On the number of nonzero digits in the beta-expansions of algebraic numbers

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Abstract — Many mathematicians have investigated the base-$b$ expansions for integral base-$b \geq 2$, and more general $\beta$-expansions for a real number $\beta > 1$. However, little is known on the $\beta$-expansions of algebraic numbers. The main purpose of this paper is to give new lower bounds for the numbers of nonzero digits in the $\beta$-expansions of algebraic numbers under the assumption that $\beta$ is a Pisot or Salem number. As a consequence of our main results, we study the arithmetical properties of power series $\sum_{n=1}^{\infty} \beta^{-\kappa(z;n)}$, where $z > 1$ is a real number and $\kappa(z;n) = \lfloor n^z \rfloor$.

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1. Normality of the digits in $\beta$-expansions

In this paper, let $\mathbb{N}$ (resp. $\mathbb{Z}^+$) be the set of nonnegative integers (resp. positive integers). We denote the integral and fractional parts of a real number $x$ by $\lfloor x \rfloor$ and $\{x\}$, respectively. Moreover, we write the minimal integer $n$ not less than $x$ by $\lceil x \rceil$. We denote the length of a nonempty finite word $W = w_1w_2\ldots w_k$ on a certain alphabet $A$ by $|W| = k$. We use the Landau symbol $O$ and the Vinogradov symbols $\gg; \ll$ with their usual meaning.

For a real number $\beta$ greater than 1, let $T_\beta : [0, 1] \to [0, 1)$ be the $\beta$-transformation defined by $T_\beta(x) := \{\beta x\}$. Using the $\beta$-transformation, Rényi [22] generalized the notion of the base-$b$ expansions of real numbers for an integral base $b$ as follows:

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Let $x$ be a real number with $0 \leq x \leq 1$. Putting $t_n(\beta, x) := \lfloor \beta T^{n-1}_\beta(x) \rfloor$ for any positive integer $n$, we have

$$x = \sum_{n=1}^{\infty} t_n(\beta, x) \beta^{-n}.$$  

The right-hand side of (1) is called the $\beta$-expansion of $x$. In what follows, we assume that $0 \leq x \leq 1$ when we consider the $\beta$-expansion of $x$. We have that $t_n(\beta, x) \leq \lfloor \beta \rfloor$. In particular, if $\beta = b$ is a rational integer, then we see $t_n(b, x) \leq b - 1$ except the only case of $t_1(b; 1) = b$.

Parry [21] showed that the digits $t_n(\beta, x)$ for $x < 1$ are characterized by the expansion of 1. Put

$$t_n(\beta, 1-) := \lim_{x \to 1-0} t_n(\beta, x)$$

for any positive integer $n$. Then we have

$$1 = \sum_{n=1}^{\infty} t_n(\beta, 1-) \beta^{-n}.$$  

For any real number $x \leq 1$, let $t(\beta, x)$ be the right-infinite sequence defined by

$$t(\beta, x) := t_1(\beta, x) t_2(\beta, x) \ldots.$$  

We also define $t(\beta, 1-)$ in the same way. Consider the case where the sequence $t(\beta, 1)$ is finite, namely, there exists a finite word $a_1 \ldots a_M$ on the alphabet $\{0, 1, \ldots, \lfloor \beta \rfloor \}$ with $a_M \neq 0$ such that

$$t(\beta, 1) = a_1 \ldots a_M 00 \ldots$$

Then it is known that

$$t(\beta, 1-) = a_1 \ldots a_{M-1}(a_M - 1)a_1 \ldots a_{M-1}(a_M - 1)a_1 \ldots.$$  

Suppose that the sequence $t(\beta, 1)$ is not finite, that is, there exist infinitely many $n$'s with $t_n(\beta, 1) \neq 0$. Then

$$t_n(\beta, 1-) = t_n(\beta, 1)$$

for any positive integer $n$. We denote by $\prec_{lex}$ the lexicographical order on the sets of the infinite sequences of nonnegative integers. Let $\sigma$ be the one-sided shift operator defined by $\sigma((s_n)_{n=1}^{\infty}) = (s_{n+1})_{n=1}^{\infty}$. Parry [21] showed for any sequence $(s_n)_{n=1}^{\infty}$ of nonnegative integers that there exists a real number $x < 1$ satisfying $s_n = t_n(\beta, x)$ for any positive integer $n$ if and only if

$$\sigma^k((s_n)_{n=1}^{\infty}) \prec_{lex} t(\beta, 1-)$$

holds for any nonnegative integer $k$. 

We review metrical results on the normality in the digits of $\beta$-expansions. We now recall the notion of $\beta$-admissibility. For any positive integers $n$ and $k$, we define the finite word $t_{n,k}(\beta, x)$ by

$$t_{n,k}(\beta, x) := t_n(\beta, x)t_{n+1}(\beta, x)\ldots t_{n+k-1}(\beta, x).$$

We call that a nonempty finite word $W$ on the alphabet $\{0, 1, \ldots, \lfloor \beta \rfloor\}$ is $\beta$-admissible if there exists a real number $x < 1$ such that

$$W = t_{1,|W|}(\beta, x).$$

If $\beta = b$ is a rational integer, then any nonempty finite word $W$ on the alphabet $\{0, 1, \ldots, b\}$ is $b$-admissible.

