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On the generalized σ-Fitting subgroup of finite groups

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Abstract – Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, and let G be a finite group. A chief factor H/K of G is said to be *σ-central* (in G) if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is a σ_i-group for some $i = i(H/K)$; otherwise, it is called *σ-eccentric* (in G). We say that G is: *σ-nilpotent* if every chief factor of G is σ-central; *σ-quasinilpotent* if for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. The product of all normal σ-nilpotent (respectively σ-quasinilpotent) subgroups of G is said to be the *σ-Fitting subgroup* (respectively the *generalized σ-Fitting subgroup*) of G and we denote it by $F_\sigma(G)$ (respectively by $F^*_\sigma(G)$). Our main goal here is to study the relations between the subgroups $F_\sigma(G)$ and $F^*_\sigma(G)$, and the influence of these two subgroups on the structure of G.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is

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an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We say that: G is *σ-primary* [1] provided it is a σ_i-group for some i; an automorphism α of G is σ_i-primary if $\langle\alpha\rangle$ is a σ_i-subgroup of $\mathrm{Aut}(G)$.

In the mathematical practice, we often deal with the following three special partitions of $\mathbb{P}$:

$\sigma^1 = \{\{2\}, \{3\}, \ldots\}, \sigma^\pi = \{\pi, \pi'\}$, and $\sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$, where $\pi = \{p_1, \ldots, p_n\}$.

The group G is called: *σ-soluble* [1] if every chief factor of G is σ-primary; *σ-decomposable* [2] or *σ-nilpotent* [3] if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$.

Remark 1.1. (i) G is: soluble if and only if G is σ^1-soluble, π-soluble if and only if G is $\sigma^{1\pi}$-soluble, π-separable if and only if G is σ^π-soluble.

(ii) Let $G \neq 1$ and $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \neq \emptyset\}$. Without loss of generality we can assume that $\sigma(G) = \{\sigma_1, \ldots, \sigma_t\}$. Then G is σ-nilpotent if and only if $G = O_{\sigma_1}(G) \times \cdots \times O_{\sigma_t}(G)$. Thus, G is: σ^1-nilpotent if and only if G is nilpotent, σ^π-nilpotent if and only if $G = O_\pi(G) \times O_{\pi'}(G)$, $\sigma^{1\pi}$-nilpotent if and only if $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$.

Let H/K be a chief factor of G. Then we say that H/K is *σ-central* (in G) [1] if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary; otherwise, it is called *σ-eccentric* (in G). A normal subgroup E of G is said to be *σ-hypercentral* (in G) if either $E = 1$ or every chief factor of G below E is σ-central in G.

The σ-nilpotent groups have many applications in the formation theory [2, 4, 5, 6] (see also the recent papers [1, 3, 7, 8, 9, 10, 11] and the survey [12]), and such groups are exactly the groups whose chief factors are σ-central (see Proposition 2.3 in [1]).

In this paper, we consider the following generalization of σ-nilpotency.

Definition 1.2. We say that G is *σ-quasinilpotent* if given any σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner (cf. [13, X, Definition 13.2]).

Note that G is called *quasinilpotent* if given any chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. Therefore G is quasinilpotent if and only if it is σ^1-quasinilpotent.

Let $Z_\sigma(G)$ denote the product of all normal σ-hypercentral subgroups of G. It is not difficult to show (see Lemma 2.7(i) below) that $Z_\sigma(G)$ is also σ-hypercentral in G. We call the subgroup $Z_\sigma(G)$ the *σ-hypercentre* of G. Dually, we define the *σ-nilpotent residual* $G^{\mathfrak{N}_\sigma}$ of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N; $G^{\mathfrak{S}_\sigma}$ is the *σ-soluble residual* of G.

Definition 1.3. (i) The product of all normal σ-nilpotent (respectively σ-quasinilpotent) subgroups of G is said to be the *σ-Fitting subgroup* [1] (respectively the *generalized σ-Fitting subgroup*) of G and denoted by $F_\sigma(G)$ (respectively by $F^*_\sigma(G)$).

(ii) We use $E_\sigma(G)$ to denote the σ-soluble residual of $F^*_\sigma(G)$, and we say that $E_\sigma(G)$ is the *σ-layer of* G (cf. [13, X, Definition 13.14]).

Note that in the case when $\sigma = \sigma^1$ the subgroups $F_\sigma(G)$, $F^*_\sigma(G)$ and $E_\sigma(G)$ coincide respectively with $F(G)$, $F^*(G)$ and $E(G)$.

Before continuing, consider some examples.

Example 1.4. Let $G = (A_5 \times A_7) \wr \langle x \rangle = K \rtimes \langle x \rangle$, where $|x| = p > 5$ is a prime and K is the base group of the regular wreath product G. Let $R = A_5^\natural$ and $L = A_7^\natural$ (we use here the terminology in [15, Ch.A]). Let $\sigma = \{\{2,3,5\}, \{2,3,5\}'\}$. Then $K = R \times L$ and so, in view of Remark 1.1(ii), $F_\sigma(G) = R$. It is clear also that $K \leq F^*_\sigma(G)$ and the automorphism of R induced by x is not inner. Hence $F^*_\sigma(G) = K$. Finally, $E_\sigma(G) = L$ and $E(G) = K$.

We say that G is: *σ-perfect* if $G^{\mathfrak{N}_\sigma} = G$; *$\sigma$-semisimple* if either $G = 1$ or $G = A_1 \times \cdots \times A_t$ is the direct product of simple non-σ-primary groups $A_1, \ldots, A_t$.

Example 1.5. Let $G = (A_5 \wr A_5) \times (A_7 \times A_{11})$ and $\sigma = \{\{2,3,5\}, \{2,3,5\}'\}$. Then G is σ-quasinilpotent but G is not σ-nilpotent. The group $A_7 \times A_{11}$ is σ-semisimple and σ-perfect.

