfrak{F} = DN/N$ by Proposition 2.2.8 in [2]. Then $(G/N)^\mathfrak{F}$ is a Hall subgroup of $G/N$ and from the $G$-isomorphism $DN/N \simeq D/(D \cap N)$ it follows that $(G/N)^\mathfrak{F}$ possesses a Sylow tower and every chief factor of $G/N$ below $(G/N)^\mathfrak{F}$ is $\mathfrak{F}$-eccentric in $G/N$. Finally, it is clear that $MN/N$ acts irreducibly on every $MN/N$-invariant Sylow subgroup of $DN/N = (G/N)^\mathfrak{F}$.
Let $H$ be a complement of $D$ in $G$. Then $H$ is a Hall subgroup of $G$. Hence $|H| = |G/D|$ by [18, Chapter A, Lemma 1.7] since every chief factor of $G$ between $D$ and $G$ is $\mathfrak{F}$-central in $G$ by the Barnes-Kegel result [18, IV, Proposition 1.5]. Therefore $M$ is an $\mathfrak{F}$-normalizer of $G$. By another well-known result on groups with soluble $G^R$, $G$ possesses an $\mathfrak{F}$-projector $V$ such that $M \leq V$ and every two $\mathfrak{F}$-projectors of $G$ are conjugate.

(4) Suppose that $\mathfrak{F} \subseteq \mathfrak{U}$. In this case $M$ (see Part (3)) is an $\mathfrak{F}$-projector of $G$.

In fact, in view of Parts (2) and (3), it is enough to prove that $M$ is $\mathfrak{F}$-maximal in $G$, that is, $M = U$ for every subgroup $U$ of $G$ such that $M \leq U \in \mathfrak{F}$. If $D = 1$, it is clear. Now assume that $D \neq 1$ and let $R$ be the non-identity normal Sylow subgroup of $D$. Then $R$ is normal in $G$ and $G/R = (D/R) \rtimes (MR/R)$ has the structure $S_{\mathfrak{F}}$, where $D/R = (G/R)^R$, by Part (2). Moreover, $MR/R \leq UR/R \cong U/(U \cap R) \in \mathfrak{F}$, so $R \rtimes M = RU$ by induction. Hence $U = M(U \cap R)$, where $U \cap R = 1$ or $U \cap R = R$ since $M$ and so also $U$ act irreducibly on $R$. In the former case from $R \times M = RU$ and $M \leq U$ we get $M = U$. Now assume that $U \cap R = R$, so $RM = U \in \mathfrak{F} \subseteq \mathfrak{U}$. Then $RM$ is supersolvable and $M$ acts irreducibly on $R$, hence $|R| = p$. It follows that $G/C_G(R)$ is cyclic, which implies that $D \leq C_G(R)$. Let $V$ be a complement of $R$ in $D$. Then $V$ is a characteristic subgroup of $D$ and $V$ is normal in $G$ and $G/V \cong RM \in \mathfrak{F}$, which implies that $D \leq V < D$. This contradiction completes the proof of the fact that $M$ is an $\mathfrak{F}$-projector of $G$.

It is clear that every subgroup $A \in \mathcal{L}_G(G)$ is $K_{\mathfrak{F}}$-subnormal in $G$ (see Lemma 3.2 below). This observation allows us to use in the proof of Theorem 1.4 the following general fact.

**Theorem 1.8.** Let $\mathcal{L}$ be the set of all $K_{\mathfrak{F}}$-subnormal subgroups of $G$. Suppose that $G^R$ is soluble. Then every subgroup of $G$ is $H_{\mathcal{L}}$-embedded in $G$ if and only if every subgroup of $\mathcal{L}$ has the structure $S_{\mathfrak{F}}$.

Note that since a subgroup $A$ of $G$ is $K_{\mathfrak{U}}$-subnormal in $G$ if and only if it is subnormal in $G$, we get from Theorem 1.8 the following known result.

**Corollary 1.9.** (See Theorem 3.3 in [12]). Let $\mathcal{L}$ be the set of all subnormal subgroups of $G$. Then every subgroup of $G$ is $H_{\mathcal{L}}$-embedded in $G$ if and only if every subgroup of $\mathcal{L}$ has the structure $S_{\mathfrak{U}}$.