Borel [7] introduced the notion of normal numbers in base-$b$ for any integer $b \geq 2$. Recall that a real number $\xi < 1$ is a normal number if, for any nonempty finite word $W$ on the alphabet $\{0, 1, \ldots, b-1\}$, we have

$$\lim_{N \to \infty} \frac{\text{Card}\{n \in \mathbb{Z}^+ \mid n \leq N, t_{n,|W|}(b, \xi) = W\}}{N} = b^{-|W|},$$

where Card denotes the cardinality.

Rényi [22] proved for any real number $\beta > 1$ that there exists a unique $T_\beta$-invariant probability measure $\mu_\beta$ on $[0,1)$ which is absolutely continuous with respect to the Lebesgue measure on $[0,1)$. Moreover, he also verified that $\mu_\beta$ is ergodic. Consequently, almost all real numbers $\xi < 1$ are normal with respect to the $\beta$-expansion, that is, for any (nonempty finite) $\beta$-admissible word $W$, we have

$$\lim_{N \to \infty} \frac{\text{Card}\{n \in \mathbb{Z}^+ \mid n \leq N, t_{n,|W|}(\beta, \xi) = W\}}{N} = \mu_\beta(\{x \in [0,1) \mid t_{1,|W|}(\beta, x) = W\}).$$

On the other hand, it is difficult to determine whether a given real number $\xi < 1$ is normal with respect to the $\beta$-expansion. For instance, if $\beta = b$ is a rational integer, then Borel [8] conjectured that every algebraic irrational number is normal in base-$b$. However, neither proof nor counterexample is known for Borel’s conjecture. The main purpose of this paper is to study the properties of digits in the $\beta$-expansions of algebraic numbers in the case where $\beta$ is a Pisot or Salem number.

We recall the definition of Pisot and Salem numbers. Let $\beta$ be an algebraic integer greater than 1. Then $\beta$ is called a Pisot number if the conjugates of $\beta$ except itself have moduli less than 1. Moreover, $\beta$ is a Salem number if the conjugates of $\beta$ except itself have absolute values not greater than 1, and there exists a conjugate of $\beta$ with absolute value 1.

In Section 2, we study the complexity of the sequence $t(\beta, \xi)$ in the case where $\beta$ is a Pisot or Salem number and $0 < \xi \leq 1$ is an algebraic number. In particular, we give new lower bounds for the numbers of nonzero digits in $t(\beta, \xi)$. The lower bounds are deduced from Theorem 2.2, which is proved in Section 3.
2. Main results

Let $\beta > 1$ and $0 < \xi \leq 1$ be algebraic numbers. Lower bounds for the numbers $\gamma(\beta, \xi; N)$ of digit changes, defined by

$$\gamma(\beta, \xi; N) := \text{Card}\{n \in \mathbb{Z}^+ | n \leq N, t_n(\beta, \xi) \neq t_{n+1}(\beta, \xi)\},$$

for positive integer $N$ were studied in [9, 11, 13, 18, 19], which gives partial results on the normality of $\xi$ with respect to the $\beta$-expansion. In particular, Bugeaud [11] proved the following: Suppose that $\beta$ is a Pisot or Salem number and that $t_n(\beta, \xi) \neq t_{n+1}(\beta, \xi)$ for infinitely many $n$. Then there exist effectively computable positive constants $C_1(\beta, \xi), C_2(\beta, \xi)$, depending only on $\beta$ and $\xi$, satisfying

$$\gamma(\beta, \xi; N) \geq C_1(\beta, \xi) \frac{(\log N)^{3/2}}{\log \log N}$$

for any $N$ with $N \geq C_2(\beta, \xi)$. Lower bounds for the block complexity $p(\beta, \xi; N)$, defined by

$$p(\beta, \xi; N) := \text{Card}\{t_n(\beta, \xi) | n \in \mathbb{Z}^+\}$$

for positive integer $N$, were also obtained in [2, 3, 10, 13, 17]. Moreover, the diophantine exponents of the sequence $t(\beta, \xi)$ were studied in [2, 15].

Bailey, Borwein, Crandall, and Pomerance [5] studied the numbers of nonzero digits in the binary expansions of algebraic irrational numbers. More generally, we estimate lower bounds for the nonzero digits in the $\beta$-expansions of algebraic numbers. Let $\beta > 1$ and $\xi \leq 1$ be real numbers. Put

$$\nu(\beta, \xi; N) := \text{Card}\{n \in \mathbb{Z}^+ | n \leq N, t_n(\beta, \xi) \neq 0\}$$

for any positive integer $N$. It is easily seen that

$$\nu(\beta, \xi; N) \geq \frac{1}{2} \gamma(\beta, \xi; N) + O(1).$$

Let $\beta$ be a Pisot or Salem number and $\xi$ an algebraic number. Assume that the digits of $t(\beta, \xi)$ change infinitely many times. Then (2) implies that

$$\nu(\beta, \xi; N) \geq \frac{C_1(\beta, \xi)}{3} \cdot \frac{(\log N)^{3/2}}{\log \log N}$$

for any sufficiently large $N$.

