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An Addendum to the Elliptic Torsion Anomalous Conjecture in codimension 2

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Abstract – The torsion anomalous conjecture states that for any variety V in an abelian variety there are only finitely many maximal V-torsion anomalous varieties. We prove this conjecture for V of codimension 2 in a product E^N of an elliptic curve E without CM, complementing previous results for E with CM. We also give an effective upper bound for the normalized height of these maximal V-torsion anomalous varieties.

Keywords. Effective Diophantine Approximation, Height, Anomalous Intersections.

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1. Introduction

This article is a complement to the result in [CVV14] by Checcoli, Veneziano and Viada. Here we prove the TAC for V of codimension 2 embedded in E^N with E an elliptic curve without CM while in [CVV14] this was proven under the assumption that E has CM. We also give a finiteness statement.

We therefore refer to [CVV14] for an introduction to the subject. We only give a short introduction to the case that we prove.

In this article, by a variety, we always mean an irreducible algebraic variety defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers and, by a *point*, we mean a $\overline{\mathbb{Q}}$-valued point.

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Torsion anomalous Conjecture (TAC). *Let V be an irreducible variety embedded in an abelian variety. Then there are only finitely many maximal V-torsion anomalous varieties.*

Unlike in the original and usual definition in [BMZ07], but like in [CVV14], for us points can be torsion anomalous, but not anomalous.

The Torsion Anomalous Conjecture is known for curves in tori and abelian varieties. In [BMZ07, Theorem 1.7] Bombieri, Masser and Zannier prove it for a variety V of codimension 2 in $\mathbb{G}_m^n$. In [CVV14] Checcoli, Veneziano and the second author prove the CM elliptic codimension 2 case, using in the proof a Lehmer type bound. Proofs using such a bound cannot be extended to the non-CM case, as Lehmer's type bounds do not exist for non-CM abelian vareties. In this respect CM and non-CM cases are different in their nature. Proofs of the TAC in non-CM cases rely typically on more classical geometry of numbers.

Discussing on what was known and how proofs work, the authors realised that the known techniques are indeed sufficient to prove also the non-CM elliptic case in codimension 2. This case is neither easier nor more difficult than others, it simply requires the right combination of well known tools in this context and some adaptation to the specific situation here. These tools are used already in [CVV14], [CVV17, §3] and [Via09], but the authors had overseen that one could also complete the case presented here.

Theorem 1.1. *For an irreducible subvariety V of a power E^N of an elliptic curve E with* $\operatorname{codim} V = 2$*, there are only finitely many maximal V-torsion anomalous varieties Y. Furthermore, the normalized height $h(Y)$ is effectively bounded in terms of E, N and V.*

Below is a sketch of the proof. We already know by [CVV14, Theorem 5.1] that there are only finitely many maximal V-torsion anomalous varieties which are not translates. To handle the case of translates, for the non CM case, we cannot use a Lehmer type bound like in [CVV14]. Instead, for points we use the approximation process used by Checcoli, Veneziano and Viada in [CVV17, §3] where one constructs a translate of a given dimension with controlled degree and height through a given point in a torsion variety of dimension one. To bound the normalized height for translates of positive dimension we use a more complicated diophantine approximation generalizing the just mentioned process. Finally, to prove the finiteness of such anomalous varieties we use an induction argument and we also use a result of Viada [Via09] on the non density of certain points of bounded height.

2. Preliminaries

The reader shall refer to the preliminaries of [CVV17]. Unfortunately there is a discrepancy of definition between [CVV17] and [CVV14] for the height of a variety, [CVV17] using the one in [Phi95] like here and [CVV14] the one in [Phi91]. Since we use results from both articles here we shall clarify. In [CVV14] we call $h(X)$ and $\mu(X)$ what we denote here $\hat{h}(X)$ and $\hat{\mu}(X)$. Here we use the notations $h(X)$, $\hat{h}(p)$ and $\hat{\mu}(X)$ as in [CVV17]. Furthermore we use the standard normalization for the Néron-Tate height $\hat{h}(p)$ and for the related $\hat{\mu}(X)$ like in [Phi91] and [Phi95], this is three times the ones used in [CVV17].

Let E be an elliptic curve over $\overline{\mathbb{Q}}$ together with a fixed Weierstrass equation

$$E : y^2 = x^3 + Ax + B$$

with A, B algebraic integers. We consider $E^N \hookrightarrow \mathbb{P}^m$ for $m = 3^N - 1$ via a composition of the natural inclusions with the Segre embedding.

