

## Nunke's Problem for Abelian $p$ -Groups of Uncountable Length

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ABSTRACT – *Nunke's Problem* asks when the torsion product of two abelian  $p$ -groups is a direct sum of countable reduced groups. In previous work the author gave a complete answer to this question when the groups involved have countable length. In this paper a complete answer is given in the case of groups of uncountable length, at least in any set-theoretic universe in which  $2^{\aleph_1} = \aleph_2$ .

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### 1. An Introduction to Nunke's Problem

All groups will be abelian  $p$ -groups for some fixed prime  $p$ . Our notation and terminology will generally follow [2]. For an ordinal  $\alpha$  we will assume some familiarity with the theory of  $p^\alpha$ -purity, which can be found, for example, in [3]. We will also use some fairly basic set theory for which we refer the reader to [1]. If  $A$  and  $B$  are groups, then we will denote their torsion product by the convenient, albeit unorthodox, notation  $A \nabla B$ .

A subgroup  $K \subseteq G$  is said to be  $p^\alpha$ -high if it is maximal with respect to the property that  $K \cap p^\alpha G = 0$ , and  $G$  is said to be a  $C_\lambda$ -group if for every  $\alpha < \lambda$ , the  $p^\alpha$ -high subgroups of  $G$  are all direct sums of countable reduced groups (or dsc groups, for short). If  $\lambda \leq \omega_1$  is a limit, then  $G$  will be a  $C_\lambda$ -group iff for every  $\alpha < \lambda$  the quotient  $G/p^\alpha G$  is dsc group [see, for example, ([5], Theorem 8)].

There is a strong connection between the torsion product and the class of dsc groups. *Nunke's Problem* asks us for a description of when  $A \nabla B$  is a dsc group of length  $\lambda \leq \omega_1$ . The following was the first important result on Nunke's problem,

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which has been rephrased using our terminology:

**THEOREM 1.1** ([11], Theorem 6). *Suppose  $A$  and  $B$  are reduced groups,  $B$  has length  $\lambda \leq \omega_1$  and  $p^\lambda A \neq 0$ . Then  $A \nabla B$  is a dsc group iff  $A$  is a  $C_\lambda$ -group and  $B$  is a dsc group.*

So the case of interest is where both  $A$  and  $B$  have length  $\lambda$ . First, it is straightforward to verify that  $A \nabla B$  is a  $C_\lambda$  group of length at least  $\lambda$  iff both  $A$  and  $B$  are  $C_\lambda$ -groups of length at least  $\lambda$  [see, for example, ([8], Proposition 4) with  $n = 2$ ]. So Nunke's problem is really a question about  $C_\lambda$ -groups.

It turns out that the countable case [i.e.,  $\lambda < \omega_1$ ] of Nunke's Problem behaves very differently from the uncountable one [i.e.,  $\lambda = \omega_1$ ]. Let  $\mathcal{R}$  be the class of all uncountable regular cardinals and  $\mathcal{R}_f$  be the class of all finite subsets of  $\mathcal{R}$ . Let  $0_{\mathcal{R}} = \emptyset \subseteq \mathcal{R}_f$ , and if  $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{R}_f$ , let  $\mathcal{X} \cdot \mathcal{Y}$  be the class of all disjoint unions  $X \cup Y$  where  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ . If  $\lambda < \omega_1$ , then in [9] for every  $C_\lambda$ -group  $G$  there was defined an invariant  $L_G^\lambda \subseteq \mathcal{R}_f$ . This definition was based on transfinite induction using filtrations on subgroups of  $G$ . Its relevance to Nunke's Problem is summarized in the following:

**THEOREM 1.2.** *Suppose  $\lambda < \omega_1$  is countable and  $G, A$  and  $B$  are  $C_\lambda$ -groups.*

(a) ([9], Theorem 1.6)  $G$  is a dsc group iff  $L_G^\lambda = 0_{\mathcal{R}}$ .

(b) ([9], Theorem 3.10)  $A \nabla B$  is a dsc group iff  $L_A^\lambda \cdot L_B^\lambda = 0_{\mathcal{R}}$ .

In other words, the elements of  $L_G^\lambda$  can be viewed as obstructions to  $G$  splitting into countable summands, and the invariant behaves well enough with respect to the torsion product to give a complete solution to Nunke's Problem in the countable case.

Turning now to the uncountable case  $\lambda = \omega_1$ , in [7] it was shown that Nunke's Problem leads us to consider a set-theoretic statement known as *Kurepa's Hypothesis* (KH). There are several equivalent ways to express KH. For example, it asserts the existence of a tree of height  $\omega_1$  having at least  $\omega_2$  branches, but whose levels are all countable. Equivalently, KH asserts the existence of a family  $\mathcal{K}$  of subsets of  $\omega_1$  such that  $|\mathcal{K}| \geq \omega_2$ , but for every countable  $\lambda < \omega_1$ ,  $\{X \cap \lambda : X \in \mathcal{K}\}$  is countable. KH is known to be true in the constructible universe, but to be undecidable over ZFC [in fact, KH is a consequence of  $\diamond^+$ , which is true in  $V=L$ ].

It is easy to see that a  $C_{\omega_1}$ -group  $G$  is a dsc group iff it is  $p^{\omega_1}$ -projective; i.e., its  $p^{\omega_1}$ -projective dimension (or  $p^{\omega_1}$ -p.d. for short) is 0. Let  $\mathcal{F}$  be the class of all  $C_{\omega_1}$ -groups whose  $p^{\omega_1}$ -p.d. is at most 1. The connection between these ideas stems from the following result:

**THEOREM 1.3** ([7], Theorem 13). *The following statements are equivalent:*

(a)  $\neg KH$ .

(b) Every  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group  $G$  is in  $\mathcal{F}$ .

(c) For all  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups  $A$  and  $B$ ,  $A \nabla B$  is a dsc group.

These are only a few of a much longer list of algebraic statements that were shown to be equivalent to  $\neg$ KH.

In [10] the following variation on Nunke's Problem was considered: If  $A$  and  $B$  are  $C_{\omega_1}$ -groups, describe exactly when  $A \nabla B \in \mathcal{F}$ . If we let  $\mathcal{Q} = \mathcal{R} - \{\aleph_1\}$  and  $\mathcal{Q}_f = \mathcal{R}_f \cap \mathcal{Q}$ , then for every  $C_{\omega_1}$ -group  $G$  an invariant  $J_G \subseteq \mathcal{Q}_f$  was defined using techniques similar to the definition of  $L_G^\lambda$  in the countable case. It answered the above variation by establishing the following:

**THEOREM 1.4.** *Suppose  $\lambda = \omega_1$  is uncountable and  $G, A$  and  $B$  are  $C_{\omega_1}$ -groups.*  
 (a) ([10], Theorem 1.12)  $G \in \mathcal{F}$  iff  $J_G = 0_{\mathcal{Q}}$ .  
 (b) ([10], Theorem 1.15)  $A \nabla B \in \mathcal{F}$  iff  $J_A \cdot J_B = 0_{\mathcal{Q}}$ .

In other words, the elements of  $J_G$  can be viewed as obstructions to  $G$  being in  $\mathcal{F}$ , and the invariant behaves well enough with respect to the torsion product to give a complete solution to this variation of Nunke's Problem. The purpose of the present paper is to combine the above two threads to answer the following questions:

- (A) *Given a  $C_{\omega_1}$ -group  $G$ ,  $J_G$  tells us when  $G \in \mathcal{F}$ ; is there an addition invariant that tells us when a given  $G \in \mathcal{F}$  is actually a dsc group?*  
 (B) *If  $A$  and  $B$  are  $C_{\omega_1}$ -groups, can we use the solution of (A) to answer Nunke's Problem in the uncountable case, at least in some reasonable versions of set theory, such as the constructible universe ( $V=L$ )?*

We will give a complete answer to (A). If  $G \in \mathcal{F}$ , then we will define an invariant  $F_G \subseteq \mathcal{Q}_f$  in a manner similar to the definition of  $L_G^\lambda$  and  $J_G$  using transfinite induction on subgroups and filtrations. We will then show that the invariant satisfies the following:

**THEOREM 1.5.** *If  $G \in \mathcal{F}$ , then  $G$  is a dsc group iff  $F_G = 0_{\mathcal{Q}}$ .*

So using Theorems 1.4(a) and 1.5 we have the following:

**COROLLARY 1.6.** *A  $C_{\omega_1}$ -group  $G$  is a dsc group iff  $J_G = 0_{\mathcal{Q}}$  and  $F_G = 0_{\mathcal{Q}}$ .*

In other words, for a  $C_{\omega_1}$ -group  $G$ , elements of  $J_G$  are obstructions to  $G$  being in  $\mathcal{F}$ , and when  $G \in \mathcal{F}$ , elements of  $F_G$  are obstructions to  $G$  being a dsc group.

Turning to question (B), if  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group, we define an invariant  $I_G \subseteq \mathcal{R}_f$  in a manner similar to that used in defining  $L_G^\lambda$ ,  $J_G$  and  $F_G$ . The difference between the definitions of these various invariants is in the base case and the type of subgroups we use in constructing our filtrations. If  $2^{\aleph_1} = \aleph_2$ , or if one of the groups  $A$  and  $B$  is actually in  $\mathcal{F}$ , then we get a complete answer to (B).

**THEOREM 1.7.** *Suppose  $A$  and  $B$  are  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups.*

- (a) If  $2^{\aleph_1} = \aleph_2$ , then  $A \nabla B$  is a dsc iff  $I_A \cdot I_B = 0_{\mathcal{R}}$ .
- (b) If  $B \in \mathcal{F}$ , then  $A \nabla B$  is a dsc iff  $J_A \cdot F_B = 0_{\mathcal{Q}}$ .