Recall that a subgroup $S$ of $G$ is called a Gaschütz subgroup of $G$ (L.A. Shemetkov [3, IV, 15.3]) if $S$ is supersolvable and for any subgroups $K \leq H$ of $G$, where $S \leq K \neq H$, the number $|H : K|$ is not prime. In the case when $G$ is soluble the class of all Gaschütz subgroups of $G$ coincides with the class of all $\mathfrak{U}$-projectors of $G$ [3, Chapter III, Remark 8.2]. Therefore, in view of Remark 1.7(4), we get from Theorem 1.8 the following

**Corollary 1.10.** Let $\mathcal{L}$ be the set of all $K_{\mathfrak{U}}$-subnormal subgroups of $G$. Then every subgroup of $G$ is $H_{\mathcal{L}}$-embedded in $G$ if and only if every subgroup of $\mathcal{L}$ has the structure $S_{\mathfrak{U}}$. Moreover, in this case the class of all Gaschütz subgroups of $G$ coincides with the class of all supersoluble normalizers of $G$. 
2. Proof of Theorem 1.8

The first lemma collects the properties of $K$-$S_\mathfrak{S}$-subnormal subgroups which we use in our proofs.

**Lemma 2.1.** Let $H$, $E$ and $R$ be subgroups of $G$, where $H$ is $K$-$S_\mathfrak{S}$-subnormal and $R$ is normal in $G$.

1. $H \cap E$ is a $K$-$S_\mathfrak{S}$-subnormal subgroup of $E$ [2, Lemma 6.1.7(2)].
2. $HR/R$ is a $K$-$S_\mathfrak{S}$-subnormal subgroup of $G/R$ [2, Lemma 6.1.6(3)].
3. If $R \leq E$ and $E/R$ is $K$-$S_\mathfrak{S}$-subnormal in $G/R$, then $E$ is $K$-$S_\mathfrak{S}$-subnormal in $G$ [2, Lemma 6.1.6(2)].
4. If $G^S \leq E$, then $E$ is $K$-$S_\mathfrak{S}$-subnormal in $G$ [2, Lemma 6.1.7(1)].

We use $L_{K_\mathfrak{S}}(G)$ to denote the set of all $K$-$S_\mathfrak{S}$-subnormal subgroups of $G$.

**Lemma 2.2.** Let $\mathcal{L} = L_{K_\mathfrak{S}}(G)$, and let $A$, $E$ and $R$ be subgroups of $G$, where $A$ is $H_{\mathcal{L}}$-embedded and $R$ is normal in $G$.

1. If $A \leq E$, then $A$ is $H_{L_{K_\mathfrak{S}}(E)}$-embedded in $E$.
2. $AR/R$ is $H_{L_{K_\mathfrak{S}}(G/R)}$-embedded in $G/R$.
3. If $|G : A|$ is a power of prime, then $A$ is either a Hall subgroup of $G$ or $K$-$S_\mathfrak{S}$-subnormal in $G$.

**Proof.** Let $V$ be a $K$-$S_\mathfrak{S}$-subnormal subgroup of $G$ such that $A$ is a Hall subgroup of $V$.

1. We have $A \leq E \cap V \leq V$, where $E \cap V$ is a $K$-$S_\mathfrak{S}$-subnormal subgroup of $E$ by Lemma 2.1(1). It is clear also that $A$ is a Hall subgroup of $E \cap V$. Hence $A$ is $H_{\mathcal{L}(E)}$-embedded in $E$.
2. In view of Lemma 2.1(2), $VR/R$ is a $K$-$S_\mathfrak{S}$-subnormal subgroup of $G/R$. It is also clear that $AR/R$ is a Hall subgroup of $VR/R$. Hence we have (2).
3. Assume that $A$ is not $K$-$S_\mathfrak{S}$-subnormal in $G$. Then $A < V$. By hypothesis, $|G : A|$ is a power of $p$ for some prime $p$. Then $|V : A|$ is a power of $p$. But $A$ is a Hall subgroup of $V$. Hence $p$ does not divides $|A|$, so $A$ is a Hall subgroup of $G$.

The lemma is proved.

**Lemma 2.3** (Knyagina and Monakhov [19]). Let $H$, $K$ and $N$ be pairwise permutable subgroups of $G$ and $H$ is a Hall subgroup of $G$. Then $N \cap HK = (N \cap H)(N \cap K)$.

The following lemma is well-known (see, for example, Lemma 3.29 in [4]).

