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On the Theriault conjecture for self homotopy equivalences

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Abstract – Our main purpose in this paper is to resolve, in a rational homotopy theory context, the following open question asked by S. Theriault : Given a topological space X, what one may say about the nilpotency of $\mathrm{aut}_1(X)$ when the cocategory of its classifying space $\mathrm{Baut}_1(X)$ is finite? Here $\mathrm{aut}_1(X)$ denotes the path component of the identity map in the set of self homotopy equivalences of X. More precisely, we prove that

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) \leqslant \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X)),$$

when X is a simply connected CW-complex of finite type and that the equality holds when $\mathrm{Baut}_1(X)$ is coformal. Many intersections with other popular open questions will be discussed.

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1. Introduction

Here we are interested to a homotopic invariant of topological spaces, the so-called *cocategory*. However "weakly" dual to the popular LS-category ([7]) introduced in 1934 by Lusternik and Schnirelmann (motivated by developing a Morse theory in the degenerate case), the cocategory as a concept is less studied.

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Maybe because it is not yet related to other invariants as well as LS-category is. In fact, for some reasonable spaces X (like CW-complexes), $\mathrm{cat}(X)$ (defined to be the least integer n, or infinite, such that X can be covered by $n+1$ open subsets contractible in X) is well related to a lot of well known other invariants like: $\mathrm{Crit}(X)$ (the minimum number of critical point for any smooth function on a smooth compact manifold), the cup-length, the topological complexity. LS-category is also required for some other popular results as Poincaré conjecture or Borsuk-Ulam Theorem.

To compensate this defect, our main purpose on this paper is to relate the cocategory as an invariant to another one, more precisely to the homotopical nilpotency.

As well as for LS-category, we know many definitions of the cocategory: $\mathrm{cocat}_{\mathbb{Q}}$ (rational version defined by Sbaï in [34]), Hcocat (Hovey version, [23]), indcocat (inductive version defined by Ganea in [17]), wcocat (Whitehead approach defined by Hopkins in [22]) and finally cocat (defined by Aniceto and Viruel in [29]).

If all known LS-category versions are equivalent (namely the original one, the Whitehead approach described in [39], the inductive version and the fibration characterization given by Ganea in [17] and [18]), nevertheless this not holds for the cocategory. In fact, according to Hovey ([23], page 225) Murillo-Viruel ([29], Proposition 3.10 and Remark 3.16), given a rational space X we have:

$$\mathrm{wcocat}(X) \leq \mathrm{cocat}(X) \leq \mathrm{indcocat}(X) \leq \mathrm{Hcocat}(X) \leq \mathrm{cocat}_{\mathbb{Q}}(X).$$

Note that cocategory and category are Eckmann-Hilton dual in many cases:

- The dual of the original definition of LS-category (that using open covering) is the one developed by Hopkins. One have to think of a covering as a (homotopy) colimit (view every inclusion as a cofibration), thus the dual of a covering has to be a (homotopy) limit (where every projection is a fibration);
- Both Hovey and Murillo-Viruel definitions of cocategory give rise to respective dual to the LS-category Whitehead version, and to the weak version one;
- Sbaï rational definition of cocategory is dual of the standard (rational) model category proposed by Félix and Halperin in [12]. However, Sbaï rational cocategory is not an algebraic model of the rationalization of Murillo-Viruel cocategory.

In this paper, we focus on the rational version of the cocategory given by Sbaï in terms of Quillen models. Before doing it, a brief overview on Quillen and Sullivan models is outlined here above. For further details on this rational homotopy theory famous gadgets, we refer the interested reader to the standard references [13] or [16]. Note first that rational homotopy theory focus on maps and spaces that are invariant under rational homotopy equivalence and that any simply connected space can be "rationally" modelled by a simply connected and rational CW-complex with no 0 or 1-cells.

A *minimal Sullivan model* is a free commutative differential graded algebra (CDGA) of the form $(\Lambda V, d)$. Here V is a graded $\mathbb{Q}$-vector space generated by a well-ordered indexed basis (v_i) verifying:

$$dv_i \in \Lambda^{\geq 2}\{v_j,\ |v_j| < |v_i|\}.$$

The following basic vocabularies will be used in the sequel.

- $(\Lambda V, d)$ is called *elliptic*, when $\dim V < \infty$ and $\dim H^*(\Lambda V, d) < \infty$;
- $(\Lambda V, d)$ is called a F_0-model, when it is elliptic with $H^{\text{odd}}(\Lambda V, d) = 0$;
- $(\Lambda V, d)$ is said to be *formal*, when $(\Lambda V, d) \simeq (H^*(\Lambda V, d), 0)$;
- $(\Lambda V, d)$ is said to be *coformal*, when $d = d_2$ (i.e., d is purely quadratic).

Any simply connected CW-complex of finite type, X, can be rationally modelled by a minimal Sullivan algebra $(\Lambda V, d)$, unique up to isomorphism, in the sense that

$$\begin{array}{rcll} H^*(X;\mathbb{Q}) & \cong & H^*(\Lambda V, d) & \text{as graded commutative algebras} \\ \pi_*(X)\otimes\mathbb{Q} & \cong & \mathrm{Hom}_{\mathbb{Q}}(V,\mathbb{Q}) & \text{as vector spaces} \end{array}.$$

X is said to be elliptic (resp. F_0-space, formal, coformal) when its model $(\Lambda V, d)$ is. X is called an *H-space*, when $d = 0$.