In particular, this gives a complete solution to all countable and uncountable cases of Nunke's Problem in any set-theoretic universe in which the generalized continuum hypothesis holds, such as the constructible universe. We need this assumption since the invariant  $I_G$  behaves like a combination of both  $J_G$  and  $F_G$  and it is not easy to see exactly how to combine them without some such restriction. In particular,  $2^{\aleph_1} = \aleph_2$  will guarantee that the torsion product of any two  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups is in  $\mathcal{F}$ , which is used frequently in our arguments.

Giving a quick outline of the paper, Section 2 is devoted to reviewing some background information, leading to a particular formulation of Shelah's Singular Compactness Theorem which will be useful for our purposes. Section 3 is devoted to proving Theorem 1.5, and Section 4 contains a proof of Theorem 1.7.

## 2. Preliminaries and Singular Compactness

We begin with a review of a few well-known properties of  $p^{\omega_1}$ -purity.

LEMMA 2.1. *Suppose  $G$  is a group and  $H$  is a dsc group of length  $\omega_1$ .*

- (a) ([8], Theorem 2)  *$G$  is a  $C_{\omega_1}$ -group iff  $G \nabla H$  is a dsc.*
- (b) ([8], Lemma 1) *A short exact sequence,  $0 \rightarrow K \rightarrow G \rightarrow (G/K) \rightarrow 0$ , is  $p^{\omega_1}$ -pure iff the corresponding sequence,  $0 \rightarrow K \nabla H \rightarrow G \nabla H \rightarrow (G/K) \nabla H \rightarrow 0$ , is splitting exact.*
- (c) *If  $G$  is a  $C_{\omega_1}$  group and  $K$  is  $p^{\omega_1}$ -pure in  $G$ , then  $K$  and  $G/K$  are  $C_{\omega_1}$ -groups.*

Part (c) is an immediate consequence of (a) and (b). We quote a couple of other simple results that are pertinent to our discussions.

LEMMA 2.2. *Suppose  $A$ ,  $B$  and  $C$  are  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups.*

- (a) ([4], Theorem 6) *If  $A$  and  $B$  have cardinality at most  $\aleph_1$ , then  $A \nabla B$  is a dsc group.*
- (b) ([5], Theorem 25) *If  $A$ ,  $B$  and  $C$  have cardinality at most  $\aleph_2$ , then  $A \nabla B \nabla C$  is a dsc group.*

We pause to review some set theoretic terminology. If  $\kappa$  is a regular cardinal, then a subset  $C \subseteq \kappa$  is a *CUB* if it is closed and unbounded in the order topology on  $\kappa$ . A subset  $S \subseteq \kappa$  is said to be *stationary* if  $C \cap S \neq \emptyset$  for all CUB subsets  $C \subseteq \kappa$ . If  $G$  is a group of regular cardinality  $\kappa$ , then a *filtration* of  $G$  is a smoothly ascending chain of subgroups,  $\{X_i\}_{i < \kappa}$ , whose union is  $G$ , such that  $|X_i| < \kappa$  for all  $i < \kappa$ .

Suppose  $C$  is a dsc group and we fix a particular decomposition of  $C$  into a collection,  $\mathcal{D}$ , of countable subgroups. A subgroup  $K$  of  $C$  will be a  *$\mathcal{D}$ -summand*

if  $K$  is the direct sum of a subcollection of these terms. If  $K$  is any summand of a dsc group  $C$ , then a decomposition of  $K$  into countable groups can be extended to a similar decomposition of  $C$ , so that  $K$  will be a  $\mathcal{D}$ -summand of  $C$ .

Now suppose  $H$  is a dsc group of length  $\omega_1$  and cardinality  $\aleph_1$  (for example,  $H_{\omega_1}$ , the “generalized Prüfer group” of length  $\omega_1$ ). If  $G$  is a  $C_{\omega_1}$ -group, then using Lemma 2.1(a), fix a decomposition  $\mathcal{D}$  of  $G \nabla H$  into countable summands. Let  $\mathcal{H}$  be the collection of subgroups  $A \subseteq G$  such that  $A \nabla H$  is a  $\mathcal{D}$ -summand of  $G \nabla H$ ; we call this the  $H$ -system for  $G$  determined by  $\mathcal{D}$ . It follows from Lemma 2.1(b) that every element of an  $H$ -system is  $p^{\omega_1}$ -pure in  $G$ . The following are straightforward consequences of this definition:

LEMMA 2.3. *Suppose  $G$  is a  $C_{\omega_1}$ -group with  $H$ -system  $\mathcal{H}$ .*

(a)  $\mathcal{H}$  is closed under unions of chains;

(b) If  $Z \subseteq G$  satisfies  $|Z| \geq \aleph_1$ , then there is a  $K \in \mathcal{H}$  containing  $Z$  such that  $|K| = |Z|$ ;

(c) If  $|G| = \kappa > \aleph_1$  and  $\kappa$  is regular, then  $G$  has a filtration  $\{X_i\}_{i < \kappa} \subseteq \mathcal{H}$ .

In this work all topological terms will be with respect to the  $\omega_1$ -topology; for a group  $G$  this uses  $\{p^\alpha G\}_{\alpha < \omega_1}$  as a neighborhood base of 0.

Suppose  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group,  $\kappa > \aleph_1$  is a cardinal and  $K$  is a  $p^{\omega_1}$ -pure subgroup of  $G$  with  $|K| = \aleph_1$ . We say  $K$  is a  $\kappa$ -Kurepa subgroup if  $\kappa \leq |\overline{K}|$  (where  $\overline{K}$  is its closure in the  $\omega_1$ -topology). In addition, we say  $K$  is  $\aleph_1$ -Kurepa if either it is  $\aleph_2$ -Kurepa, or else it is closed in  $G$  but fails to be a dsc. Let  $\nu_G$  be the least cardinal  $\kappa \geq \aleph_1$  such that  $G$  does not have a  $\kappa$ -Kurepa subgroup. Let  $\nu$  be the supremum of  $\nu_G$  over all  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups  $G$ ; clearly  $\nu_G \leq \nu \leq (2^{\aleph_1})^+$ . By ([6], Theorem 5),  $\kappa < \nu$  iff there is a family  $\mathcal{K}$  of subsets of  $\omega_1$  with  $|\mathcal{K}| \geq \kappa$  satisfying KH. We will primarily be concerned with the situation where  $\nu \leq \aleph_3$ , which will follow if we have  $2^{\aleph_1} = \aleph_2$ . In particular, in the constructible universe,  $V=L$ , both KH and the generalized continuum hypothesis hold, so that  $\nu = \aleph_3$ .

Suppose  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group and  $\mathcal{H}$  is an  $H$ -system for  $G$ . Consider the set  $\mathcal{C}$  of elements of  $\mathcal{H}$  that are closed; we call this the  $C$ -system for  $G$  determined by  $\mathcal{H}$ . The next statement then follows directly from Lemma 2.3 and some results from [6].

LEMMA 2.4. *Suppose  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group and  $\mathcal{C}$  is a  $C$ -system for  $G$ .*

(a) ([6], Theorem 3)  $\mathcal{C}$  is closed under unions of chains;

(b) ([6], Theorem 8) If  $|G| = \kappa \geq \nu_G$  and  $\kappa$  is regular, then  $G$  has a filtration  $\{X_i\}_{i < \kappa} \subseteq \mathcal{C}$ .

(c) If  $K \subseteq G$  is  $p^{\omega_1}$ -pure and  $\kappa := |K| \geq \nu_G$ , then  $|\overline{K}| = \kappa$  and there is an  $X \in \mathcal{C}$  such that  $K \subseteq X$  and  $|X| = \kappa$ .

PROOF. Regarding (c), if  $|\overline{K}| > \kappa$ , we could find  $\tilde{G} \in \mathcal{H}$  of cardinality  $\kappa^+$  containing  $K$  such that  $|\tilde{G} \cap \overline{K}| = \kappa^+$ . This, however, would contradict (b).

To construct our  $X \in \mathcal{C}$ , find  $\{X_m\}_{m < \mathfrak{o}} \subseteq \mathcal{H}$ , all of cardinality  $\kappa$ , such that  $K \subseteq X_0 \subseteq \overline{X_0} \subseteq X_1 \subseteq \overline{X_1} \subseteq X_2 \subseteq \dots$ . By ([6], Theorem 3),  $X := \cup_{m < \mathfrak{o}} X_m$  is a closed element of  $\mathcal{H}$ , i.e.,  $X \in \mathcal{C}$ . And clearly  $K \subseteq X$  and  $|X| = \kappa$ .  $\square$

Our next objective is to prove a variant on the Singular Compactness Theorem of S. Shelah. It is similar to other versions (see, for example, [1]), but it appears to be somewhat simpler than many.

Suppose  $\gamma$  is a singular cardinal. If  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group, then a collection  $\mathcal{S}$  of  $p^{\omega_1}$ -pure subgroups will be called a  $\gamma$ -dsc system if it satisfies the following:

- (1)  $0 \in \mathcal{S}$ ;
- (2) every  $S \in \mathcal{S}$  is a dsc group with  $|S| < \gamma$ ;
- (3) if  $Z \subseteq G$  with  $\aleph_1 \leq |Z| < \gamma$ , then there is an  $S \in \mathcal{S}$  such that  $Z \subseteq S$  and  $|S| = |Z|$ ;
- (4) if  $\mathcal{S}' \subseteq \mathcal{S}$  is a chain with union  $S$ , then  $|S| < \gamma$  implies that  $S \in \mathcal{S}$ .

If  $\aleph_1 \leq \kappa < \gamma$  is a cardinal, let  $\mathcal{S}_\kappa = \{S \in \mathcal{S} : |S| \leq \kappa\}$ . We note one consequence of the above conditions.

- (5) If  $Z \subseteq G$  with  $|Z| \leq \kappa$ , then there is an  $S \in \mathcal{S}_\kappa$  such that  $Z \subseteq S$  and whenever  $S' \in \mathcal{S}_\kappa$  with  $S \subseteq S'$ , then  $S$  is a summand of  $S'$ .