The main purpose of this paper is to improve lower bound (3). Bailey, Borwein, Crandall, and Pomerance [5] proved for any algebraic irrational number $\xi \leq 1$ of degree $D$ that there exist positive constants $C_3(\xi)$ and $C_4(\xi)$, depending only on $\xi$, satisfying

$$\nu(2, \xi; N) \geq C_3(\xi)N^{1/D}.$$
for any integer \( N \) with \( N \geq C_4(\xi) \). Note that \( C_3(\xi) \) is effectively computable but \( C_4(\xi) \) is not. Rivoal [23] improved the constant \( C_3(\xi) \) for certain classes of algebraic irrational numbers.

Adamczewski, Faverjon [4] and Bugeaud [12] independently verified for each integral base \( b \geq 2 \) and any algebraic irrational number \( \xi \) of degree \( D \) that there exist effectively computable positive constants \( C_5(b, \xi) \) and \( C_6(b, \xi) \), depending only on \( b \) and \( \xi \), satisfying

\[
\nu(b, \xi; N) \geq C_5(b, \xi) N^{1/D}
\]

for any integer \( N \) with \( N \geq C_6(b, \xi) \).

Let again \( \beta \) be a Pisot or Salem number and \( \xi = 1 \) an algebraic number. Put \( [Q(\beta, \xi) : Q(\beta)] = D \), where \( [L : K] \) denotes the degree of the field extension \( L/K \). Suppose that there exist infinitely many nonzero digits in the sequence \( t(\beta, \xi) \). Then we have [20]

\[
\nu(\beta, \xi; N) \geq C_7(\beta, \xi) \left( \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}} \right)
\]

for any integer \( N \) with \( N \geq C_8(\beta, \xi) \), where \( C_7(\beta, \xi) \) and \( C_8(\beta, \xi) \) are effectively computable positive constants depending only on \( \beta \) and \( \xi \). The inequality (5) follows from Theorem 2.1 in [20], which we introduce as follows: For any sequence \( s = (s_n)_{n=0}^\infty \) of integers, we set

\[
\Gamma(s) = \{ n \in \mathbb{N} \mid s_n \neq 0 \}
\]

and

\[
f(s; X) := \sum_{n=0}^\infty s_n X^n.
\]

Moreover, for any nonnegative integer \( N \) and any nonempty set \( A \) of nonnegative integers, we put

\[
\lambda(A; N) := \text{Card}([0, N] \cap A).
\]

**Theorem 2.1 ([20, Theorem 2.1]).** Let \( \beta \) be a Pisot or Salem number and \( \xi \) an algebraic number with \( [Q(\beta, \xi) : Q(\beta)] = D \). Suppose that there exists a sequence \( s = (s_n)_{n=0}^\infty \) of integers satisfying the following two assumptions:

1. There exists a positive integer \( B \) such that, for any \( n \in \mathbb{N} \), we have \( 0 \leq s_n \leq B \). Moreover, there exist infinitely many \( n \) such that \( s_n > 0 \).
2. \( \xi = f(s; \beta^{-1}) \).

Then there exist effectively computable positive constants \( C_9 = C_9(\beta, \xi, B) \) and \( C_{10} = C_{10}(\beta, \xi, B) \), depending only on \( \beta, \xi \) and \( B \), such that, for any integer \( N \) with \( N \geq C_{10} \), we have

\[
\lambda(\Gamma(s); N) \geq C_9 \left( \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}} \right).
\]
In what follows, we improve Theorem 2.1 under the same assumptions.

**Theorem 2.2.** Let $\beta$ be a Pisot or Salem number and $\xi$ an algebraic number with $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$. Suppose that there exists a sequence $s = (s_n)_{n=0}^\infty$ of integers satisfying the following two assumptions:

1. There exists a positive integer $B$ such that, for any $n \in \mathbb{N}$, we have $0 \leq s_n \leq B$. Moreover, there exist infinitely many $n$ such that $s_n > 0$.

2. $\xi = f(s; \beta^{-1}).$

Then there exist effectively computable positive constants $C_{11} = C_{11}(\beta, \xi, B)$ and $C_{12} = C_{12}(\beta, \xi, B)$, depending only on $\beta, \xi$ and $B$, such that, for any integer $N$ with $N \geq C_{12}$, we have

$$\lambda(\Gamma(s); N) \geq C_{11} \frac{N^{1/D}}{\log N}.$$

We note that Theorems 2.1 and 2.2 are applicable not only to the $\beta$-expansion but also to a general $\beta$-representation

$$\xi = \sum_{n=0}^\infty t_n \beta^{-n},$$

where $(t_n)_{n=0}^\infty$ is a bounded sequence of nonnegative integers.