2.1 – *Height for points*

For a point $p = (p_0 : \ldots : p_m) \in \mathbb{P}^m(\overline{\mathbb{Q}})$, we use the absolute logarithmic Weil height

$$h_a(p) = \sum_v \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \max_i \{|p_i|_v\}$$

and the height

$$h_2(p) = \sum_{v \text{ finite}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \max_i \{|p_i|_v\} + \sum_{v \text{ infinite}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \left(\sum_i |p_i|_v^2 \right)^{1/2}$$

where K is a field of definition for p and $\mathcal{M}_K$ is its set of places.

We also consider the canonical *Néron-Tate height* $\hat{h}$ induced by our fixed embedding of E^N in $\mathbb{P}^m$. Several authors, e.g., Zimmer [Zim76] and Silverman [Sil90], gave explicit bounds for the difference of $\hat{h}$ and the logarithmic Weil height h_a. Therefore we conclude by the definitions of

h_a and h_2 and $m = 3^N - 1$ for p in E^N

$$h_2(p) - \frac{N}{2}\log 3 \le h_a(p) \le \hat{h}(p) + c_1, \tag{1}$$

$$\hat{h}(p) \le h_a(p) + c_2 \le h_2(p) + c_2 \tag{2}$$

with some explicit positive constants $c_1 = c_1(E, N)$ and $c_2 = c_2(E, N)$ only depending on E and N.

2.2 – *Heights of varieties*

For subvarieties $V \subset \mathbb{P}^m$, we denote by $h(V)$ the normalized height of V defined by Philippon [Phi95] in terms of the Chow form of the ideal of V. We note that Philippon [Phi91, §2.B] previously defined another height on subvarieties $V \subset \mathbb{P}^m$ which we denote by $h_c(V)$ here. When the variety V reduces to a point p, then by [BGS94, (3.1.6)] the height $h(p)$ of p coincides with $h_2(p)$ and by [Lan83, §3.1] the height $h_c(p)$ is equal to the absolute logarithmic Weil height $h_a(p)$. These two heights for subvarieties of $\mathbb{P}^m$ are related by the inequalities

$$h_c(V) \le h(V) \le h_c(V) + (\dim V + 1)\deg V \sum_{i=1}^{m} \frac{1}{2i}, \tag{3}$$

see for example the remarks after the definition of $h(V)$ in [Phi95].

For subvarieties $V \subset E^N$ we also consider the canonical height $\hat{h}(V)$ associated to our fixed embedding of E^N in $\mathbb{P}^m$ defined by Philippon [Phi95, §3] which extends the Néron-Tate height of points in E^N. By Proposition 9 in [Phi91] there is a constant c_3 only depending on E^N embedded in $\mathbb{P}^m$ such that

$$|\hat{h}(V) - h_c(V)| \le c_3 \cdot \deg V. \tag{4}$$

2.3 – *The arithmetic Bézout theorem*

One of the central theorems of arithmetic intersection is the arithmetic Bézout theorem [Phi95, Theorem 3]. This theorem plays a crucial role in our proof.

Theorem 2.1. *Let X and Y be irreducible subvarieties of E^N and $Z_1, \dots, Z_g$ the irreducible components of $X \cap Y$. Then we have*

$$\sum_{i=1}^{g} h(Z_i) \le \deg(X)h(Y) + \deg(Y)h(X) + C_0(\dim X, \dim Y, m)\deg(X)\deg(Y)$$

where

$$C_0(d_1, d_2, m) = \left(\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\frac{1}{2(i+j+1)}\right) + \left(m - \frac{d_1+d_2}{2}\right)\log 2. \tag{5}$$

2.4 – *The Zhang inequality*

For an irreducible subvariety $X \subset \mathbb{P}^m$, David and Philippon [DP97] proved the inequality

$$\mu_c(X) \le \frac{h_c(X)}{\deg(X)} \le (1 + \dim X)\mu_c(X) \tag{6}$$

with the *essential minimum*

$$\mu_c(X) := \inf\left\{\theta \in \mathbb{R}_{\ge 0} \colon \{P \in X \colon h_a(P) \le \theta\} \text{ is Zariski dense in } X\right\}.$$

We refer this as Zhang's inquality because Zhang [Zha95, Theorem 5.2] proved a similar result in a more general context.