A subgroup A of G is *σ-subnormal* in G [1] if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$. Note that A is subnormal in G if and only if it is σ^1-subnormal in G.

In this paper, we study properties and relations between the subgroups $F_\sigma(G)$, $F^*_\sigma(G)$ and $E_\sigma(G)$. Our main observations here are the following two results which, in particular, show that the subgroup $F^*_\sigma(G)$ has properties similar to the properties of the generalized Fitting subgroup $F^*(G)$ of G (see Section 4 below and Ch.X in [13]).

Theorem A. *The following statements hold:*

(i) *$F_\sigma(G)$ is the largest normal σ-nilpotent subgroup of G and $F^*_\sigma(G)$ is the largest normal σ-quasinilpotent subgroup of G.*

(ii) *A σ-subnormal subgroup A of G is contained in $F^*_\sigma(G)$ (respectively in $F_\sigma(G)$) if and only if A is σ-quasinilpotent (respectively σ-nilpotent). Hence $F^*_\sigma(G) \cap A = F^*_\sigma(A)$ and $F_\sigma(G) \cap A = F_\sigma(A)$.*

In the case when $\sigma = \sigma^1$, we get from Theorem A(i)(ii) the following

Corollary 1.6 (See [13, X, Theorem 13.10]). *$F^*(G)$ is quasinilpotent and every subnormal quasinilpotent subgroup of G is contained in $F^*(G)$.*

Theorem B. *Let $F = F_\sigma(G)$, $F^* = F^*_\sigma(G)$, and $E = E_\sigma(G)$. Then the following statements hold:*

(i) *$F = Z_\sigma(F^*)$ and F^*/F is σ-semisimple.*

(ii) *$F^* = EF$ and $F = C_{F^*}(E)$, so $F^* = C_{F^*}(F)F$. Also $E \cap F = Z(E)$, E is σ-perfect and $E/Z(E)$ is σ-semisimple.*

(iii) *$F/Z_\sigma(G) = F_\sigma(G/Z_\sigma(G))$ and $F^*/Z_\sigma(G) = F^*_\sigma(G/Z_\sigma(G))$.*

(iv) *Every σ-perfect σ-quasinilpotent σ-subnormal subgroup H of G is contained in $E_\sigma(G)$. Moreover, $E_\sigma(E_\sigma(G)) = E_\sigma(G)$.*

As a first application of Theorems A and B, we prove also the following

Theorem C. *G is σ-quasinilpotent if and only if given any σ-eccentric chief* *factor H/K of G below $F^*_\sigma(G)$, every automorphism of H/K induced by an element of G is inner.*

In the case when $\sigma = \sigma^1$, we get from Theorem C the following

Corollary 1.7. *G is quasinilpotent if and only if given any chief factor H/K of G below $F^*(G)$, every automorphism of H/K induced by an element of G is inner.*

Let H/K be a chief factor of G. We define the *σ-centralizer* $C^\sigma_G(H/K)$ of H/K in G: $C^\sigma_G(H/K) = C_G(H/K)$ if H/K is not σ-primary, and $C^\sigma_G(H/K) = O_{\sigma_i}(G)C_G(H/K)$ in the case when H/K is σ_i-primary.

Now, by analogy with the *inneriser* of H/K (see [6, p.41]), we define the *σ-inneriser* $C^{*\sigma}_G(H/K)$ of H/K in G: $C^{*\sigma}_G(H/K) = HC^\sigma_G(H/K)$ if H/K is not σ-primary, and $C^{*\sigma}_G(H/K) = C^\sigma_G(H/K)$ in the case when H/K is σ-primary.

As one more application of Theorems A and B we prove the following

Theorem D. (i) *The subgroup $F_\sigma(G)$ coincides with the intersection of the σ-centralizers of the chief factors of G.*

(ii) *The subgroup $F^*_\sigma(G)$ coincides with the intersection of the σ-innerisers of the chief factors of G.*

Corollary 1.8 (Ballester-Bolinches and Ezquerro [6, p.97]). *The subgroup $F^*(G)$ coincides with the intersection of the innerisers of the chief factors of G.*

In Section 4 we discuss further applications of Theorems A and B.

2. Preliminaries

Lemma 2.1. (i) *If $K \leq L < T \leq H \leq E \trianglelefteq G$, where H/K is a chief factor of G and T/L is a chief factor of E, and an element $x \in E$ induces an inner automorphism on H/K, then x induces an inner automorphism on T/L. Moreover, if $H/K = (H_1/K) \times \cdots \times (H_t/K)$, where $H_1/K, \dots, H_t/K$ are normal subgroups of E/K and x induces inner automorphisms on these factors, then x induces an inner automorphism on H/K.*

(ii) *If G is a σ-quasinilpotent group and N is a normal subgroup of G, then N and G/N are σ-quasinilpotent.*

(iii) *If G/N and G/L are σ-quasinilpotent, then $G/(N \cap L)$ is also σ-quasinilpotent.*

Proof. (i) See the proof of Lemma 13.1 in [13, X].

(ii), (iii) See the proof of Lemma 13.3 in [13, X].

Lemma 2.2. *Let H/K be a chief factor of G. Then every automorphism of H/K induced by an element of G is inner if and only if $G/K = (H/K)C_{G/K}(H/K)$.*

Proof. See the proof of Lemma 13.4 in [13, X].

