On the generalized $\sigma$-Fitting subgroup of finite groups

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ABSTRACT – Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, and let $G$ be a finite group. A chief factor $H/K$ of $G$ is said to be $\sigma$-central (in $G$) if the semidirect product $(H/K) \rtimes (G/C_{G}(H/K))$ is a $\sigma_i$-group for some $i \in \{H/K\}$; otherwise, it is called $\sigma$-eccentric (in $G$). We say that $G$ is: $\sigma$-nilpotent if every chief factor of $G$ is $\sigma$-central; $\sigma$-quasinilpotent if for every $\sigma$-eccentric chief factor $H/K$ of $G$, every automorphism of $H/K$ induced by an element of $G$ is inner. The product of all normal $\sigma$-nilpotent (respectively $\sigma$-quasinilpotent) subgroups of $G$ is said to be the $\sigma$-Fitting subgroup (respectively the generalized $\sigma$-Fitting subgroup) of $G$ and we denote it by $F_{\sigma}(G)$ (respectively by $F_{\sigma}^{*}(G)$). Our main goal here is to study the relations between the subgroups $F_{\sigma}(G)$ and $F_{\sigma}^{*}(G)$, and the influence of these two subgroups on the structure of $G$.


KEYWORDS. finite group, $\sigma$-nilpotent group, $\sigma$-quasinilpotent group, $\sigma$-Fitting subgroup, generalized $\sigma$-Fitting subgroup.

1. Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is

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an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$.

In what follows, $\sigma = \{\sigma_i|i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We say that: $G$ is $\sigma$-primary [1] provided it is a $\sigma_i$-group for some $i$; an automorphism $\alpha$ of $G$ is $\sigma_i$-primary if $\langle \alpha \rangle$ is a $\sigma_i$-subgroup of $\text{Aut}(G)$.

In the mathematical practice, we often deal with the following three special partitions of $\mathbb{P}$:

- $\sigma^1 = \{\{2\}, \{3\}, \ldots\}$, $\sigma^\pi = \{\pi, \pi'\}$, and $\sigma^1\pi = \{\{p_i\}, \ldots, \{p_n\}, \pi\}$, where $\pi = \{p_1, \ldots, p_n\}$.

The group $G$ is called: $\sigma$-soluble [1] if every chief factor of $G$ is $\sigma$-primary; $\sigma$-decomposable [2] or $\sigma$-nilpotent [3] if $G = G_1 \times \cdots \times G_n$ for some $\sigma$-primary groups $G_1, \ldots, G_n$.

**Remark 1.1.** (i) $G$ is: soluble if and only if $G$ is $\sigma^1$-soluble, $\pi$-soluble if and only if $G$ is $\sigma^\pi$-soluble.

(ii) Let $G \neq 1$ and $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$. Without loss of generality we can assume that $\sigma(G) = \{\sigma_1, \ldots, \sigma_t\}$. Then $G$ is $\sigma$-nilpotent if and only if $G = O_{\sigma_1}(G) \times \cdots \times O_{\sigma_t}(G)$. Thus, $G$ is: $\sigma^1$-nilpotent if and only if it is nilpotent, $\sigma^\pi$-nilpotent if and only if $G = O_{\sigma_1}(G) \times O_{\sigma}(G)$, $\sigma^1\pi$-nilpotent if and only if $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\sigma}(G)$.

Let $H/K$ be a chief factor of $G$. Then we say that $H/K$ is $\sigma$-central (in $G$) [1] if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is $\sigma$-primary; otherwise, it is called $\sigma$-eccentric (in $G$). A normal subgroup $E$ of $G$ is said to be $\sigma$-hypercentral (in $G$) if either $E = 1$ or every chief factor of $G$ below $E$ is $\sigma$-central in $G$.

The $\sigma$-nilpotent groups have many applications in the formation theory [2, 4, 5, 6] (see also the recent papers [1, 3, 7, 8, 9, 10, 11] and the survey [12]), and such groups are exactly the groups whose chief factors are $\sigma$-central (see Proposition 2.3 in [1]).

In this paper, we consider the following generalization of $\sigma$-nilpotency.

**Definition 1.2.** We say that $G$ is $\sigma$-quasinilpotent if given any $\sigma$-eccentric chief factor $H/K$ of $G$, every automorphism of $H/K$ induced by an element of $G$ is inner (cf. [13, X, Definition 13.2]).

Note that $G$ is called quasinilpotent if given any chief factor $H/K$ of $G$, every automorphism of $H/K$ induced by an element of $G$ is inner. Therefore $G$ is quasinilpotent if and only if it is $\sigma^1$-quasinilpotent.

Let $Z_\sigma(G)$ denote the product of all normal $\sigma$-hypercentral subgroups of $G$. It is not difficult to show (see Lemma 2.7(i) below) that $Z_\sigma(G)$ is also $\sigma$-hypercentral in $G$. We call the subgroup $Z_\sigma(G)$ the $\sigma$-hypercentre of $G$. Dually, we define the $\sigma$-nilpotent residual $G^{\sigma\text{res}}$ of $G$, that is, the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$; $G^{\sigma\text{res}}$ is the $\sigma$-soluble residual of $G$.

**Definition 1.3.** (i) The product of all normal $\sigma$-nilpotent (respectively $\sigma$-quasinilpotent) subgroups of $G$ is said to be the $\sigma$-Fitting subgroup [1] (respectively the generalized $\sigma$-Fitting subgroup) of $G$ and denoted by $F_\sigma(G)$ (respectively by $F^*_\sigma(G)$).
(ii) We use $E_\sigma(G)$ to denote the $\sigma$-soluble residual of $F^*_\sigma(G)$, and we say that
$E_\sigma(G)$ is the $\sigma$-layer of $G$ (cf. [13, X, Definition 13.14]).

Note that in the case when $\sigma = \sigma^1$ the subgroups $F_\sigma(G)$, $F^*_\sigma(G)$ and $E_\sigma(G)$
coincide respectively with $F(G)$, $F^*(G)$ and $E(G)$.

Before continuing, consider some examples.

Example 1.4. Let $G = (A_5 \times A_7) \wr \langle x \rangle = K \times \langle x \rangle$, where $|x| = p > 5$ is a
prime and $K$ is the base group of the regular wreath product $G$. Let $R = A_5^2$ and
$L = A_7^p$ (we use here the terminology in [15, Ch.A]). Let $\sigma = \{[2, 3, 5], \{2, 3, 5\}\}'$. Then
$K = R \times L$ and so, in view of Remark 1.1(ii), $F_\sigma(G) = R$. It is clear also that
$K \leq F^*_\sigma(G)$ and the automorphism of $R$ induced by $x$ is not inner. Hence
$F^*_\sigma(G) = K$. Finally, $E_\sigma(G) = L$ and $E(G) = K$.

