Galois lines for a canonical curve of genus 4, II: Skew cyclic lines

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Abstract – Let \( C \subset \mathbb{P}^3 \) be a canonical curve of genus 4 over an algebraically closed field \( k \) of characteristic zero. For a line \( l \), we consider the projection \( \pi_l : C \to \mathbb{P}^1 \) with center \( l \) and the extension of the function fields \( \pi_l^* : k(\mathbb{P}^1) \to k(C) \). A line \( l \) is referred to as a cyclic line if the extension \( k(C)/\pi_l^*(k(\mathbb{P}^1)) \) is cyclic. A line \( l \subset \mathbb{P}^3 \) is said to be skew if \( C \cap l = \emptyset \). We prove that the number of skew cyclic lines is equal to 0, 1, 3 or 9. We determine curves that have nine skew cyclic lines.

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1. Introduction and Theorem

Yoshihara [10] investigated various properties of skew Galois lines (for the definition, see below) for nondegenerate nonsingular curves \( C \) in \( \mathbb{P}^3 \). He proved that the number of skew Galois lines for an irrational \( C \) is finite, and that the number of skew Galois lines for \( C \) is at most one if \( \deg C \) is a prime and \( \deg C \geq 5 \). He also studied the defining equations of curves \( C \) of low degrees that have skew Galois lines. In addition, Yoshihara et al. [2, 7, 11], studied the number and arrangement of skew Galois lines for elliptic space curves. Fukasawa and Higashine [4] and subsequent work by Fukasawa [3] determined the arrangement of all the Galois lines for the Giulietti–Korchmáros curve and for the Artin–Schreier–Mumford curve, respectively. More recently, in [8], we

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studied the number of non-skew cyclic lines for canonical curves of genus 4. As a continuation of this work [8], in this study, we investigate the number of skew cyclic lines for canonical curves of genus 4. We would like to note Kuribayashi et al. [9], however we will not use it in the present paper. By giving generators with respect to linear representations in the vector space of holomorphic differentials, they presented a complete classification of automorphism groups for compact Riemann surfaces of genera 3 and 4.

Let \( C \subset \mathbb{P}^3 \) be a canonical curve of genus 4 over an algebraically closed field \( k \) of characteristic 0, which is a \((2, 3)\)-complete intersection in \( \mathbb{P}^3 \). A line \( l \subset \mathbb{P}^3 \) is said to be skew if \( C \cap l = \emptyset \). For a line \( l \), we consider the projection \( \pi_l : C \to \mathbb{P}^1 \) with center \( l \) and the extension of the function fields \( \pi_l^* : k(\mathbb{P}^1) \hookrightarrow k(C) \). Because \( \deg C = 6 \), we have \( \deg \pi_l \leq 6 \), and if \( l \) is skew, then we have \( \deg \pi_l = 6 \). We refer to a line \( l \) as a Galois line if the extension is Galois. We refer to the Galois line \( l \) as a \( C_6 \)-line (resp. \( S_3 \)-line) if the Galois group is isomorphic to the cyclic group \( C_6 \) of order 6 (resp. the symmetric group \( S_3 \) on 3 letters). We note that \( l \) is a skew cyclic line if and only if \( l \) is a \( C_6 \)-line, in the setting of this paper. In [8], we explicitly gave the equations of \( C \) in the particular case in which \( C \) has two cyclic trigonal morphisms; we prove that the number of cyclic lines with \( \deg \pi_l = 4 \) is at most 1; and the number of cyclic lines with \( \deg \pi_l = 5 \) is at most 1. Our main theorem of the present paper is as follows:

**Theorem.** Let \( C \subset \mathbb{P}^3 \) be a canonical curve of genus 4 over an algebraically closed field of characteristic 0. Then, the number of \( C_6 \)-lines equals 0, 1, 3, or 9. Moreover, if there exist nine \( C_6 \)-lines for \( C \), then \( C \) is projectively equivalent to the curve defined by one of the following:

\[
\begin{align*}
(1) & & XY - Z^2 &= 0 \\
& & X^3 + Y^3 + W^3 &= 0
\end{align*}
\]

or

\[
\begin{align*}
(2) & & X^2 + Y^2 + Z^2 &= 0 \\
& & XYZ + W^3 &= 0
\end{align*}
\]

where \((X : Y : Z : W)\) are homogeneous coordinates on \( \mathbb{P}^3 \).

In Section 2, we present selected preliminary results. The proof of the theorem is provided in Section 3. In Sections 4 and 5, we determine all the \( C_6 \)-lines for curves defined by Equations (1) and (2). Section 6 presents examples of curves that have only one or three \( C_6 \)-lines.

In the present paper, we assume that the base field \( k \) is algebraically closed and \( \text{char}(k) = 0 \). For a line \( l \), “skew” means “skew with respect to \( C \)”, and also \( C_6 \)-line
means “with respect to $C$”, and the reference to $C$ will always be tacitly assumed. For the Galois line $l$, we denote \( \{ \sigma \in \text{Aut}(C) \mid \pi_l \circ \sigma = \pi_l \} \) by $G_l$, which is isomorphic to the Galois group. We denote by $C_m$ the cyclic group of order $m$; by $D_m$ the dihedral group of order $2m$; by $A_m$ the alternating group on $m$ letters; by $S_m$ and the symmetric group on $m$ letters.

2. Preliminaries

Let $C \subset \mathbb{P}^3$ be a canonical curve of genus $4$. Let $(X : Y : Z : W)$ be homogeneous coordinates on $\mathbb{P}^3$. The following are well-known facts:

PROPOSITION 2.1 (p. 118 of [1], p. 298 of [6]). The curve $C$ is a $(2, 3)$-complete intersection; that is, the homogeneous ideal $I(C) \subset k[X, Y, Z, W]$ of $C$ is generated by a quadratic form $Q$ and cubic form $F$. The degree of $C$ is $6$. The surface \( \{ Q = 0 \} \) is a unique quadric surface that contains $C$. The gonality $\text{gon}(C)$ of $C$ is equal to $3$. If $\text{rank } Q = 3$, then $C$ has a unique trigonal morphism $C \rightarrow \mathbb{P}^1$, which is given by the projection from the vertex of the surface \( \{ Q = 0 \} \). If $\text{rank } Q = 4$, then $C$ has exactly two trigonal morphisms $C \rightarrow \mathbb{P}^1$.

Let $l \subset \mathbb{P}^3$ be a line and $\pi_l : C \rightarrow \mathbb{P}^1$ the projection with center $l$. Because $\deg C = 6$ and $C$ is not hyperelliptic, we have $3 \leq \deg \pi_l \leq 6$. A line $l$ is skew if and only if $\deg \pi_l = 6$. If $\deg \pi_l \geq 4$, then $\pi_l$ uniquely determines the center $l$.

PROPOSITION 2.2 ([8]). Assume $\deg \pi_l \geq 4$. Then, $\pi_l = \pi_{l'}$ (up to an isomorphism of the codomains $\mathbb{P}^1$ of $\pi_l$ and $\pi_{l'}$), if and only if $l = l'$.

We have a canonical representation $\text{Aut}(C) \hookrightarrow GL(\Gamma(C, \Omega^1)) \cong GL(4, k)$, where $\Omega^1$ is the sheaf of regular $1$-forms on $C$. As $C \subset \mathbb{P}^3$ is a canonical curve, we also have $\text{Aut}(C) \hookrightarrow \text{Aut}(\mathbb{P}^3) \cong PGL(4, k)$. That is, for every $\sigma \in \text{Aut}(C)$, there exists a unique projective transformation $T : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ such that $T(C) = C$ and $T|_C = \sigma$. We express the elements in $\text{Aut}(C)$ as the projective transformations of $\mathbb{P}^3$.

PROPOSITION 2.3. There exists a quadratic form $Q \in k[X, Y, Z, W]$ and cubic form $F \in k[X, Y, Z, W]$ with $I(C) = (Q, F)$ such that $\sigma(Q = 0) = (\sigma(Q = 0)$ and $\sigma(F = 0) = (F = 0)$ for any $\sigma \in \text{Aut}(C) \subset \text{Aut}(\mathbb{P}^3)$.