Lemma 2.3 (see Proposition 2.3 in [1]). *The following are equivalent:*

(i) *G is σ-nilpotent.*

(ii) *G has a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$.*

(iii) *Every chief factor of G is σ-central in G.*

Lemma 2.4. *Let N be a normal σ_i-subgroup of G. Then $N \leq Z_\sigma(G)$ if and only if $O^{\sigma_i}(G) \leq C_G(N)$.*

Proof. If $O^{\sigma_i}(G) \leq C_G(N)$, then for every chief factor H/K of G below N both groups H/K and $G/C_G(H/K)$ are σ_i-group since $G/O^{\sigma_i}(G)$ is a σ_i-group. Hence $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary. Thus $N \leq Z_\sigma(G)$.

Now assume that $N \leq Z_\sigma(G)$. Let $1 = Z_0 < Z_1 < \ldots < Z_t = N$ be a chief series of G below N and $C_i = C_G(Z_i/Z_{i-1})$. Let $C = C_1 \cap \cdots \cap C_t$. Then G/C is a σ_i-group. On the other hand, $C/C_G(N) \simeq A \leq \mathrm{Aut}(N)$ stabilizes the series $1 = Z_0 < Z_1 < \ldots < Z_t = N$, so $C/C_G(N)$ is a $\pi(N)$-group by Theorem 0.1 in [14]. Hence $G/C_G(N)$ is a σ_i-group and so $O^{\sigma_i}(G) \leq C_G(N)$.

The lemma is proved.

The next two lemmas are evident.

Lemma 2.5. *$G^{\mathfrak{S}_\sigma}$ is σ-perfect.*

Lemma 2.6. *If H/K and T/L are G-isomorphic chief factors of G, then:*

(i) $(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L))$, and

(ii) $C_G(H/K) = C_G(T/L)$.

(iii) $C_G^\sigma(H/K) = C_G^\sigma(T/L)$.

We write $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$, and we say that G is a Π-*group* provided $\sigma(G) \subseteq \Pi \subseteq \sigma$.

Lemma 2.7. *Let $Z = Z_\sigma(G)$. Let A, B and N be subgroups of G, where N is normal in G.*

(i) *Z is σ-hypercentral in G.*

(ii) *$Z_\sigma(A)N/N \leq Z_\sigma(AN/N)$.*

(iii) *$Z_\sigma(B) \cap A \leq Z_\sigma(B \cap A)$.*

(iv) *If $N \leq Z$ and N is a Π-group, then N is σ-nilpotent and $G/C_G(N)$ is a σ-nilpotent Π-group.*

(v) *If G/Z is σ-nilpotent, then G is also σ-nilpotent.*

(vi) *If $N \leq Z$, then $Z/N = Z_\sigma(G/N)$.*

(vii) *If $G = A \times B$, then $Z = Z_\sigma(A) \times Z_\sigma(B)$.*

Proof. (i) It is enough to consider the case when $Z = A_1A_2$, where A_1 and A_2 are normal σ-hypercentral subgroups of G. Moreover, in view of the Jordan-Hölder theorem, it is enough to show that if $A_1 \leq K < H \leq A_1A_2$, then H/K is σ-central. But in this case we have $H = A_1(H \cap A_2)$, where evidently $H \cap A_2 \nleq K$, so we have the G-isomorphism $(H \cap A_2)/(K \cap A_2) \simeq (H \cap A_2)K/K = H/K$, and hence H/K is σ-central in G by Lemma 2.6.

(ii) First assume that $A = G$, and let H/K be a chief factor of G such that $N \leq K < H \leq NZ$. Then H/K is G-isomorphic to the chief factor $(H \cap Z)/(K \cap Z)$ of G below Z. Therefore H/K is σ-central in G by Assertion (i) and Lemma 2.6. Consequently, $ZN/N \leq Z_\sigma(G/N)$.

Now let A be any subgroup of G, and let $f : A/A \cap N \to AN/N$ be the canonical isomorphism from $A/A \cap N$ onto AN/N. Then $f(Z_\sigma(A/A \cap N)) = Z_\sigma(AN/N)$ and

$$f(Z_\sigma(A)(A \cap N)/(A \cap N)) = Z_\sigma(A)N/N.$$

Hence, in view of the preceding paragraph, we have

$$Z_\sigma(A)(A \cap N)/(A \cap N) \leq Z_\sigma(A/A \cap N).$$

Hence $Z_\sigma(A)N/N \leq Z_\sigma(AN/N)$.

(iii) First assume that $B = G$, and let $1 = Z_0 < Z_1 < \ldots < Z_t = Z$ be a chief series of G below Z and $C_i = C_G(Z_i/Z_{i-1})$. Now consider the series

$$1 = Z_0 \cap A \leq Z_1 \cap A \leq \ldots \leq Z_t \cap A = Z \cap A.$$

We can assume without loss of generality that this series is a chief series of A below $Z \cap A$.

Let $i \in \{1, \ldots, t\}$. Then, by Assertion (i), Z_i/Z_{i-1} is σ-central in G, $(Z_i/Z_{i-1}) \rtimes (G/C_i)$ is a σ_k-group say. Hence $(Z_i \cap A)/(Z_{i-1} \cap A)$ is a σ_k-group. On the other hand, $A/A \cap C_i \simeq C_iA/C_i$ is a σ_k-group and

$$A \cap C_i \leq C_A((Z_i \cap A)/(Z_{i-1} \cap A)).$$

Thus $(Z_i \cap A)/(Z_{i-1} \cap A)$ is σ-central in A. Therefore, in view of the Jordan-Hölder theorem for the chief series, we have $Z \cap A \leq Z_\sigma(A)$.

Now assume that B is any subgroup of G. Then, in view of the preceding paragraph, we have

$$Z_\sigma(B) \cap A = Z_\sigma(B) \cap (B \cap A) \leq Z_\sigma(B \cap A).$$

(iv) By Assertion (iii) and Lemma 2.3, N is σ-nilpotent, and it has a complete Hall σ-set $\{H_1, \ldots, H_t\}$ such that $N = H_1 \times \cdots \times H_t$. Then

$$C_G(N) = C_G(H_1) \cap \cdots \cap C_G(H_t).$$

It is clear that $H_1, \ldots, H_t$ are normal in G. We can assume without loss of generality that H_i is a σ_i-group. Then, by Assertion (i) and Lemma 2.4, $G/C_G(H_i)$ is a σ_i-group. Hence

$$G/C_G(N) = G/(C_G(H_1) \cap \cdots \cap C_G(H_t))$$

is a σ-nilpotent Π-group.