We say that $G$ is: $\sigma$-perfect if $G^{\sigma^p} = G$; $\sigma$-semisimple if either $G = 1$ or
$G = A_1 \times \cdots \times A_t$ is the direct product of simple non-$\sigma$-primary groups $A_1, \ldots, A_t$.

Example 1.5. Let $G = (A_5 \wr A_7) \times (A_7 \times A_{11})$ and $\sigma = \{[2, 3, 5], \{2, 3, 5\}\}'$. Then
$G$ is $\sigma$-quasinilpotent but $G$ is not $\sigma$-nilpotent. The group $A_7 \times A_{11}$ is
$\sigma$-semisimple and $\sigma$-perfect.

A subgroup $A$ of $G$ is $\sigma$-subnormal in $G$ [1] if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \leq A_i$ or $A_i/(A_{i-1}A_i)$ is $\sigma$-primary
for all $i = 1, \ldots, n$. Note that $A$ is normal in $G$ if and only if it is $\sigma^1$-subnormal
in $G$.

In this paper, we study properties and relations between the subgroups $F_\sigma(G)$,
$F^*_\sigma(G)$ and $E_\sigma(G)$. Our main observations here are the following two results
which, in particular, show that the subgroup $F^*_\sigma(G)$ has properties similar to the
properties of the generalized Fitting subgroup $F^*(G)$ of $G$ (see Section 4 below
and Ch.X in [13]).

Theorem A. The following statements hold:

(i) $F_\sigma(G)$ is the largest normal $\sigma$-nilpotent subgroup of $G$ and $F^*_\sigma(G)$ is the
largest normal $\sigma$-quasinilpotent subgroup of $G$.

(ii) A $\sigma$-subnormal subgroup $A$ of $G$ is contained in $F^*_\sigma(G)$ (respectively in
$F_\sigma(G)$) if and only if $A$ is $\sigma$-quasinilpotent (respectively $\sigma$-nilpotent). Hence
$F^*_\sigma(G) \cap A = F^*_\sigma(A)$ and $F_\sigma(G) \cap A = F_\sigma(A)$.

In the case when $\sigma = \sigma^1$, we get from Theorem A(ii) the following

Corollary 1.6 (See [13, X, Theorem 13.10]). $F^*(G)$ is quasinilpotent and
every subnormal quasinilpotent subgroup of $G$ is contained in $F^*(G)$.

Theorem B. Let $F = F_\sigma(G)$, $F^* = F^*_\sigma(G)$, and $E = E_\sigma(G)$. Then the
following statements hold:

(i) $F = Z_\sigma(F^*)$ and $F^*/F$ is $\sigma$-semisimple.

(ii) $F^* = EF$ and $F = C_{F^*}(E)$, so $F^* = C_{F^*}(F)F$. Also $E \cap F = Z(E)$, $E$
is $\sigma$-perfect and $E/Z(E)$ is $\sigma$-semisimple.

(iii) $F/Z_\sigma(G) = F_\sigma(G/Z_\sigma(G))$ and $F^*/Z_\sigma(G) = F^*_\sigma(G/Z_\sigma(G))$.

(iv) Every $\sigma$-perfect $\sigma$-quasinilpotent $\sigma$-subnormal subgroup $H$ of $G$ is con-
tained in $E_\sigma(G)$. Moreover, $E_\sigma(E_\sigma(G)) = E_\sigma(G)$.

As a first application of Theorems A and B, we prove also the following

Theorem C. $G$ is $\sigma$-quasinilpotent if and only if given any $\sigma$-eccentric chief
factor $H/K$ of $G$ below $F^*(G)$, every automorphism of $H/K$ induced by an element of $G$ is inner.

In the case when $\sigma = \sigma^1$, we get from Theorem C the following

**Corollary 1.7.** $G$ is quasinilpotent if and only if given any chief factor $H/K$ of $G$ below $F^*(G)$, every automorphism of $H/K$ induced by an element of $G$ is inner.

Let $H/K$ be a chief factor of $G$. We define the $\sigma$-centralizer $C^\sigma_G(H/K)$ of $H/K$ in $G$: $C^\sigma_G(H/K) = C_G(H/K)$ if $H/K$ is not $\sigma$-primary, and $C^\sigma_G(H/K) = O_{\sigma_i}(G)C_G(H/K)$ in the case when $H/K$ is $\sigma$-primary.

Now, by analogy with the inneriser of $H/K$ (see [6, p.41]), we define the $\sigma$-inneriser $C^\sigma_G(H/K)$ of $H/K$ in $G$: $C^\sigma_G(H/K) = HC_G^\sigma(H/K)$ if $H/K$ is not $\sigma$-primary, and $C^\sigma_G(H/K) = C^\sigma_G(H/K)$ in the case when $H/K$ is $\sigma$-primary.

As one more application of Theorems A and B we prove the following

**Theorem D.** (i) The subgroup $F_\sigma(G)$ coincides with the intersection of the $\sigma$-centralizers of the chief factors of $G$.

(ii) The subgroup $F^*_\sigma(G)$ coincides with the intersection of the $\sigma$-innerisers of the chief factors of $G$.

**Corollary 1.8** (Ballester-Bolinches and Ezquerro [6, p.97]). The subgroup $F^*(G)$ coincides with the intersection of the innerisers of the chief factors of $G$.

In Section 4 we discuss further applications of Theorems A and B.

2. Preliminaries

**Lemma 2.1.** (i) If $K \leq L < T = H \leq E \leq G$, where $H/K$ is a chief factor of $G$ and $T/L$ is a chief factor of $E$, and an element $x \in E$ induces an inner automorphism on $H/K$, then $x$ induces an inner automorphism on $T/L$. Moreover, if $H/K = (H_1/K) \times \cdots \times (H_t/K)$, where $H_1/K, \ldots, H_t/K$ are normal subgroups of $E/K$ and $x$ induces inner automorphisms on these factors, then $x$ induces an inner automorphism on $H/K$.

(ii) If $G$ is a $\sigma$-quasinilpotent group and $N$ is a normal subgroup of $G$, then $N$ and $G/N$ are $\sigma$-quasinilpotent.

(iii) If $G/N$ and $G/L$ are $\sigma$-quasinilpotent, then $G/(N \cap L)$ is also $\sigma$-quasinilpotent.

**Proof.** (i) See the proof of Lemma 13.1 in [13, X].

(ii), (iii) See the proof of Lemma 13.3 in [13, X].

**Lemma 2.2.** Let $H/K$ be a chief factor of $G$. Then every automorphism of $H/K$ induced by an element of $G$ is inner if and only if $G/K = (H/K)C_G(K)H/K$.