PROOF. There exists a unique quadric $Q = 0$ that contains $C$. Clearly, $\sigma(Q = 0) = (Q = 0)$. Let

\[ I_3 := \{ F \in k[X, Y, Z, W] \mid F \text{ is a cubic form, } C \subset (F = 0) \} \cup \{0\}, \]
\( J := \{(aX + bY + cZ + dW)Q \mid a, b, c, d \in k\}, \) and let \( G \subset GL(4, k) \) be a finite group isomorphic to \( \text{Aut}(C) \) via the natural quotient map \( GL(4, k) \to PGL(4, k) \). Then, \( \dim_k I_3 = 5, \dim_k J = 4, J \subseteq I_3, \) and \( G \) acts linearly on \( I_3 \) and \( J \). Because \( \text{char}(k) = 0 \), according to Maschke’s theorem, the representation \( G \to GL(I_3) \) is completely reducible. Thus, there exists \( F \in I_3 \setminus J \) such that \( (A^*F)/F \in k \setminus \{0\} \) for any \( A \in G \).

**Proposition 2.4 ([10]).** Assume that there exists a \( C_6 \)-line \( l \). Then, by taking a suitable projective transformation of \( \mathbb{P}^3 \), we may assume that \( l \) is defined by \( X = Y = 0 \), and a generator \( \sigma \) of \( G_l \subset Aut(\mathbb{P}^3) \) is expressed by a diagonal matrix with diagonal components \( 1, 1, \alpha, \beta \) (\( \alpha, \beta \in k \setminus \{0\} \)), and \( (\text{ord}(\alpha), \text{ord}(\beta)) = (3, 6), (2, 3), \) or \( (2, 6) \). That is, we may assume

\[
\sigma = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \zeta^2 & 0 \\
0 & 0 & \zeta & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & \zeta^2
\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & \zeta
\end{pmatrix},
\]

where \( \zeta \) is a primitive 6th root of the unity.

**Proof.** Most of the claims are proved in the proof of Theorem 4.5 in [10] (see Claim 7 on pp. 466–467 of [10]). We only have to verify the following: the diagonal matrix with diagonal components \( 1, 1, \zeta^4, \zeta \) is unsuitable for a generator \( \sigma \) of \( G_l \). Indeed, if \( \sigma \) is such an automorphism, then using Proposition 2.3, \( Q \) will be reducible.

In Proposition 2.4, we note that the position of the line \( l \) and the form of the generator \( \sigma \) are specified simultaneously. From the following argument we see that it is possible: first we fix the position of the line \( l \) to be \( X = Y = 0 \); next from \( \pi_l = \pi_l \circ \sigma \), we find the conditions that the representation matrix of \( \sigma \) must satisfy; finally by using a projective transformation that does not change the position of \( l \), we diagonalize the representation matrix of \( \sigma \).

**Definition 2.5.** We say that a \( C_6 \)-line \( l \) is of type \( (3, 6) \) (resp. of type \( (2, 3) \), of type \( (2, 6) \)) if a generator of \( G_l \subset Aut(\mathbb{P}^3) \) can be represented as a matrix with eigenvalues \( 1, 1, \alpha, \beta \) with \( (\text{ord}(\alpha), \text{ord}(\beta)) = (3, 6) \) (resp. \( (2, 3) \), \( (2, 6) \)).

**Corollary 2.6.** We assume that there exists a \( C_6 \)-line \( l \). Let \( Q \in k[X, Y, Z, W] \) be a quadratic form such that the quadric surface \( Q = 0 \) contains \( C \). Then, \( \text{rank} Q = 3 \). Hence, there exists only one trigonal morphism \( g^1_3 : C \to \mathbb{P}^1 \), which is given by the projection from the vertex of \( Q = 0 \).
Proof. The quadric $Q = 0$ containing $C$ satisfies $\sigma(Q = 0) = (Q = 0)$ for any $\sigma \in G_I$. From Proposition 2.4, we see that $\text{rank } Q = 3$.

For $\sigma$ stated in Proposition 2.4, we note that $\text{Fix}(\sigma) := \{ P \in \mathbb{P}^3 \mid \sigma(P) = P\}$ consists of a line $Z = W = 0$ and two points $(0 : 0 : 1 : 0), (0 : 0 : 0 : 1)$, and $l : X = Y = 0$ passes through these two points. Hence, we can immediately see the following:

**Proposition 2.7.** Let $l_1$ and $l_2$ be distinct $C_6$-lines for $C$. Then, $G_{l_1} \neq G_{l_2}$ as subgroups of $\text{Aut}(C)$.

On $S_3$-lines, we have the following proposition. Proposition 2.8 is not used in the proof of our main theorem, but is required for the calculations in Sections 4 and 5. In Sections 4 and 5, we will determine not only $C_6$-lines but also $S_3$-lines for curves concretely defined by Equations (1) and (2).

**Proposition 2.8 (The proof of Theorem 4.5 in [10]).** Let $l$ be an $S_3$-line for $C$. Then, by taking a suitable projective transformation, we may assume that $l : X = Y = 0$, and $G_l$ is generated by the following two elements:

$$
\sigma := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \omega & 0 \\
0 & 0 & 0 & \omega^2
\end{pmatrix}
\quad \text{and} \quad
\tau := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
$$

where $\omega$ is a primitive cubic root of the unity.

Since the proof of Proposition 2.8 is not stated in [10] as it is obvious, we present the proof here.

Proof. Let $\sigma$ and $\tau$ be automorphisms of $C$ such that $G_l = \langle \sigma, \tau \rangle$, where $\sigma^3 = \tau^2 = \text{id}_C$ and $\tau \sigma \tau = \sigma^2$. By taking a suitable projective transformation, we may assume that $l$ is defined by $X = Y = 0$. Because $\pi_l \circ \sigma = \pi_l$ and $\pi_l \circ \tau = \pi_l$, we have that $\sigma$ and $\tau$ are represented as

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}.
$$

Because $\sigma^3 = \text{id}_C$, $\sigma$ is diagonalizable. We may assume that

$$
\sigma = \begin{pmatrix}
I & O \\
O & A
\end{pmatrix}
\quad \text{and} \quad
\tau = \begin{pmatrix}
I & O \\
L & M
\end{pmatrix},
$$

where $L$ and $M$ are some $2 \times 2$ matrices,
\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \text{or} \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}. \]

By using \( \tau^2 = \text{id}_C \) and \( \tau \sigma \tau = \sigma^2 \), we infer that \( L + ML = O \), \( M^2 = I \), \( L + MAL = O \) and \( MAM = A^2 \). We have

\[ A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad L = O, \quad \text{and} \quad M = \begin{pmatrix} 0 & c \\ 1/c & 0 \end{pmatrix} \text{for some} \ c \in k \setminus \{0\}. \]

By taking the projective transformation that is represented by the diagonal matrix with diagonal elements \( 1, 1, c \) and \( 1 \), we have the representations of \( \sigma \) and \( \tau \) as stated in the proposition.

For \( \sigma \) and \( \tau \) stated in Proposition 2.8, we note that \( \text{Fix}(\sigma) := \{ P \in \mathbb{P}^3 \mid \sigma(P) = P \} \) consists of a line \( Z = W = 0 \) and two points \( (0 : 0 : 1 : 0), (0 : 0 : 0 : 1) \), and \( l \) passes through these two points. \( \text{Fix}(\tau) := \{ P \in \mathbb{P}^3 \mid \tau(P) = P \} \) consists of a hyperplane \( Z - W = 0 \) and a point \( (0 : 0 : -1 : 1) \), and \( l \) passes through the point.