(v), (vi) These assertions are corollaries of Assertion (i) and the Jordan-Hölder theorem.

(vii) Let $Z_1 = Z_\sigma(A)$ and $Z_2 = Z_\sigma(B)$. Since Z_1 is characteristic in A, it is normal in G.

First assume that $Z_1 \neq 1$ and let R be a minimal normal subgroup of G contained in Z_1. Then R is σ-primary, R is a σ_i-group say, by Assertion (iv). Hence $A/C_A(R)$ is a σ_i-group by Lemma 2.4. But $C_G(R) = B(C_G(R) \cap A) = BC_A(R)$, so

$$G/C_G(R) = AB/C_A(R)B \simeq A/(A \cap C_A(R)B) = A/C_A(R)(A \cap B) = A/C_A(R)$$

is a σ_i-group and hence R is σ-central in G. Then $R \leq Z_\sigma(G)$, so $Z_\sigma(G)/R = Z_\sigma(G/R)$ by Assertion (vi). On the other hand, we have $Z_1/R = Z_\sigma(A/R)$ and $Z_2R/R = Z_\sigma(BR/R)$, so by induction we have

$$Z_\sigma(G/R) = Z_\sigma((A/R) \times (BR/R)) = Z_\sigma(A/R) \times Z_\sigma(BR/R)$$

$$= (Z_1/R) \times (Z_2R/R) = Z_1Z_2/R = Z/R.$$

Hence $Z = Z_1 \times Z_2$.

Finally, suppose that $Z_1 = 1 = Z_2$. Assume that $Z_\sigma(G) \neq 1$ and let R be a minimal normal subgroup of G contained in $Z_\sigma(G)$. Then, in view of Assertions (i) and (iii), $R \cap A = 1 = R \cap B$ and hence $G = A \times B \leq C_G(R)$. Thus $R \leq Z(G) = Z(A) \times Z(B) = 1$, a contradiction. Hence we have (vii).

The lemma is proved.

Lemma 2.8. *Given a group G the following are equivalent:*

(i) *G is σ-quasinilpotent.*

(ii) *$G/Z_\sigma(G)$ is σ-semisimple.*

(iii) *$G = E_\sigma(G)F_\sigma(G)$ and $[E_\sigma(G), F_\sigma(G)] = 1$. Hence $E_\sigma(G)/(E_\sigma(G) \cap F_\sigma(G)) = E_\sigma(G)/Z(E_\sigma(G))$ is σ-semisimple.*

(iv) *$G/F_\sigma(G)$ is σ-semisimple and $G = F_\sigma(G)C_G(F_\sigma(G))$.*

Proof. Let $Z = Z_\sigma(G)$, $F = F_\sigma(G)$ and $E = E_\sigma(G)$.

(i) $\Rightarrow$ (ii) Assume that this is false and let G be a counterexample of minimal order. Then the hypothesis holds for G/Z by Lemma 2.1(ii). On the other hand, $Z_\sigma(G/Z) = 1$ by Lemma 2.7(vi). Hence in the case when $Z \neq 1$, $G/Z_\sigma(G)$ is σ-semisimple by the choice of G.

Now assume that $Z = 1$ and let R be any minimal normal subgroup of G. Then $R/1$ is a σ-eccentric chief factor of G, so $G = RC_G(R)$ by Lemma 2.2. Therefore, since $Z(G) \leq Z = 1$, $C_G(R) \neq G$ and hence R is σ-semisimple. Thus $G = R \times C_G(R)$. Therefore $Z_\sigma(R) \times Z_\sigma(C_G(R)) = Z_\sigma(G) = 1$ by Lemma 2.7(vii). Moreover, the choice of G implies that $C_G(R)$ is σ-semisimple, so $G \simeq G/Z = G/1$ is σ-semisimple and hence Assertion (ii) is true, a contradiction.

(ii) $\Rightarrow$ (i) Let H/K be a chief factor of G. If $H \leq Z_\sigma(G)$, then H/K is σ-central in G by Lemma 2.7(i). Now suppose that $Z_\sigma(G) \leq K$. Since $G/Z_\sigma(G)$ is σ-semisimple by hypothesis, every automorphism of H/K induced by an element of G is inner by Lemma 2.2. Hence applying the Jordan-Hölder theorem, for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner and so G is σ-quasinilpotent.

(ii) $\Rightarrow$ (iii) First note that $Z \leq F$ by Lemma 2.7(iv), so $Z = F$ since G/Z is σ-semisimple by hypothesis. But then $G = EF$ and, by Lemma 2.7(iv), $G/C_G(F)$ is σ-nilpotent. Hence $E \leq C_G(F)$, so $[E, F] = 1$. Lemma 2.7(iii) implies that $Z \cap E = F \cap E \leq Z_\sigma(E)$, so $E/F \cap E$ is σ-semisimple and $F \cap E = Z(E)$.

(iii) $\Rightarrow$ (iv) This implication is evident.