**Proof.** See the proof of Lemma 13.4 in [13, X].

**Lemma 2.3** (see Proposition 2.3 in [1]). The following are equivalent:

(i) $G$ is $\sigma$-nilpotent.

(ii) $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$.

(iii) Every chief factor of $G$ is $\sigma$-central in $G$.

**Lemma 2.4.** Let $N$ be a normal $\sigma_i$-subgroup of $G$. Then $N \leq Z_\sigma(G)$ if and only if $O^{\sigma_i}(G) \leq C_G(N)$. 
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**Proof.** If $O^{\sigma}(G) \leq C_G(N)$, then for every chief factor $H/K$ of $G$ below $N$ both groups $H/K$ and $G/C_G(H/K)$ are $\sigma$-group since $G/O^{\sigma}(G)$ is a $\sigma$-group. Hence $(H/K) \rtimes (G/C_G(H/K))$ is $\sigma$-primary. Thus $N \leq Z_\sigma(G)$.

Now assume that $N \leq Z_\sigma(G)$. Let $1 = Z_0 < Z_1 < \ldots < Z_t = N$ be a chief series of $G$ below $N$ and $C_1 = C_G(Z_t/Z_{t-1})$. Let $C = C_1 \cap \cdots \cap C_t$. Then $G/C$ is a $\sigma$-group. On the other hand, $C/C_G(N) \simeq A \leq \text{Aut}(N)$ stabilizes the indices $1 = Z_0 < Z_1 < \ldots < Z_t = N$, so $C/C_G(N)$ is a $\pi(N)$-group by Theorem 0.1 in [14]. Hence $G/C_G(N)$ is a $\sigma$-group and so $O^{\sigma}(G) \leq C_G(N)$.

The lemma is proved.

The next two lemmas are evident.

**Lemma 2.5.** $G^{\sigma_\sigma}$ is $\sigma$-perfect.

**Lemma 2.6.** If $H/K$ and $T/L$ are $G$-isomorphic chief factors of $G$, then:

(i) $(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L))$, and

(ii) $C_G(H/K) = C_G(T/L)$.

(iii) $C_G^{\sigma}(H/K) = C_G^\sigma(T/L)$.

We write $\sigma(G) = \{\sigma_i \cap \pi(G) \neq \emptyset\}$, and we say that $G$ is a $\Pi$-group provided $\sigma(G) \subseteq \Pi \subseteq \sigma$.

**Lemma 2.7.** Let $Z = Z_\sigma(G)$. Let $A$, $B$ and $N$ be subgroups of $G$, where $N$ is normal in $G$.

(i) $Z$ is $\sigma$-hypercentral in $G$.

(ii) $Z_\sigma(A)/N \leq Z_\sigma(AN/N)$.

(iii) $Z_\sigma(B) \cap A \leq Z_\sigma(B \cap A)$.

(iv) If $N \leq Z$ and $N$ is a $\Pi$-group, then $N$ is $\sigma$-nilpotent and $G/C_G(N)$ is a $\sigma$-nilpotent $\Pi$-group.

(v) If $G/Z$ is $\sigma$-nilpotent, then $G$ is also $\sigma$-nilpotent.

(vi) If $N \leq Z$, then $Z/N \simeq Z_\sigma(G/N)$.

(vii) If $G = A \times B$, then $Z = Z_\sigma(A) \times Z_\sigma(B)$.

**Proof.** (i) It is enough to consider the case when $Z = A_1A_2$, where $A_1$ and $A_2$ are normal $\sigma$-hypercentral subgroups of $G$. Moreover, in view of the Jordan-Hölder theorem, it is enough to show that if $A_1 \leq K \leq H \leq A_1A_2$, then $H/K$ is $\sigma$-central. But in this case we have $H = A_1(H \cap A_2)$, where evidently $H \cap A_2 \not\leq K$, so we have the $G$-isomorphism $(H \cap A_2)/(K \cap A_2) \simeq (H \cap A_2)/K = H/K$, and hence $H/K$ is $\sigma$-central in $G$ by Lemma 2.6.

(ii) First assume that $G = A$, and let $H/K$ be a chief factor of $G$ such that $N \leq K < H \leq NZ$. Then $H/K$ is $G$-isomorphic to the chief factor $(H \cap Z)/(K \cap Z)$ of $G$ below $Z$. Therefore $H/K$ is $\sigma$-central in $G$ by Assertion (i) and Lemma 2.6. Consequently, $ZN/N \leq Z_\sigma(G/N)$.

Now let $A$ be any subgroup of $G$, and let $f : A/AN \to AN/N$ be the canonical isomorphism from $A/A \cap N$ onto $AN/N$. Then $f(Z_\sigma(A/A \cap N)) = Z_\sigma(AN/N)$ and $f(Z_\sigma(A/A \cap N)/(A \cap N)) = Z_\sigma(AN/N)$. Hence, in view of the preceding paragraph, we have $Z_\sigma(A/A \cap N)/(A \cap N) \leq Z_\sigma(A/A \cap N)$. 

Hence \( Z_\sigma(A)N/N \leq Z_\sigma(AN/N) \).

(iii) First assume that \( B = G \), and let \( 1 = Z_0 < Z_1 < \ldots < Z_t = Z \) be a chief series of \( G \) below \( Z \) and \( C_i = C_G(Z_i/Z_{i-1}) \). Now consider the series

\[
1 = Z_0 \cap A \leq Z_1 \cap A \leq \ldots \leq Z_t \cap A = Z \cap A.
\]

We can assume without loss of generality that this series is a chief series of \( A \) below \( Z \cap A \).

Let \( i \in \{1, \ldots, t\} \). Then, by Assertion (i), \( Z_i/Z_{i-1} \) is \( \sigma \)-central in \( G \), \( (Z_i/Z_{i-1}) \times (G/C_i) \) is a \( \sigma_k \)-group say. Hence \( (Z_i \cap A)/(Z_{i-1} \cap A) \) is a \( \sigma_k \)-group. On the other hand, \( A/A \cap C_i \asymp C_iA/C_i \) is a \( \sigma_k \)-group and

\[
A \cap C_i \leq C_A((Z_i \cap A)/(Z_{i-1} \cap A)).
\]

Thus \( (Z_i \cap A)/(Z_{i-1} \cap A) \) is \( \sigma \)-central in \( A \). Therefore, in view of the Jordan-Hölder theorem for the chief series, we have \( Z \cap A \leq Z_\sigma(A) \).