**Lemma 2.4.** Let $H/K$ be an abelian chief factor of $G$ and let $M$ be a maximal subgroup of $G$ with $K \leq M$ and $HM = G$. Then $G/MG \simeq (H/K) \rtimes (G/C_G(H/K))$.

**Proof of Theorem 1.8.** First we show that if every subgroup of $G$ is $H_{\mathcal{L}}$-embedded in $G$, then every $K$-$S_\mathfrak{S}$-subnormal subgroup $V$ of $G$ such that $V \not\in \mathfrak{S}$ and $V$ has the structure $S_\mathfrak{S}$. Assume that this is false and let $G$ be a counterexample of minimal order. Then every subgroup of $G$ is $H_{\mathcal{L}}$-embedded in $G$ and there exists a subgroup $V \in \mathcal{L}$ such that $V$ fails to have the structure $S_\mathfrak{S}$. Obviously, $G \not\in \mathfrak{S}$. Otherwise such $V$ cannot exist since $\mathfrak{S}$ is hereditary.

1. If $E$ is any proper subgroup of $G$, then every $K$-$S_\mathfrak{S}$-subnormal subgroup $H$ of $E$ such that $H \not\in \mathfrak{S}$ has the structure $S_\mathfrak{S}$. Hence $V = G$. 

Since
\[ E/(E \cap G^3) \simeq G^3 E/G^3 \leq G/G^3 \in \mathfrak{F}, \]
\( E^3 \leq G^3 \in \mathfrak{S} \) and so the hypothesis holds for \( E \) by Lemma 2.2(1). Hence we have (1) by the choice of \( G \).

(2) If \( H \) is a subgroup of \( G \) such that \( |G : H| \) is power of prime and \( H \) is not a Hall subgroup of \( G \), then \( H \) is \( K^-\mathfrak{F}\)-subnormal in \( G \) (This directly follows from Lemma 2.2(3)).

(3) If \( N \) is any non-identity normal subgroup of \( G \), then every \( K^-\mathfrak{F}\)-subnormal subgroup \( H/N \) of \( G/N \) such that \( H/N \notin \mathfrak{F} \) has the structure \( S_3 \).

Since
\[ (G/N)^3 = G^3 N/N \simeq G^3 / (G^3 \cap N) \in \mathfrak{S} \]
by Proposition 2.2.8 in [2], this follows from Lemma 2.2(2) and the choice of \( G \).

(4) Write \( D = G^3 \) and \( \pi = \pi(D) \). Then \( D \) is a Hall \( \pi \)-subgroup of \( G \). Hence every Hall \( \pi'\)-subgroup of \( G \) is a complement of \( D \) in \( G \).

Recall that \( G \notin \mathfrak{F} \), so \( D \neq 1 \). Suppose that this claim is false and let \( P \) be a Sylow \( p \)-subgroup of \( D \) such that \( 1 < P < G_p \), where \( G_p \) is a Sylow \( p \)-subgroup of \( G \). Let \( R \) be a minimal normal subgroup of \( G \) contained in \( D \). Then \( R \) is a \( q \)-group for some prime \( q \) since \( D \) is soluble by hypothesis. Moreover, \( D/R = (G/R)^3 \) is a Hall subgroup of \( G/R \) by Claim (3). Suppose that \( PR/R \neq 1 \). Then \( PR/R \) is a Sylow \( p \)-subgroup of \( G/R \). If \( q \neq p \), then \( P \) is a Sylow \( p \)-subgroup of \( G \). This contradicts the fact that \( P < G_p \). Hence \( q = p \), so \( R \leq P \) and therefore \( P/R \) is a Sylow \( p \)-subgroup of \( G/R \). It follows that \( P \) is a Sylow \( p \)-subgroup of \( G \). This contradiction shows that \( PR/R = 1 \), which implies that \( P = R \) is a Sylow \( p \)-subgroup of \( D \). Therefore \( R \) is the unique minimal normal subgroup of \( G \) contained in \( D \). It is also clear that a \( p \)-complement \( U \) of \( D \) is a Hall subgroup of \( G \).

Now we show that \( R \notin \Phi(G) \). Indeed, assume that \( R \leq \Phi(G) \). Then \( D \neq R \) since \( D = G^3 \). Hence \( U \neq 1 \). Since \( D \) is soluble, every two \( p \)-complements of \( D \) are conjugate in \( D \) and so the Frattini Argument implies that \( G = DN_G(U) = (RU)N_G(U) = RN_G(U) = N_G(U) \) since \( R \leq \Phi(G) \). Therefore \( G \) has a minimal normal subgroup \( L \) such that \( L \neq R \) and \( L \leq D \). This contradiction shows that \( R \notin \Phi(G) \).