If (5) failed, then we could construct a smoothly ascending chain

$$Z \subseteq S_0 \subseteq S_1 \subseteq \dots \subseteq S_i \subseteq S_{i+1} \subseteq \dots \quad (i < \kappa^+)$$

of elements of  $\mathcal{S}_\kappa$  such that for each  $i < \kappa^+$ ,  $S_i$  fails to be a summand of  $S_{i+1}$ . Now, if we let  $S = \cup_{i < \kappa^+} S_i$ , then  $|S| \leq \kappa^+ < \gamma$ . So by (4),  $S \in \mathcal{S}$ , and in particular,  $S$  is a dsc group. Note that  $\{S_i\}_{i < \kappa^+}$  will be a filtration of  $S$ , so for a CUB subset  $C \subseteq \kappa^+$ ,  $S_i$  is a summand of  $S$  for each  $i \in C$ . However, this contradicts that this  $S_i$  is not even a summand of  $S_{i+1}$ , let alone  $S$ .

Often, when we mention condition (5) we will assume that we extend a decomposition of  $S$  to a decomposition of  $S'$ ; in this way  $S$  will be a  $\mathcal{D}$ -summand of  $S'$ . More generally, if  $S^0 \subseteq S^1 \subseteq \dots \subseteq S^n$  is an ascending sequence of elements of  $\mathcal{S}_\kappa$  satisfying (5), then we can successively build up our decompositions so that each  $S^k$  is a  $\mathcal{D}$ -summand of  $S^{k+1}$ .

This brings us to our version of the Singular Compactness Theorem.

**THEOREM 2.5.** *Suppose  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group of cardinality  $\gamma$ , where  $\gamma$  is a singular cardinal. Then  $G$  is a dsc group iff it has a  $\gamma$ -dsc system.*

**PROOF.** One direction is clear, so assume that  $\mathcal{S}$  is a  $\gamma$ -dsc system for  $G$ . Let  $\lambda$  be the cofinality of  $\gamma$  and let  $\{\kappa_i\}_{i < \lambda}$  be a smoothly increasing sequence of cardinals greater than  $\aleph_1$  whose limit is  $\gamma$ . We start by defining  $S_i = 0$  for all  $i < \lambda$  and letting  $\{C_i^0\}_{i < \lambda}$  be a smoothly ascending chain of subgroups whose union is  $G$ , such that each  $|C_i^0| \leq \kappa_i$ . Next, for each  $i < \lambda$  let  $\mathcal{T}_i \subseteq \mathcal{S}_{\kappa_i}$  be a collection of subgroups that satisfy (5) for  $\kappa = \kappa_i$ .

For some positive integer  $n$  suppose we have defined

- (1)  $\{S_i^{n-1}\}_{i < \lambda}$  where each  $S_i^{n-1} \in \mathcal{T}_i$  has a decomposition  $\mathcal{D}_i^{n-1}$  into countable subgroups;
- (2) a smoothly ascending chain of subgroups  $\{C_i^{n-1}\}_{i < \lambda}$  whose union is  $G$  such that  $S_i^{n-1} \subseteq C_i^{n-1}$  and  $|C_i^{n-1}| \leq \kappa_i$  for all  $i < \lambda$ .

We now define  $\{S_i^n\}_{i < \lambda}$  so that for all  $i < \lambda$ :

- (3)  $S_i^{n-1}$  is a summand of  $S_i^n \in \mathcal{T}_i$  and the decomposition  $\mathcal{D}_i^{n-1}$  for  $S_i^{n-1}$  extends to a decomposition  $\mathcal{D}_i^n$  for  $S_i^n$ ;
- (4)  $S_i^{n-1} \subseteq C_i^{n-1} \subseteq P_i^n \subseteq S_i^n$ , where  $P_i^n$  is a  $\mathcal{D}_{i+1}^n$ -summand of  $S_{i+1}^n$ .

For all  $i < \lambda$  let  $U_i^n \in \mathcal{T}_i$  such that  $|U_i^n| \leq \kappa_i$  and  $S_i^{n-1} \subseteq C_i^{n-1} \subseteq U_i^n$ ; extend the decomposition  $\mathcal{D}_i^{n-1}$  on each  $S_i^{n-1}$  to a decomposition  $\mathcal{E}_i^n$  for each  $U_i^n$ . Now  $C_i^{n-1} \subseteq C_{i+1}^{n-1} \subseteq U_{i+1}^n$ , so there is an  $\mathcal{E}_{i+1}^n$ -summand  $P_i^n$  of  $U_{i+1}^n$  containing  $C_i^{n-1}$  for which  $|P_i^n| \leq \kappa_i$ . We then find  $S_i^n \in \mathcal{T}_i$  such that  $U_i^n + P_i^n \subseteq S_i^n$ . If we extend the decomposition  $\mathcal{E}_i^n$  of each  $U_i^n$  to a decomposition  $\mathcal{D}_i^n$  for each  $S_i^n$ , then (3) and (4) follow.

Now, given  $\{S_i^n\}_{i < \lambda}$  as above, we show how to define a smoothly ascending chain of subgroups  $\{C_i^n\}_{i < \lambda}$  with union  $G$  so that:

- (5)  $|C_i^n| \leq \kappa_i$  and  $S_i^n \subseteq C_i^n$ .

For each  $i < \lambda$ , let our decomposition  $\mathcal{D}_i^n$  of  $S_i^n$  be written  $\bigoplus_{j < \kappa_i} X_{i,j}$ . For each  $i < \lambda$ , define

$$C_i^n = \langle X_{k,j} : k < \lambda, j < \min\{\kappa_i, \kappa_k\} \rangle.$$

Clearly  $|C_i^n| \leq \kappa_i$ , and  $\{C_i^n\}_{i < \lambda}$  is a smoothly ascending chain with union  $G$ . In addition,  $S_i^n = \bigoplus_{j < \kappa_i} X_{i,j} \subseteq C_i^n$ .

So by induction, for all  $n < \omega$  and  $i < \lambda$  we have defined  $S_i^n$ ,  $C_i^n$  and  $P_i^n$ . Observe that if we let  $S_i = \bigcup_{n < \omega} S_i^n$ , then  $S_i$  is a dsc group with a decomposition  $\mathcal{D}_i$  determined by its inductive construction. Next, since  $S_i^n \subseteq C_i^n \subseteq S_i^{n+1}$ , we can conclude that  $S_i = \bigcup_{n < \omega} C_i^n$ , so that  $\{S_i\}_{i < \lambda}$  is a smoothly increasing chain of subgroups with union  $G$ .

Finally, by (4),  $S_i^{n-1} \subseteq P_i^n \subseteq S_i^n$  and  $P_i^n$  is a  $\mathcal{D}_{i+1}^n$  summand of  $S_{i+1}^n$ , and so a  $\mathcal{D}_{i+1}$  summand of  $S_{i+1}$ . Therefore,  $S_i = \bigcup_{n < \omega} P_i^n$  will also be a  $\mathcal{D}_{i+1}$ -summand of  $S_{i+1}$ . This readily implies that  $G$  is a dsc group.  $\square$

### 3. When is $G \in \mathcal{F}$ actually a dsc group?

For a  $C_{\omega_1}$ -group  $G$  we now review how  $J_G \subseteq \mathcal{Q}_f$  was defined in [10]. Suppose  $T \in \mathcal{Q}_f$ ; if  $T = \emptyset$ , let  $\mu(T) = \aleph_1$ , and if  $T \neq \emptyset$ , let  $\mu(T)$  be its greatest element. Let  $T' = T - \{\mu(T)\}$ , and if  $i < \mu(T)$ , let  $T_i = (T' \cup \{i\}) \cap \mathcal{Q}$  (that is,  $T_i = T'$  if  $i \notin \mathcal{Q}$  and  $T_i = T' \cup \{i\}$  when  $i \in \mathcal{Q}$ ). In particular, if  $T \in \mathcal{Q}_f$  is non-empty and  $i < \mu(T)$ , then  $\mu(T_i) < \mu(T)$ .

For every  $C_{\omega_1}$ -group  $G$  and every  $T \in \mathcal{Q}_f$  we decide whether  $T \in J_G$  by induction on  $\mu(T) := \kappa$ . First the base case:

(J-0) If  $\kappa = \aleph_1$  (i.e.,  $T = \emptyset$ ), then  $T \in J_G$  iff  $p^{\omega_1}G \neq 0$ .

Suppose now that  $\kappa \geq \aleph_2$ , and that for all  $S \in \mathcal{Q}_f$  with  $\mu(S) < \kappa$  and for all  $C_{\omega_1}$ -groups  $\widehat{G}$  we have defined when  $S \in J_{\widehat{G}}$ . Then  $T \in J_G$  iff one of following conditions holds:

(J-1)  $\Upsilon_T^J(G) = \{i < \kappa : T_i \in J_G\}$  is stationary in  $\kappa$ ; or

(J-2)  $G$  has a  $p^{\omega_1}$ -pure subgroup  $A$  of cardinality  $\kappa$  with a  $p^{\omega_1}$ -pure filtration  $\{X_i\}_{i < \gamma}$  such that  $\Lambda_T^J(A) = \{i < \kappa : T_i \in J_{A/X_i}\}$  is stationary in  $\kappa$ .

We include the following example to give a simple illustration of the computation of this invariant:

**PROPOSITION 3.1** ([10], Corollary 1.20). *Suppose  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group, and  $\kappa \in \mathcal{Q}$  is a successor cardinal. Then  $T := \{\kappa\} \in J_G$  iff  $G$  has a  $\kappa$ -Kurepa subgroup.*

**PROOF.** If  $\gamma$  is a cardinal with  $\kappa = \gamma^+$ , then  $C := \{i < \kappa : i > \gamma\}$  is certainly a CUB in  $\kappa$ . Since  $T_i = \emptyset$  for all  $i \in C$ , our definition is simplified considerably.