Now assume that \( B \) is any subgroup of \( G \). Then, in view of the preceding paragraph, we have

\[
Z_\sigma(B) \cap A = Z_\sigma(B) \cap (B \cap A) \leq Z_\sigma(B \cap A).
\]

(iv) By Assertion (iii) and Lemma 2.3, \( N \) is \( \sigma \)-nilpotent, and it has a complete Hall \( \sigma \)-set \( \{H_1, \ldots, H_t\} \) such that \( N = H_1 \times \cdots \times H_t \). Then

\[
C_G(N) = C_G(H_1) \cap \cdots \cap C_G(H_t).
\]

It is clear that \( H_1, \ldots, H_t \) are normal in \( G \). We can assume without loss of generality that \( H_i \) is a \( \sigma_i \)-group. Then, by Assertion (i) and Lemma 2.4, \( G/C_G(H_i) \) is a \( \sigma_i \)-group. Hence

\[
G/C_G(N) = G/(C_G(H_1) \cap \cdots \cap C_G(H_t))
\]

is a \( \sigma \)-nilpotent II-group.

(v), (vi) These assertions are corollaries of Assertion (i) and the Jordan-Hölder theorem.

(vii) Let \( Z_1 = Z_\sigma(A) \) and \( Z_2 = Z_\sigma(B) \). Since \( Z_1 \) is characteristic in \( A \), it is normal in \( G \).

First assume that \( Z_1 \neq 1 \) and let \( R \) be a minimal normal subgroup of \( G \) contained in \( Z_1 \). Then \( R \) is \( \sigma \)-primary, \( R \) is a \( \sigma \)-group say, by Assertion (iv). Hence \( A/C_A(R) \) is a \( \sigma \)-group by Lemma 2.4. But \( C_G(R) = B(C_G(R) \cap A) = BC_A(R) \), so

\[
G/C_G(R) = AB/C_A(R)B \simeq A/(A \cap C_A(R)B) = A/C_A(R)(A \cap B) = A/C_A(R)
\]

is a \( \sigma \)-group and hence \( R \) is \( \sigma \)-central in \( G \). Then \( R \leq Z_\sigma(G) \), so \( Z_\sigma(G)/R = Z_\sigma(G/R) \) by Assertion (vi). On the other hand, we have \( Z_1/R = Z_\sigma(A/R) \) and \( Z_2R/R = Z_\sigma(BR/R) \), so by induction we have

\[
Z_\sigma(G/R) = Z_\sigma((A/R) \times (BR/R)) = Z_\sigma(A/R) \times Z_\sigma(BR/R)
\]
Moreover, the choice of $\cap$ is $\sigma$-semisimple by hypothesis. Then $R/\cap = G/\cap = 1$ is a $\sigma$-semisimple group, so $Z(G/\cap) = 1$, a contradiction. Hence we have (vii).

The lemma is proved.

**Lemma 2.8.** Given a group $G$ the following are equivalent:

(i) $G$ is $\sigma$-quasinilpotent.

(ii) $G/\sigma(G)$ is $\sigma$-semisimple.

(iii) $G = E_\sigma(G)F_\sigma(G)$ and $[E_\sigma(G), F_\sigma(G)] = 1$. Hence $E_\sigma(G)/(E_\sigma(G) \cap F_\sigma(G)) = E_\sigma(G)/Z(E_\sigma(G))$ is $\sigma$-semisimple.

(iv) $G/F_\sigma(G)$ is $\sigma$-semisimple and $G = F_\sigma(G)C_G(F_\sigma(G))$.

**Proof.** Let $Z = Z_\sigma(G)$, $F = F_\sigma(G)$ and $E = E_\sigma(G)$.

(i) $\Rightarrow$ (ii) Assume that this is false and let $G$ be a counterexample of minimal order. Then the hypothesis holds for $G/Z$ by Lemma 2.1(ii). On the other hand, $Z_\sigma(G/Z) = 1$ by Lemma 2.7(vi). Hence in the case when $Z \neq 1$, $G/Z_\sigma(G)$ is $\sigma$-semisimple by the choice of $G$.

Now assume that $Z = 1$ and let $R$ be any minimal normal subgroup of $G$. Then $R/1$ is a $\sigma$-eccentric chief factor of $G$, so $G = RC_G(R)$ by Lemma 2.2. Therefore, since $Z(G) \leq 1$, $C_G(R) \neq G$ and hence $R$ is $\sigma$-semisimple. Thus $G = R \times C_G(R)$. Therefore $Z_\sigma(R) \times Z_\sigma(C_G(R)) = Z_\sigma(G) = 1$ by Lemma 2.7(vii). Moreover, the choice of $G$ implies that $C_G(R)$ is $\sigma$-semisimple, so $G \simeq G/Z = G/1$ is $\sigma$-semisimple and hence Assertion (ii) is true, a contradiction.

(ii) $\Rightarrow$ (i) Let $H/K$ be a chief factor of $G$. If $H \leq Z_\sigma(G)$, then $H/K$ is $\sigma$-central in $G$ by Lemma 2.7(i). Now suppose that $Z_\sigma(G) \leq K$. Since $G/Z_\sigma(G)$ is $\sigma$-semisimple by hypothesis, every automorphism of $H/K$ induced by an element of $G$ is inner by Lemma 2.2. Hence applying the Jordan-Hölder theorem, for every $\sigma$-eccentric chief factor $H/K$ of $G$, every automorphism of $H/K$ induced by an element of $G$ is inner and so $G$ is $\sigma$-quasinilpotent.

(iii) $\Rightarrow$ (iv) First note that $Z \leq F$ by Lemma 2.7(iv), so $Z = F$ since $G/Z$ is $\sigma$-semisimple by hypothesis. Then $G = E_\sigma(G)F_\sigma(G)$ is $\sigma$-nilpotent. Hence $E \leq C_G(F)$, so $[E, F] = 1$. Lemma 2.7(iii) implies that $Z \cap E = F \cap E \leq Z_\sigma(E)$, so $E/F \cap E$ is $\sigma$-semisimple and $F \cap E = Z(E)$.

(iii) $\Rightarrow$ (iv) This implication is evident.

(iv) $\Rightarrow$ (i) Let $H/K$ be a chief factor of $G$. If $F_\sigma(G) \leq K$, then every automorphism of $H/K$ induced by an element of $G$ is inner by Lemma 2.2 since $G/F_\sigma(G)$ is $\sigma$-semisimple by hypothesis. Now suppose that $H \leq F_\sigma(G)$. Then

$$C_G(H/K) = C_G(H/K) \cap F_\sigma(G)C_G(F_\sigma(G)) = C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K),$$

so

$$G/C_G(H/K) = F_\sigma(G)C_{F_\sigma(G)}(H/K)\cap C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K) \simeq F_\sigma(G)/F_\sigma(G) \cap C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K) = F_\sigma(G)/C_{F_\sigma(G)}(H/K)Z(F_\sigma(G))$$
is $\sigma$-primary by Lemma 2.4. Therefore $H/K$ is $\sigma$-central in $G$. Now applying the Jordan-Hölder theorem, we get that for every $\sigma$-eccentric chief factor $H/K$ of $G$, every automorphism of $H/K$ induced by an element of $G$ is inner. Hence $G$ is $\sigma$-quasinilpotent.