Let \( S \) be a maximal subgroup of \( G \) such that \( RS = G \). Then \( |G : S| \) is a power of \( p \). On the other hand, since \( R \) is not a Sylow \( p \)-subgroup of \( G \), \( p \) divides \( |S| \). Then \( S \) is not a Hall subgroup of \( G \) and so \( S \) is \( K^-\mathfrak{F}\)-subnormal in \( G \) by Claim (2). Therefore \( G/S_G \in \mathfrak{F} \), which implies that \( R \leq D \leq S_G \leq S \). This contradiction completes the proof of (4).

(5) If \( E \) is any proper subgroup of \( G \) containing a complement of \( D \) in \( G \), then \( E \) is a Hall subgroup of \( G \) and \( E \) is not \( K^-\mathfrak{F}\)-subnormal in \( G \).

It is enough to show that \( E \) is not \( K^-\mathfrak{F}\)-subnormal in \( G \). Assume that there is a subgroup chain
\[ E = E_0 \leq E_1 \leq \cdots \leq E_r = G \]
such that either \( E_{i-1} \leq E_i \) or \( E_i / (E_{i-1}) E_i \in \mathfrak{F} \) for all \( i = 1, \ldots, r \). Let \( W = E_{r-1} \). We can assume without loss of generality that \( W \neq G \) since \( E < G \).
First assume that $W$ is normal in $G$. Then $G/W \cong DW/W \cong D/(D \cap W)$ is soluble, so there exists a normal maximal subgroup $U$ of $G$ such that $W \leq U$. Then $G/U \in \mathfrak{F}$ since $\mathfrak{F}$ contains all nilpotent groups by our assumption on $\mathfrak{F}$, hence $G = DE \leq U < G$. This contradiction implies that $G/W_G \in \mathfrak{F}$, so $D \leq W_G$ and hence $G = DE \leq W < G$. This contradiction shows that every proper subgroup $E$ of $G$ containing a complement of $D$ in $G$ is not $K$-$\mathfrak{F}$-subnormal in $G$. Therefore the claim is true.

(6) For every $p \in \pi$ and any complement $M$ of $D$ in $G$, there is a Sylow $p$-subgroup $P$ of $D$ such that $M \leq N_G(P)$ and $M \not\leq N_G(L)$ for each non-identity proper subgroup $L$ of $P$, that is, $M$ acts irreducibly on $P$.

The Frattini Argument and Claim (4) imply that for some Sylow $p$-subgroup $P$ of $G$ we have $M \leq N_G(P)$. Moreover, if for some non-identity proper subgroup $L$ of $P$ we have $M \leq N_G(L)$, then $ML$ is not a Hall subgroup of $G$, contrary to Claim (5). Hence we have (6).

(7) $D$ possesses a Sylow tower.

Let $R$ be a minimal normal subgroup of $G$ contained in $D$. Then $R$ is a $p$-group for some prime $p$ since $D$ is soluble. Moreover, Claim (6) implies that $R$ is a Sylow $p$-subgroup of $D$. On the other hand, by Claim (3), $D/R$ possesses a Sylow tower. Hence we have (7).

(8) Every chief factor of $G$ below $D$ is $\mathfrak{F}$-eccentric.

Let $R$ be a minimal normal subgroup of $G$ contained in $D$. Then $R$ is a Sylow $p$-subgroup of $G$ for some prime $p$ by Claim (6). Let $U$ be a $p$-complement of $G$. We have $G = RU$ and, by Lemma 2.4,

$$G/U_G = (RU_G/U_G) \times (U/U_G) \simeq R \times (G/C_G(R)).$$

If $R/1$ were $\mathfrak{F}$-central, then $D \leq U_G$. But then $G = U$, a contradiction. Hence $R/1$ is $\mathfrak{F}$-eccentric in $G$.

If $G/R \in \mathfrak{F}$, then $R = D$ and we are done. Otherwise $G/R \not\in \mathfrak{F}$ and, by Claim (3), every chief factor of $G/R$ below $D/R$ is $\mathfrak{F}$-eccentric in $G/R$. Thus, every chief factor of $G$ below $D$ is $\mathfrak{F}$-eccentric in $G$ by the Jordan-Hölder theorem for the chief series.