Assume that \( \text{rank} \; Q = 3 \), where the quadric \( Q = 0 \) contains \( C \). Because the trigonal morphism \( g_1^3 : C \to \mathbb{P}^1 \) is unique, for any \( \sigma \in \text{Aut}(C) \), there exists \( A_{\sigma} \in \text{Aut}(\mathbb{P}^1) \) such that \( g_3^1 \circ \sigma = A_{\sigma} \circ g_1^3 \). Let \( G \) be a subgroup of \( \text{Aut}(C) \). Let \( \varphi : G \to \text{Aut}(\mathbb{P}^1) \) be the map \( \sigma \mapsto A_{\sigma} \), which is a homomorphism between the groups. Let \( \text{Ker} \; \varphi \) and \( \text{Im} \; \varphi \) be the kernel and image of \( \varphi \), respectively. We denote the inclusion \( \text{Ker} \; \varphi \hookrightarrow G \) as \( \psi \). We have a short exact sequence

\[ 1 \to \text{Ker} \; \varphi \xrightarrow{\psi} G \xrightarrow{\varphi} \text{Im} \; \varphi \to 1. \]

The short exact sequence (3) and Proposition 2.9 play central roles in the proof of our main theorem.

**Proposition 2.9.** We have the following:

(I) The group \( \text{Im} \; \varphi \) is isomorphic to one of the following groups: \( C_m \; (m \in \mathbb{Z}_{>0}), \; D_m \; (m \in \mathbb{Z}_{>0}), \; A_4, \; S_4 \) or \( A_5 \).

(II) The three conditions “\( \text{Ker} \; \varphi \not= 1 \),” “\( \text{Ker} \; \varphi \cong C_3 \),” and “\( g_3^1 \) is cyclic” are equivalent.

**Proof.** Because \( \text{Im} \; \varphi \subset \text{Aut}(\mathbb{P}^1) \) is finite, (I) is well-known. As \( \text{Ker} \; \varphi = \{ \sigma \in G \mid g_3^1 \circ \sigma = g_3^1 \} \), we see that (II) holds.

On automorphism groups of a plane quadric curve, we have the following proposition. Proposition 2.10 is required in the proof of our main theorem.
Proposition 2.10. Let $V \subset \mathbb{P}^2$ be the curve defined by $XY = Z^2$, which is isomorphic to $\mathbb{P}^1$.

(I) Let $S_4 \subset \text{Aut}(V) \subset \text{Aut}(\mathbb{P}^2)$ be the symmetric group on four letters. Then, by taking a suitable projective transformation, we can assume that $S_4 = \langle \rho, \tau \rangle$,

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix},$$

where $i$ is a primitive 4th root of the unity.

(II) Let $D_m \subset \text{Aut}(V) \subset \text{Aut}(\mathbb{P}^2)$ ($m \geq 2$) be the dihedral group of order $2m$. Then, by taking a suitable projective transformation, we can assume that $D_m = \langle \rho, \tau \rangle$,

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_m^2 & 0 \\ 0 & 0 & \zeta_m \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\zeta_m$ is a primitive $m$th root of the unity.

Proof. We may assume that the group $S_4 \subset \text{Aut}(\mathbb{P}^1)$ (resp. $D_m \subset \text{Aut}(\mathbb{P}^1)$) is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ (resp. } \begin{pmatrix} 1 & 0 \\ 0 & \zeta_m \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

The form of the matrices in the proposition comes from the images of these generators via the embedding $\mathbb{P}^1 \ni (x_0 : x_1) \mapsto (x_0^2 : x_1^2 : x_0x_1) \in \mathbb{P}^2$. ■

3. Proof of Theorem

In this section, we prove the main theorem. Note that, if there exists a $C_6$-line, then $C$ has a unique trigonal morphism $g_3 : C \to \mathbb{P}^1$ by Corollary 2.6. Let us consider the short exact sequence (3) for

$$G := \langle \{\sigma \in \text{Aut}(C) \mid \sigma \in G_I \text{ for some } C_6\text{-line } l\} \rangle.$$

The map $\varphi$ defined just before the sequence (3) will be used many times with the group $G$ defined here.

We give an overview of the proof. We will assume that there exists at least two $C_6$-lines, and discuss the proof in the following two cases: there exists at least one $C_6$-line of type $(3, 6)$; there does not exist a $C_6$-line of type $(3, 6)$. It will be important
that $g_3^1$ is cyclic in both cases (Propositions 3.1 and 3.2). In the case that there exists a $C_6$-line of type $(3, 6)$, we can determine the defining equations of the curve $C$ concretely (Lemma 3.4). Once the curve $C$ is given by the concrete equations, it is possible to find all the Galois lines completely (Section 4). In the case that there does not exist a $C_6$-line of type $(3, 6)$, we will consider the short exact sequence (3). The group $\text{Ker} \varphi$ and homomorphisms $\varphi$ and $\psi$ are easy to understand, and it is known what groups can be isomorphic to the group $\text{Im} \varphi$ (Proposition 2.9). We will discuss the proof for each group that may be $\text{Im} \varphi$, and we will find $\text{Im} \varphi \cong D_2, D_3$ or $S_4$ (Lemmas 3.6–3.10).

In the case that $\text{Im} \varphi \cong S_4$, we can determine the defining equations of the curve $C$ concretely (Lemma 3.12), and find all the Galois lines completely (Section 5). In the cases that $\text{Im} \varphi \cong D_2, D_3$, we can determine the defining equations of $C$ roughly, and we will see that the number of $C_6$-lines is equal to 3 (Lemma 3.13).

The two Propositions below provide sufficient conditions for $g_3^1$ to be cyclic.

**Proposition 3.1.** Assume that there exists a $C_6$-line $l$ of type $(2, 3)$ or $(2, 6)$. Let $\sigma_1$ be a generator of $G_l$. Then, $\text{Ker} \varphi = \langle \sigma_1^2 \rangle$, and $\text{ord}(\varphi(\sigma_1)) = 2$. In particular, the trigonal morphism $g_3^1$ is cyclic.

**Proof.** By Proposition 2.4, using a suitable projective transformation, we may assume that $\sigma_1$ is expressed as the diagonal matrix with diagonal components $1, 1, -1, \xi^2$ or $1, 1, -1, \zeta$, where $\zeta$ is a primitive 6th root of the unity. The quadric $Q = 0$ that contains $C$ has the vertex $R := (0 : 0 : 0 : 1)$. The trigonal morphism $g_3^1$ is given by the projection $\pi_R$ with center $R$. Because $\pi_R \circ \sigma_1^2 = \pi_R$, we have $\sigma_1^2 \in \text{Ker} \varphi$. Use Proposition 2.9.

**Proposition 3.2.** We assume that there exist two $C_6$-lines. Then, the trigonal morphism $g_3^1$ is cyclic.

**Proof.** Let $l_1$ and $l_2$ be two $C_6$-lines for $C$. We assume that $\text{Ker} \varphi = 1$. Then, $G \cong \text{Im} \varphi \cong C_m, D_m, A_4, S_4$, or $A_5$. This contradicts the fact that $G$ includes two cyclic groups, $G_{l_1}$ and $G_{l_2}$, of order 6. Therefore, $\text{Ker} \varphi \neq 1$. Use Proposition 2.9.

We assume that there exist two $C_6$-lines for $C$. Let $P_1, \ldots, P_6$ be all the ramification points of the cyclic trigonal morphism $g_3^1$.

**Lemma 3.3.** There exists a hyperplane $H \subset \mathbb{P}^3$ such that $\{P_1, \ldots, P_6\} \subset H$.

**Proof.** By Proposition 3.1 in [8], there exist $x, y \in k(C)$ such that $k(C) = k(x, y)$ and $y^3 = \prod_{j=1}^5 (x - c_j)$. We can assume that $x(P_j) = c_j$ ($j = 1, \ldots, 5$) and $x(P_6) = \infty$. Then, $(x - c_j) = 3P_j - 3P_6$ ($j = 1, \ldots, 5$) and $(y) = P_1 + \cdots + P_5 - 5P_6$. By using the Riemann–Roch theorem, it is clear that $K_C \sim 6P_6$. Thus, $K_C \sim P_1 + \cdots + P_6$. Because $C \subset \mathbb{P}^3$ is a canonical curve, this concludes the lemma.
Lemma 3.4. Assume that there exists a $C_6$-line of type $(3, 6)$ and that the trigonal morphism $g_3^1$ is cyclic. Then, $C$ is projectively equivalent to the curve defined by Equations (1).