The lemma is proved.

**Lemma 2.9** (See Lemma 2.6 in [1]). Let $A$, $K$ and $N$ be subgroups of $G$. Suppose that $A$ is $\sigma$-subnormal in $G$ and $N$ is normal in $G$.

1. $A \cap K$ is $\sigma$-subnormal in $K$.
2. If $K$ is $\sigma$-subnormal in $G$, then $K \cap A \leq \langle A, K \rangle$ are $\sigma$-subnormal in $G$.
3. If $A$ is a $\sigma_i$-group, then $A \leq O_{\sigma_i}(G)$. Hence if $A$ is $\sigma$-nilpotent, then $A \leq F_\sigma(G)$.
4. $AN/N$ is $\sigma$-subnormal in $G/N$.

**Lemma 2.10** (See Corollary 2.4 and Lemma 2.5 in [1]). The class of all $\sigma$-nilpotent groups $\mathfrak{R}_{\sigma}$ is closed under taking products of normal subgroups, homomorphic images and subgroups.

**Lemma 2.11.** If $G$ is $\sigma$-semisimple and $A$ is a $\sigma$-subnormal subgroup of $G$, then $A$ is $\sigma$-semisimple.

**Proof.** Suppose that this lemma is false and let $G$ be a counterexample of minimal order. Then $G = A_1 \times \cdots \times A_t$ for some simple non-$\sigma$-primary groups $A_1, \ldots, A_t$. Then $A_1, \ldots, A_t$ are non-abelian.

By hypothesis, there is a chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ of subgroups of $G$ such that either $A_i/A_{i-1}$ is normal in $A_i$ or $A_i/(A_{i-1})A_i$ is $\sigma$-primary for all $i = 1, \ldots, t$. Let $M = A_{t-1}$. Without loss of generality we can assume that $M < G$. Suppose that $A \leq M_G$. Then $A$ is $\sigma$-subnormal in $M_G$ by Lemma 2.9(1). On the other hand, $M_G$ is $\sigma$-semisimple by [15, Ch.A, 4.13(b)], and so $A$ is $\sigma$-semisimple by the choice of $G$.

This contradiction shows that $A \not\leq M_G$, so $G/M_G$ is $\sigma$-primary. But each chief factor of $G$ is not $\sigma$-primary by the Jordan-Hölder theorem. This contradiction completes the proof of the lemma.

3. Proofs of Theorems A, B, C and D

**Proof of Theorem A.** (i) From Lemma 2.10, it follows that $F_\sigma(G)$ is the largest normal $\sigma$-nilpotent subgroup of $G$. In order to prove that $F_\sigma(G)$ is the largest normal $\sigma$-quasinilpotent subgroup of $G$, it is enough to show if $G = AB$, where $A$ and $B$ are normal $\sigma$-quasinilpotent subgroups of $G$, then $G$ is $\sigma$-quasinilpotent. Assume that this is false and let $G$ be a counterexample of minimal order. Let $R$ be a minimal normal subgroup of $G$ and $C = C_G(R)$. By Lemma 2.1(ii), the hypothesis holds for $G/R$, so the choice of $G$ implies that $G/R$ is $\sigma$-quasinilpotent. Therefore in view of Lemma 2.1(iii), $R$ is a unique minimal normal subgroup of $G$.

Let $Z_1 = Z_\sigma(A)$ and $Z_2 = Z_\sigma(B)$. If $A \cap B = 1$, then $Z_\sigma(G) = Z_1 \times Z_2$ by
Lemma 2.7(vii). On the other hand, $A/Z_1$ and $B/Z_2$ are $\sigma$-semisimple by Lemma 2.8, so

$$G/Z = (A \times B)/(Z_1 \times Z_2) \simeq (A/Z_1) \times (B/Z_2)$$

is $\sigma$-semisimple. Hence $G$ is $\sigma$-quasinilpotent by Lemma 2.8. Therefore $A \cap B \neq 1$, so $R \leq A \cap B$. First assume that $R$ is $\sigma$-primary, $R$ is a $\sigma_i$-group say. Then by Lemma 2.8, $R \leq Z_1 \cap Z_2$ and so $AC/C \simeq A/A \cap C$ and $BC/C \simeq B/B \cap C$ are $\sigma_i$-groups by Lemma 2.4. Hence $G/C = (AC/C)(BC/C)$ is a $\sigma_i$-group, so $R$ is $\sigma$-central in $G$. Therefore $R \leq Z_\sigma(G)$ and so $Z_\sigma(G/R) = Z_\sigma(G)/R$ by Lemma 2.7(vi). Thus $G$ is $\sigma$-quasinilpotent by Lemma 2.8.

Thus $R$ is not $\sigma$-primary. Hence $R$ is non-abelian, so $C = 1$. Then $R = R_1 \times \cdots \times R_t$, where $R_1, \ldots, R_t$ are minimal normal subgroups of $A$, so all these groups are simple by Lemma 2.8 and hence $R_1, \ldots, R_t$ are minimal normal subgroups of $B$. But then, by Lemma 2.2, $R_1 = R = A = B = G$ is $\sigma$-semisimple. Hence $G$ is $\sigma$-quasinilpotent.

(ii) Let $A$ be any $\sigma$-subnormal subgroup of $G$. First note that in view of Lemmas 2.9(3) and 2.10, $A$ is contained in $F_\sigma(G)$ if and only if $A$ is $\sigma$-nilpotent.

Now we show that if $A$ is $\sigma$-quasinilpotent, then it is contained in $F_\sigma^*(G)$.

Suppose that this is false and let $G$ be a counterexample with $|G| + |A|$ minimal. Then for each $\sigma$-quasinilpotent $\sigma$-subnormal subgroup $S$ of $G$ such that $S < A$ we have $S \leq F_\sigma^*(G)$. Therefore the choice of $G$ implies that if $A = NK$, where $N$ and $K$ are normal subgroups of $A$, then either $N = A$ or $K = A$. Lemma 2.8 implies that $A = A^\sigma N F_\sigma(A)$. Then, in view of Lemma 2.1(ii), either $F_\sigma(A) = A$ or $A^\sigma = A$. But in the former case we have $A \leq F_\sigma(G) \leq F_\sigma^*(G)$ by Lemma 2.9(3), so $A^\sigma = A$.

