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# Centralizers of finite subgroups in Hall's universal group

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Abstract – The structure of the centralizers of elements and finite abelian subgroups in Hall's universal group is studied by B. Hartley by using the property of existential closed structure of Hall's universal group in the class of locally finite groups. The structure of the centralizers of arbitrary finite subgroups were an open question for a long time. Here by using basic group theory and the construction of P. Hall we give a complete description of the structure of centralizers of arbitrary finite subgroups in Hall's universal group. Namely we prove the following. Let $U$ be the Hall's universal group and $F$ be a finite subgroup of $U$. Then the centralizer $C_U(F)$ is isomorphic to an extention of $Z(F)$ by $U$.

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## 1. Introduction

A locally finite group $U$ satisfying,

**(i)** Every finite group can be embedded into $U$,
**(ii)** Any two isomorphic finite subgroups of $U$ are conjugate in $U$ is called a universal group.

Philip Hall proved the existence and uniqueness of universal groups in the countable case in (1959), see [3]. This group is referred to as Hall's universal group.

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Hall constructed his group as a union of a tower of finite symmetric groups;

$$U_1 \leq U_2 \leq \dots$$

where $U_1$ is a symmetric group of order greater than 2 and if $U_n$ is given, then $U_{n+1}$ is the symmetric group on the set $U_n$ and the group $U_n$ is embedded into $U_{n+1}$ by right regular representation.

Hence

$$U = \lim_{\rightarrow} U_i$$

Hall's universal group $U$ satisfies the following properties some of which are quite unusual; for the proofs, see [3] and [7, Chapter 6].

Proposition 1.1. *Let $U$ be Hall's universal group.*

***(a)*** *Let $C_m$ denote the set of all elements of order $m > 1$ of $U$. Then $C_m$ is a single class of conjugate elements and $U = C_m C_m$. In particular $U$ is simple.*

*The automorphism $\alpha$ of the group $G$ is called locally inner if for every finite set $F$ of elements of $G$, there is an element $g = g_F$ of $G$ such that $f^\alpha = f^g$ for every element $f \in F$.*

***(b)*** *If $G$ is any locally finite universal group, then every automorphism of $G$ is locally inner.*

***(c)*** *The cardinality of the automorphism group of Hall's universal group $U$ is $|Aut(U)| = 2^{\aleph_0}$.*

***(d)*** *Every countably infinite locally finite group can be embedded into $U$.*

Hall's universal group can be written as a direct limit of alternating groups. Then the question of whether Hall's universal group can be written as a direct limit of other families of finite simple groups is answered by F. Leinen in [8]. He proved that Hall's universal group can be constructed as a direct limit of simple linear groups $\{PSL(n_i, \mathbf{F}_q)\}$, $\{PSU(n_i, \mathbf{F}_q)\}$, $\{PSp(2n_i, \mathbf{F}_q)\}$, $\{P\Omega^+(2n_i, \mathbf{F}_q)\}$, $\{P\Omega(2n_i+1, \mathbf{F}_q)\}$, $\{P\Omega^-(2n_i+2, \mathbf{F}_q)\}$.

## 2. Main result

About the centralizers of elements (subgroups) in Hall's universal group, the following results were announced by Hartley in [4, Proposition 1.8].

Proposition 2.1. ***(a)*** *If $F$ is a finite subgroup of $U$ with trivial center, then $C_U(F)$ is isomorphic to $U$.*

***(b)*** *If $A$ is a finite abelian subgroup of $U$, then $C_U(A)/A$ is an infinite simple group.*

For the structure of the centralizers of subgroups in permutation groups one can see [1, Chapter 4] and [9, Chapter 6]. For the centralizers of subgroups in algebraically closed groups see; [5] and [6, Chapter 2].

Is it possible to find the structure of centralizers of finite subgroups in $U$ by using basic group theory?

The answer is positive, but first, we recall some of the facts on centralizers of subgroups in finite symmetric groups. Since $U$ is a direct limit of finite symmetric groups, we may use the centralizers of subgroups in symmetric groups to find the centralizers of finite subgroups in Hall's universal group. In the following some of the results are well known but for the reader's convenience we give the proof here.

Both the right and left regular representations of $G$ are subgroups of the group $Sym(G)$ and they commute with each other elementwise in $Sym(G)$.