Now, if  $\widetilde{G}$  is any  $C_{\omega_1}$  group, then  $\emptyset \in J_{\widetilde{G}}$  iff  $p^{\omega_1}\widetilde{G} \neq 0$ . In particular, for all  $i \in C$ ,  $T_i = \emptyset \notin \Upsilon_T^J(G)$ . Therefore,  $T \in J_G$  iff (J-2) holds for  $T$ .

Suppose, then, that  $G$  has a  $\kappa$ -Kurepa subgroup,  $K$ . Let  $A \subseteq G$  be a  $p^{\omega_1}$ -pure subgroup containing  $K$  such that  $\kappa = |A| = |A \cap \overline{K}|$ . So, if  $\{X_i\}_{i < \kappa}$  is a  $p^{\omega_1}$ -pure filtration of  $A$ , then  $A/X_i$  is never  $p^{\omega_1}$ -bounded. In other words,  $\Lambda_T^J(A) = \kappa$ , so that (J-2) holds and  $T \in J_G$ .

On the other hand, suppose that  $G$  does not have a  $\kappa$ -Kurepa subgroup and  $A \subseteq G$  is any  $p^{\omega_1}$ -pure subgroup of cardinality  $\kappa$ . Certainly,  $A$  cannot have a  $\kappa$ -Kurepa subgroup, so by Lemma 2.4(b), we can find a closed  $p^{\omega_1}$ -pure filtration  $\{X_i\}_{i < \kappa}$  of  $A$ . Since  $A/X_i$  is  $p^{\omega_1}$ -bounded for every  $i < \kappa$ , we can conclude that  $\Lambda_T^J(A) \cap C = \emptyset$ . Since  $A$  was arbitrary, (J-2) does not hold.  $\square$

A little more intuitively, a statement such as  $\{\kappa, \kappa'\} \in J_G$  (where  $\aleph_2 \leq \kappa < \kappa'$ ) indicates that either  $G$  has a  $\kappa$ -Kurepa subgroup, or it has some  $p^{\omega_1}$ -pure subgroup  $A \subseteq G$  of cardinality  $\kappa'$  such that when  $\{X_i\}_{i < \kappa'}$  is any filtration of  $A$ , at least one of the quotients  $A/X_i$  has a  $\kappa$ -Kurepa subgroup. More generally, the more elements an element of  $J_G$  contains, the more we are looking at filtrations of subgroups that are themselves embedded in filtrations of subgroups, etc. If in this kind of dissection it is inevitable that we encounter a  $\kappa$ -Kurepa subgroup, then by Theorem 1.4(a) we have an obstruction to  $G$  being a member of  $\mathcal{F}$ .

So in computing  $J_G$  we are searching deep inside of  $G$  for  $\kappa$ -Kurepa subgroups with  $\kappa \geq \aleph_2$ . If  $J_G = 0_{\mathcal{Q}}$ , then there are no such embedded subgroups and  $G \in \mathcal{F}$ . And when this happens we will need an additional invariant to detect the presence of similarly embedded  $\aleph_1$ -Kurepa subgroups. The elements of this invariant will



then be obstructions to  $G \in \mathcal{F}$  actually being a dsc group. This will require some preliminary work.

The group  $M$  is an  $\omega_1$ -elementary  $S$ -group if there is a dsc group  $H$  containing  $M$  as a  $p^{\omega_1}$ -pure subgroup such that  $H/M \simeq \mathbb{Z}_{p^\infty}$ . From now on we will let  $M$  be some fixed  $\omega_1$ -elementary  $S$ -group of cardinality  $\omega_1$  (it does not matter exactly which one is chosen). If  $G$  is any group, then  $0 \rightarrow G \nabla M \rightarrow G \nabla H \rightarrow G [\simeq G \nabla \mathbb{Z}_{p^\infty}] \rightarrow 0$  is a  $p^{\omega_1}$ -projective resolution of  $G$ .

The following summarizes some well-known properties of the class  $\mathcal{F}$ .

LEMMA 3.2. *Suppose  $G$ ,  $A$  and  $B$  are  $C_{\omega_1}$ -groups.*

(a)  $G \in \mathcal{F}$  iff  $G \nabla M$  is a dsc group.

(b) If  $G \in \mathcal{F}$ , then  $G$  is  $p^{\omega_1}$ -bounded.

(c) If  $|G| \leq \aleph_1$ , then  $G \in \mathcal{F}$  iff it is  $p^{\omega_1}$ -bounded.

(d) If  $B \in \mathcal{F}$ , then  $A \nabla B \in \mathcal{F}$ .

(e) If  $G \in \mathcal{F}$  and  $K \subseteq G$  is a  $p^{\omega_1}$ -pure subgroup, then  $K \in \mathcal{F}$ .

(f) If  $A$  and  $B$  are  $p^{\omega_1}$ -bounded and of cardinality at most  $\aleph_2$ , then  $A \nabla B \in \mathcal{F}$ .

(g) ([6], Theorem 19 with  $n = 2$ ) If  $\nu \leq \aleph_3$  and  $A$  and  $B$  are  $p^{\omega_1}$ -bounded, then  $A \nabla B \in \mathcal{F}$ .

PROOF. (a): Follows from the above resolution of  $G$ . (b): If  $G \nabla M$  is a dsc group, then  $G$  must be  $p^{\omega_1}$ -bounded by Theorem 1.1. (c): One direction is (b), the other follows from Lemma 2.2(a) with  $A = G$  and  $B = M$ . (d): If  $B \in \mathcal{F}$ , then  $B \nabla M$  is a dsc group so that  $A \nabla B \nabla M$  is also a dsc group. (e): By (d),  $(G \nabla M)/(K \nabla M) \simeq (G/K) \nabla M \in \mathcal{F}$ . So, since  $G \nabla M$  is a dsc, so is  $K \nabla M$ . (f): Follows from Lemma 2.2(b) with  $C = M$ .  $\square$

If  $G \in \mathcal{F}$ , then a  $p^{\omega_1}$ -pure subgroup  $A \subseteq G$  will be  $\mathcal{F}$ -pure if  $G/A \in \mathcal{F}$ . By Lemma 3.2(b) any  $\mathcal{F}$ -pure subgroup is closed. In addition, if  $G$  is a dsc and  $A \subseteq G$  is an  $\mathcal{F}$ -pure subgroup, then since  $G/A \in \mathcal{F}$ , we can conclude that  $A$  is also a dsc group, i.e., an  $\mathcal{F}$ -pure subgroup of a dsc group is also a dsc group.

Now if  $G \in \mathcal{F}$ , then  $G \nabla M$  will be a dsc group. So  $A \subseteq G$  is  $\mathcal{F}$ -pure iff  $A \subseteq G$  is  $p^{\omega_1}$ -pure and  $(G/A) \nabla M$  is a dsc group iff the sequence  $0 \rightarrow A \nabla M \rightarrow G \nabla M \rightarrow (G/A) \nabla M \rightarrow 0$  splits. [Since  $M$  has a summand that is a dsc group of length  $\omega_1$ , if this latter sequence splits, then  $A$  will automatically be a  $p^{\omega_1}$ -pure subgroup of  $G$ .] This characterization clearly implies that  $\mathcal{F}$ -purity is transitive; i.e., if  $B$  is  $\mathcal{F}$ -pure in  $A$  and  $A$  is  $\mathcal{F}$ -pure in  $G$ , then  $B$  is  $\mathcal{F}$ -pure in  $G$ .

If  $G \in \mathcal{F}$ , then fix a decomposition  $\mathcal{D}$  of  $G \nabla M$  into countable groups. Consider the collection  $\mathcal{M}$  of all subgroups  $A \subseteq G$  such that  $A \nabla M \subseteq G \nabla M$  is a  $\mathcal{D}$ -summand. We call this the  $F$ -system determined by  $\mathcal{D}$ . As in Lemma 2.4, we have the following:

LEMMA 3.3. *Suppose  $G \in \mathcal{F}$  and  $\mathcal{M}$  is an  $F$ -system for  $G$ .*

(a)  $\mathcal{M}$  is closed under unions of chains;

(b) If  $A \subseteq G$  has cardinality  $\kappa \geq \aleph_1$ , then there is an  $X \in \mathcal{M}$  such that  $A \subseteq X$  and  $|X| = \kappa$ .

(c) If  $|G| = \kappa \geq \aleph_2$  and  $\kappa$  is regular, then  $G$  has a filtration  $\{X_i\}_{i < \lambda} \subseteq \mathcal{M}$ .

Note that Lemma 3.3(b) implies that a group  $G \in \mathcal{F}$  cannot have a  $\kappa$ -Kurepa subgroup for any  $\kappa > \aleph_1$ . We also note the following simple idea.

LEMMA 3.4. *If  $G \in \mathcal{F}$ , then  $G$  has an  $\aleph_1$ -Kurepa subgroup iff it has an  $\mathcal{F}$ -pure subgroup  $A$  of cardinality  $\aleph_1$  that fails to be a dsc group.*

PROOF. Since an  $\mathcal{F}$ -pure subgroup is closed, sufficiency is obvious. Conversely, if  $A \subseteq G$  is an  $\aleph_1$ -Kurepa subgroup, then by Lemma 3.3(b) we can find a  $\mathcal{F}$ -pure subgroup  $X \subseteq G$  containing  $A$  with  $|X| = \aleph_1$ . Since  $A$  is closed in  $X$ , by Lemma 3.2(c),  $X/A \in \mathcal{F}$ . Therefore,  $X$  also fails to be a dsc group, as desired.  $\square$

If  $G \in \mathcal{F}$ , we define our invariant  $F_G \subseteq \mathcal{Q}_f$  by induction on  $\mu(T) := \kappa$  as follows:

(F-0) If  $\kappa = \aleph_1$  (i.e.,  $T = \emptyset$ ), then  $T \in F_G$  iff  $G$  has an  $\mathcal{F}$ -pure subgroup  $A$  of cardinality  $\omega_1$  that is not a dsc group; or equivalently, an  $\aleph_1$ -Kurepa subgroup.