The final contradiction for the necessity. From Claims (4), (6)–(8) it follows that $V = G$ has the structure $S_\mathfrak{F}$, against our assumption. Therefore the necessity of the condition of the theorem is proved.

Now suppose that every $K$-$\mathfrak{F}$-subnormal subgroup $E$ of $G$ such that $E \not\in \mathfrak{F}$ has the structure $S_{\mathfrak{F}}$. We will prove by induction on $|G|$ that in this case every subgroup of $G$ is $H_\mathcal{L}$-embedded in $G$. Assume that this is false and let $G$ be a counterexample of minimal order. Then $D \neq 1$ by Remark 1.5(1). By hypothesis, $G = D \rtimes M$, where $D = G^\mathfrak{F}$ is a Hall subgroup of $G$, $D$ possesses a Sylow tower and $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$. Let $\pi = \pi(D)$. Then $G$ is $\pi$-soluble.

First we show that every subgroup $A$ of $G$ containing $M$ is a Hall subgroup of $G$. If $A = M$, it is clear. Now assume that $M < A$ and let $D_0 = D \cap A$. Then $A = D_0 \rtimes M$ and $D_0 \neq 1$. Let $p \in \pi(D_0) \subseteq \pi$, $\pi_0 = \{p \cup \pi'\}$ and $\pi_1 = \pi \setminus \{p\}$. Thus, if $H < D_0 \rtimes M$ is a $p$-complement of $A$ in $G$, then $H \not\in D_0 \rtimes \mathfrak{F}$, hence $H \not\in D_0 \rtimes \mathcal{F}$. Therefore $H \not\in D_0 \rtimes \mathfrak{F}$, contrary to our assumption on $H$.

Hence $G$ is $K$-$\mathfrak{F}$-subnormal in $G$. By Claim (4), $G$ possesses a Sylow tower. Therefore $G$ is $H_\mathcal{L}$-embedded in $G$.
The Frattini Argument implies that for some Sylow $p$-subgroup $P_0$ of $D_0$ we have $M \leq N_G(P_0)$. Moreover, $G$ is evidently $\pi_1$-soluble and so, by Theorem 3.6 in [20, Chapter 6], for some Sylow $p$-subgroup $P$ of $D$ we have $P_0M \leq PM = MP$, where $P = PM \cap D$, so $M \leq N_G(P)$. But $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$, so $P_0 = P$. Therefore every Sylow subgroup of $A$ is a Sylow subgroup of $G$. Hence $A$ is a Hall subgroup of $G$.

Now let $A$ be any subgroup of $G$. First assume that $DA \leq G$. Since $G/D \in \mathfrak{F}$ and the formation $\mathfrak{F}$ is hereditary by our assumption on $\mathfrak{F}$, the subgroup $DA/D$ is $K$-$\mathfrak{F}$-subnormal in $G/D$ and so $DA$ is $K$-$\mathfrak{F}$-subnormal in $G$ by Lemma 2.1(3). Therefore $DA$ has the structure $S_3$ by hypothesis and so $A$ is a Hall subgroup of some $K$-$\mathfrak{F}$-subnormal subgroup $W$ of $DA$ by induction. But then $W$ is a $K$-$\mathfrak{F}$-subnormal subgroup of $G$ and so $A$ is $H_C$-embedded in $G$.

Finally, assume that $DA = G$. Since $G$ and $A$ are $\pi$-soluble, for some $x$ we have $M \leq A^x$ and so from the above we get that $A^x$ is a Hall subgroup of $G$, so $A^x$ and $A$ are $H_C$-embedded in $G$. Therefore the sufficiency of the condition of the theorem is proved.

The theorem is proved.

3. Proof of Theorem 1.2

We use $Z_\mathfrak{F}(G)$ to denote the product of all normal subgroups $A$ of $G$ such that either $A = 1$ or every chief factor of $G$ below $A$ is $\mathfrak{F}$-central in $G$.

**Lemma 3.1** (See Theorem 2.7 in [21, Ch. 1]). Let $Z = Z_\mathfrak{F}(G)$ and let $N$ be a normal subgroup of $G$ contained in $Z$. Then

1. $N \in \mathfrak{F}$ and $Z/N = Z_\mathfrak{F}(G/N)$.
2. Moreover, if $G/Z \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

The following lemma is a corollary of Lemmas 2.1(3) and 3.1(1).