Proof. Let $l$ be a $C_6$-line of type $(3, 6)$. We assume that $G_l = \langle \sigma_l \rangle$ and

$$
\sigma_l = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \zeta^2 & 0 \\
0 & 0 & 0 & \zeta
\end{pmatrix},
$$

where $\zeta$ denotes a primitive 6th root of the unity. By using Proposition 2.3 and considering a suitable projective transformation, we can determine the defining equation of $C$ as follows:

\begin{align*}
Q &= b(X, Y)Z + W^2 = 0 \\
F &= X^3 + Y^3 + Z^3 = 0,
\end{align*}

where $b(X, Y) = X - aY$ ($a \in k$) or $Y$. If $b(X, Y) = Y$, then $C$ is projectively equivalent to the curve defined by Equations (1). Assume that $b(X, Y) = X - aY$. Let us show $a = 0$. The vertex of quadric $Q = 0$ is $R := (a : 1 : 0 : 0)$. The trigonal morphism $g_3^1 : C \rightarrow \mathbb{P}^1$ is given by the projection $\pi_R : (X : Y : Z : W) \mapsto (X - aY : Z : W)$. Let $P \in C$ be a ramification point of $g_3^1$. Then, $Z(P) \neq 0$. Indeed, if $Z(P) = 0$, then $P = (\zeta^{2j + 1} : 1 : 0 : 0)$, where $j = 0, 1$ or 2. However, $(\zeta^{2j + 1} : 1 : 0 : 0)$ is not a ramification point of $g_3^1$. Let $\pi_R(P) = (c : 1 : \sqrt{-c})$, where $c \in k$. A point in $C \cap \pi_R^{-1}(\pi_R(P))$ is $(ay + c : y : 1 : \sqrt{-c})$, where $y \in k$ satisfies

\begin{equation}
(ay + c)^3 + y^3 + 1 = 0.
\end{equation}

Note that $a^3 + 1 \neq 0$, because $C$ is nonsingular. As $P$ is a total ramification point of $g_3^1$, Equation (5) has a triple root. In other words, there exists $\beta \in k$ such that

\begin{equation}
(a^3 + 1)(y - \beta)^3 = (a^3 + 1)y^3 + 3a^2c y^2 + 3ac^2y + c^3 + 1.
\end{equation}

Then, we have

\begin{align*}
-3\beta(1 + a^3) &= 3a^2c \\
3\beta^2(1 + a^3) &= 3ac^2 \\
-\beta^3(1 + a^3) &= c^3 + 1
\end{align*}

If $a \neq 0$, then Equations (7) do not have a root $\beta$. Hence, $a = 0$ and $C$ is projectively equivalent to the curve defined by Equations (1). □
We note that as in the proof of Lemma 3.4, for the curve defined by Equations (1), there exists a $C_6$-line of type $(3, 6)$ and $g^1_3$ is cyclic. The number of $C_6$-lines of the curve defined by Equations (1) will be calculated later in Section 4. In the discussion of Section 4 we do not use the results in Section 3. From Proposition 3.2, Lemma 3.4, and Section 4, we have the following:

**Proposition 3.5.** Assume that there exist two $C_6$-lines and one of them is of type $(3, 6)$. Then, $C$ is projectively equivalent to the curve defined by Equations (1). There are exactly nine $C_6$-lines and exactly one $S_3$-line for $C$. We have that $\text{Aut}(C) \cong C_3 \times D_6$.

**Proof.** From the assumption that there exist two $C_6$-lines, by using Proposition 3.2, the trigonal morphism $g^1_3$ is cyclic. Combining this with the assumption that there exists a $C_6$-line of type $(3, 6)$, by using Lemma 3.4, we have that $C$ is projectively equivalent to the curve defined by Equations (1). By the results in Section 4, we have $\text{Aut}(C)$ and the number of skew Galois lines.

Hereafter, in this section, we continue to prove our main theorem, except in the case that $C$ is projectively equivalent to the curve defined by Equations (1). That is, we assume that there exist at least two $C_6$-lines for $C$, and every $C_6$-line is not of type $(3, 6)$.

**Lemma 3.6.** $\text{Im } \varphi \not\cong A_5$.

**Proof.** Assume that $\text{Im } \varphi \cong A_5$. Then, $|G| = 180$. However, the Hurwitz theorem states $|G| = 84(g - 1), 48(g - 1), 40(g - 1), \ldots = 252, 144, 120, \ldots$; thus, this is a contradiction.

**Lemma 3.7.** $\text{Im } \varphi \not\cong A_4$ or $C_m$.

**Proof.** From Proposition 3.1, $\text{Im } \varphi$ is generated by some elements of order 2. However, $A_4$ and $C_m (m \geq 3)$ are not generated by elements of order 2. If $\text{Im } \varphi \cong C_2$, then, $G$ does not include two $C_6$ subgroups, because the order of $G$ equals 6.

**Lemma 3.8.** If $\text{Im } \varphi \cong D_m$, then $m \leq 6$.

**Proof.** Let $Q = 0$ be the quadric that contains $C$, where the rank of the quadratic $Q$ equals 3, and $R$ be the vertex of the quadric $Q = 0$. Then, the cyclic trigonal morphism $g^1_3$ is given by the projection $\pi_R$ with center $R$. All the ramification points $P_1, \ldots, P_6$ of $g^1_3$ are on a hyperplane $H = 0$. Because $g^1_3 = \Phi|_{P_j}$ $(j = 1, \ldots, 6)$, for any $\sigma \in \text{Aut}(C)$, $\sigma(\{P_1, \ldots, P_6\}) = \{P_1, \ldots, P_6\}$. Thus, $\sigma((Q = H = 0)) = (Q = H = 0)$, where $Q = H = 0$ is a plane quadric curve. We can regard that $g^1_3 = \pi_R|_C : C \to (Q = H = 0) \cong \mathbb{P}^1$ and $\varphi : G \ni \sigma \mapsto \sigma|_{Q=H=0} \in \text{Im } \varphi \subset \text{Aut}(Q = H = 0)$. Because $\text{Im } \varphi$ acts on the set...
\( \{P_1, \ldots, P_6\} \subset (Q = H = 0) \) faithfully, we determine that the order of each element in \( \text{Im} \varphi \) is at most 6. This concludes that \( m \leq 6 \).

By Lemmas 3.6, 3.7, and 3.8, we have \( \text{Im} \varphi \cong D_m \) (\( 2 \leq m \leq 6 \)) or \( S_4 \).

**Lemma 3.9.** The maximum number of \( C_6 \)-lines is nine. If there exist nine \( C_6 \)-lines, then \( \text{Im} \varphi \cong S_4 \).

**Proof.** Let \( l_1, l_2, \ldots \) be all the \( C_6 \)-lines for \( C \), which are of types (2, 3) or (2, 6). Let \( \sigma_j \) (\( j = 1, 2, \ldots \)) be a generator of \( G_{l_j} \). By Propositions 2.7 and 3.1, \( \varphi(\sigma_1), \varphi(\sigma_2), \ldots \) are mutually distinct elements in \( \text{Im} \varphi \) and are of order 2. The number of elements of order 2 in \( S_4 \) (resp. \( D_6, D_5, D_4, D_3, D_2 \)) equals 9 (resp. 7, 5, 5, 3, 3). This now concludes the lemma.

**Lemma 3.10.** \( \text{Im} \varphi \not\cong D_4, D_5, D_6 \).

**Proof.** Assume that \( \text{Im} \varphi \cong D_4 \). Because the rank of the quadric \( Q = 0 \) that contains \( C \) equals 3, by taking a suitable projective transformation, we may assume that \( Q = XY - Z^2 \). From Lemma 3.3, all the ramification points \( P_1, \ldots, P_6 \) of the cyclic trigonal morphism \( g_3 \) are on some hyperplane \( H = 0 \). By taking a suitable projective transformation that does not change \( Q \), we may assume \( H = W \). Note that we can take such a projective transformation because \( (0 : 0 : 0 : 1) \not\in H \). By using Proposition 2.10 and the same argument as in the proof of Lemma 3.8, we may assume that:

\[
G = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & \lambda_1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \lambda_2
\end{pmatrix},
\]

where \( \omega \) (resp. \( i \)) is a primitive cubic (resp. 4th) root of the unity and \( \lambda_1, \lambda_2 \in k \setminus \{0\} \).