Suppose now that  $\kappa \geq \aleph_2$ , and that for all  $S \in \mathcal{Q}_f$  with  $\mu(S) < \kappa$  and for all groups  $\tilde{G} \in \mathcal{F}$  we have defined when  $S \in F_{\tilde{G}}$ . Then  $T \in F_G$  iff one of the following two conditions holds:

(F-1)  $\Upsilon_T^F(G) = \{i < \kappa : T_i \in F_G\}$  is stationary in  $\kappa$ ; or

(F-2)  $G$  has an  $\mathcal{F}$ -pure subgroup  $A$  of cardinality  $\kappa$  with an  $\mathcal{F}$ -pure filtration  $\{X_i\}_{i < \gamma}$  such that  $\Lambda_T^F(A) = \{i < \kappa : T_i \in F_{A/X_i}\}$  is stationary in  $\kappa$ .

The following gives some basic properties of this invariant. It is similar to results for  $L_G^\lambda$  and  $J_G$  given in earlier works.

LEMMA 3.5 (cf. [9], Lemma 1.4). *Suppose  $G \in \mathcal{F}$ ,  $T \in \mathcal{Q}_f$  and  $\kappa = \mu(T)$ .*

(a) *If  $T \in F_G$ ,  $S \in \mathcal{Q}_f$  and  $T \subseteq S$ , then  $S \in F_G$ .*

(b) *If  $K$  is an  $\mathcal{F}$ -pure subgroup of  $G$ , then  $F_K \subseteq F_G$ .*

(c) *If  $G = G_1 \oplus G_2$ , then  $F_G = F_{G_1} \cup F_{G_2}$ .*

(d) *If  $T \in F_G$ , then there is an  $\mathcal{F}$ -pure subgroup  $K \subseteq G$  such that  $|K| \leq \kappa$  and  $T \in F_K$ .*

(e) *If  $T \in F_G$  is minimal under inclusion, then  $\mu(T) \leq |G|$ .*

(f) *If  $K$  is an  $\mathcal{F}$ -pure subgroup of  $G$  and  $|K| = |G| = \kappa \geq \aleph_2$ , then there are  $\mathcal{F}$ -pure filtrations of  $K$  and  $G$  such that  $\Lambda_T^F(K) \subseteq \Lambda_T^F(G)$ .*

PROOF. As in [9], all but (f) follow from a straightforward induction. For example, consider (e). If  $\kappa = \aleph_1$  and  $T \in F_G$ , then it follows immediately from (F-0) that  $G$  has cardinality at least  $\aleph_1$ . Suppose now that  $\kappa > \aleph_1$ . If (F-1) holds, consider any  $i \in \mathcal{E} := \Upsilon_T^F(G)$ . Find  $S_i \subseteq T_i$  with  $S_i \in F_G$  minimal under inclusion. Since  $T$  is also minimal under inclusion, we must have  $i \in S_i$ . Therefore by induction,  $|G| \geq \sup\{\mu(S_i)\}_{i \in \mathcal{E}} = \{i\}_{i \in \mathcal{E}} = \kappa$ . If (F-2) holds, we know that  $G$  has an  $\mathcal{F}$ -pure subgroup  $A$  of cardinality  $\kappa$ , giving the result.

Finally, consider (f). Note that  $G/K \in \mathcal{F}$ . Let  $\{Z_i \subseteq G/K\}_{i < \kappa}$  be defined as follows: If  $|G/K| = \kappa$ , then let it be an  $\mathcal{F}$ -pure filtration of  $G/K$ ; otherwise, let each  $Z_i = G/K$ . Now, let  $\{X_i\}_{i < \kappa}$  be an  $\mathcal{F}$ -pure filtration of  $K$  and  $\{Y_i\}_{i < \kappa}$  be an  $\mathcal{F}$ -pure filtration of  $G$ . It is clear that the set of  $i < \kappa$  such that  $X_i = Y_i \cap K$  and  $Z_i = [Y_i + K]/K$  is a CUB in  $\kappa$ . Restricting our filtration to the members of this CUB, the result easily follows from (b).  $\square$

We will have use for the following elementary property:

LEMMA 3.6. *If  $A$  and  $B$  are  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups, then  $A \nabla B$  does not have a  $\kappa$ -Kurepa subgroup for any cardinal  $\kappa$ .*

PROOF. If  $K$  is any  $p^{\omega_1}$ -pure subgroup of  $A \nabla B$  of cardinality  $\aleph_1$ , then we can find  $p^{\omega_1}$ -pure subgroups  $X \subseteq A$  and  $Y \subseteq B$  of cardinality  $\aleph_1$  such that  $K \subseteq X \nabla Y$ . It easily follows that  $X \nabla Y$  is closed in  $X \nabla B$ , which is closed in  $A \nabla B$ . This implies that  $|\overline{K}| = \aleph_1$ ; so  $A \nabla B$  does not have  $\kappa$ -Kurepa subgroups for any  $\kappa \geq \aleph_2$ . So assume  $\kappa = \aleph_1$ . Lemma 2.2(a) implies that  $X \nabla Y$  is a dsc group. If  $K$  were closed in  $A \nabla B$ , then by Lemma 3.2(c),  $(X \nabla Y)/K \in \mathcal{F}$ . This would imply that  $K$  would be a dsc group. Therefore,  $A \nabla B$  has no  $\aleph_1$ -Kurepa subgroups.  $\square$

Before we state and prove the main result of this section, we mention a useful variation on Fodor's Lemma. Suppose  $\gamma$  is a regular cardinal and  $\mathcal{V} \subseteq \gamma$  is a stationary subset. A function  $f : \mathcal{V} \rightarrow \mathcal{Q}_f$  such that  $f(i) \subseteq i$  for all  $i \in \mathcal{V}$  will be called *regressive*.

LEMMA 3.7 ([9], Lemma 1.5). *Suppose  $\gamma \in \mathcal{Q}$  and  $\mathcal{V} \subseteq \gamma$  is a stationary subset. If  $f : \mathcal{V} \rightarrow \mathcal{Q}_f$  is a regressive function, then there is a stationary subset  $\mathcal{V}' \subseteq \mathcal{V}$  such that  $f(i) = f(j)$  for all  $i, j \in \mathcal{V}'$ .*

So we have arrived at the solution to the question in the title of this section and one of the main results of the paper.

PROOF. - **of Theorem 1.5:** Suppose first that  $G$  is a dsc group. We show  $T \notin F_G$  for each  $T \in \mathcal{Q}_f$  by induction on  $\kappa = \mu(T)$ . First, if  $\kappa = \aleph_1$ , then  $T = \emptyset$  and we are in the base case (F-0). However, we observed that any  $\mathcal{F}$ -pure subgroup of a dsc group is actually another dsc group, so that  $T \notin F_G$ . Now, if  $\kappa > \aleph_1$ , then we show (F-1) cannot hold: By induction, for all  $i < \kappa$ ,  $\mu(T_i) < \kappa$ , so that  $T_i \notin F_G$ . Therefore,  $\Upsilon_T^F(G) = \emptyset$  and (F-1) fails. Finally, suppose  $A \subseteq G$  is as in (F-2). There is clearly a decomposition  $G = G_1 \oplus G_2$ , where  $A \subseteq G_1$  and  $|G_1| = \kappa$ . It follows from Lemma 3.5(f) that  $\Lambda_T^I(A) \subseteq \Lambda_T^I(G_1)$ . But  $G_1$  has a filtration by summands, so by induction,  $\Lambda_T^I(G_1)$  is empty. So (F-2) fails, and this completes one direction of the argument.

Conversely, we show by induction on  $\gamma := |G|$  that  $F_G = 0_{\mathcal{Q}}$  implies that  $G$  is a dsc group. Consider first the case where  $\gamma = \aleph_1$ . Since  $\emptyset \notin F_G$ ,  $G$  cannot have an  $\mathcal{F}$ -pure subgroup of cardinality  $\aleph_1$  which is not a dsc group. In particular,  $G$  itself must be a dsc group.

So, suppose the result holds for all groups in  $\mathcal{F}$  of cardinality strictly less than  $\gamma > \aleph_1$ . Suppose  $K$  is an  $\mathcal{F}$ -pure subgroup of  $G$  with  $|K| < \gamma$ . By Lemma 3.2(e),  $K \in \mathcal{F}$ , and by Corollary 3.5(b),  $F_K \subseteq F_G = 0_{\mathcal{Q}}$ ; and by induction, this implies that  $K$  is a dsc group. In particular, every group in an  $F$ -system for  $G$  of strictly smaller cardinality than  $\gamma$  will be a dsc. We now divide the argument into two cases.

Suppose first that  $\gamma$  is singular. If  $\mathcal{M}$  is an  $F$ -system for  $G$ , then clearly  $\mathcal{S} = \{S \in \mathcal{M} : |S| < \gamma\}$  is a  $\gamma$ -dsc system for  $G$ . So by Theorem 2.5,  $G$  is a dsc group, as desired.

Suppose now that  $\gamma$  is regular and  $\{X_i\}_{i < \gamma} \subseteq \mathcal{M}$  is a filtration of  $G$ ; so each  $X_i$  is a dsc group. Let  $\mathcal{U} \subseteq \gamma$  be the collection of all  $i < \gamma$  such that  $X_i$  is a summand of  $X_j$  whenever  $i \leq j < \gamma$ . If  $\mathcal{U}$  contains a CUB subset  $\mathcal{E}$ , then replacing our filtration by restricting to the elements of  $\mathcal{E}$ , we may assume that each  $X_i$  is a summand of  $X_{i+1}$ . This easily implies that  $G$  itself is a dsc group, as required.