Indeed as $(g_1^{-1}x)g_2 = g_1^{-1}(xg_2)$, we have $xl(g_1)r(g_2) = xr(g_2)l(g_1)$ for any $x \in G$. So

$$l(g_1)r(g_2) = r(g_2)l(g_1).$$

It follows that $l(G) \leq C_{Sym(G)}((r(G))$ and $r(G) \leq C_{Sym(G)}((l(G))$.

Lemma 2.2. [2, Page 86] *The centralizer of the right regular representation $r(G)$ in $Sym(G)$ is the left regular representation $l(G)$. Similarly $C_{Sym(G)}(l(G)) = r(G)$.*

Proof. Let $\pi$ be a permutation in $Sym(G)$ belonging to the centralizer of $r(G)$. Let $(1)\pi = g^{-1}$. We show that $\pi = l(g)$. Let $\pi l(g)^{-1} = \pi^*$. Then $\pi^*$ belongs to the centralizer of $r(G)$ and fixes the identity i.e. $(1)\pi^* = 1$. Here $(1)\pi^* r(x) = x$. By $\pi^* r(x) = r(x)\pi^*$ we have $(1)r(x)\pi^* = (x)\pi^* = x$, and so $(x)\pi^* = x$ for any $x \in G$, whence $\pi^*$ is the identity permutation and so $\pi l(g)^{-1} = \pi^* = id$, we obtain $\pi = l(g)$. Hence the centralizer of $r(G)$ is $l(G)$. Moreover there exists a permutation $t$ of $Sym(G)$ of order 2 namely the permutation taking every element to its inverse in $G$ satisfying $t^{-1}r(G)t = l(G)$. As for any $x \in G$ we have,

$$x.t^{-1}r(g)t = x^{-1}r(g)t = (x^{-1}g)t = (x^{-1}g)^{-1} = g^{-1}x = xl(g) \text{ for all } x \in G.$$

So $t^{-1}r(g)t = l(g)$. Observe that $t$ has order two in $Sym(G)$ and so
$C_{Sym(G)}(r(G))^t = C_{Sym(G)}(l(G)) = r(G)$.

□

Lemma 2.3. *$l(G) \cap r(G) = l(Z(G)) = r(Z(G)) \cong Z(G)$.*

Proof. Let $x \in l(G) \cap r(G)$. Then $x = l(h) = r(s)$ for some $h, s \in G$. Then $1.l(h) = 1.r(s)$ implies that $h^{-1} = s$. Then for any $g \in G$, we have $g.l(h) = g.r(h^{-1})$ i.e. $h^{-1}g = gh^{-1}$ for all $g \in G$. It follows that $h \in Z(G)$ and we have

$$l(G) \cap r(G) \subseteq r(Z(G)) = l(Z(G))$$

Conversely for any element $z \in Z(G)$ we have $l(z) = r(z^{-1}) \in l(G) \cap r(G)$. Hence

$$l(Z(G)) = r(Z(G)) \leq l(G) \cap r(G).$$

So

$$l(Z(G)) = r(Z(G)) = l(G) \cap r(G) \cong Z(G)$$

□

We now prove the main theorem.

THEOREM 2.4. *Let $U$ be the Hall's universal group and $F$ be a finite subgroup of $U$. Then the centralizer $C_U(F)$ is isomorphic to an extention of $Z(F)$ by $U$.*

PROOF. Let $F$ be a finite subgroup of $U$. As $U = \bigcup_{i=1}^{\infty} U_i$ is a direct limit of finite symmetric groups $U_i$ where $U_i$ is embedded by right regular representation into $U_{i+1} = Sym(U_i)$, we may assume that $F \leq U_i$ for some $i \in \mathbb{N}$. As $F$ is also contained in $U_{i+1}$ the orbits of $F$ under right regular representation in $U_{i+1}$ are the left cosets of $F$ in $U_i$. The action of $F$ on each of its orbits are equivalent to the action of $F$ on itself by right multiplication. So we have $m_{i+1} = \frac{|U_i|}{|F|}$ permutationally equivalent orbits. By Lemma 2.2 the centralizer of the right regular representation $r(F)$ in $Sym(F)$ is the left regular representation $l(F)$. The structure of centralizers of intransitive subgroups in symmetric groups is well known; see [1, Page 109]. Hence

$$C_{Sym(U_{i+1})}(r(F)) \cong l(F)^{x_1^{-1}} \times \ldots \times l(F))^{x_{m_{i+1}}^{-1}} \rtimes Sym(m_{i+1})$$

where $x_i$'s are left coset representatives of $F$ in $U_i$ and the elements of $Sym(m_{i+1})$ permutes the permutationally isomorphic pairs $l(F)^{x_i^{-1}}$ for $i = 1, 2, \ldots, m_{i+1}$. In fact $C_{Sym(U_{i+1})}(r(F)) \cong l(F) \wr Sym(m_{i+1})$ where the wreath product is the permutational wreath product.