As a consequence of Theorem 2.2, we improve (5) as follows:

**Corollary 2.3.** Let $\beta$ be a Pisot or Salem number and $\xi \leq 1$ an algebraic number with $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$. Suppose that there exist infinitely many nonzero digits in $t(\beta, \xi)$. Then there exist effectively computable positive constants $C_{13}(\beta, \xi)$ and $C_{14}(\beta, \xi)$, depending only on $\beta$ and $\xi$, satisfying

$$\nu(\beta, \xi; N) \geq C_{13}(\beta, \xi) \frac{N^{1/D}}{\log N^{1/D}}$$

for any integer $N$ with $N \geq C_{14}(\beta, \xi)$.

We apply Theorem 2.2 to the arithmetical properties on certain values of power series at algebraic points. Let $(v_n)_{n=1}^\infty$ be a sequence of nonnegative integers such that $v_{n+1} > v_n$ for sufficiently large $n$. Bugeaud [9, 11] posed a problem on the transcendence of $\sum_{n=1}^\infty \alpha^{v_n}$, where $\alpha$ is an algebraic number with $0 < |\alpha| < 1$, under the assumption that $(v_n)_{n=1}^\infty$ increases sufficiently rapidly. Corvaja and Zannier [14] proved for any algebraic number $\alpha$ with $0 < |\alpha| < 1$ that if

$$\liminf_{n \to \infty} \frac{v_{n+1}}{v_n} > 1$$
holds, then $\sum_{n=1}^{\infty} \alpha^{v_n}$ is transcendental. In particular, consider the case of $\alpha = \beta^{-1}$, where $\beta$ is a Pisot or Salem number. Adamczewski [1] proved that if

$$\limsup_{n \to \infty} \frac{v_{n+1}}{v_n} > 1,$$

then $\sum_{n=1}^{\infty} \beta^{-v_n}$ is transcendental. However, if

$$\lim_{n \to \infty} \frac{v_{n+1}}{v_n} = 1,$$

then it is generally difficult to determine whether $\sum_{n=1}^{\infty} \alpha^{v_n}$ is transcendental. For instance, put, for any real number $z > 1$ and any positive integer $n$, $\kappa(z; n) := \lfloor n^z \rfloor$. Moreover, set $\psi(z; X) := \sum_{n=1}^{\infty} X^{\kappa(z; n)}$. Then the transcendence of $\psi(z; \alpha)$ is unknown except the case where $\psi(2; \alpha)$ is transcendental for any algebraic number $\alpha$ with $0 < |\alpha| < 1$, which was proved by Duverney, Nishioka, Nishioka, Shiokawa [16], and Bertrand [6] independently.

Using Theorem 2.1 or Theorem 2.2, we obtain that if

$$\limsup_{n \to \infty} \frac{v_{n+1}}{v_n} = \infty$$

for any positive real number $R$, then, for any Pisot or Salem number $\beta$, we have $\sum_{n=1}^{\infty} \beta^{-w_n}$ is transcendental. This criterion for transcendence is applicable to certain sequences $(v_n)_{n=1}^{\infty}$ satisfying (9). For instance, let, for any positive integer $n$,

$$w_n := \lfloor n^{\log n} \rfloor = \lfloor \exp ((\log n)^2) \rfloor.$$

Then $(w_n)_{n=1}^{\infty}$ fulfills (9). Since $(w_n)_{n=1}^{\infty}$ satisfies (10), we see that $\sum_{n=1}^{\infty} \beta^{-w_n}$ is transcendental.

Moreover, Using Theorem 2.1, we get for real number $z > 1$ and any Pisot or Salem number $\beta$ that $\psi(z; \beta^{-1})$ cannot be algebraic of small degree over $\mathbb{Q}(\beta)$, precisely

$$[\mathbb{Q}(\psi(z; \beta^{-1}), \beta) : \mathbb{Q}(\beta)] \geq \left[ \frac{z + 1}{2} \right].$$

In fact, we put

$$\psi(z; X) =: \sum_{n=0}^{\infty} s_n X^n.$$

Then a bounded sequence $s = (s_n)_{n=0}^{\infty}$ of nonnegative integers satisfies

$$\lim_{N \to \infty} \frac{\lambda(\Gamma(s); N)}{N^{1/x}} = 1.$$

If $\psi(z; \beta^{-1})$ is transcendental, then (11) is clear because the left-hand side is equal to infinity. Assume that $\psi(z; \beta^{-1})$ is an algebraic number satisfying

$$[\mathbb{Q}(\psi(z; \beta^{-1}), \beta) : \mathbb{Q}(\beta)] = D.$$
Then (6) holds only in the case of \( z \leq 2D - 1 \). Similarly, using Theorem 2.2, we deduce that

\[
\left[ \mathbb{Q}(\psi(z; \beta^{-1}), \beta) : \mathbb{Q}(\beta) \right] \geq \lfloor z \rfloor,
\]

which improves (11).