(iv) $\Rightarrow$ (i) Let H/K be a chief factor of G. If $F_\sigma(G) \leq K$, then every automorphism of H/K induced by an element of G is inner by Lemma 2.2 since $G/F_\sigma(G)$ is σ-semisimple by hypothesis. Now suppose that $H \leq F_\sigma(G)$. Then

$$C_G(H/K) = C_G(H/K) \cap F_\sigma(G)C_G(F_\sigma(G)) = C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K),$$

so

$$G/C_G(H/K) = F_\sigma(G)C_G(F_\sigma(G))/C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K)$$

$$\simeq F_\sigma(G)/F_\sigma(G) \cap C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K) = F_\sigma(G)/C_{F_\sigma(G)}(H/K)Z(F_\sigma(G))$$

$$\simeq (F_\sigma(G)/C_{F_\sigma(G)}(H/K))/(C_{F_\sigma(G)}(H/K)Z(F_\sigma(G))/C_{F_\sigma(G)}(H/K))$$

is σ-primary by Lemma 2.4. Therefore H/K is σ-central in G. Now applying the Jordan-Hölder theorem, we get that for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. Hence G is σ-quasinilpotent.

The lemma is proved.

Lemma 2.9 (See Lemma 2.6 in [1]). *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *$A \cap K$ is σ-subnormal in K.*

(2) *If K is σ-subnormal in G, then $K \cap A$ and $\langle A, K\rangle$ are σ-subnormal in G.*

(3) *If A is a σ_i-group, then $A \leq O_{\sigma_i}(G)$. Hence if A is σ-nilpotent, then $A \leq F_\sigma(G)$.*

(4) *AN/N is σ-subnormal in G/N.*

Lemma 2.10 (See Corollary 2.4 and Lemma 2.5 in [1]). *The class of all σ-nilpotent groups $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups.*

Lemma 2.11. *If G is σ-semisimple and A is a σ-subnormal subgroup of G, then A is σ-semisimple.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Then $G = A_1 \times \cdots \times A_t$ for some simple non-σ-primary groups $A_1, \ldots, A_t$. Then $A_1, \ldots, A_t$ are non-abelian.

By hypothesis, there is a chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ of subgroups of G such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. Without loss of generality we can assume that $M < G$. Suppose that $A \leq M_G$. Then A is σ-subnormal in M_G by Lemma 2.9(1). On the other hand, M_G is σ-semisimple by [15, Ch.A, 4.13(b)], and so A is σ-semisimple by the choice of G.

This contradiction shows that $A \nleq M_G$, so G/M_G is σ-primary. But each chief factor of G is not σ-primary by the Jordan-Hölder theorem. This contradiction completes the proof of the lemma.

3. Proofs of Theorems A, B, C and D

Proof of Theorem A. (i) From Lemma 2.10, it follows that $F_\sigma(G)$ is the largest normal σ-nilpotent subgroup of G. In order to prove that $F^*_\sigma(G)$ is the largest normal σ-quasinilpotent subgroup of G, it is enough to show if $G = AB$, where A and B are normal σ-quasinilpotent subgroups of G, then G is σ-quasinilpotent. Assume that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G and $C = C_G(R)$. By Lemma 2.1(ii), the hypothesis holds for G/R, so the choice of G implies that G/R is σ-quasinilpotent. Therefore in view of Lemma 2.1(iii), R is a unique minimal normal subgroup of G.

Let $Z_1 = Z_\sigma(A)$ and $Z_2 = Z_\sigma(B)$. If $A \cap B = 1$, then $Z_\sigma(G) = Z_1 \times Z_2$ by

Lemma 2.7(vii). On the other hand, A/Z_1 and B/Z_2 are σ-semisimple by Lemma 2.8, so

$$G/Z = (A \times B)/(Z_1 \times Z_2) \simeq (A/Z_1) \times (B/Z_2)$$

is σ-semisimple. Hence G is σ-quasinilpotent by Lemma 2.8. Therefore $A \cap B \neq 1$, so $R \leq A \cap B$. First assume that R is σ-primary, R is a σ_i-group say. Then by Lemma 2.8, $R \leq Z_1 \cap Z_2$ and so $AC/C \simeq A/A \cap C$ and $BC/C \simeq B/B \cap C$ are σ_i-groups by Lemma 2.4. Hence $G/C = (AC/C)(BC/C)$ is a σ_i-group, so R is σ-central in G. Therefore $R \leq Z_\sigma(G)$ and so $Z_\sigma(G/R) = Z_\sigma(G)/R$ by Lemma 2.7(vi). Thus G is σ-quasinilpotent by Lemma 2.8.

Thus R is not σ-primary. Hence R is non-abelian, so $C = 1$. Then $R = R_1 \times \cdots \times R_t$, where $R_1, \ldots, R_t$ are minimal normal subgroups of A, so all these groups are simple by Lemma 2.8 and hence $R_1, \ldots, R_t$ are minimal normal subgroups of B. But then, by Lemma 2.2, $R_1 = R = A = B = G$ is σ-semisimple. Hence G is σ-quasinilpotent.

(ii) Let A be any σ-subnormal subgroup of G. First note that in view of Lemmas 2.9(3) and 2.10, A is contained in $F_\sigma(G)$ if and only if A is σ-nilpotent.

Now we show that if A is σ-quasinilpotent, then it is contained in $F^*_\sigma(G)$. Suppose that this is false and let G be a counterexample with $|G| + |A|$ minimal. Then for each σ-quasinilpotent σ-subnormal subgroup S of G such that $S < A$ we have $S \leq F^*_\sigma(G)$. Therefore the choice of G implies that if $A = NK$, where N and K are normal subgroups of A, then either $N = A$ or $K = A$. Lemma 2.8 implies that $A = A^{\mathfrak{N}_\sigma} F_\sigma(A)$. Then, in view of Lemma 2.1(ii), either $F_\sigma(A) = A$ or $A^{\mathfrak{N}_\sigma} = A$. But in the former case we have $A \leq F_\sigma(G) \leq F^*_\sigma(G)$ by Lemma 2.9(3), so $A^{\mathfrak{N}_\sigma} = A$.