We show that if  $\mathcal{U}$  fails to contain a CUB, then we are led to a contradiction. Let  $\mathcal{V} = \gamma - \mathcal{U}$ ; so we are assuming that  $\mathcal{V}$  is stationary. Again replacing our filtration by restricting our attention to the terms in some CUB, we may assume that for every  $i \in \mathcal{V}$ , that  $X_i$  fails to be a summand of  $X_{i+1}$ . In addition, since  $\mathcal{W} := \{i < \gamma : |X_i| = |i|\}$  is a CUB, replacing  $\mathcal{V}$  by the stationary set  $\mathcal{V} \cap \mathcal{W}$ , we may assume that  $|X_i| = |i|$  for all  $i \in \mathcal{V}$ .

If  $i \in \mathcal{V}$ , then there is a decomposition  $X_{i+1} = Y \oplus Z$ , where  $X_i \subseteq Y$  and  $|Y| = |X_i| = |i| < \gamma$ . Now  $Y/X_i \in \mathcal{F}$ , but since  $X_i$  is not a summand of  $Y$ ,  $Y/X_i$  will not be a dsc group. It follows from induction and Lemma 3.5(e) that there is a  $T^i \in F_{Y/X_i} \subseteq F_{X_{i+1}/X_i} \subseteq F_{G/X_i}$  such that  $\mu(T^i) \leq |i| \leq i$ .

The mapping  $i \mapsto T^i - \{i\}$  will be regressive, so by Lemma 3.7, there is a stationary subset  $\mathcal{V}' \subseteq \mathcal{V}$  on which this assignment is constant. If  $\hat{T}$  is this constant value, then we let  $T = \hat{T} \cup \{\gamma\}$ . It follows that  $\mathcal{V}' \subseteq \Lambda_T^F(G)$  so that by (F-2),  $T \in F_G$ . This means that  $F_G \neq 0_{\mathcal{Q}}$  and this contradiction completes the proof.  $\square$

#### 4. Nunke's Problem when $2^{\aleph_1} = \aleph_2$ .

We would like to somehow combine  $J_G$  and  $F_G$  into a single invariant to address Nunke's problem. Our construction needs to detect  $\kappa$ -Kurepa subgroups, both for  $\kappa \geq \aleph_2$ , as does  $J_G$ , but also for  $\kappa = \aleph_1$ , as does  $F_G$  in the case where  $G \in \mathcal{F}$ .

Recall  $\mathcal{R} = \{\aleph_1\} \cup \mathcal{Q}$ . Given a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group  $G$ , we define a collection of *non-empty* finite subsets of  $\mathcal{R}$ ,  $I_G \subseteq \mathcal{R}_f$ , by induction on  $\mu(T) := \kappa$  as follows:

(I-0) If  $\kappa < \nu$  and  $\eta$  is the least element of  $T$ , then  $T \in I_G$  iff  $G$  has an  $\eta$ -Kurepa subgroup.

Suppose now that  $\kappa \geq \nu$ , and that for all  $\emptyset \neq S \in \mathcal{R}_f$  with  $\mu(S) < \kappa$  and for all  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups  $\hat{G}$  we have defined when  $S \in I_{\hat{G}}$ . Then  $T \in I_G$  iff one of the following two conditions holds:

- (I-1)  $\Upsilon_T^I(G) = \{i < \kappa : T_i \in I_G\}$  is stationary in  $\kappa$ ; or  
(I-2)  $G$  has a closed  $p^{\omega_1}$ -pure subgroup  $A$  of cardinality  $\kappa$  with a closed  $p^{\omega_1}$ -pure filtration  $\{X_i\}_{i < \gamma}$  such that  $\Lambda_T^I(A) = \{i < \kappa : T_i \in I_{A/X_i}\}$  is stationary in  $\kappa$ .

The following summary of the basic properties of  $I_G$  is proved in an almost identical manner to the corresponding result for  $F_G$  (that is, Lemma 3.5).

LEMMA 4.1. *Suppose  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group,  $T \in \mathcal{R}_f$  and  $\kappa = \mu(T)$ .*

- (a) *If  $T \in I_G$ ,  $S \in \mathcal{R}_f$  and  $T \subseteq S$ , then  $S \in I_G$ .*  
(b) *If  $K$  is a closed  $p^{\omega_1}$ -pure subgroup of  $G$ , then  $I_K \subseteq I_G$ .*  
(c) *If  $G = G_1 \oplus G_2$ , then  $I_G = I_{G_1} \cup I_{G_2}$ .*  
(d) *If  $T \in I_G$  and  $\kappa \geq \nu$ , then there is a closed  $p^{\omega_1}$ -pure subgroup  $K \subseteq G$  such that  $|K| \leq \kappa$  and  $T \in I_K$ .*  
(e) *If  $T \in I_G$  is minimal under inclusion, then  $\mu(T) \leq |G|$ .*  
(f) *If  $K$  is a closed  $p^{\omega_1}$ -pure subgroup of  $G$  and  $|K| = |G| = \kappa \geq \nu$ , then there are closed  $p^{\omega_1}$ -pure filtrations of  $K$  and  $G$  such that  $\Lambda_T^I(K) \subseteq \Lambda_T^I(G)$ .*

It also follows easily from induction that if  $G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group and  $T \in I_G$  has least element  $\eta$ , then  $\eta < \nu$  (since this holds for the base case).

We now turn to our main objective, addressing the uncountable case of Nunke's problem. The following result will be the base cases of our inductions:

LEMMA 4.2. *If  $A$  and  $B$  are  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups of cardinality at most  $\aleph_2$ , then  $A \nabla B \in \mathcal{F}$  and the following are equivalent:*

- (a)  *$A \nabla B$  is not a dsc;*  
(b)  *$\{\aleph_2\} \in F_{A \nabla B}$ ;*  
(c) *One of  $A, B$  has an  $\aleph_1$ -Kurepa subgroup and the other has an  $\aleph_2$ -Kurepa subgroup;*  
(d)  *$\{\aleph_1, \aleph_2\} \in I_A \cdot I_B$ .*

PROOF. By Lemma 3.2(f) we know that  $A \nabla B \in \mathcal{F}$ . Clearly, (b) implies (a) by Theorem 1.5. Assuming (a), by Lemma 3.5(e) there is  $T \in F_{A \nabla B}$  with  $\mu(T) \leq \aleph_2$ . By Lemma 3.6,  $\emptyset \notin F_{A \nabla B}$ . Therefore,  $T = \{\aleph_2\}$ , so that (b) follows. The equivalence of (c) and (d) is due to the definitions of  $I_A, I_B$  and the fact that every element of these invariants is non-empty. Finally, the equivalence of (a) and (c) is due to the  $n = 2$  case of ([6], Theorem 15).  $\square$

If  $T \in \mathcal{Q}_f$ , let  $\tilde{T} = \{\aleph_1\} \cup T \in \mathcal{R}_f$ . This brings us to half our main result:

THEOREM 4.3. *Suppose  $A$  and  $B$  are  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -groups.*

- (a) *If  $\nu \leq \aleph_3$ , then  $A \nabla B \in \mathcal{F}$  and  $I_A \cdot I_B \subseteq \tilde{F}_{A \nabla B} := \{\tilde{T} : T \in F_{A \nabla B}\}$ .*  
(b) *If  $B \in \mathcal{F}$ , then  $A \nabla B \in \mathcal{F}$  and  $J_A \cdot F_B \subseteq F_{A \nabla B}$ .*

PROOF. Observe that in case (a), if  $S \in I_A \cdot I_B$ , then since the least elements of the sets in  $I_A$  and  $I_B$  are at most  $\aleph_2$ , we can conclude that  $\{\aleph_1, \aleph_2\} \subseteq S$ , and in particular, that  $S = \tilde{T}$  for some unique  $T \in \mathcal{Q}_f$ .

We begin with a crucial observation. We will often consider  $p^{\omega_1}$ -pure subgroup  $X \subseteq A$  and  $Y \subseteq B$ . We have containments  $X \nabla Y \subseteq X \nabla B \subseteq A \nabla B$  with quotients  $Q_1 := X \nabla (B/Y)$  and  $Q_2 := (A/X) \nabla B$ . In the context of part (a) we will further suppose  $X$  and  $Y$  are closed, i.e.,  $A/X$  and  $B/Y$  are  $p^{\omega_1}$ -bounded. Therefore by Lemma 3.2(g),  $Q_1, Q_2 \in \mathcal{F}$ . In the context of part (b) we will further suppose that  $Y$  is  $\mathcal{F}$ -pure in  $B$ , i.e.,  $B/Y \in \mathcal{F}$ . Since both  $B$  and  $B/Y$  are both in  $\mathcal{F}$ , by Lemma 3.2(d) we will have  $Q_1, Q_2 \in \mathcal{F}$ . In either case, it follows that  $X \nabla Y$  is  $\mathcal{F}$ -pure in  $A \nabla B$ . In particular, if  $X = Y = 0$ , we can conclude that  $A \nabla B \simeq (A \nabla B)/(0 \nabla 0) \in \mathcal{F}$ .

By induction on  $\kappa := \mu(T)$ , for each  $T \in \mathcal{Q}_f$  we show that if (a)  $\tilde{T} \in I_A \cdot I_B$ ; or (b)  $T \in J_A \cdot F_B$ , then  $T \in F_{A \nabla B}$ . For our base case, suppose  $\kappa = \mu(T) \leq \aleph_2$ .

(a): Since every element of  $I_A$  and  $I_B$  is non-empty,  $\tilde{T} \in I_A \cdot I_B$  exactly if  $\tilde{T} = \{\aleph_1, \aleph_2\} \in I_A \cdot I_B$ . We can find closed  $p^{\omega_1}$ -pure subgroups  $X \subseteq A$  and  $Y \subseteq B$  of cardinality at most  $\aleph_2$  such that  $\{\aleph_1, \aleph_2\} \in I_X \cdot I_Y$ . And since  $X \nabla Y$  is  $\mathcal{F}$ -pure in  $A \nabla B$ , by Lemma 4.2,  $T = \{\aleph_2\} \in F_{X \nabla Y} \subseteq F_{A \nabla B}$ .