By using Proposition 2.3, we find a cubic form \( F \in k[X, Y, Z, W] \setminus \{0\} \), such that the cubic surface \( F = 0 \) contains \( C \). By the condition \( \sigma(F) = (F) \) for any \( \sigma \in G \), we have \( F = a(X^2 + Y^2)Z + W^3 \), \( F = a(X^2 - Y^2)Z + W^3 \), or \( F = aXYZ + bZ^3 + W^3 \), where \( a, b \in k \). The curves defined by \( Q = XY - Z^2 = 0 \) and \( F = a(X^2 + Y^2)Z + W^3 = 0 \) are projectively equivalent to the curve defined by Equations (2), and thus, \( \text{Im} \varphi \cong S_4 \).

The curves defined by \( Q = XY - Z^2 = 0 \) and \( F = a(X^2 - Y^2)Z + W^3 = 0 \) are also projectively equivalent to the curve defined by Equations (2). The curves defined by \( Q = XY - Z^2 = 0 \) and \( F = aXYZ + bZ^3 + W^3 = 0 \) have singular points \( (1 : 0 : 0 : 0) \) and \( (0 : 1 : 0 : 0) \). Hence, We see that \( \text{Im} \varphi \not\cong D_4 \).

By using the same argument as above, we also see that \( \text{Im} \varphi \not\cong D_5 \).
Assume that $\text{Im } \varphi \cong D_6$. From the same argument as above, we see that $C$ must be projectively equivalent to the curve defined by Equations (1). Then, there exists a $C_6$-line for $C$ of type $(3, 6)$. However, this is a contradiction. This concludes $\text{Im } \varphi \not\cong D_6$. ■

Remark 3.11. To prove our main theorem, we have discussed the proof above with the assumption that there is no $C_6$-line of type $(3, 6)$, which is stated just after Proposition 3.5. If we allow the existence of $C_6$-lines of type $(3, 6)$, then by the same argument as in the proof of Lemma 3.10, we see the following: if a canonical curve $C \subset \mathbb{P}^3$ of genus 4 satisfies the conditions “there exists a unique trigonal morphism $g_3^1$,” “$g_3^1$ is cyclic,” and “$\text{Im } \varphi \cong D_6$,” then $C$ is projectively equivalent to the curve defined by Equations (1).

Hence, $\text{Im } \varphi \cong D_2, D_3$, or $S_4$.

Lemma 3.12. Assume that $\text{Im } \varphi \cong S_4$. Then, $C$ is projectively equivalent to the curve defined by Equations (2). Hence, there exist nine $C_6$-lines (see Section 5).

Proof. We may assume that the ramification points $P_1, \ldots, P_6$ of the trigonal morphism $g_3^1$ are on the hyperplane $W = 0$ and the quadric $Q = 0$ that contains $C$ is $XY - Z^2 = 0$. By using Proposition 2.10, we can assume that

$$G = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix} \left\langle \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & \lambda_1
\end{pmatrix}, \begin{pmatrix}
1 & 1 & 2 & 0 \\
1 & 1 & -2 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 0 & \lambda_2
\end{pmatrix} \right\rangle,$$

where $\omega$ (resp. $i$) is a primitive cubic (resp. 4th) root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By the same argument as in the proof of Lemma 3.10, $C$ must be defined by

$$
\begin{cases}
XY - Z^2 = 0 \\
c(X^2 - Y^2)Z + W^3 = 0
\end{cases},
$$

where $c \in k$. Then, $C$ is projectively equivalent to the curve defined by Equations (2). ■

Note that we do not use the results in Section 3 in the discussion of Section 5.

Lemma 3.13. If $\text{Im } \varphi \cong D_2$ or $D_3$, then the number of $C_6$-lines equals 3.

Proof. If $\text{Im } \varphi \cong D_2$ or $D_3$, then the number of $C_6$-lines is at most three because the group $\text{Im } \varphi$ contains only three elements of order 2.

Assume that $\text{Im } \varphi \cong D_2$. We may assume that all the ramification points $P_1, \ldots, P_6$ of the trigonal morphism $g_3^1$ are on the hyperplane $W = 0$ and the quadric $Q = 0$ that
contains $C$ is $XY - Z^2 = 0$. By using Proposition 2.10, we can assume that

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \frac{1}{\omega} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & \omega \end{pmatrix}, \frac{1}{\omega} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \frac{1}{\omega} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \frac{1}{\omega} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \right\},$$

where $\omega$ is a primitive cubic root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By the same argument as in the proof of Lemma 3.10, $C$ must be projectively equivalent to the curve defined by

$$\begin{cases} XY - Z^2 = 0 \\ (X^3 + Y^3) + c(X + Y)Z^2 + W^3 = 0 \end{cases} \quad (9)$$

or

$$\begin{cases} XY - Z^2 = 0 \\ (X^2 + Y^2)Z + cZ^3 + W^3 = 0 \end{cases} \quad (10),$$

where $c \in k$. Then, the three lines $X = Y = 0$, $X + Y = Z = 0$, $X - Y = Z = 0$ are $C_6$-lines. Indeed, if $C$ is defined by Equations (9) or (resp. Equations (10)), then we have automorphisms of order 6 as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

(resp. $$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \right\}.$$

Thus, the number of $C_6$-lines is at least three.

Assume that $\text{Im} \varphi \cong D_3$. According to the above argument, $C$ must be projectively equivalent to the curve defined by

$$\begin{cases} XY - Z^2 = 0 \\ (X^3 + Y^3) + cZ^3 + W^3 = 0 \end{cases} \quad (11),$$

where $c \in k$. Then, the three lines $X = Y = 0$, $X + Y = Z = 0$, $X - Y = Z = 0$ are $C_6$-lines. Indeed, if $C$ is defined by Equations (9) or (resp. Equations (10)), then we have automorphisms of order 6 as follows:
where \( c \in k \). Then, the three lines \( X + Y = Z = 0, X + \omega Y = Z = 0, X + \omega^2 Y = Z = 0 \) are \( C_6 \)-lines. Indeed, we have automorphisms of order 6 as follows:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \omega & 0 & 0 \\
\omega^2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \omega^2 & 0 & 0 \\
\omega & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix}.
\]

Thus, the number of \( C_6 \)-lines is at least three.

The proof of our main theorem is now complete.

4. Example: Galois lines for the curve defined by Equations (1)

In this section, let \( C \) be the nonsingular projective curve such that \( k(C) = k(x, y) \), and

\[
(12) \quad x^6 + y^3 + 1 = 0.
\]

The polynomial on the left-hand side of Equation (12) is irreducible. Let \( g^1_3 : C \to \mathbb{P}^1 \) be the tringular morphism given by the function \( x \). Then, \( g^1_3 \) is a cyclic triple covering, and there exist 6 branch points. By using the Riemann–Hurwitz formula, we have that the genus of \( C \) is equal to 4. Let \( (x)_\infty = D \) be the divisor of poles of \( x \). Then, \( (x^2)_\infty = (y)_\infty = 2D \). Therefore, \( \dim_k H^0(C, O_C(2D)) \geq 4 \). By using the Riemann–Roch theorem, we have that \( K_C \sim 2D \). The morphism \( C \ni P \mapsto (1 : x^2(P) : x(P) : y(P)) \in \mathbb{P}^3 \) is a canonical embedding. The image of this canonical embedding is expressed as Equations (1). We regard \( C \) as the canonical curve defined by Equations (1).

We can identify nine \( C_6 \)-lines and one \( S_3 \)-line, as indicated in Tables 1 and 2. Because \( \deg \pi_{l_j} = 6, \sigma_j \in \text{Aut}(C), \text{ord}(\sigma_j) = 6, \) and \( \pi_{l_j} \circ \sigma_j = \pi_{l_j} \) \((j = 1, \ldots, 9)\), it is clear that the lines \( l_1, \ldots, l_9 \) are \( C_6 \)-lines. As \( \deg \pi_{l_{10}} = 6, \sigma_{10}, \tau_{10} \in \text{Aut}(C), \langle \sigma_{10}, \tau_{10} \rangle \cong S_3, \pi_{l_{10}} \circ \sigma_{10} = \pi_{l_{10}}, \) and \( \pi_{l_{10}} \circ \tau_{10} = \pi_{l_{10}} \), the line \( l_{10} \) is clearly an \( S_3 \)-line.