By hypothesis, there is a chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ of subgroups of G such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. Without loss of generality we can assume that $M < G$. Suppose that $A \leq M_G$. Then A is σ-subnormal in M_G by Lemma 2.9(1), so $A \leq F^*_\sigma(M_G)$ by the choice of G. Since $F^*_\sigma(M_G)$ is characteristic in M_G, it is normal in G and so $A \leq F^*_\sigma(M_G) \leq F^*_\sigma(G)$. This contradiction shows that $A \nleq M_G$, so G/M_G is σ-primary. Hence $A/M_G \cap A \simeq AM_G/M_G$ is σ-primary and so $A = A^{\mathfrak{N}_\sigma} \leq M_G \cap A \leq M_G$. This contradiction shows that $A \leq F^*_\sigma(G)$.

Next we show that if $A \leq F^*_\sigma(G)$, then A is σ-quasinilpotent. Let $Z = Z_\sigma(F^*_\sigma(G))$. Lemma 2.8 implies that $F^*_\sigma(G)/Z$ is σ-semisimple. On the other hand, ZA/Z is σ-subnormal in $F^*_\sigma(G)/Z$ by Lemma 2.9(4). Hence ZA/Z is σ-semisimple by Lemma 2.11. Finally, $A/A \cap Z \simeq ZA/Z$ and $A \cap Z \leq Z_\sigma(A)$ by Lemma 2.7(iii). Hence A is σ-quasinilpotent by Lemma 2.8.

Part (i) implies that $F^*_\sigma(A)$ is σ-quasinilpotent, so $F^*_\sigma(A) \leq F^*_\sigma(G) \cap A$. On the other hand, Lemma 2.9(1) and (2) implies that $F^*_\sigma(G) \cap A$ is σ-subnormal in A, so $F^*_\sigma(G) \cap A \leq F^*_\sigma(A)$. Thus $F^*_\sigma(G) \cap A = F^*_\sigma(A)$. Similarly, it can be proved that $F_\sigma(G) \cap A = F_\sigma(A)$.

The theorem is proved.

Proof of Theorem B. Let $Z = Z_\sigma(G)$. Then $Z \leq F \leq F^*$. Indeed, the first of these two inclusions follows from Lemma 2.7(iv). The second inclusion is evident.

(i) This follows from Theorem A(i) and Lemma 2.8.

(ii) Since F^* is σ-quasinilpotent by Theorem A(i), Lemma 2.5 implies that E is σ-perfect. Moreover, Lemma 2.8 implies that the following hold: $F^* = EF$, $[E, F] = 1$ and $E/E \cap F = E/Z(E)$ is σ-semisimple. It follows that $F \leq C_{F^*}(E)$, so $C_{F^*}(E) = C_{F^*}(E) \cap EF = F(C_{F^*}(E) \cap E) = FZ(E) = F$.

(iii) Let $V/Z = F_\sigma(G/Z)$. By Theorem A(i) and Lemma 2.10, F/Z is σ-nilpotent. Hence $F/Z \leq V/Z$, so $F \leq V$. Theorem A(i) implies that V/Z is σ-nilpotent. On the other hand, Lemma 2.7(iii) implies that $Z \leq Z_\sigma(V)$ and so V is σ-nilpotent by Lemma 2.7(v), which implies that $V \leq F$. Hence $F = V$, so $F/Z = F_\sigma(G/Z)$.

Let $V^*/Z = F^*_\sigma(G/Z)$. By Theorem A(i) and Lemma 2.1(ii), F^*/Z is σ-quasinilpotent. Hence $F^*/Z \leq V^*/Z$, so $F^* \leq V^*$. Now let $V_0/Z = Z_\sigma(V^*/Z)$. Lemma 2.7(iii) implies that $Z \leq Z_\sigma(V^*)$ and so $V_0 = Z_\sigma(V^*)$ by Lemma 2.7(vi). Hence

$$(V^*/Z)/Z_\sigma(V^*/Z) = (V^*/Z)/(V_0/Z) \simeq V^*/V_0$$

is σ-semisimple by Lemma 2.8. Therefore, again by Lemma 2.8, V^* is σ-quasinilpotent and so $V^* \leq F^* \leq V^*$. Hence $F^*/Z = F^*_\sigma(G/Z)$.

(iv) By Theorem A(ii), $H \leq F^*$. On the other hand, since F^*/E is σ-nilpotent by Lemma 2.10 and H is σ-perfect by hypothesis, $H/H \cap E \simeq HE/E_\sigma(G)$ is identity. Hence $H \leq E$. Finally, E is σ-quasinilpotent by Theorem A(ii) and so $E_\sigma(E) = E$ since E is σ-perfect by Part (ii).

The theorem is proved.

Proof of Theorem C. It is enough to prove that if given any σ-eccentric chief factor H/K of G below $F^*_\sigma(G)$, every automorphism of H/K induced by an element of G is inner, then G is σ-quasinilpotent. Suppose that this is false and let G be a counterexample of minimal order.

(1) *If R is a minimal normal subgroup of G, then $R \leq F^*_\sigma(G)$* (This directly follows from the evident fact that every minimal normal subgroup of G is σ-quasinilpotent).

(2) *Every proper normal subgroup V of G is σ-quasinilpotent. Hence $G/F^*_\sigma(G)$ is a simple group.*

By Theorem A(ii), $F^*_\sigma(V) = F^*_\sigma(G) \cap V$. Hence for every σ-eccentric chief factor H/K of G below $F^*_\sigma(V)$, every automorphism of H/K induced by an element of G is inner.