As the groups $U_i$ are embedded in $U_{i+1}$ by right regular representation, the centralizers $C_{U_i}(F)$ are also embedded into $C_{U_{i+1}}(F)$ by right regular representation and hence the group $C_U(F) = \bigcup C_{U_i}(F)$ will be the direct limits of the centralizers $C_{U_i}(F)$. This is one of the differences between the diagonal embedding and the regular embedding, since during the diagonal embedding usually the group theoretical properties are preserved but by regular embedding some of these properties are not preserved like cycle structure of a permutation.

But the subgroup $l(F)^{x_1^{-1}}$ and also the copies of it, in the centralizer $C_{U_{i+1}}(F)$ will not contribute new elements when we take their right regular representation in $U_{i+1}$ in the next step, as the centralizer of $F$ in the next step is known and there are no elements coming from the right regular representation. By Lemma 2.3 $l(G) \cap r(G) = l(Z(G)) = r(Z(G))$, the only elements goes from the base group of $C_{U_i}(F)$ to $C_{U_{i+1}}(F)$ are the $r(Z(F))$. The group $Z(F)$ which is contained in each $U_i$ will consists of diagonal elements in the wreath product, but when we take the direct limit of these, as the subgroup $r(F)$ the center $Z(r(F))$ will be diagonally embedded.

In the semidirect product part, the embedding of symmetric groups $Sym(m_i)$ into $Sym(m_{i+1})$ are embedded by right regular representations and the sequence of integers $m_i$ is an infinite increasing sequence. Moreover when we take right regular representation of $F$ on $U_i$ and then on $U_{i+1}$ the only thing we have, more cosets than before, modulo identification. But the cosets of $F$ in $U_i$ can be preserved, only new cosets comes. The permutations in $U_i$ which permutes the cosets will permute them as elements of $U_{i+1}$ the only thing that, they are embedded into $U_{i+1}$ by regular embeddings. Since countable locally finite universal group is unique up to isomorphism [7, Theorem 6.4] from the direct limit of the symmetric groups by right regular representation, we obtain an isomorphic copy of $U$. So $C_U(F)$ is isomorphic to an extention of $Z(F)$ by $U$.

In particular if $Z(F) = \{1\}$, then $C_U(F) \cong U$.

$\square$

Then the results of Hartley in Proposition 2.1 will be a corollary of the above theorem. Moreover we have the following corollary.

Corollary 2.5. *The centralizer $C_U(F)$ of every finite subgroup $F$ of $U$ has $U$ as an epimorphic image.*

## References


[1] J. D. Dixon – B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, Springer -Verlag, 1991.

[2] M. Hall, Jr, The Theory of Groups, The Macmillan Company, New York, 1963.

[3] P. Hall, *Some construction for locally finite groups,* J. London Math. Soc **34**, (1959), pp. 305–319.

[4] B. Hartley, *Simple locally finite groups,* NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 471, Kluwer Acad. Publ., Dordrecht, 1995.

[5] K. Hickin – A. Macintyre, *Algebraically closed groups: embeddings and centralizers.* Word problems, Stud. Logic Foundations Math., **95**, North-Holland, Amsterdam-New York, (1980), pp. 141-155.

[6] G. Higman – E. Scott, *Existentially closed groups*, LMS Monographs no 3, Oxford University Press, 1988.

[7] O. H. Kegel – B. A. F. Wehrfritz, *Locally Finite Groups*, North-Holland Publishing Company - Amsterdam, 1973.

[8] F. Leinen, *Lokale systeme in universellen gruppen,* Arch. Math. **41**, (1983), pp. 401–403.

[9] A. Seress, *Permutation group algorithms*, cambridge tracts in Math. **152**, Cambridge university press, 2003.


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