**Lemma 3.2.** Every subgroup $A \in L_{\mathfrak{F}}(G)$ is $K$-$\mathfrak{F}$-subnormal in $G$.

**Proof of Theorem 1.2.** (1) Let $p$ be a prime dividing $|D|$. Suppose that for a Sylow $p$-subgroup $P$ of $D$ we have $p < |P|$. By hypothesis, $G$ possesses an $H_C$-embedded subgroup $A$ of order $p|G|^{1/p}$. Then $A$ is a Hall subgroup of some subgroup $E \in L_{\mathfrak{F}}(G)$. It is clear that $A = E$, so $A^G/A_G \leq Z_\mathfrak{F}(G/A_G)$. On the other hand, $G/A^G$ is a $p$-group and hence $G/A^G \in \mathfrak{F}$ since $\mathfrak{F}$ contains all nilpotent groups by our assumption on $\mathfrak{F}$, so $G/A_G \in \mathfrak{F}$ by Lemma 3.1. It follows that $D \leq A_G \leq A$ and so $|P| \leq p$, a contradiction. Hence $|D|$ is square free. Therefore $D$ is supersoluble by [22, IV, Satz 2.8], so the Sylow $p$-subgroup $P$ of $D$, where $p$ is the largest prime dividing $|D|$, is normal and so characteristic in $D$. Hence $P$ is normal in $G$ and $G/C_G(P)$ is cyclic since $|P|$ is a prime, which implies $D \leq C_G(P)$. Therefore $P \leq Z(D)$, so for a $p$-complement $D_0$ of $D$ we have $D = P \times D_0$ in $D$, where $D_0$ is characteristic in $D$ and so normal in $G$. Similarly, we can show that for some prime $q$ dividing $|D_0|$ and for a Sylow $q$-subgroup $Q$ of $D_0$ we have $Q \leq Z(D_0)$, which implies that $Q$ has a normal complement in $D_0$ and so on. Therefore $D$ is a cyclic group of square free order. Finally, if 2 divides $|D|$ and $Q$ is the Sylow 2-subgroup of $D$ and $L$ is the 2-complement of $D$, then $C_G(Q) = G$.
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since $|Q| = 2$ and so $G/L \in \mathcal{F}$ by Lemma 3.1(2) since $\mathcal{F}$ contains all nilpotent groups. But then $D \leq L < D$, a contradiction. Therefore $|D|$ is odd.

(2) We prove this statement by induction on $|G|$. First note that if $M$ is a minimal supplement to $D$ in $G$, then $M \cap D \leq \Phi(M)$ and so $M \in \mathcal{F}$ since $\mathcal{F}$ is saturated and $M/(M \cap D) \simeq G/D \in \mathcal{F}$. On the other hand, for the complement $L$ of $M \cap D$ in $D$ we have $L \leq G$ (since $L$ is cyclic of square free order) and $G/L = DM/L = L(M \cap D)/M/L \simeq M \in \mathcal{F}$, so $L = D$ and hence $M$ is a complement to $D$ in $G$. Therefore for each integer $r$ dividing $|M|$ there exists a subgroup of $M$ of order $r$.

Let $d$ be any integer dividing $|G|$, $\pi = \pi(d)$ and $\pi_1 = \pi \cap \pi(D)$. Assume that $\pi_1 = \emptyset$. Then $d$ divides $|M| = [G/D]$ and a subgroup $H$ of $M$ of order $d$ is a Hall subgroup in $D H$, where $D H \in \mathcal{L}$ since $D \leq (D H G)G$ and $G/D \in \mathcal{F}$. Hence $H$ is $H_{\mathcal{L}}$-embedded in $G$.

Now assume that $\pi_1 \neq \emptyset$. Suppose that for some $p \in \pi_1$ the prime $p$ divides $d : p$ and let $L$ be the subgroup of order $p$ in $D$. Then, by induction, $G/L$ possesses an $H_{\mathcal{L}}$-embedded subgroup $V/L$ of order $d : p$. Let $E/L$ be a subgroup of $G/L$ such that $E/L \in \mathcal{L}_G(G/L)$ and $V/L$ is a Hall subgroup of $E/L$. Then, in view of the $G$-isomorphism

$$E^{G}/E_G \simeq (E^{G}/L)/((E_G/L) = (E/L)^{G/L}/(E/L)_{G/L},$$

$E \in \mathcal{L}_G(G)$, and $V$ is a Hall subgroup of $E$ since $\pi(V/L) = \pi(V)$. Hence $|V| = d$ and $V$ is $H_{\mathcal{L}}$-embedded in $G$.