Now let $K \leq L < T \leq H$, where H/K is a chief factor of G below $F^*_\sigma(V)$ and T/L is a chief factor of V. Suppose that T/L is σ-eccentric in V. Then H/K is σ-eccentric in G. Indeed, assume that H/K is σ-central in G. Then H/K and $G/C_G(H/K)$ are σ_i-groups for some i. Hence T/L is a σ_i-group. On the other hand, $C_G(H/K) \cap V \leq C_V(T/L)$ and also we have $V/C_V(T/L) \simeq (V/C_V(H/K))/(C_V(T/L)/C_G(H/K))$, where $V/C_V(H/K) \simeq VC_G(H/K)/C_G(H/K)$ is a σ_i-group. Hence $V/C_V(T/L)$ is a σ_i-group and so T/L is σ-central in V, a contradiction. Thus H/K is σ-eccentric in G. Hence, by hypothesis, every element of

G induces an inner automorphism on H/K. Therefore every automorphism of T/L induced by an element of V is inner by Lemma 2.1(i). Thus V is σ-quasinilpotent.

(3) *If R is a minimal normal subgroup of G, then R is not σ-central in G.*

Suppose that R is σ-central in G. Then $R \leq Z = Z_\sigma(G)$ and, by Theorem B(iii), $F^*_\sigma(G/Z) = F^*_\sigma(G)/Z$. Now let $(H/Z)/(K/Z)$ be a chief factor of G/Z below $F^*_\sigma(G/Z)$. Then H/K is a chief factor of G below $F^*_\sigma(G)$. Moreover, if $(H/Z)/(K/Z)$ is σ-eccentric in G/Z, then H/K is σ-eccentric in G and so every element $x \in G$ induces an inner automorphism on H/K. Then xZ induces an inner automorphism on $(H/Z)/(K/Z)$. Therefore the hypothesis holds for G/Z, so the choice of G implies that G/Z is σ-quasinilpotent. But then G is σ-quasinilpotent by Lemmas 2.7(vi) and 2.8, contrary to the choice of G. Hence we have (3).

Final contradiction. Let R be a minimal normal subgroup of G. Then $R \leq F^*_\sigma(G)$ by Claim (1). Moreover, R is σ-eccentric in G by Claim (3), so every automorphism of R induced by an element of G is inner by hypothesis. Hence $G = RC_G(R)$ by Lemma 2.2, where evidently $C_G(R) \neq G$. Then, by Claim (2), $C_G(R) \leq F^*_\sigma(G)$, so $G = F^*_\sigma(G)$ is σ-quasinilpotent by Theorem A(i). This contradiction completes the proof of the result.

Proof of Theorem D. Let D be the intersection of the σ-centralizers of the chief factors of G. First we show that $F_\sigma(G) \leq D$, that is, for any chief factor H/K of G we have $F_\sigma(G) \leq C^\sigma_G(H/K)$. If $F_\sigma(G) \leq K$, it is evident. Now assume that $H \leq F_\sigma(G)$. Then H/K is σ-primary, H/K is a σ_i-group say. Hence $C^\sigma_G(H/K) = O_{\sigma_i}(G)C_G(H/K)$. By Theorem A(i), $F_\sigma(G)$ is σ-nilpotent, so $F_\sigma(G) = O_{\sigma_i}(F_\sigma(G)) \times O_{\sigma_i'}(F_\sigma(G))$ by Lemma 2.3. Moreover, $O_{\sigma_i}(F_\sigma(G)) = O_{\sigma_i}(G) \leq C^\sigma_G(H/K)$. On the other hand, Lemma 2.4 implies that $O_{\sigma_i'}(F_\sigma(G)) \leq C_{F_\sigma(G)}(H/K)$. Hence $F_\sigma(G) \leq C^\sigma_G(H/K)$. Therefore for any chief factor H/K of G we have $F_\sigma(G) \leq C^\sigma_G(H/K)$ by the Jordan-Hölder theorem and Lemma 2.6.

Now we show that D is σ-nilpotent. Let H/K be a chief factor of G such that $H \leq D$. Let $C = C^\sigma_G(H/K)$. Then $H \leq D \leq C$, so H/K is a σ_i-group for some i. Hence $C = O_{\sigma_i}(G)C_G(H/K)$. Therefore $C/C_G(H/K) \simeq O_{\sigma_i}(G)/(O_{\sigma_i}(G) \cap C_G(H/K))$ is a σ_i-group, so H/K is σ-hypercentral in C/K by Lemma 2.4. Thus H/K is σ-hypercentral in D/K by Lemma 2.7(iii). Therefore all factors of some chief series of D are σ-central in D and so D is σ-nilpotent by the Jordan-Hölder theorem, which implies that $D \leq F_\sigma(G)$. Hence $D = F_\sigma(G)$.

Now let D^* be the intersection of the σ-innerisers of the chief factors of G. First we show that $D^* \leq F^*_\sigma(G)$. Let H/K be a chief factor of G such that $H \leq D^*$, and let $C = C^{*\sigma}_G(H/K)$. Then $H \leq D^* \leq C$. If H/K is not σ-primary, then $C = HC^\sigma_G(H/K) = HC_G(H/K)$ and so every element of C induces an inner automorphism on H/K. Hence every element of D^* induces an inner automorphism on T/L for every chief factor T/L of D^* such that $K \leq L < T \leq H$ by Lemma 2.1(i). Now suppose that H/K is a σ_i-group for some i. Then $C = O_{\sigma_i}(G)C_G(H/K)$, so every chief factor T/L of C such that $K \leq L < T \leq H$ is σ-central in C by Lemma 2.4. Therefore D^* is σ-quasinilpotent. Hence $D^* \leq F^*_\sigma(G)$.