By hypothesis, there is a chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ of subgroups of $G$ such that either $A_{i-1}$ is normal in $A_i$ or $A_i/(A_{i-1})_{A_i}$ is $\sigma$-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. Without loss of generality we can assume that $M < G$. Suppose that $A \leq M_G$. Then $A$ is $\sigma$-subnormal in $M_G$ by Lemma 2.9(1), so $A \leq F_\sigma^*(M_G)$ by the choice of $G$. Since $F_\sigma^*(M_G)$ is characteristic in $M_G$, it is normal in $G$ and so $A \leq F_\sigma^*(M_G) \leq F_\sigma^*(G)$. This contradiction shows that $A \not\leq M_G$, so $G/M_G$ is $\sigma$-primary. Hence $A/M_G \cap A \simeq AM_G/M_G$ is $\sigma$-primary and so $A = A^\sigma N \leq M_G \cap A \leq M_G$. This contradiction shows that $A \leq F_\sigma^*(G)$.

Next we show that if $A \leq F_\sigma^*(G)$, then $A$ is $\sigma$-quasinilpotent. Let $Z = Z_\sigma(F_\sigma^*(G))$. Lemma 2.8 implies that $F_\sigma^*(G)/Z$ is $\sigma$-semisimple. On the other hand, $ZA/Z$ is $\sigma$-subnormal in $F_\sigma^*(G)/Z$ by Lemma 2.9(4). Hence $ZA/Z$ is $\sigma$-semisimple by Lemma 2.11. Finally, $A/A \cap Z \simeq ZA/Z$ and $A \cap Z \leq Z_\sigma(A)$ by Lemma 2.7(iii). Hence $A$ is $\sigma$-quasinilpotent by Lemma 2.8.

Part (i) implies that $F_\sigma^*(A)$ is $\sigma$-quasinilpotent, so $F_\sigma^*(A) \leq F_\sigma^*(G) \cap A$. On the other hand, Lemma 2.9(1) and (2) implies that $F_\sigma^*(G) \cap A$ is $\sigma$-subnormal in $A$, so $F_\sigma^*(G) \cap A \leq F_\sigma^*(A)$. Thus $F_\sigma^*(G) \cap A = F_\sigma^*(A)$. Similarly, it can be proved that $F_\sigma(G) \cap A = F_\sigma(A)$.

The theorem is proved.

**Proof of Theorem B.** Let $Z = Z_\sigma(G)$. Then $Z \leq F \leq F^*$. Indeed, the first of these two inclusions follows from Lemma 2.7(iv). The second inclusion is
(i) This follows from Theorem A(i) and Lemma 2.8.
(ii) Since $F^*$ is $\sigma$-quasinilpotent by Theorem A(i), Lemma 2.5 implies that $E$ is $\sigma$-perfect. Moreover, Lemma 2.8 implies that the following hold: $[E, F] = 1$ and $E/E \cap F = E/Z(E)$ is $\sigma$-semisimple. It follows that $F \leq C_{F^*}(E)$, so $C_{F^*}(E) = C_{F'}(E) \cap EF = F(C_{F'}(E) \cap E) = FZ(E) = F$.
(iii) Let $V/Z = F_\sigma(G/Z)$. By Theorem A(i) and Lemma 2.10, $F/Z$ is $\sigma$-nilpotent. Hence $F/Z \leq V/Z$, so $F \leq V$. Theorem A(i) implies that $V/Z$ is $\sigma$-nilpotent. On the other hand, Lemma 2.7(iii) implies that $Z \leq Z_\sigma(V)$ and so $V$ is $\sigma$-nilpotent by Lemma 2.7(v), which implies that $V \leq F$. Hence $F = V$, so $F/Z = F_\sigma(G/Z)$.

Let $V^*/Z = F_\sigma^*(G/Z)$. By Theorem A(i) and Lemma 2.1(iii), $F^*/Z$ is $\sigma$-quasinilpotent. Hence $F^*/Z \leq V^*/Z$, so $F^* \leq V^*$. Now let $V_0/Z = Z_\sigma(V^*/Z)$. Lemma 2.7(iii) implies that $Z \leq Z_\sigma(V^*)$ and so $V_0 = Z_\sigma(V^*)$ by Lemma 2.7(vi). Hence $$(V^*/Z)/Z_\sigma(V^*/Z) = (V^*/Z)/(V_0/Z) \simeq V^*/V_0$$ is $\sigma$-semisimple by Lemma 2.8. Therefore, again by Lemma 2.8, $V^*$ is $\sigma$-quasinilpotent and so $V^* \leq F^* \leq V^*$. Hence $F^*/Z = F_\sigma^*(G/Z)$.

(iv) By Theorem A(ii), $H \leq F^*$. On the other hand, since $F^*/E$ is $\sigma$-nilpotent by Lemma 2.10 and $H$ is $\sigma$-perfect by hypothesis, $H/H \cap E \simeq HE/E_\sigma(E)$ is identity. Hence $H \leq E$. Finally, $E$ is $\sigma$-quasinilpotent by Theorem A(ii) and so $E_\sigma(E) = E$ since $E$ is $\sigma$-perfect by Part (ii).

The theorem is proved.

**Proof of Theorem C.** It is enough to prove that if given any $\sigma$-eccentric chief factor $H/K$ of $G$ below $F_\sigma^*(G)$, every automorphism of $H/K$ induced by an element of $G$ is inner, then $G$ is $\sigma$-quasinilpotent. Suppose that this is false and let $G$ be a counterexample of minimal order.

1. If $R$ is a minimal normal subgroup of $G$, then $R \leq F_\sigma^*(G)$ (This directly follows from the evident fact that every minimal normal subgroup of $G$ is $\sigma$-quasinilpotent).
2. Every proper normal subgroup $V$ of $G$ is $\sigma$-quasinilpotent. Hence $G/F_\sigma^*(G)$ is a simple group.

By Theorem A(ii), $F_\sigma^*(V) = F_\sigma^*(G) \cap V$. Hence for every $\sigma$-eccentric chief factor $H/K$ of $G$ below $F_\sigma^*(V)$, every automorphism of $H/K$ induced by an element of $G$ is inner.

Now let $K \leq L < T \leq H$, where $H/K$ is a chief factor of $G$ below $F_\sigma^*(V)$ and $T/L$ is a chief factor of $V$. Suppose that $T/L$ is $\sigma$-eccentric in $V$. Then $H/K$ is $\sigma$-eccentric in $G$. Indeed, assume that $H/K$ is $\sigma$-central in $G$. Then $H/K$ and $G/C_V(H/K)$ are $\sigma_i$-groups for some $i$. Hence $T/L$ is a $\sigma_i$-group. On the other hand, $C_V(H/K) \cap V \leq C_V(T/L)$ and also we have $V/C_V(H/K) \simeq (V/C_V(H/K))/((C_V(T/L))/C_V(H/K))$, where $V/C_V(H/K) \simeq V/C_V(H/K)/C_V(H/K)$ is a $\sigma_i$-group. Hence $V/C_V(T/L)$ is a $\sigma_i$-group and so $T/L$ is $\sigma_i$-central in $V$, a contradiction. Thus $H/K$ is $\sigma$-eccentric in $G$. Hence, by hypothesis, every element of
G induces an inner automorphism on \( H/K \). Therefore every automorphism of \( T/L \) induced by an element of \( V \) is inner by Lemma 2.1(i). Thus \( V \) is \( \sigma \)-quasinilpotent.