3. Proof of Theorem 2.2

For the proof of Theorem 2.2, we recall the following Liouville type inequality deduced from Theorem 11 in [24, p. 34].

**Lemma 3.1 ([20, Proposition 3.1]).** Let \( z \) and \( \xi \) be algebraic numbers. Suppose that there exists a sequence \( s = (s_n)_{n=0}^{\infty} \) of integers satisfying the following three assumptions:

1. There exists a positive integer \( B \) such that, for any \( n \in \mathbb{N} \), we have \( 0 \leq s_n \leq B \).
2. \( \xi = f(s; z) \).
3. For any \( M \in \mathbb{N} \), we have

\[
\sum_{n=0}^{M} s_n z^n \neq \xi.
\]

Let \( (w(m))_{n=0}^{\infty} \) be a strictly increasing sequence of nonnegative integers defined by

\[
\{ n \in \mathbb{N} \mid s_n \neq 0 \} =: \{ w(0) < w(1) < \cdots \}.
\]

Then there exist effectively computable positive constants \( C_{15} = C_{15}(z, \xi, B) \) and \( C_{16} = C_{16}(z, \xi, B) \), depending only on \( z, \xi \) and \( B \), such that, for any integer \( m \) with \( m \geq C_{16} \), we have

\[
\frac{w(m + 1)}{w(m)} < C_{15}.
\]

If \( D = 1 \), then (8) is deduced from (6). Thus, we may assume that \( D \geq 2 \). For simplicity, put

\[
\Gamma := \Gamma(s), \lambda(N) := \lambda(\Gamma; N).
\]

We may assume that \( s_0 \neq 0 \), that is,

\[
0 \in \Gamma.
\]

In what follows, the implied constants in the symbol \( \ll \) and the constants \( C_{17}, C_{18}, \ldots \) are effectively computable positive ones depending only on \( \beta, \xi \) and \( B \). We see for
any $M \in \mathbb{N}$ that $\sum_{n=0}^{M} s_n \beta^{-n} \neq \xi$ by (7) and the first assumption of Theorem 2.2. Thus, using Lemma 3.1, we get that there exist $C_{17}$ and $C_{18}$ satisfying
\begin{equation}
\Gamma \cap [x; C_{17}x) \neq \emptyset
\end{equation}
for any real number $x$ with $x \geq C_{18}$. By $[Q(\beta, \xi) : Q(\beta)] = D$, there exists a polynomial $P(X) = A_{D}X^{D} + A_{D-1}X^{D-1} + \cdots + A_{0} \in \mathbb{Z}[\beta][X]$ with $A_{D} > 0$ such that $P(\xi) = 0$. In the same way as the proof of Theorem 2.1 in [20], we see for any $k$ with $1 \leq k \leq D$ that
\begin{equation}
\xi^{k} = \left( \sum_{m \in \Gamma} s_{m} \beta^{-m} \right)^{k} = \sum_{m=0}^{\infty} \beta^{-m} \rho(k; m),
\end{equation}
where
$$\rho(k; m) = \sum_{m_{1}, \ldots, m_{k} \in \Gamma, m_{1} + \cdots + m_{k} = m} s_{m_{1}} \cdots s_{m_{k}}.$$ Note for any nonnegative integer $m$ that $\rho(k; m)$ is a nonnegative integer. Moreover, putting
$$k\Gamma := \{ m_{1} + \cdots + m_{k} \mid m_{1}, \ldots, m_{k} \in \Gamma \},$$
we get that $\rho(k; m)$ is positive if and only if $m \in k\Gamma$. By (12), we have
\begin{equation}
(0 \in \Gamma \subset 2\Gamma \subset \cdots \subset (D-1)\Gamma \subset D\Gamma).
\end{equation}
Observe that
\begin{equation}
\lambda(k\Gamma; N) = \text{Card}([0, N] \cap k\Gamma) \leq \text{Card}([0, N] \cap \Gamma)^{k} = \lambda(N)^{k}
\end{equation}
and that
\begin{equation}
\rho(k; m) \leq B^{k} \sum_{m_{1}, \ldots, m_{k} \in \Gamma, m_{1} + \cdots + m_{k} = m} 1 \leq B^{k}(m+1)^{k}.
\end{equation}
We see that
\begin{equation}
0 = P(\xi) = A_{0} + \sum_{k=1}^{D} A_{k} \xi^{k}
= A_{0} + \sum_{k=1}^{D} A_{k} \sum_{m=0}^{\infty} \beta^{-m} \rho(k; m)
\end{equation}
by (14). Let $R$ be a nonnegative integer. Then, multiplying (18) by $\beta^{R}$, we get
\begin{equation}
0 = A_{0} \beta^{R} + \sum_{k=1}^{D} A_{k} \sum_{m=-R}^{\infty} \beta^{-m} \rho(k; m + R).
\end{equation}
In particular, putting

\[ Y_R := \sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + R), \]

we obtain

\[ \sum_{m=-R}^{0} \beta^{-m} \rho(k; m + R). \]

Note that \( Y_R \) is an algebraic integer by (19) because \( \beta \) is a Pisot or Salem number.