Finally, we show that $F^*_\sigma(G) \leq C^{*\sigma}_G(H/K)$ for every chief factor H/K of G. In view of the Jordan-Hölder theorem, it is only enough to consider the case when $H \leq F^*_\sigma(G)$. If H/K is σ_i-primary for some i, then $F^*_\sigma(G)/C_{F^*_\sigma(G)}(H/K)$ is σ_i-

primary by Theorem A(i) and Lemmas 2.4 and 2.8. Moreover, $C_G^{*\sigma}(H/K) = O_{\sigma_i}(G)C_G(H/K)$. Hence $E_\sigma(G) \leq C_{F_\sigma^*(G)}(H/K)$, and

$$O_{\sigma_i'}(F_\sigma(G)) = O_{\sigma_i'}(F_\sigma(F^*(G))) \leq C_{F_\sigma^*(G)}(H/K).$$

Thus

$$F_\sigma^*(G) = E_\sigma(G)F_\sigma(G) \leq C_G^{*\sigma}(H/K)$$

by Theorem B(ii). Now assume that H/K is not σ-primary. Then $C_G^{*\sigma}(H/K) = HC_G(H/K)$. Lemma 2.8 implies that $F_\sigma^*(G)/F_\sigma(G)$ is a direct product of some simple non-abelian groups. Hence $F_\sigma^*(G)/F_\sigma(G) = (H_1/F_\sigma(G)) \times \cdots \times (H_t/F_\sigma(G))$ for some minimal normal subgroups $H_1/F_\sigma(G), \ldots, H_t/F_\sigma(G)$ of $G/F_\sigma(G)$ by [15, Ch.A, 4.14]. In view of the Jordan-Hölder theorem and Lemma 2.6, we can assume without loss of generality that $H/K = H_1/F_\sigma(G)$, so $H_2 \cdots H_t \leq C_G(H/K)$. But then $F_\sigma^*(G) = HC_{F_\sigma^*(G)}(H/K) \leq C_G^{*\sigma}(H/K)$. Hence $F_\sigma^*(G) \leq D^*$, so $F_\sigma^*(G) = D^*$.

The result is proved.

4. Further applications

First consider the following

Corollary 4.1. $C_G(F_\sigma^*(G)) \leq F_\sigma^*(G)$.

Proof. Let $F^* = F_\sigma^*(G)$ and $C = C_G(F^*)$. Suppose that $C \nleq F^*$ and let H/F^* be a chief factor of G, where $H \leq CF^*$. Then $H = F^*(H \cap C)$, where $H \cap C$ is a normal σ-quasinilpotent subgroup of G by Lemma 2.8 since $(H \cap C)/((H \cap C) \cap F^*) \simeq H/F^*$ and $(H \cap C) \cap F^* \leq Z(H \cap C)$. Thus $H \leq F^*$ by Theorem A(i). This contradiction completes the proof of the corollary.

From corollary 4.1 and Theorem B we get

Corollary 4.2. *If G is σ-soluble, then $C_G(F_\sigma(G)) \leq F_\sigma(G)$.*

In the case when $\sigma = \sigma^1$ we get from Corollary 4.2 the following

Corollary 4.3 (See [16, Ch.6, Theorem 1.3]). *If G is soluble, then $C_G(F(G)) \leq F(G)$.*

In view of Remark 1.1, in the case when $\sigma = \sigma^\pi$, we get from Corollary 4.2 the following

Corollary 4.4. *If G is π-separable, then $C_G(O_\pi(G) \times O_{\pi'}(G)) \leq O_\pi(G) \times O_{\pi'}(G)$.*

Now note that if G is π-separable and $O_{\pi'}(G) = 1$, then $F_{\sigma^\pi}(G) = O_\pi(G)$ and so from Corollary 4.4 we get the following

Corollary 4.5 (See [16, Ch.6, Theorem 3.2]). *If G is π-separable, then*

$$C_{G/O_{\pi'}(G)}(O_\pi(G/O_{\pi'}(G))) \leq O_\pi(G/O_{\pi'}(G)).$$

In view of Remark 1.1, in the case when $\sigma = \sigma^{1\pi}$ and $O_{\pi'}(G) = 1$, we have $F_\sigma(G) = O_{p_1}(G) \times \cdots \times O_{p_n}(G) = F(G)$ and so we get from Corollary 4.4 the following

Corollary 4.6. *If G is π-soluble, then:*
(1)

$$C_G(O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)) \leq O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$$
$$= F(O_\pi(G)) \times O_{\pi'}(G).$$

(2) *If $O_{\pi'}(G) = 1$, then $C_G(F(G)) \leq F(G)$.*

Note that since $F(O_\pi(G)) = O_{p_1}(G) \times \cdots \times O_{p_n}(G)$, we get from Corollary 4.6 the following its special case.

Corollary 4.7 (Monakhov and Shpyrko [17]). *If G is π-soluble group, then:*
(1) $C_G(O_\pi(G) \times O_{\pi'}(G)) \leq F(O_\pi(G)) \times O_{\pi'}(G)$.
(2) *If $O_{\pi'}(G) = 1$, then $C_G(F(G)) \leq F(G)$.*

Corollary 4.8. *Let H be a σ-soluble subgroup of G. If $E_\sigma(G) \leq N_G(H)$, then $E_\sigma(G) \leq C_G(H)$. Hence $E_\sigma(G)$ centralizes each normal σ-soluble subgroup of G.*

Proof. Since $E_\sigma(G) \leq N_G(H)$, $[E_\sigma(G), H] \leq E_\sigma(G) \cap H$ and $E_\sigma(G) \cap H$ is a σ-soluble normal subgroup of $E_\sigma(G)$. Hence $E_\sigma(G) \cap H \leq Z(E_\sigma(G))$ since $E_\sigma(G)/Z(E_\sigma(G))$ is σ-semisimple by Theorem B(ii). Hence $[E_\sigma(G), H, E_\sigma(G)] = 1$, so $[E_\sigma(G), H] = [E_\sigma(G), E_\sigma(G), H] = 1$ by the lemma on three subgroups [18, III, 1.10]. The corollary is proved.

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