Let \( R := (0 : 0 : 0 : 1) \), which is the vertex of the quadric \( XY - Z^2 = 0 \). The projection \( \pi_R : C \to (XY - Z^2 = W = 0) \cong \mathbb{P}^1 \subset (W = 0) \cong \mathbb{P}^2 \) yields the unique tringular morphism \( g^1_3 \). We have that \( g^1_3 \) is cyclic,

\[
\rho := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix} \in \text{Aut}(C), \text{ord}(\rho) = 3 = \deg g^1_3, \text{ and } \pi_R \circ \rho = \pi_R.
\]
Table 1. \(C_6\)-lines for the curve defined by Equations (1)

<table>
<thead>
<tr>
<th>line (l)</th>
<th>def. eq. of (l)</th>
<th>(G_l)</th>
<th>generators of (G_l)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_1)</td>
<td>(X = Y = 0)</td>
<td>(C_6)</td>
<td>(\sigma_1 = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_2)</td>
<td>(X + Y = Z = 0)</td>
<td>(C_6)</td>
<td>(\sigma_2 = \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_3)</td>
<td>(X + \omega Y = Z = 0)</td>
<td>(C_6)</td>
<td>(\sigma_3 = \begin{pmatrix} 0 &amp; \omega &amp; 0 &amp; 0 \ \omega^2 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_4)</td>
<td>(X + \omega^2 Y = Z = 0)</td>
<td>(C_6)</td>
<td>(\sigma_4 = \begin{pmatrix} 0 &amp; \omega^2 &amp; 0 &amp; 0 \ \omega &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_5)</td>
<td>(X - Y = Z = 0)</td>
<td>(C_6)</td>
<td>(\sigma_5 = \begin{pmatrix} 0 &amp; -1 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -\omega \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_6)</td>
<td>(X - \omega Y = Z = 0)</td>
<td>(C_6)</td>
<td>(\sigma_6 = \begin{pmatrix} 0 &amp; -\omega &amp; 0 &amp; 0 \ -\omega^2 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -\omega \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_7)</td>
<td>(X - \omega^2 Y = Z = 0)</td>
<td>(C_6)</td>
<td>(\sigma_7 = \begin{pmatrix} 0 &amp; -\omega^2 &amp; 0 &amp; 0 \ -\omega &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -\omega \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_8)</td>
<td>(X = W = 0)</td>
<td>(C_6)</td>
<td>(\sigma_8 = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; \omega &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -\omega^2 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>(l_9)</td>
<td>(Y = W = 0)</td>
<td>(C_6)</td>
<td>(\sigma_9 = \begin{pmatrix} \omega &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -\omega^2 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix} )</td>
</tr>
</tbody>
</table>

\(\omega\) is a primitive cubic root of the unity.
The ramification points of $g^1_3$ are $P_1 := (1 : -1 : i : 0)$, $P_2 := (1 : -1 : -i : 0)$, $P_3 := (1 : -\omega : i\omega^2 : 0)$, $P_4 := (1 : -\omega : -i\omega^2 : 0)$, $P_5 := (1 : -\omega^2 : i\omega : 0)$, and $P_6 := (1 : -\omega^2 : -i\omega : 0)$, where $\omega$ (resp. $i$) is a primitive cubic (resp. 4th) root of the unity. Because $g^1_3 = \Phi_{[3P_j]}$ ($j = 1, \ldots, 6$), we have $\text{Aut}(C)$ acts on $\{P_1, \ldots, P_6\}$. Thus, $\sigma(W = 0) = (W = 0)$ for any $\sigma \in \text{Aut}(C) \subset \text{Aut}(\mathbb{P}^3)$.

Because $g^1_3$ is a unique trigonal morphism, a unique $A_{\sigma} \in \text{Aut}(\mathbb{P}^1)$ exists for any $\sigma \in \text{Aut}(C)$ such that $g^1_3 \circ \sigma = A_{\sigma} \circ g^1_3$. We denote the map $\sigma \mapsto A_{\sigma}$ as $\varphi : \text{Aut}(C) \rightarrow \text{Aut}(\mathbb{P}^1)$, which is a homomorphism between the groups. Note that $\sigma(W = 0) = (W = 0)$, and $g^1_3$ is obtained by using the projection $\pi_R : (X : Y : Z : W) \mapsto (X : Y : Z)$. By considering $\varphi(\sigma) = A_{\sigma}$ as an automorphism of the quadric plane curve $(XY - Z^2 = W = 0) \subset (W = 0) \cong \mathbb{P}^2$, we see that $\varphi$ is expressed as follows:

$$
\sigma = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix} \mapsto \sigma' = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix},
$$

where $\sigma'$ is regarded as an element of $\text{Aut}(XY - Z^2 = W = 0) \subset \text{Aut}(\mathbb{P}^2)$. Let $\text{Ker} \varphi$ and $\text{Im} \varphi$ be the kernel and image of $\varphi$, respectively. We have the short exact sequence (3) for $\text{Aut}(C)$, and $\text{Ker} \varphi = \langle \rho \rangle$.

**Claim 4.1.** $\text{Im} \varphi \cong D_6$, which is the dihedral group of order 12.

**Proof.** From Proposition 2.9, $\text{Im} \varphi$ is isomorphic to $C_m$, $D_m$, $A_4$, $S_4$, or $A_5$. Let $\sigma_j$ ($j = 1, \ldots, 10$) be the automorphism provided in Tables 1 and 2. Because the order of $\varphi(\sigma_8)$ is equal to 6, we see that $\text{Im} \varphi \cong C_m$ or $D_m$, where $m$ is a multiple of 6. Because $\varphi(\sigma_1) \neq \varphi(\sigma_2)$, and the orders of both $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ are equal to 2, we...
have $\text{Im } \varphi \cong D_6$. Note that $\text{Aut}(C)$ acts on the set $\{P_1, \ldots, P_6\}$. Let $\sigma \in \text{Aut}(C)$. If $\sigma(P_j) = P_j$ for every $P_j$ ($j = 1, \ldots, 6$), then $\varphi(\sigma)$ is the identity. Thus, the order of $\varphi(\sigma)$ is at most 6. This concludes that $\text{Im } \varphi \cong D_6$. 

We have an exact sequence $1 \rightarrow C_3 \xrightarrow{\psi} \text{Aut}(C) \xrightarrow{\varphi} D_6 \rightarrow 1$. The order of $\text{Aut}(C)$ is 36. Let $G := \langle \rho, \sigma_2, \sigma_8 \rangle$.

**Claim 4.2.** $\text{Aut}(C) = G \cong C_3 \times D_6$.

**Proof.** Because of the exact sequence $1 \rightarrow C_3 \xrightarrow{\psi} G \xrightarrow{\varphi} D_6 \rightarrow 1$, $G = \text{Aut}(C)$. We show that there is a left-inverse of $\psi$. For $\sigma \in G$, we have a unique matrix representation $M_{\sigma}$ such that $M_{\sigma} (XY - Z^2) = XY - Z^2$ and the $(4, 4)$-component of $M_{\sigma}$ is 1, $\omega$, or $\omega^2$. We denote the $(4, 4)$-component of $M_{\sigma}$ as $\lambda_{\sigma}$. Let $\psi' : G \rightarrow \text{Ker } \varphi \cong C_3$ be as follows:

$$\sigma = M_{\sigma} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma} \end{pmatrix}.$$  

Because $\psi'$ is a homomorphism between groups, and $\psi' \circ \psi = \text{id}$, this concludes that $G \cong C_3 \times D_6$. 