In the same way as the proof of Lemma 4.1 in [20], we deduce the following: There exists positive integers \( C_{19} \) and \( C_{20} \) such that if \( R \) is an integer with \( R \geq C_{20} \), then we have

\[ Y_R = 0 \text{ or } |Y_R| \geq R^{-C_{19}}. \]

In the case of \( \beta = 2 \), Bailey, Borwein, Crandall, and Pomerance [5] investigated the numbers of positive \( Y_R \) to prove (4). More precisely, they estimated upper and lower bounds for the value

\[ \text{Card} \{ R \in \mathbb{N} \mid R \leq N; Y_R > 0 \} \]

for a nonnegative integer \( N \). However, if \( \beta \) is a general Pisot or Salem number, then it is difficult to obtain upper bounds. So we modify their definition, that is, we consider the value

\[ y_N := \text{Card} \{ R \in \mathbb{N} \mid R \leq N; Y_R \geq C_{21} \} \]

for a integer \( N \) with \( N \gg 1 \), where \( C_{21} = \min \{ 1/\beta, A_D/\beta \} \). We give upper bounds for \( y_N \) in Lemma 3.2, using the function \( \lambda(N) \). Note that we modify the definition of \( y_N \) to get (22), which is the key inequality for the proof of Lemma 3.2. On the other hand, we estimate upper bounds for \( y_N \) in Lemma 3.5. The main tool for the proof of Lemma 3.5 is Lemma 3.4, which is deduced from Liouville type inequality (20).

In what follows, we assume that \( N \) is a sufficiently large integer satisfying

\[ \left( 1 + \frac{1}{N} \right)^D < \frac{1 + \beta}{2}. \]

**Lemma 3.2.**

\[ y_N \ll \log N + \lambda(N)^D. \]

for any integer \( N \) with \( N \gg 1 \).
Proof. Putting \( K := \lceil (D + 1) \log_\beta N \rceil \), we get

\[
y_N \leq K + y_{N-K} = K + \sum_{0 \leq R < N-K} 1 \leq K + \frac{1}{C_{21}} \sum_{R=0}^{N-K} |Y_R|.
\]

Observe that

\[
\sum_{R=0}^{N-K} |Y_R| \leq \sum_{R=0}^{N-K} \sum_{k=1}^{D} \sum_{m=1}^{\infty} |A_k| \beta^{-m} \rho(k; m + R) = \sum_{k=1}^{D} |A_k| \sum_{R=0}^{N-K} \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + R)
\]

(23)

\[
=: \sum_{k=1}^{D} |A_k| z_N^{(k)},
\]

where

\[
z_N^{(k)} = \sum_{R=0}^{N-K} \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + R)
\]

for any \( N \) and \( k \) with \( N \geq 0 \) and \( 1 \leq k \leq D \). By (22) and (23), it suffices to show

\[
z_N^{(k)} \ll \lambda(N)^D
\]

(24)

for any \( N \) and \( k \) with \( N \gg 1 \) and \( 1 \leq k \leq D \). We see that

\[
z_N^{(k)} = \sum_{m=1}^{K} \beta^{-m} \sum_{R=0}^{N-K} \rho(k; m + R) + \sum_{m=K+1}^{\infty} \beta^{-m} \sum_{R=0}^{N-K} \rho(k; m + R)
\]

(25)

\[=: S_1(k) + S_2(k).\]

Using the first assumption of Theorem 2.2 and the definition of \( \rho(k; R), \lambda(N) \), we obtain

\[
S_1(k) \leq \sum_{m=1}^{K} \beta^{-m} \sum_{R=0}^{N} \rho(k; R) \leq \sum_{m=1}^{\infty} \beta^{-m} \sum_{R=0}^{N} \rho(k; R) \ll \sum_{R=0}^{N} \rho(k; R) = \sum_{R=0}^{N} \sum_{m_1, \ldots, m_k \in \mathbb{N}} s_{m_1} \cdots s_{m_k}
\]

\[\leq \sum_{m_1, \ldots, m_k \in \mathbb{N}} s_{m_1} \cdots s_{m_k} \leq B^k \sum_{m_1, \ldots, m_k \leq N} 1 \leq B^D \lambda(N)^D \ll \lambda(N)^D.
\]

(26)
On the other hand, (17) implies by $k \leq D$ that
\[
S_2(k) \ll \sum_{m=R+1}^{\infty} \beta^{-m} \sum_{R=0}^{N-K} (m + R + 1)^D \leq N \sum_{m=R+1}^{\infty} \beta^{-m} (m + N)^D.
\]
Thus, using (21), we obtain for any integer $N$ with $N \gg 1$ that
\[
S_2(k) \ll N \beta^{-1-K} (1 + K + N)^D \sum_{m=0}^{\infty} \beta^{-m} \left( \frac{1 + \beta}{2} \right)^m
\]
\[
\ll \beta^{-K} N^{D+1} \leq 1.
\]
Therefore, combining (25), (26), and (27), we deduce (24).