We also consider the *Néron-Tate essential minimum* $\hat{\mu}(X)$ for $X \subset E^N$ defined as

$$\hat{\mu}(X) := \inf\left\{\theta \in \mathbb{R}_{\ge 0} \colon \{P \in X \colon \hat{h}(P) \le \theta\} \text{ is Zariski dense in } X\right\}.$$

Then (1) and (2) give

$$\mu_c(X) \le \hat{\mu}(X) + c_1, \tag{7}$$

$$\hat{\mu}(X) \le \mu_c(X) + c_2. \tag{8}$$

2.5 – *The auxiliary translate*

As mentioned in the introduction, part of the method used here relies on an approximation process for a point in a torsion variety. We need a translate of given dimension and controlled degree and height passing through the point. This was one of the main ingredients of [CVV17] where Checcoli, Veneziano and the second author prove a bound for C-torsion anomalous points of relative codimension one on a curve C. We are going to use their approximation theorem.

THEOREM 2.2 ([CVV17] Proposition 3.1 and 3.2). *Let P be a point in a torsion variety $B \subset E^N$ and k, s integers with $k, s \in \{1, \dots, N\}$ and $k \geq \dim B$. Then there are effective positive constants c_4, c_5, c_6 depending only on E, N, k and s such that for every real $T \geq 1$ there exists an abelian subvariety $H \subset E^N$ of codimension s such that*

$$\begin{aligned} \deg(H+P) &\leq c_4 T, \\ h(H+P) &\leq c_5 T^{1-\frac{N}{ks}} \hat{h}(P) + c_6 T. \end{aligned}$$

If E is non CM, then the constants c_4, c_5, c_6 are explicit.

Note that, in [CVV17], the constants are explicit if E is non CM. In [Via18], E. Viada give explicit constants also in the CM case.

3. Proof of the main theorem

In this section, we denote by V a weak-transverse subvariety of E^N of codimension 2. To prove our main Theorem 1.1, we have to show that V contains only finitely many maximal V-torsion anomalous varieties and that the normalized height of maximal V-torsion anomalous varieties is effectively bounded from above in terms of V, E and N. Note that it is enough to show this for V weak-transverse because an irreducible varieties $V \subset E^N$ which is not weak-transverse is itself V-torsion anomalous. We show that there are only finitely many maximal V-torsion anomalous varieties of the following types:

(1) maximal torsion varieties contained in V,

(2) maximal V-torsion anomalous varieties which are not translates,

(3) maximal V-torsion anomalous points which are not torsion points,

(4) maximal V-torsion anomalous translates of positive dimension which are not torsion varieties.

Clearly this covers all possible cases.

Note that, for a V-torsion anomalous variety Y, there is a unique minimal torsion variety $B \subset E^N$ such that Y is an irreducible component of $V \cap B$. This is easily seen because every non-empty intersection of two torsion varieties in E^N is again a torsion variety. Recall that the *relative codimension* of Y is $\operatorname{codim}_Y B = \dim B - \dim Y$. Since Y is V-torsion anomalous, we have

$$\dim B - \dim Y = \operatorname{codim} Y - \operatorname{codim} B < \operatorname{codim} V = 2.$$

This means that any V-torsion anomalous subvariety Y in V is of relative codimension 0 or 1. Relative codimension 0 only occurs if Y is a torsion variety (type (1)). By the Manin-Mumford Conjecture, we see that the maximal torsion subvarieties of E^N contained in V are exactly the irreducible components of the Zariski closure of the set of torsion points in V. Therefore, there are only finitely many Y of type (1) and their normalized height is clearly trivially bounded.

For Y of type (2), (3) or (4) the relative codimension is 1. Previous results of Checcoli, Veneziano and the second author show that there are only finitely many Y of type (2) and that their canonical height $\hat{h}(Y)$ is effectively bounded [CVV14, Theorem 5.1]. Adapting the proof to the definitions of height used here, we get an effective bound for $h(Y)$. In a subsequent paper the three authors cover the case of Y of type (3). They show that the normalized height of the maximal V-torsion anomalous points of relative codimension one is effectively bounded [CVV17, Theorem 1.1] and that these points are finitely many [CVV17, Corollary 1.2].