(3) If \( R \) is a minimal normal subgroup of \( G \), then \( R \) is not \( \sigma \)-central in \( G \).

Suppose that \( R \) is \( \sigma \)-central in \( G \). Then \( R \leq Z = Z_{\sigma}(G) \) and, by Theorem B(iii), \( F_{\sigma}^{*}(G/Z) = F_{\sigma}^{*}(G)/Z \). Now let \( (H/Z)/(K/Z) \) be a chief factor of \( G/Z \) below \( F_{\sigma}^{*}(G/Z) \). Then \( H/K \) is a chief factor of \( G \) below \( F_{\sigma}^{*}(G) \). Moreover, if \( (H/Z)/(K/Z) \) is \( \sigma \)-eccentric in \( G/Z \), then \( H/K \) is \( \sigma \)-eccentric in \( G \) and so every element \( x \in G \) induces an inner automorphism on \( H/K \). Then \( xZ \) induces an inner automorphism on \( (H/Z)/(K/Z) \). Therefore the hypothesis holds for \( G/Z \), so the choice of \( G \) implies that \( G/Z \) is \( \sigma \)-quasinilpotent. But then \( G \) is \( \sigma \)-quasinilpotent by Lemmas 2.7(vi) and 2.8, contrary to the choice of \( G \). Hence we have (3).

Final contradiction. Let \( R \) be a minimal normal subgroup of \( G \). Then \( R \leq F_{\sigma}^{*}(G) \) by Claim (1). Moreover, \( R \) is \( \sigma \)-eccentric in \( G \) by Claim (3), so every automorphism of \( R \) induced by an element of \( G \) is inner by hypothesis. Hence \( G = R C_{G}(R) \) by Lemma 2.2, where evidently \( C_{G}(R) \neq G \). Then, by Claim (2), \( C_{G}(R) \leq F_{\sigma}^{*}(G) \), so \( G = F_{\sigma}^{*}(G) \) is \( \sigma \)-quasinilpotent by Theorem A(i). This contradiction completes the proof of the result.

Proof of Theorem D. Let \( D \) be the intersection of the \( \sigma \)-centralizers of the chief factors of \( G \). First we show that \( F_{\sigma}(G) \leq D \), that is, for any chief factor \( H/K \) of \( G \) we have \( F_{\sigma}(G) \leq C_{G}^{\sigma}(H/K) \). If \( F_{\sigma}(G) \leq K \), it is evident. Now assume that \( H \leq F_{\sigma}(G) \). Then \( H/K \) is \( \sigma \)-primary, \( H/K \) is a \( \sigma \)-group say. Hence \( C_{G}^{\sigma}(H/K) = O_{\sigma_{1}}(G) C_{G}(H/K) \). By Theorem A(i), \( F_{\sigma}(G) \) is \( \sigma \)-nilpotent, so \( F_{\sigma}(G) = O_{\sigma_{1}}(F_{\sigma}(G)) \times O_{\sigma_{1}}(F_{\sigma}(G)) \) by Lemma 2.3. Moreover, \( O_{\sigma_{1}}(F_{\sigma}(G)) = O_{\sigma_{1}}(G) \leq C_{G}(H/K) \). On the other hand, Lemma 2.4 implies that \( O_{\sigma_{1}}(F_{\sigma}(G)) \leq C_{F_{\sigma}(G)}/H/K \). Hence \( F_{\sigma}(G) \leq C_{G}^{\sigma}(H/K) \). Therefore for any chief factor \( H/K \) of \( G \) we have \( F_{\sigma}(G) \leq C_{G}^{\sigma}(H/K) \) by the Jordan-Hölder theorem and Lemma 2.6.

Now we show that \( D \) is \( \sigma \)-nilpotent. Let \( H/K \) be a chief factor of \( G \) such that \( H \leq D \). Let \( C = C_{G}^{\sigma_{i}}(H/K) \). Then \( H \leq D \leq C \), so \( H/K \) is a \( \sigma \)-group for some \( i \). Hence \( C = O_{\sigma_{i}}(G) C_{G}(H/K) \). Therefore \( C / C_{G}(H/K) \simeq O_{\sigma_{i}}(G) / (O_{\sigma_{i}}(G) \cap C_{G}(H/K)) \) is a \( \sigma \)-group, so \( H/K \) is \( \sigma \)-hypercentral in \( C/K \) by Lemma 2.4. Thus \( H/K \) is \( \sigma \)-hypercentral in \( D/K \) by Lemma 2.7(iii). Therefore all factors of some chief series of \( D \) are \( \sigma \)-central in \( D \) and so \( D \) is \( \sigma \)-nilpotent by the Jordan-Hölder theorem, which implies that \( D \leq F_{\sigma}(G) \). Hence \( D = F_{\sigma}(G) \).

Now let \( D^{*} \) be the intersection of the \( \sigma \)-innerisers of the chief factors of \( G \). First we show that \( D^{*} \leq F_{\sigma}^{*}(G) \). Let \( H/K \) be a chief factor of \( G \) such that \( H \leq D^{*} \), and let \( C = C_{G}^{\sigma_{i}}(H/K) \). Then \( H \leq D^{*} \leq C \). If \( H/K \) is not \( \sigma \)-primary, then \( C = H C_{G}^{\sigma_{i}}(H/K) = H C_{G}(H/K) \) and so every element of \( C \) induces an inner automorphism on \( H/K \). Hence every element of \( D^{*} \) induces an inner automorphism on \( T/L \) for every chief factor \( T/L \) of \( D^{*} \) such that \( K \leq L < T \leq H \) by Lemma 2.1(i). Now suppose that \( H/K \) is a \( \sigma \)-group for some \( i \). Then \( C = O_{\sigma_{i}}(G) C_{G}(H/K) \), so every chief factor \( T/L \) of \( C \) such that \( K \leq L < T \leq H \) is \( \sigma \)-central in \( C \) by Lemma 2.4. Therefore \( D^{*} \) is \( \sigma \)-quasinilpotent. Hence \( D^{*} \leq F_{\sigma}^{*}(G) \).