Recalling that $0 \in (D - 1)\Gamma$ by (15), set
\[
[0, N) \cap (D - 1)\Gamma =: \{0 = i(1) < i(2) < \cdots < i(\tau)\},
\]
where
\[
\tau = \tau(N) \leq \lambda(N)^{D-1}
\]
by (16). Put $i(1 + \tau) := N$.

Let $1 \leq h \leq \tau$. We define the interval $I_h$ by
\[
I_h := \begin{cases} 
[i(h), i(1 + h)) & (1 \leq h \leq -1 + \tau), \\
i(\tau), i(1 + \tau)] & (h = \tau).
\end{cases}
\]
Moreover, let $|I_h| := i(1 + h) - i(h)$ and
\[
y_N(h) := \text{Card} \{R \in I_h \mid Y_R \geq C_{21}\}.
\]
Then we have
\[
\sum_{h=1}^{\tau} |I_h| = N
\]
and
\[
\sum_{h=1}^{\tau} y_N(h) = y_N.
\]
Consider the case where $I_h$ satisfies
\[
|I_h| > 8D(1 + C_{17})C_{19} \log \beta N =: C_{22} \log \beta N.
\]
If $N \gg 1$, then applying (13) with $x = |I_h|/(1 + C_{17})$, we see by (31) that there exists $\theta(h) \in \Gamma$ with
\[
\frac{1}{1 + C_{17}} |I_h| \leq \theta(h) < \frac{C_{17}}{1 + C_{17}} |I_h|.
\]
Putting $M(h) := i(h) + \theta(h)$, we get

$$i(h) + \frac{1}{1 + C_{17}} |I_h| \leq M(h) < i(h) + \frac{C_{17}}{1 + C_{17}} |I_h|. \tag{32}$$

Moreover, we obtain $M(h) \in D\Gamma$, by $i(h) \in (D - 1)\Gamma$ and $\theta(h) \in \Gamma$.

**Lemma 3.3.** Let $N, h$ be integers with $N \gg 1$ and $1 \leq h \leq \tau$. Assume that (31) holds. Then $Y_R > 0$ for any integer $R$ with $i(h) \leq R < M(h)$.

**Proof.** We prove the lemma by induction on $R$. We first consider the case where $R = -1 + M(h)$. Observe that

$$Y_{-1 + M(h)} = A_D \sum_{m=1}^{\infty} \beta^{-m} \rho(D; m + M(h) - 1)$$

$$+ \sum_{k=1}^{D-1} A_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + M(h) - 1)$$

$$=: S_3 + S_4. \tag{33}$$

By $M(h) \in D\Gamma$, we get

$$S_3 \geq \frac{A_D}{\beta} \rho(D; M(h)) \geq \frac{A_D}{\beta}. \tag{34}$$

We estimate upper bounds for $|S_4|$. Let $k, m$ be integers with $1 \leq k \leq D - 1$ and $1 \leq m \leq -1 + [2D \log_\beta N]$. Observe that, by (32), (31), and $C_{19} \geq 1$,

$$i(1 + h) - M(h) \geq i(1 + h) - i(h) - \frac{C_{17}}{1 + C_{17}} |I_h|$$

$$= \frac{1}{1 + C_{17}} |I_h| > 8D \log_\beta N > m$$

Hence, we see

$$i(h) < m + M(h) - 1 < i(1 + h),$$

by $i(h) < M(h) \leq m + M(h) - 1$. Consequently, $m + M(h) - 1 \not\in (D - 1)\Gamma$. In particular, by (15) we obtain $m + M(h) - 1 \not\in k\Gamma$. Therefore, we deduce that

$$\rho(k; m + M(h) - 1) = 0$$

for any $k, m$ with $1 \leq k \leq D - 1$ and $1 \leq m \leq -1 + [2D \log_\beta N]$. 
Using (17), we obtain
\[ |S_4| \leq \sum_{k=1}^{D-1} |A_k| \sum_{m \geq [2D \log_3 N]} \beta^{-m} \rho(k; m + M(h) - 1) \]
\[ \leq \sum_{k=1}^{D-1} |A_k| \sum_{m \geq [2D \log_3 N]} \beta^{-m} B^D (m + N)^D \]
\[ \ll \sum_{m \geq [2D \log_3 N]} \beta^{-m} (m + N)^D. \]

Consequently, (21) implies that
\[ |S_4| \ll \beta^{-[2D \log_3 N]} (\lfloor 2D \log_3 N \rfloor + N)^D \sum_{m=0}^{\infty} \beta^{-m} \left( \frac{1 + \beta}{2} \right)^m \]
\[ \ll N^{-D}. \]

If \( N \gg 1 \), then
\[ (35) \quad |S_4| < \frac{A_D}{2\beta}. \]

Combining (33), (34), and (35), we deduce \( Y_{-1+M(h)} > 0 \).