It remains to show that there are only finitely many Y of type (4) and that there is an effective upper bound for their normalized height only depending on V, E and N.

¿From now on, we denote by Y a maximal V-torsion anomalous variety of type (4), that is

- of the form $Y = H + p$ for a non-trivial abelian subvariety H and a point p in E^N,
- an irreducible component of $V \cap B$ where $B \subset E^N$ is an irreducible torsion variety with $\dim B = \dim H + 1$.

3.1 – *Bounded Height*

We first prove the following

PROPOSITION 3.1. *Under the same assumption as in Theorem 1.1 we have that the maximal V-torsion anomalous varieties Y of type (4) are of the form $Y = H + p_1$ with p_1 lying in an irreducible torsion subvariety of dimension 1. In addition, the Néron-Tate height of p_1 is bounded as*

$$\hat{h}(p_1) \leq C(\deg V + h(V))(\deg V)^{N-1}$$

and the height $h(Y)$ of Y is bounded as

$$h(Y) \leq \frac{3C}{2}(\deg V + h(V))(\deg V)^{2^{\dim V}+N-1}$$

for some effective constant C depending only on E and N. The constant can be made explicit.

Proof. Note that $Y = H + p \subset B = B_0 + \zeta$ with B_0 an abelian subvariety and ζ a torsion point. Thus $0 + p - \zeta$ and H are contained in B_0. Moreover, there is a unique abelian subvariety $B_1 \subset E^N$ of dimension 1 such that we have $B_0 = H + B_1$ and the Lie algebras of B_1 and H are orthogonal to each other as subspaces of $\mathbb{C}^N$ with the canonical Hermitian structure (we fix an isomorphism $\mathrm{Lie}(E(\mathbb{C})) \cong \mathbb{C}$ inducing an identification $\mathrm{Lie}(E^N(\mathbb{C})) = \mathbb{C}^N$). This abelian subvariety B_1 is equal to the identity component of $H^\perp \cap B_0$ where $H^\perp$ is the orthogonal complement of H in E^N. Since p lies in $B = H + B_1 + \zeta$, we can write

$$p = h + p_1$$

with $h \in H$ and $p_1 \in B_1 + \zeta$. In particular, we have $Y = H + p_1$. We apply Theorem 2.2 for the torsion variety $B_1 + \zeta$, the point p_1 lying in the torsion variety $B_1 + \zeta$ and the integers $k = 1$ and $s = N - 1$. By this theorem, for each $T \geq 1$, there is an abelian subvariety $H_1 \subset E^N$ of dimension 1 such that

$$\deg(H_1 + p_1) \leq c_4 T, \tag{9}$$

$$h(H_1 + p_1) \leq c_5 \frac{\hat{h}(p_1)}{T^{\frac{1}{N-1}}} + c_6 T. \tag{10}$$

We claim that Y is an irreducible component of the intersection $V \cap (H + H_1 + p_1)$. To prove this, we note that we have $\dim Y \geq \dim(H + H_1 + p_1) - 1$, where equality holds if and only if H_1 is not contained in H. Therefore, if Y is no irreducible component of $V \cap (H + H_1 + p_1)$, then H_1 is not contained in H and $H + H_1 + p_1$ is contained in V. In this case, the translate $H + H_1 + p_1$ would be contained in the intersection $V \cap (B + H_1 + \zeta)$ with

$$\begin{aligned} \operatorname{codim}(H + H_1 + p_1) &= \operatorname{codim} Y - 1 = \operatorname{codim} B \\ &\leq \operatorname{codim}(B + H_1) + 1 < \operatorname{codim}(B + H_1) + \operatorname{codim} V, \end{aligned}$$

hence $H + H_1 + p_1 = Y + H_1$ would be contained in a V-torsion anomalous subvariety contradicting the maximality of Y. This proves that Y is an irreducible component of $V \cap (H + H_1 + p_1)$.