Finally, we show that \( F_{\sigma}^{*}(G) \leq C_{G}^{\sigma_{i}}(H/K) \) for every chief factor \( H/K \) of \( G \). In view of the Jordan-Hölder theorem, it is only enough to consider the case when \( H \leq F_{\sigma}^{*}(G) \). If \( H/K \) is \( \sigma \)-primary for some \( i \), then \( F_{\sigma}^{*}(G) / C_{F_{\sigma}^{*}(G)}/H/K \) is \( \sigma \)-
primary by Theorem A(i) and Lemmas 2.4 and 2.8. Moreover, $C_{G}^\sigma(H/K) = O_{\sigma}(G)C_{G}(H/K)$. Hence $E_{\sigma}(G) \leq C_{F_{2}(G)}(H/K)$, and

$$O_{\sigma}(F_{\sigma}(G)) = O_{\sigma}(F_{\sigma}(F^{*}(G))) \leq C_{F_{2}(G)}(H/K).$$

Thus

$$F_{\sigma}^{*}(G) = E_{\sigma}(G)F_{\sigma}(G) \leq C_{G}^\sigma(H/K)$$

by Theorem B(ii). Now assume that $H/K$ is not $\sigma$-primary. Then $C_{G}^\sigma(H/K) = HC_{G}(H/K)$. Lemma 2.8 implies that $F_{\sigma}^{*}(G)/F_{\sigma}(G)$ is a direct product of some simple non-abelian groups. Hence $F_{\sigma}^{*}(G)/F_{\sigma}(G) = \left((H_{1}/F_{\sigma}(G)) \times \cdots \times (H_{1}/F_{\sigma}(G))\right)$ for some minimal normal subgroups $H_{1}/F_{\sigma}(G), \ldots, H_{1}/F_{\sigma}(G)$ of $G/F_{\sigma}(G)$ by [15, Ch.A, 4.14]. In view of the Jordan-Hölder theorem and Lemma 2.6, we can assume without loss of generality that $H/K = H_{1}/F_{\sigma}(G)$, so $H_{2} \cdots H_{t} \leq C_{G}(H/K)$. But then $F_{\sigma}^{*}(G) = HC_{F_{2}(G)}(H/K) \leq C_{G}^\sigma(H/K)$. Hence $F_{\sigma}^{*}(G) \leq D^{*}$, so $F_{\sigma}^{*}(G) = D^{*}$.

The result is proved.

4. Further applications

First consider the following

**Corollary 4.1.** $C_{G}(F_{\sigma}^{*}(G)) \leq F_{\sigma}^{*}(G)$.

**Proof.** Let $F^{*} = F_{\sigma}^{*}(G)$ and $C = C_{G}(F^{*})$. Suppose that $C \not\leq F^{*}$ and let $H/F^{*}$ be a chief factor of $G$, where $H \leq CF^{*}$. Then $H = F^{*}(H \cap C)$, where $H \cap C$ is a normal $\sigma$-quasinilpotent subgroup of $G$ by Lemma 2.8 since $(H \cap C)/(H \cap C \cap F^{*}) \simeq H/F^{*}$ and $(H \cap C) \cap F^{*} \leq Z(H \cap C)$. Thus $H \leq F^{*}$ by Theorem A(i). This contradiction completes the proof of the corollary.

From corollary 4.1 and Theorem B we get

**Corollary 4.2.** If $G$ is $\sigma$-soluble, then $C_{G}(F_{\sigma}(G)) \leq F_{\sigma}(G)$.

In the case when $\sigma = \sigma^{*}$ we get from Corollary 4.2 the following

**Corollary 4.3** (See [16, Ch.6, Theorem 1.3]). If $G$ is soluble, then $C_{G}(F(G)) \leq F(G)$.

In view of Remark 1.1, in the case when $\sigma = \sigma^{*}$, we get from Corollary 4.2 the following

**Corollary 4.4.** If $G$ is $\pi$-separable, then $C_{G}(O_{\pi}(G) \times O_{\pi'}(G)) \leq O_{\pi}(G) \times O_{\pi'}(G)$.

Now note that if $G$ is $\pi$-separable and $O_{\pi'}(G) = 1$, then $F_{\pi'}(G) = O_{\pi}(G)$ and so from Corollary 4.4 we get the following

**Corollary 4.5** (See [16, Ch.6, Theorem 3.2]). If $G$ is $\pi$-separable, then

$$C_{G/O_{\pi'}(G)}(O_{\pi}(G/O_{\pi'}(G))) \leq O_{\pi}(G/O_{\pi'}(G)).$$

In view of Remark 1.1, in the case when $\sigma = \sigma^{1\pi}$ and $O_{\pi'}(G) = 1$, we have $F_{\sigma}(G) = O_{p_{1}}(G) \times \cdots \times O_{p_{n}}(G) = F(G)$ and so we get from Corollary 4.4 the following
Corollary 4.6. If $G$ is $\pi$-soluble, then:
\begin{equation}
\sigma(\calD_{O_p(G)}(G) \times \cdots \times O_p(G) \times O_{\pi}(G)) \leq \calD_{O_p(G)}(G) \times \cdots \times O_p(G) \times O_{\pi}(G)
= \calD(O_{\pi}(G)) \times O_{\pi}(G).
\end{equation}
\begin{enumerate}
\item If $O_{\pi}(G) = 1$, then $C_G(F(G)) \leq F(G)$.
\end{enumerate}

Corollary 4.7 (Monakhov and Shypko [17]). If $G$ is $\pi$-soluble group, then:
\begin{enumerate}
\item $C_G(O_{\pi}(G) \times O_{\pi}(G)) \leq F(O_{\pi}(G)) \times O_{\pi}(G)$.
\item If $O_{\pi}(G) = 1$, then $C_G(F(G)) \leq F(G)$.
\end{enumerate}

Corollary 4.8. Let $H$ be a $\sigma$-soluble subgroup of $G$. If $E_{\sigma}(G) \leq N_G(H)$, then $E_{\sigma}(G) \leq C_G(H)$. Hence $E_{\sigma}(G)$ centralizes each normal $\sigma$-soluble subgroup of $G$.

Proof. Since $E_{\sigma}(G) \leq N_G(H)$, $[E_{\sigma}(G), H] \leq E_{\sigma}(G) \cap H$ and $E_{\sigma}(G) \cap H$ is a $\sigma$-soluble normal subgroup of $E_{\sigma}(G)$. Hence $E_{\sigma}(G) \cap H \leq Z(E_{\sigma}(G))$ since $E_{\sigma}(G)/Z(E_{\sigma}(G))$ is $\sigma$-semisimple by Theorem B(ii). Hence $[E_{\sigma}(G), H, E_{\sigma}(G)] = 1$, so $[E_{\sigma}(G), H] = [E_{\sigma}(G), E_{\sigma}(G), H] = 1$ by the lemma on three subgroups [18, III, 1.10]. The corollary is proved.

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