Next we assume \( Y_R > 0 \) for some \( R \) with \( i(h) < R < M(h)(< i(1 + h)) \). Using \( \rho(k; R) = 0 \) for \( k = 1, \ldots, D - 1 \) by (15), we see
\[ Y_{R-1} = \sum_{k=1}^{D} A_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + R - 1) \]
\[ = \frac{1}{\beta} A_D \rho(D; R) + \frac{1}{\beta} \sum_{k=1}^{D} A_k \sum_{m=2}^{\infty} \beta^{-(m-1)} \rho(k; m - 1 + R) \]
\[ = \frac{1}{\beta} A_D \rho(D; R) + \frac{1}{\beta} Y_R \geq \frac{1}{\beta} Y_R > 0 \]
by the inductive hypothesis. Therefore, we proved the lemma. \( \square \)

**Lemma 3.4.** Let \( N, h \) be integers with \( N \gg 1 \) and \( 1 \leq h \leq \tau \). Assume that (31) holds. Let \( R \) be an integer with
\[ i(h) + 4C_{19} \log_3 N \leq R < M(h). \]
Then we have
\[ R - \max \{ R' \in \mathbb{N} \mid R' < R, Y_{R'} \geq C_{21} \} \leq 2C_{19} \log_3 N. \]
Proof. Let 

\[ R_1 := \max \{ R' \in \mathbb{N} \mid R' < R, Y_{R'} \geq C_{21} \} . \]

In the same way as the proof of (36), we see for any integer \( r \) with \( i(h) < r < i(1 + h) \) that

\[ Y_{r-1} = \frac{1}{\beta} A_D \rho(D; r) + \frac{1}{\beta} Y_r. \]

(37)

For the proof of the lemma, we may assume that \( Y_R < 1 \). In fact, if \( Y_R \geq 1 \), then we have \( Y_{R-1} \geq 1/\beta \geq C_{21} \) by (37) and \( R - R_1 = 1 \leq 2C_{19} \log_3 N \).

Put \( S := \lceil C_{19} \log_3 N \rceil \). Assume for any integer \( m \) with \( 0 \leq m \leq S \) that

\[ \rho(D; R - m) = 0. \]

Since \( M(h) > R > R - 1 > \cdots > R - S > i(h) \), we get by (37) that

\[ 1 > Y_R = \beta Y_{R-1} = \cdots = \beta^S Y_{R-S} = \beta^{1+S} Y_{R-S-1} > 0. \]

In fact, Lemma 3.3 implies \( Y_{R-S-1} > 0 \) by \( R - S - 1 \geq i(h) \). Consequently, we see

\[ \beta^{S+1} < Y_{R-S-1} = |Y_{R-S-1}|^{-1}. \]

If \( N \gg 1 \), then we have \( R - S - 1 \geq 2C_{19} \log_\beta N \geq C_{20} \). Thus, using (20), we obtain

\[ \beta^{S+1} < |Y_{R-S-1}|^{-1} \leq (R - S - 1)^{C_{19}} < N^{C_{19}}. \]

Hence, we deduce that

\[ [C_{19} \log_3 N] + 1 = S + 1 < C_{19} \log_3 N, \]

a contradiction. Therefore, there exists an integer \( m' \) with \( 0 \leq m' \leq S \) such that \( \rho(D; R - m') \geq 1 \). Finally, using (37) and \( Y_{R-m'} > 0 \) by Lemma 3.3, we obtain

\[ Y_{R-m'-1} \geq \frac{A_D}{\beta} \geq C_{21} \]

and

\[ R - R_1 \leq m' + 1 \leq 2C_{19} \log_3 N. \]

Lemma 3.5. There exists \( C_{23} \) satisfying the following: If \( N \gg 1 \), then, for any integer \( h \) with \( 1 \leq h \leq \tau \), we have

\[ y_N(h) \geq \frac{|I_h|}{C_{23} \log_3 N}. \]

(38)
**Proof.** If (31) holds, then (38) follows from Lemma 3.4. In what follows, we suppose that \(|I_h| \leq C_{22} \log_3 N\). If necessary, increasing \(C_{23}\), we may assume that 
\[ C_{23} > C_{22}. \]
Thus, (38) holds by 
\[
\left\lfloor \frac{|I_h|}{C_{23} \log_3 N} \right\rfloor = 0.
\]

If \(N \gg 1\), then, combining (30), Lemma 3.5, and (29), (28), we deduce that
\[
y_N = \sum_{h=1}^{\tau} y_N(h) \geq \sum_{h=1}^{\tau} \left( \frac{|I_h|}{C_{23} \log_3 N} - 1 \right)
\]
\[
\geq \frac{N}{C_{23} \log_3 N} - \tau \gg \frac{N}{\log N} - \lambda(N)^{D-1}.
\]
On the other hand, Lemma 3.2 implies that
\[
\log N + \lambda(N)^D \gg y_N \gg \frac{N}{\log N} - \lambda(N)^{D-1}.
\]
Therefore, we proved Theorem 2.2.

**References**


On the number of nonzero digits in the beta-expansions of algebraic numbers


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