Finally, assume that for each $p \in \pi_1$ the prime $p$ does not divide $d : p$, that is, $d = p_1 \cdots p_l q_1^{\alpha_1} \cdots q_n^{\alpha_n}$, where $\pi_1 = \{p_1, \ldots, p_l\}$ and $\{q_1, \ldots, q_n\} = \pi \setminus \pi_1$. Let $A$ be a subgroup of $M$ of order $q_1^{\alpha_1} \cdots q_n^{\alpha_n}$, $B$ the subgroup of $D$ of order $p_1 \cdots p_l$ and $H = AB$. Then $|H| = d$ and $H$ is a Hall subgroup of the subgroup $DA \in \mathcal{L}$. Hence $H$ is $H_{\mathcal{L}}$-embedded in $G$. Therefore the statement (2) holds for $G$.

(3) First we show by induction on $|G|$ that if every subgroup $H$ of $G$ is $H_{\mathcal{L}}$-embedded in $G$, then $D$ is a Hall cyclic subgroup of odd square free order. If $G \in \mathcal{F}$, it is true. Now assume that $G \notin \mathcal{F}$. Then, in view of Lemma 3.2 and Theorem 1.8, $G = D \times M$, where $D$ is a Hall subgroup of $G$ and every chief factor of $G$ below $D$ is $\mathcal{F}$-eccentric. Moreover, $D$ possesses a Sylow tower and $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$. Let $p$ be a prime dividing $|D|$ such that a Sylow $p$-subgroup $P$ of $D$ is normal in $D$ and so it is normal in $G$. Then $P$ is a minimal normal subgroup of $G$ since $M$ acts irreducibly on $P$.

We show that the hypothesis holds for $G/P$. Let $H/P$ be any subgroup of $G/P$. Then $H$ is a Hall subgroup of some subgroup $E \in \mathcal{L}$ by hypothesis, so $H/P$ is a Hall subgroup of $E/P$. Moreover, in view of the $G$-isomorphism $E^{G}/E_G \simeq (E/P)^{G/P}/((E/P)_{G/P}$ we get that every chief factor of $G/P$ between $(E/P)_{G/P}$ and $(E/P)^{G/P}$ is $\mathcal{F}$-central in $G/P$, so $E/P \in \mathcal{L}_G(G/P)$. Therefore $H/P$ is $H_{\mathcal{L}_{G/P}}$-embedded in $G/P$, where $G/P \in \mathcal{L}_{G/P}$. Therefore the hypothesis holds for $G/P$. On the other hand, $D/P = (G/P)^{G/P}$. Therefore $D/P$ is a cyclic group of odd square free order by induction.

Assume that $|P| > p$ and let $L$ be a subgroup of order $p$ in $P$. Then $L$ is a Sylow $p$-subgroup of some subgroup $E \in \mathcal{L}$, so $L = P \cap E$. Hence $P \notin E_G$, so $P \cap E_G = 1$
since $P$ is a minimal normal subgroup of $G$. On the other hand, $L \leq E \leq E^G$ and so $P \leq E^G$. From the $G$-isomorphism $P \cong E^G P / E^G$ it follows that $P / 1$ is $\mathfrak{F}$-central in $G$. This contradiction shows that $|P| = p$, so $D$ is of square free order. Then $G/C_G(P)$ is cyclic, so $D \leq C_G(P)$ and hence $P \leq Z(D)$. It follows that $D$ is nilpotent and so it is a cyclic subgroup of square free order. Moreover, $|D|$ is odd (see the proof of Statement (1)).

Now assume that $D$ is a cyclic Hall subgroup of square free order of $G$ and let $A$ be any subgroup of $G$. Let $M$ be a complement of $D$ in $G$. Then $A = (D \cap A)(M^x \cap A)$ for some $x \in G$, so $A$ is a Hall subgroup of $E := D(M^x \cap A)$. On the other hand, $D \leq E_G$ and so $E \in \mathcal{L}$. Hence $A$ is $H_L$-embedded in $G$.

The theorem is proved.

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