Now we apply the arithmetic Bézout theorem for $Y \subset V \cap (H + H_1 + p_1)$ and get

$$\begin{aligned} h(Y) \leq{}& \deg(V) h(H + H_1 + p_1) + \deg(H + H_1 + p_1) h(V) \\ &+ C_0(N - 2, \dim(H + H_1), m) \cdot \deg V \cdot \deg(H + H_1 + p_1). \end{aligned} \tag{11}$$

where we can estimate the constant C_0 defined by (5) by

$$C_0(N-2, \dim(H+H_1), m) \leq \frac{N-1}{2}(1+\log N) + \left(3^N - \frac{N}{2}\right)\log 2.$$

We recall that we have $Y = H + p_1 = H + p_1' + \zeta$ with $p_1' \in B_1 \subset H^\perp$ and ζ the torsion point specified above. By Philippon [Phi12], and more precisely the version in [CVV14, Lemma 7.2] we know that

$$\hat{\mu}(H + p_1') = \hat{h}(p_1').$$

Therefore, using also the Zhang Inequality (6) and the relation (8), we get the inequality

$$\hat{h}(p_1) = \hat{h}(p_1') = \hat{\mu}(H+p_1') = \hat{\mu}(Y) \leq \mu_c(Y) + c_2 \leq \frac{h_c(Y)}{\deg(H)} + c_2. \tag{12}$$

We now bound the right hand side of this inequality from above using (11) by estimating $h(H+H_1+p_1)$ and $\deg(H+H_1+p_1)$ from above. By (3), (6), (7), we get

$$\begin{aligned} h(H+H_1+p_1) &\leq h_c(H+H_1+p_1) + \deg(H+H_1)(1+\dim(H+H_1))b_m \\ &\leq N \cdot \deg(H+H_1) \cdot (\hat{\mu}(H+H_1+p_1) + c_1 + b_m)\,. \end{aligned}$$

with $b_m := \sum_{i=1}^m \frac{1}{2i}$.

Note that we have $\hat{h}(\zeta + P) = \hat{h}(P)$ for all $P \in H_1 + p_1$ and torsion points ζ in H. Since the torsion points of H are Zariski dense in H and by the definition of the Néron-Tate essential minimum we get

$$\hat{\mu}(H+H_1+p_1) \leq \hat{\mu}(H_1+p_1).$$

Together with the previous inequality and the Zhang Inequality (6) and the relations (8) and (3), we get

$$h(H+H_1+p_1) \leq N \cdot \deg(H+H_1) \cdot \left(\frac{h(H_1+p_1)}{\deg(H_1)} + c_2 + c_1 + b_m\right).$$

By [MW93, Lemma 1.2], we can easily derive the estimate

$$\deg(H+H_1) \leq c_7 \deg H \deg H_1. \tag{13}$$

with $c_7 := 2^N$. Then, using also (10) and (9), we obtain

$$\frac{h(H+H_1+p_1)}{\deg(H)} \leq c_7 N \left(c_5 \frac{\hat{h}(p_1)}{T^{\frac{1}{N-1}}} + (c_6 + c_4\,(c_2 + c_1 + b_m))\,T\right). \tag{14}$$

Furthermore, by (13) and (9), we have

$$\frac{\deg(H+H_1+p_1)}{\deg(H)} = \frac{\deg(H+H_1)}{\deg(H)} \le c_4c_7T. \tag{15}$$

Combining (14), (15) with (11), we get

$$\frac{h(Y)}{\deg(H)} \le \frac{c_8 \deg V}{T^{\frac{1}{N-1}}}\hat{h}(p_1) + (c_9 + c_{10}\deg V + c_{11}h(V))T \tag{16}$$

with the effective positive constants

$$\begin{aligned}
c_8 &\coloneqq \max\{1, c_5c_7N\},\\
c_9 &\coloneqq c_6c_7,\\
c_{10} &\coloneqq c_4c_7\left(N\left(c_2+c_1+b_{3^N-1}\right) + \frac{N-1}{2}(1+\log N) + \left(3^N - \frac{N}{2}\right)\log 2\right),\\
c_{11} &\coloneqq c_4c_6c_7
\end{aligned}$$

depending only on E and N. Together with (12) this gives

$$\left(1 - \frac{c_8\deg V}{T^{\frac{1}{N-1}}}\right)\hat{h}(p_1) \le (c_9 + c_{10}\deg V + c_{11}h(V))T + c_2. \tag{17}$$

We now choose $T \coloneqq (2c_8)^{N-1}(\deg V)^{N-1} \ge 1$ and get

$$\hat{\mu}(Y) = \hat{h}(p_1) \le C(\deg V + h(V))(\deg V)^{N-1} \tag{18}$$

with $C \coloneqq (2c_8)^{N-1}(\max\{c_{10}, c_{11}\} + c_9 + c_2)$. This concludes the proof of the first part of the proposition.