The group $\text{Aut}(C) \cong C_3 \times D_6$ has only ten $C_6$ subgroups:

$$\langle \sigma_1, \ldots, \sigma_9 \rangle, \quad \tilde{\sigma} := \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

As $\tilde{\sigma}$ has no multiple eigenvalues, $\langle \tilde{\sigma} \rangle$ is not a Galois group associated with a Galois line. Therefore, the number of $C_6$-lines is equal to 9. The group $\text{Aut}(C) \cong C_3 \times D_6$ has only six $S_3$ subgroups: $\langle \sigma_m, \tau_n \rangle$, where

$$\sigma_m = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^m \end{pmatrix} \quad (m = 0, 1, 2), \quad \tau_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & (-1)^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (n = 0, 1).$$

By Proposition 2.8, the lines that might be $S_3$-lines are $l_8 : X = W = 0$, $l_9 : Y = W = 0$, and $l_{10} : Z = W = 0$. However, $l_8$ and $l_9$ are $C_6$-lines. The line $l_{10}$ is the only one $S_3$-line.

**Remark 4.3.** Let $P' := (1 : 0 : 0 : 0)$, which is the point at which lines $l_9$ and $l_{10}$ intersect. By the projection $\pi_{P'} : (X : Y : Z : W) \mapsto (Y : Z : W)$ with center $P'$, we
have a singular plane curve $T_6 : Y^6 + Z^6 + Y^3 W^3 = 0$ as the image $\pi_{P'}(C)$. The points $(0 : 1 : 0) = \pi_{P'}(l_9)$ and $(1 : 0 : 0) = \pi_{P'}(l_{10})$ are outer Galois points for $T_6$ with Galois groups $C_6$ and $S_3$, respectively. The plane curves $T_{2m} : Y^{2m} + Z^{2m} + Y^m W^m = 0$ are examples of curves that are known to have two outer Galois points with Galois groups $C_{2m}$ and $D_m$ (See [5]).

5. Example: Galois lines for the curve defined by Equations (2)

In this section, let $C$ be the nonsingular projective curve such that $k(C) = k(x, y)$, and

\begin{equation}
 y^6 + x^2(x^2 + 1) = 0.
\end{equation}

The polynomial on the left-hand side of Equation (13) is irreducible. Let $g^1 : C \to \mathbb{P}^1$ be the cyclic morphism of degree 6 given by the function $x$. By using the Riemann–Hurwitz formula, we have that the genus of $C$ is equal to 4. Let $P_\infty, P_{\infty'}, P_0, P_{0'}, P_i, P_{-i}$ be six points on $C$ such that $x(P_\infty) = x(P_{\infty'}) = \infty, x(P_0) = x(P_{0'}) = 0, x(P_i) = i$, and $x(P_{-i}) = -i$, where $i$ is a primitive 4th root of the unity. Because $(x) = 3P_0 + 3P_{0'} - 3P_\infty - 3P_{\infty'}$, $(y) = P_0 + P_{0'} + P_i + P_{-i} - 2P_\infty - 2P_{\infty'}$, and $(x - i) = 6P_i - 3P_\infty - 3P_{\infty'}$, we have

\begin{align*}
 \left( \frac{y^3}{x(x-i)} \right)_\infty &= 3P_i \quad \text{and} \quad \left( \frac{y}{x-i} \right)_\infty = 5P_i.
\end{align*}

Hence, the Weierstrass semigroup of $P_i$ is $H(P_i) = \langle 3, 5 \rangle$ (for the definition of Weierstrass semigroup, see [8, Equation (2)]). Thus, $C$ is not hyperelliptic and $K_C \sim 6P_i - 3P_\infty + 3P_{\infty'}$. Because $1, y^3/(x(x-i)), y/(x-i), 1/(x-i)$ are linearly independent over $k$, the morphism $C \ni P \mapsto (x^2(P) : y^3(P) : x(P) : -x(P)y(P)) \in \mathbb{P}^3$ is a canonical embedding. The image of this embedding is expressed as Equations (2). We regard $C$ as the canonical curve defined by Equations (2).

We can find nine $C_6$-lines and four $S_3$-lines, as in Tables 3 and 4. Because $\deg \pi_{l_j} = 6$, $\sigma_j \in \text{Aut}(C)$, $\ord(\sigma_j) = 6$ and $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ ($j = 1, \ldots, 9$), we see that the lines $l_1, \ldots, l_9$ are $C_6$-lines. As $\deg \pi_{l_j} = 6$, $\sigma_j, \tau_j \in \text{Aut}(C)$, $\langle \sigma_j, \tau_j \rangle \cong S_3$, $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ and $\pi_{l_j} \circ \tau_j = \pi_{l_j}$ ($j = 10, \ldots, 13$), we see that the lines $l_{10}, \ldots, l_{13}$ are $S_3$-lines.

**Claim 5.1.** $\text{Aut}(C) \cong C_3 \times S_4$.

**Proof.** We have the following automorphisms of $C$:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix},
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Table 3. $C_6$-lines for the curve defined by Equations (2)

<table>
<thead>
<tr>
<th>line $l$</th>
<th>def. eq. of $l$</th>
<th>$G_l$</th>
<th>generators of $G_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1$</td>
<td>$X = Y = 0$</td>
<td>$C_6$</td>
<td>$\sigma_1 = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -\omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>$Y = Z = 0$</td>
<td>$C_6$</td>
<td>$\sigma_2 = \begin{pmatrix} -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -\omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_3$</td>
<td>$X = Z = 0$</td>
<td>$C_6$</td>
<td>$\sigma_3 = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; -1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -\omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_4$</td>
<td>$X + Y = Z = 0$</td>
<td>$C_6$</td>
<td>$\sigma_4 = \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_5$</td>
<td>$X - Y = Z = 0$</td>
<td>$C_6$</td>
<td>$\sigma_5 = \begin{pmatrix} 0 &amp; -1 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_6$</td>
<td>$X + Z = Y = 0$</td>
<td>$C_6$</td>
<td>$\sigma_6 = \begin{pmatrix} 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_7$</td>
<td>$X - Z = Y = 0$</td>
<td>$C_6$</td>
<td>$\sigma_7 = \begin{pmatrix} 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_8$</td>
<td>$X = Y + Z = 0$</td>
<td>$C_6$</td>
<td>$\sigma_8 = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_9$</td>
<td>$X = Y - Z = 0$</td>
<td>$C_6$</td>
<td>$\sigma_9 = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; -1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; \omega \end{pmatrix}$</td>
</tr>
</tbody>
</table>

$\omega$ is a primitive cubic root of the unity.
Table 4. $S_3$-lines for the curve defined by Equations (2)

<table>
<thead>
<tr>
<th>line $l$</th>
<th>def. eq. of $l$</th>
<th>$G_l$</th>
<th>generators of $G_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_{10}$</td>
<td>$X + Y + Z = W = 0$</td>
<td>$S_3$</td>
<td>$\sigma_{10} = \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$, $\tau_{10} = \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_{11}$</td>
<td>$X - Y + Z = W = 0$</td>
<td>$S_3$</td>
<td>$\sigma_{11} = \begin{pmatrix} 0 &amp; -1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$, $\tau_{11} = \begin{pmatrix} 0 &amp; -1 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_{12}$</td>
<td>$-X + Y + Z = W = 0$</td>
<td>$S_3$</td>
<td>$\sigma_{12} = \begin{pmatrix} 0 &amp; -1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$, $\tau_{12} = \begin{pmatrix} 0 &amp; -1 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$l_{13}$</td>
<td>$X + Y - Z = W = 0$</td>
<td>$S_3$</td>
<td>$\sigma_{13} = \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$, $\tau_{13} = \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
</tr>
</tbody>
</table>
where \( \omega \) is a primitive cubic root of the unity. The group generated by these four elements, which is a subgroup of \( \text{Aut}(C) \), is isomorphic to \( C_3 \times S_4 \). By considering the short exact sequence (3) for \( G = \text{Aut}(C) \), we have that \( 1 \rightarrow C_3 \rightarrow \text{Aut}(C) \xrightarrow{\varphi} \text{Im} \varphi \rightarrow 1 \) and \( \text{Im} \varphi \cong C_m, D_m, A_4, S_4 \), or \( A_5 \). By using the same argument as in the proof of Lemma 3.8 or Claim 4.1, if \( \text{Im} \varphi \cong C_m \) or \( D_m \), then \( m \leq 6 \). Because \( C_3 \times S_4 \subset \text{Aut}(C) \), we see that \( \text{Im} \varphi \cong S_4 \) and \( \text{Aut}(C) \cong C_3 \times S_4 \).