(b): Since  $A$  is  $p^{\omega_1}$ -bounded, every element of  $J_A$  is non-empty. So if  $T \in J_A \cdot F_B$  and  $\kappa \leq \aleph_2$ , then we must have  $\{\aleph_2\} \in J_A$  and  $\emptyset \in F_B$ . By Lemma 3.5(d) and Proposition 3.1, this means that  $A$  has a  $p^{\omega_1}$ -pure subgroup  $X$  of cardinality  $\aleph_2$  that has an  $\aleph_2$ -Kurepa subgroup; and  $B$  has an  $\mathcal{F}$ -pure subgroup  $Y$  of cardinality  $\aleph_1$ , that is not a dsc group. Since  $X \nabla Y$  is an  $\mathcal{F}$ -pure subgroup of  $A \nabla B$ , by Lemma 4.2 we can conclude that  $T = \{\aleph_2\} \in F_{X \nabla Y} \subseteq F_{A \nabla B}$ .

So we may assume  $\kappa > \aleph_2$ . For the remainder of the proof we will concentrate on part (a), indicating parenthetically the very minor changes needed to establish part (b). Suppose  $\tilde{T} \in I_A \cdot I_B$ . It follows that  $\tilde{T}$  will be a disjoint union of the non-empty sets  $U \in I_A$  and  $V \in I_B$ . Suppose first that  $\kappa \in U$ . If  $U$  satisfies (I-1), then  $\Upsilon_U^I(A)$  is stationary. However, if  $i \in \Upsilon_U^I(A)$  and  $\mu(V) < i < \kappa$ , then by induction  $\tilde{T}_i = U_i \cup V \in I_A \cdot I_B$  implies that  $T_i \in F_{A \nabla B}$ . In other words,  $\Upsilon_U^I(A) \subseteq \Upsilon_T^F(A \nabla B)$ , and since the left is stationary, so is the right; i.e.,  $T \in F_{A \nabla B}$ . The case where  $\kappa \in V$  is identical. [(b): We assume  $T$  is the disjoint union of  $U \in J_A$  and  $V \in F_B$ . If  $\kappa \in U$  and  $U$  satisfies (J-1), it follows that  $\Upsilon_U^J(A) \subseteq \Upsilon_T^F(A \nabla B)$  and if  $\kappa \in V$  and  $V$  satisfies (F-1), it follows that  $\Upsilon_V^F(B) \subseteq \Upsilon_T^F(A \nabla B)$ .]

Now, if  $\kappa \in U$  and  $U \in I_A$  satisfies (I-2), then we can find a closed  $p^{\omega_1}$ -pure subgroup  $X \subseteq A$  such that  $|X| = \kappa$  and  $\Lambda_U^I(X)$  is stationary. By Lemma 3.5(d) we can find a closed  $p^{\omega_1}$ -pure subgroup  $Y \subseteq B$  such that  $V \in I_Y$  and  $|Y| \leq \mu(V) < \kappa$ . If  $\{Z_i\}_{i < \kappa}$  is a closed  $p^{\omega_1}$ -pure filtration of  $X$ , then  $\{Z_i \nabla Y\}_{i < \kappa}$  will be an  $\mathcal{F}$ -pure filtration of  $X \nabla Y$ . So if  $i \in \Lambda_U^I(X)$  with  $i > \mu(V)$ , then by induction  $T_i = U_i \cup V \in F_{(X/Z_i) \nabla Y} = F_{(X \nabla Y)/(Z_i \nabla Y)}$ . This means that  $\Lambda_U^I(X) \subseteq \Lambda_T^F(X \nabla Y)$ , so that  $T \in F_{X \nabla Y} \subseteq F_{A \nabla B}$ . By symmetry, this also works when  $\kappa \in V$  and  $V \in I_B$  satisfies (I-2). [(b): Again, supposing  $T$  is the disjoint union  $U \cup V$ , if  $\kappa \in U$  and  $U \in J_A$  satisfies (J-2), then we can find a  $p^{\omega_1}$ -pure subgroup  $X \subseteq A$  and an  $\mathcal{F}$ -pure subgroup  $Y \subseteq B$  such that  $|Y| < |X| = \kappa$ ,  $U \in I_X$ ,  $V \in I_Y$ . This time using a  $p^{\omega_1}$ -pure filtration  $\{Z_i\}_{i < \kappa}$  of  $X$  shows that  $\Lambda_U^J(X) \subseteq \Lambda_T^F(X \nabla Y)$ , so that  $T \in F_{X \nabla Y} \subseteq F_{A \nabla B}$ . Similarly, if  $\kappa \in V$  and  $V \in F_B$  satisfies (F-2), in the above,  $|X| < |Y| = \kappa$ . If  $\{Z_i\}_{i < \kappa}$  is an  $\mathcal{F}$ -pure filtration of  $Y$ , we can conclude

that  $\Lambda_V^F(Y) \subseteq \Lambda_T^F(X \nabla Y)$ , so that  $T \in F_{X \nabla Y} \subseteq F_{A \nabla B}$ . ]  $\square$

We now conclude the proof of the main result of this paper. For simplicity, in the statement of Theorem 1.7(a) we included  $2^{\aleph_1} = \aleph_2$  as a hypothesis; what we really need is that  $\nu \leq \aleph_3$ , as in Theorem 4.3(a).

**PROOF. - Theorem 1.7:** Suppose first that  $A \nabla B$  is a dsc. In part (a), by Theorems 1.5 and 4.3(a) we have  $I_A \cdot I_B \subseteq \tilde{F}_{A \nabla B} = \tilde{0}_{\mathcal{Q}} = 0_{\mathcal{R}}$ ; and in part (b), by Theorem 4.3(b) we have  $J_A \cdot F_B \subseteq F_{A \nabla B} = 0_{\mathcal{Q}}$ .

To prove the converse, in either case, since  $A \nabla B \in \mathcal{F}$ , we need to show that  $F_{A \nabla B} = 0_{\mathcal{Q}}$ . In (a) we are assuming  $I_A \cdot I_B = 0_{\mathcal{Q}}$  and in (b) we are assuming that  $J_A \cdot F_B = 0_{\mathcal{Q}}$ . As above, these two arguments are similar and we will concentrate on (a), indicating parenthetically the very minor changes needed to address (b).

We prove the following using induction on  $\kappa := \mu(T)$ : If  $T \in F_{A \nabla B}$ , then there is an  $S \in \mathcal{Q}_f$  such that  $\tilde{S} \in I_A \cdot I_B$  and  $\mu(S) \leq \kappa$ . [(b):  $S \in J_A \cdot F_B$ .]

Suppose first that  $\kappa = \aleph_1$ , i.e.,  $T = \emptyset$ . By Lemma 3.6,  $T = \emptyset \notin F_{A \nabla B}$ , so the implication is vacuously true.

Suppose next that  $\mu(T) = \aleph_2$ , i.e.,  $T = \{\aleph_2\}$ . Using Lemmas 3.5(d) and 2.4(c) we can find closed  $p^{\omega_1}$ -pure subgroups  $X \subseteq A$  and  $Y \subseteq B$  such that  $|X \nabla Y| \leq \kappa$  and  $T \in F_{X \nabla Y}$ . So by Lemma 4.2,  $\tilde{T} = \{\aleph_1, \aleph_2\} \in I_X \cdot I_Y \subseteq I_A \cdot I_B$ , as required. [(b): Here  $X \subseteq A$  would be  $p^{\omega_1}$ -pure and  $Y \subseteq B$  would be  $\mathcal{F}$ -pure. We would still have  $T = \{\aleph_2\} \in F_{X \nabla Y}$ . Since  $Y \in \mathcal{F}$  cannot have an  $\aleph_2$ -Kurepa subgroup, by Lemma 4.2 we could conclude that  $X$  has an  $\aleph_2$ -Kurepa subgroup and  $Y$  an  $\aleph_1$ -Kurepa subgroup. So again, by Proposition 3.1,  $\tilde{T} = \{\aleph_2\} \cup \{\aleph_1\} \in J_X \cdot F_Y \subseteq J_A \cdot F_B$ , as required.]

Now suppose  $\kappa > \aleph_2$ , so  $T \in F_{A \nabla B}$  must satisfy (F-1) or (F-2). Suppose first that  $T$  satisfies (F-1), i.e.,  $\Upsilon_T^F(A \nabla B)$  is stationary. For any  $i \in \Upsilon_T^F(A \nabla B)$ , by induction  $T_i \in F_{A \nabla B}$  implies that there is an element  $\tilde{S} \in I_A \cdot I_B$  with  $\mu(S) \leq \mu(T_i) < \kappa$ . [(b): Here  $S \in J_A \cdot F_B$ .]

Finally suppose  $T$  satisfies (F-2). We can find closed  $p^{\omega_1}$ -pure subgroups  $A' \subseteq A$ ,  $B' \subseteq B$  such that  $|A' \nabla B'| = \kappa$  and  $T \in F_{A' \nabla B'}$ . Replacing  $A$  by  $A'$  and  $B$  by  $B'$ , from Lemma 3.5(f) we can conclude that  $\mathcal{E} := \Lambda_T^F(A \nabla B)$  is stationary. Define  $\{X_i\}_{i < \kappa}$  as follows: If  $|A| = \kappa$ , let it be a closed  $p^{\omega_1}$ -pure filtration of  $A$ ; otherwise, let each  $X_i = A$ . And if  $|B| = \kappa$ , let  $\{Y_i\}_{i < \kappa}$  be a closed  $p^{\omega_1}$ -pure filtration of  $B$ ; otherwise let each  $Y_i = B$ . It follows that  $\{X_i \nabla Y_i\}_{i < \kappa}$  is an  $\mathcal{F}$ -pure filtration of  $A \nabla B$ . [(b): We assume that  $A'$  and each  $X_i$  is  $p^{\omega_1}$ -pure in  $A$ ; and  $B'$  and each  $Y_i$  is  $\mathcal{F}$ -pure in  $B$ .]