For the normalized height $h(Y)$, combining (16), (18) and our choice of T, we get the upper bound

$$h(Y) \le \frac{3C}{2}\cdot \deg Y \cdot (\deg V + h(V))(\deg V)^{N-1}.$$

Note that Y is a maximal translate contained in V because all maximal V-torsion anomalous translates are maximal translates contained in V by [CVV14, Lemma 7.1]. Using Lemma 2 from [BZ96] and the inductive construction in its proof, we can therefore bound uniformly the degree of Y in terms of V by

$$\deg Y \le (\deg V)^{2^{\dim V}}. \tag{19}$$

Hence we have

$$h(Y) \le \frac{3C}{2}(\deg V + h(V))(\deg V)^{2^{\dim V}+N-1}.$$

In addition, the constant C can be made explicit, because the constants c_i for $i = 1, 2, 4, \ldots, 7$ are explicit in the non-CM case, and in [Via18] are made explicit also for the CM case. $\square$

It remains to show the finiteness.

3.2 – *Finiteness*

Proposition 3.2. *Under the same assumption as in Theorem 1.1, we have that there exist only finitely many maximal V-torsion anomalous varieties of type (4).*

Proof of the Proposition. We recall that all maximal V-torsion anomalous translates in E^N are of the form $Y = H + p$ with H an abelian subvariety and

$$\deg H = \deg Y \leq (\deg V)^{2^{\dim V}}.$$

Since there are only finitely many abelian subvarieties H of E^N with bounded degree, to prove the proposition it is sufficient to show that, for a fixed abelian subvariety H of E^N, there are only finitely many translates of H which are maximal V-torsion anomalous.

The proof relies on the following non-density theorem by E. Viada.

Theorem 3.3 ([Via09] Theorem 1.1 (i)). *Let $V \subset E^N$ be a weak-transverse subvariety of dimension d and $T \geq 0$ a real number. Then the set*

$$S_{d+1}(V_T) := \{p \in V : \hat{h}(p) \leq T\} \cap \bigcup_{\operatorname{codim} B \geq d+1} B$$

where B runs over all irreducible torsion varieties of codimension at least $d+1$ is not Zariski dense in V.

We use induction on the dimension of V. If V is a weak-transverse curve of codimension 2, then there are no V-torsion anomalous translates of positive dimension, therefore the statement follows by part (1), (2) and (3) discussed at the beginning of this section.

Assume it is proven for $\dim V = d - 1$ and $\operatorname{codim} V = 2$, we then show it for $\dim V = d$ and $\operatorname{codim} V = 2$. We assume by contradiction that there are infinitely many maximal V-torsion anomalous translates $Y_i = H + p_i$ of H. By Proposition 3.1 we can take $p_i \in B_i$ for an irreducible torsion variety B_i of dimension one. In addition, $\hat{h}(p_i) \leq T$

for $T := 3C(\deg V + h(V))(\deg V)^{N-1}$. Therefore, the union of all points in $p_i + \mathrm{Tor}_H$ is contained in $S_{N-1}(V_T)$ and by Theorem 3.3 it is not Zariski dense in V. Thus the closure X of $\cup_i(p_i + H) = \cup_i Y_i$ is strictly contained in V, so $\dim X < \dim V$. Since we assumed that there are infinitely many Y_i, we also have $\dim X > \dim H$. Let A be the minimal torsion variety containing X. Since the union $\cup_i (p_i + \mathrm{Tor}_H)$ is dense in X and a subset of $S_{N-1}(X_T) \subset S_{\dim X+1}(X_T)$, by Theorem 3.3 the subvariety X cannot be weak-tranverse in E^N. Thus $A \neq E^N$. Now we consider an irreducible component Z of $V \cap A$ which contains X. By the maximality of $H + p_i$, the variety Z cannot be V-torsion anomalous thus $\mathrm{codim}_A Z = 2$ and $\dim Z = \dim A - 2 < N - 2 = \dim V = d$. Consider the weak-transverse variety Z embedded in A. Note that the infinitely many $H + p_i$ are subvarieties of Z. Moreover, by an easy check of the codimensional equation, we see that the $H + p_i$ are Z-torsion anomalous in A, and by inductive hypothesis finitely many in contradiction to our assumption. □

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