Because the group \( C_3 \times S_4 \) contains exactly nine \( C_6 \) subgroups and exactly four \( S_3 \) subgroups, this concludes that the lines in Tables 3 and 4 are all the \( C_6 \)-lines and all the \( S_3 \)-lines, respectively.

### 6. Other Examples

In this section, we present two examples of canonical curves of genus 4, which have exactly one \( C_6 \)-line and exactly three \( C_6 \)-lines, respectively.

**Example 6.1.** Let \( C \subset \mathbb{P}^3 \) be the curve defined by

\[
\begin{align*}
Q & := YZ - W^2 = 0 \\
F & := X^3 - X^2Y - XY^2 + Z^3 = 0
\end{align*}
\]

Then, \( C \) is a canonical curve of genus 4. The line \( l : X = Y = 0 \) is a \( C_6 \)-line. Indeed,

\[
\sigma := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \omega & 0 \\
0 & 0 & 0 & -\omega^2
\end{pmatrix}
\]

(where \( \omega \) is a primitive cubic root of the unity)

satisfies \( \sigma \in \text{Aut}(C) \), \( \pi_l \circ \sigma = \pi_l \), and \( \text{ord}(\sigma) = 6 = \deg \pi_l \). Because \( \text{rank } Q = 3 \), the trigonal morphism \( g_3^1 \) is unique, and \( g_3^1 \) is obtained by the projection \( \pi_R \) with center \( R := (1 : 0 : 0 : 0) \), which is the vertex of \( Q = 0 \). Because \( \pi_R^{-1}((1 : 1 : 1)) \) consists of only two points \((1 : 1 : 1 : 1)\) and \((-1 : 1 : 1 : 1)\), we see that \( g_3^1 \) is not Galois. From Proposition 3.2, the number of \( C_6 \)-lines equals one.

**Example 6.2.** Let \( C \subset \mathbb{P}^3 \) be the curve defined by

\[
\begin{align*}
Q & := XY - Z^2 = 0 \\
F & := X^3 + Y^3 + Z^3 + W^3 = 0
\end{align*}
\]
Then, $C$ is a canonical curve of genus $4$. The lines $l_1 : X + Y = Z = 0$, $l_2 : X + \omega Y = Z = 0$, and $l_3 : X + \omega^2 Y = Z = 0$ are $C_6$-lines of type $(2,3)$. Indeed,

$$
\sigma_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & \omega & 0 & 0 \\ \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \quad \text{and} \quad \sigma_3 := \begin{pmatrix} 0 & \omega^2 & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}
$$

(where $\omega$ is a primitive cubic root of the unity) satisfy $\sigma_j \in \text{Aut}(C)$, $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ and $\text{ord}(\sigma_j) = 6 = \deg \pi_{l_j}$ ($j = 1, 2, 3$).

We show that all the $C_6$-lines for $C$ of types $(2,3)$ or $(2,6)$ are the three lines $l_1$, $l_2$ and $l_3$. Here, we explain how to find the $C_6$-lines of types $(2,3)$ or $(2,6)$. Let $P_1 := (1 : \zeta^2 : \zeta : 0)$, $P_2 := (1 : \zeta^4 : \zeta^2 : 0)$, $P_3 := (1 : \zeta^8 : \zeta^4 : 0)$, $P_4 := (1 : \zeta : \zeta^5 : 0)$, $P_5 := (1 : \zeta^5 : \zeta^7 : 0)$, and $P_6 := (1 : \zeta^7 : \zeta^8 : 0)$, where $\zeta$ is a primitive 9th root of the unity. Because rank $Q = 3$, from Proposition 2.1, there exists a unique trigonal morphism $g_1^3 : C \to \mathbb{P}^1$. From Proposition 3.1 (or 3.2), $g_1^3$ is cyclic. Points $P_1, \ldots, P_6$ are all the ramification points of $g_1^3$. Let $H(3P_m + 3P_n) \subset \mathbb{P}^3$ (resp. $H(6P_m) \subset \mathbb{P}^3$) ($P_m, P_n \in \{P_1, \ldots, P_6\}$) be the hyperplane that defines the divisor $3P_m + 3P_n$ (resp. $6P_m$) on $C$. Let $l$ be a $C_6$-line of type $(2,3)$ or $(2,6)$. From Proposition 3.1, the projection $\pi_l : C \to \mathbb{P}^1$ is the composition of $g_1^3 : C \to \mathbb{P}^1$ and some morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2. Thus, $P_1, \ldots, P_6$ are ramification points of $\pi_l$. At least two fibers of $\pi_l$ are formed as $3P_m + 3P_n$, where $P_m \neq P_n$ and $P_m, P_n \in \{P_1, \ldots, P_6\}$. In other words, there exist four mutually distinct points $P_{m_1}, P_{m_2}, P_{m_3}, P_{m_4} \in \{P_1, \ldots, P_6\}$ such that $H(3P_{m_1} + 3P_{m_2}) \cap H(3P_{m_3} + 3P_{m_4}) = l$. Moreover, we have $l \subset H(3P_{m_5} + 3P_{m_6})$ or $l \subset H(6P_m) \cap H(6P_{m_6})$, where $\{P_{m_1}, \ldots, P_{m_6}\} = \{P_1, \ldots, P_6\}$. By using this fact, we search for lines that might be $C_6$-lines of types $(2,3)$ or $(2,6)$.

For example, let $l_{1234} \subset \mathbb{P}^3$ be the line $H(3P_1 + 3P_2) \cap H(3P_3 + 3P_4)$. Because $H(3P_1 + 3P_2)$ and $H(3P_3 + 3P_4)$ are defined by $\zeta^2 X + Y - (\zeta + \zeta^2)Z = 0$ and $X + Y - (\zeta^4 + \zeta^5)Z = 0$, respectively, we have $R_{1234} := (-\zeta(1 + \zeta) : \zeta(1 + \zeta)(1 + \zeta^2) : 1 : 0) \in l_{1234}$. The hyperplanes $H(3P_5 + 3P_6)$, $H(6P_5)$, and $H(6P_6)$ are defined by $\zeta^6 X + Y - (\zeta^7 + \zeta^8)Z = 0$, $\zeta^5 X + Y - 2\zeta^7 Z = 0$, and $\zeta^7 X + Y - 2\zeta^8 Z = 0$, respectively. We see that $R_{1234} \notin H(3P_5 + 3P_6)$, $R_{1234} \notin H(6P_5)$, and $R_{1234} \notin H(6P_6)$. Thus, $l_{1234} \notin H(3P_5 + 3P_6)$, $l_{1234} \notin H(6P_5)$, and $l_{1234} \notin H(6P_6)$. This concludes that $l_{1234}$ is not a $C_6$-line of type $(2,3)$ or $(2,6)$. By using the same argument as above and computer calculations, we check whether $l_{m_1m_2m_3m_4}$ can be a $C_6$-line of type $(2,3)$ or $(2,6)$ for every line $l_{m_1m_2m_3m_4} := H(3P_{m_1} + 3P_{m_2}) \cap H(3P_{m_3} + 3P_{m_4})$. Then, we see that only three lines $l_{1236}, l_{1423}, l_{1625}$ might be $C_6$-lines of types $(2,3)$ or $(2,6)$, which are $C_6$-lines $l_3, l_2, l_1$, respectively.

According to Sections 4 and 5, seven $C_6$-lines of types $(2,3)$ or $(2,6)$ exist for the curve defined by Equations (1), and nine $C_6$-lines of types $(2,3)$ or $(2,6)$ exist for the
curve defined by Equations (2). Thus, C is not projectively equivalent to the curves defined by Equations (1) or (2). From our main theorem, we see that all the $C_6$-lines for C are the three lines $l_1$, $l_2$, and $l_3$.

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References