For each  $i < \kappa$  there is a commutative diagram with  $\mathcal{F}$ -pure rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & X_i \nabla Y_i & \rightarrow & X_i \nabla B & \rightarrow & X_i \nabla (B/Y_i) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & A \nabla Y_i & \rightarrow & A \nabla B & \rightarrow & A \nabla (B/Y_i) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & (A/X_i) \nabla Y_i & \rightarrow & (A/X_i) \nabla B & \rightarrow & (A/X_i) \nabla (B/Y_i) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

So the following push-out of both the right column and bottom row is  $\mathcal{F}$ -pure:

$$0 \rightarrow (A \nabla B)/(X_i \nabla Y_i) \rightarrow ([A/X_i] \nabla B) \oplus (A \nabla [B/Y_i]) \rightarrow [A/X_i] \nabla [B/Y_i] \rightarrow 0$$

For any  $i \in \mathcal{E}$ ,  $T_i \in F_{(A \nabla B)/(X_i \nabla Y_i)}$ . And by Lemma 3.5(c) and the above sequence we can conclude that

$$T_i \in F_{[A/X_i] \nabla B} \cup F_{A \nabla [B/Y_i]}.$$

Since there are only two choices, there is a stationary subset  $\mathcal{E}' \subseteq \mathcal{E}$  such that either  $T_i \in F_{[A/X_i] \nabla B}$  for all  $i \in \mathcal{E}'$ , or  $T_i \in F_{A \nabla [B/Y_i]}$  for all  $i \in \mathcal{E}'$ . Assume  $T_i \in F_{[A/X_i] \nabla B}$  for all  $i \in \mathcal{E}'$  (the other case, again, is symmetric). By induction we can conclude that whenever  $i \in \mathcal{E}'$ , there is a disjoint union  $\tilde{S}^i = U^i \cup V^i$ , where  $U^i \in I_{A/X_i}$  and  $V^i \in I_B$ , with  $\mu(S^i) \leq \mu(T_i) \leq i < \kappa$ . [(b): We will have  $S^i = U^i \cup V^i$ , with our two possibilities being  $U^i \in J_{A/X_i}$  and  $V^i \in F_B$ ; or else  $U^i \in J_A$  and  $V^i \in F_{B/Y_i}$  - in either case with  $\mu(S^i) \leq \mu(T_i) \leq i < \kappa$ .]

The map  $i \mapsto U^i - \{i\}$  is regressive, so by Lemma 3.7 there is a stationary subset  $\mathcal{E}'' \subseteq \mathcal{E}'$  such that for all  $i, j \in \mathcal{E}''$ ,  $U^i - \{i\} = U^j - \{j\} := \hat{U}$ . If we define  $U = \hat{U} \cup \{\kappa\}$ , it follows that  $\mathcal{E}'' \subseteq \Lambda_{\hat{U}}^I(A)$ , so that  $U \in I_A$ . If we choose any  $i \in \mathcal{E}''$  with  $\mu(\hat{U}) < i < \kappa$ , then it follows that  $\tilde{S} := U \cup V^i \in I_A \cdot I_B$  and  $\mu(S) = \kappa$ , completing the argument. [(b): Again, replace  $I$  by the appropriate  $J$  or  $F$ .]  $\square$

## 5. Examples and Further Discussion

We start with a straightforward observation.

**PROPOSITION 5.1.** *If  $K$  is a closed  $p^{\omega_1}$ -pure subgroup of the dsc group  $G$ , then  $I_K = 0_{\mathcal{R}}$ .*

**PROOF.** If we can show  $I_G = 0_{\mathcal{R}}$ , then this follows from Lemma 4.1(b). As in the first paragraph of Theorem 1.5, we show  $T \notin F_G$  for each  $T \in \mathcal{R}_f$  by induction on  $\kappa = \mu(T)$ . First, if  $\kappa < \nu$ , then we are in the base case (I-0). As was noted previously, a dsc group cannot have a  $\kappa$ -Kurepa subgroup for any  $\kappa < \nu$ , so  $T \notin I_G$ .



Now, if  $\kappa \geq \nu$ , then the fact that (I-1) cannot hold follows easily by induction. Finally, suppose  $A \subseteq G$  is as in (I-2). There is clearly a decomposition  $G = G_1 \oplus G_2$ , where  $A \subseteq G_1$  and  $|G_1| = \kappa$ . It follows from Lemma 4.1(f) that  $\Lambda_T^I(G_1)$  is stationary. But  $G_1$  has a filtration by summands, so by induction,  $\Lambda_T^I(G_1)$  is empty. This contradiction completes the argument.  $\square$

If  $\nu = \aleph_3$ , then there is a  $p^{\omega_1}$ -pure short exact sequence  $0 \rightarrow G \rightarrow H \rightarrow H/G \rightarrow 0$  where  $H$  is a dsc group of cardinality  $\aleph_2$  and  $H/G$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group with  $p^{\omega_1}$ -p.d. equaling 2. So  $G$  is a closed  $p^{\omega_1}$ -pure subgroup of  $H$ , but not an  $\mathcal{F}$ -pure subgroup. It follows that this  $G$  is not a dsc group, but by Proposition 5.1, we know that  $I_G = 0_{\mathcal{R}}$ . In other words, the invariant  $I_G$  does not necessarily tell us when an individual  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group is a dsc group; this is in contrast to Theorems 1.2(a) and 1.4(a). In addition, for every  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group  $A$  we will have  $I_A \cdot I_G = I_A \cdot 0_{\mathcal{Q}} = 0_{\mathcal{Q}}$ ; so by Theorem 1.7(a),  $A \nabla G$  is always a dsc group even though  $G$  itself is not [this fact was previously observed in ([6], Theorem 20)].

Staying with this  $G \subseteq H$ , note that  $G \in \mathcal{F}$ , and since it fails to be a dsc group,  $F_G \neq 0_{\mathcal{Q}}$ . Since  $I_G = 0_{\mathcal{Q}}$ , we know that  $G$  cannot have an  $\aleph_1$ -Kurepa subgroup. Therefore, we can conclude that  $\emptyset \notin F_G$ . If  $T \in F_G$  is minimal under inclusion, then by Lemma 3.5(e) we must have  $\mu(T) \leq \aleph_2$ ; so  $T = \{\aleph_2\}$  is the only minimal set and  $F_G = \{S \in \mathcal{Q}_f : \aleph_2 \in S\}$ . Notice that this shows that when  $G \subseteq H$  is  $\mathcal{F}$ -pure we can conclude that  $F_G \subseteq F_H$ , but this does not necessarily hold if  $G \subseteq H$  is only assumed to be a closed  $p^{\omega_1}$ -pure subgroup.

We now show that a naive generalization of Theorem 1.7(a) for  $\nu = \aleph_4$  does not hold. Suppose  $A$  is a  $p^{\omega_1}$ -bounded  $C_{\omega_1}$ -group of cardinality  $\aleph_3$  with an  $\aleph_3$ -Kurepa subgroup. Let  $B = A \nabla A$ . It follows from ([6], Theorem 15 with  $n = 3$ ) that  $A \nabla B = A \nabla A \nabla A$  is not a dsc. On the other hand, if  $T \in I_B$  is minimal under inclusion, then  $\mu(T) \leq \aleph_3 < \nu = \aleph_4$ . However, by Lemma 3.6,  $B$  does not have a  $\kappa$ -Kurepa subgroup for any  $\kappa$ ; so no such  $T$  exists. Therefore,  $I_A \cdot I_B = I_A \cdot 0_{\mathcal{R}} = 0_{\mathcal{R}}$ . So in any possible generalization of Theorem 1.7(a) for  $\nu \geq \aleph_4$  it will be necessary to amend the definition of  $I_G$ .

We next observe that Theorem 1.7(b) holds even when  $A$  is  $p^{\omega_1}$ -unbounded. In this case, then  $J_A = 1_{\mathcal{Q}} := \mathcal{Q}_f$ . So by Theorems 1.1 and 1.5, if  $B \in \mathcal{F}$ , then  $A \nabla B$  is a dsc group iff  $B$  is a dsc group iff  $J_A \cdot F_B = 1_{\mathcal{Q}} \cdot F_B = F_B = 0_{\mathcal{Q}}$ . In fact, Theorem 1.5 is simply one case of Theorem 1.7(b), where  $A = \mathbb{Z}_{p^\infty}$  and  $B = G$ , so that  $A \nabla B \simeq G \in \mathcal{F}$  and  $J_A \cdot F_B = F_G$ .

In Theorems 4.3 and 1.7 we would clearly prefer the following hold: (a)  $I_A \cdot I_B = \tilde{F}_{A \nabla B}$  (for  $\nu \leq \aleph_3$ ); and (b)  $J_A \cdot F_B = F_{A \nabla B}$ . In both cases Theorem 4.3 gives one containment, the question is the reverse containment. The problems come in the presence of regular cardinals  $\kappa$  that are *weakly Mahlo*, i.e.,  $\mathcal{S} = \{\gamma \in \mathcal{Q} : \gamma < \kappa\}$  is stationary in  $\kappa$ . As in ([9], Theorem 2.3), these inclusions will in fact be equalities when we restrict to the class of non-weakly Mahlo regular cardinals.

Finally, when  $2^{\aleph_1} = \aleph_2$  we have stated that Theorem 1.7(a) solves Nunke's Problem. It would perhaps be more accurate to say that the result shifts the problem from a question regarding the torsion product, to a question of computing invariants whose values, even in easy cases, depend upon undecidable statements from set theory. Still, it gives a way to analyze the problem by considering each group as a separate entity without any reference to their torsion product.

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