Minimal Quillen model is the Eckmann-Hilton dual to the Sullivan minimal model and involves free differential graded Lie algebras (DGLA) $(\mathbb{L}_W, \partial)$, in opposite of CDGA. Here W is a graded vector space W, and $\mathbb{L}_W$ is equipped with a decomposable differential (i.e., $\partial : \mathbb{L}_W \to \mathbb{L}_W^{\geq 2}$), where $\mathbb{L}_W^k$ designates the set of brackets of length k. Any simply connected and rational CW-complex of finite type, X, admits a minimal Quillen model $(\mathbb{L}_W, \partial)$, unique up to isomorphism, which encodes the rational homotopy type as follows:

$$\begin{array}{rcl} H_*(\mathbb{L}_W, d) & \cong & \pi_{*+1}(X)\otimes\mathbb{Q} \\ W & \cong & \tilde{H}_{*+1}(X;\mathbb{Q}) \end{array}.$$

In particular, $\partial\mathbb{L}_W \subset \mathbb{L}_W^2$ when X is formal, and $\mathbb{L}_W \simeq \pi_{*+1}(X)\otimes\mathbb{Q}$ in the coformal case.

Félix and Halperin proved in [12] that the rational LS-category of X is the smallest integer (or infinite) such that the projection

$$(\Lambda V, d) \to (\Lambda V/\Lambda^{\geq n+1}, d)$$

admits a retract. Dually, Sbaï defined in [34] the rational cocategory of X, denoted troughout this paper by $\mathrm{cocat}_{\mathbb{Q}}(X)$, to be the smallest integer (or infinite) such that the projection

$$(\mathbb{L}_W, \partial) \to (\mathbb{L}_W/\mathbb{L}_W^{\geq n+1}, \partial)$$

admits a retract. In particular we have:

- $\mathrm{cocat}_{\mathbb{Q}}(X) = 0$ if and only if X is contractible;

- $\mathrm{cocat}_{\mathbb{Q}}(X) = 1$ if and only if X has the rational homotopy type of Eilenberg-MacLane space;
- $\mathrm{cocat}_{\mathbb{Q}}(S^{2n+1}) = 1$ and $\mathrm{cocat}_{\mathbb{Q}}(S^{2n}) = 2$.

Given X, a topological space, $\mathrm{aut}(X)$ denotes the set of its self homotopy equivalences, that are maps $f : X \to X$ which admits a homotopy inverse (i.e., $\mathrm{aut}(X)$ is the set of automorphism of X in the pointed homotopy category). $\mathrm{aut}_1(X)$ denotes the identity path component. S. Theriault asked the following:

Conjecture 1.1 (Theriault Open Question). *Is it true that* $\mathrm{Hnil}(\mathrm{aut}_1(X))$ *is finite whenever* $\mathrm{cocat}(\mathrm{Baut}_1(X))$ *is?*

Here, $\mathrm{Hnil}(\mathrm{aut}_1(X))$ denotes the *homotopical nilpotency* of $aut_1(X)$, viewed as a connected grouplike space. That is the invariant defined by Berstein and Ganea (see [5]) to be the least integer n such that the $(n+1)$-th commutator c_{n+1} is nullhomotopic. Note that the iterated commutators $c_n : G^n \to G$ are inductively defined, using the homotopy inverse, as follows: c_1 is the identity, $c_2(a, b) := aba^{-1}b^{-1}$ and $c_n := c_2 \circ (c_{n-1}, c_1)$.

We will answer positively to this Theriault's question in a rational homotopy theory setting. More precisely we prove that:

Theorem 1.2. *Let X be a simply connected CW-complex of finite type. If* $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ *is finite, then* $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X))$ *is also. Moreover, we have*

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) \leqslant \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X)).$$

and that

Proposition 1.3. *Let X be a simply connected CW-complex of finite type, such that* $\mathrm{Baut}_1(X)$ *is coformal. If* $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ *is finite, then* $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X))$ *is also. Moreover, we have*

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) = \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X)).$$

The formal case will be discussed in

Proposition 1.4. *Let X be a simply connected CW-complex of finite type, such that* $\mathrm{Baut}_1(X)$ *is formal. If* $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ *is finite, then* $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X))$ *is also. Moreover, we have*

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) \leqslant 2.$$

The remaining of the paper is organized as follows: Section 2 is devoted to prove our main results (namely Theorem 1.2, Propositions 1.3 and 1.4). In section 3 we will discuss intersections with some other well known open problems, like the still open question if $\mathrm{Baut}_1(X)$ is a rational H-space (see Problem 3.2, [36]). By the end, we will ask, in term of fibrations, what about Theriault's conjecture?

2. Proofs

$\mathrm{aut}_1(X)$ and its classifying space $\mathrm{Baut}_1(X)$ play a crucial role in topology and geometry (Stasheff's classification for fibration over a given fiber [37], fake Lie groups [30], the homotopy type of the space of diffeomorphisms on a smooth manifold ([4]).

The respective Sullivan and Quillen minimal models of both $\mathrm{aut}_1(X)$ and $\mathrm{Baut}_1(X)$ are well and deeply described in terms of derivations (see [14] and [6]). Indeed, by a *derivation* of degree n on a CDGA (A, d), we mean any linear self map $\theta : A^* \to A^{*-n}$ (i.e., reducing degrees by n) satisfying

$$\theta(ab) = \theta(a)b - (-1)^{|\theta|.|a|} a\theta(b).$$

The graded space of all derivations on A, denoted throughout this paper by $\mathrm{Der}(A)$, has a DGLA structure. The commutator bracket and the differential are given by

$$\begin{array}{rcl} [\theta_1, \theta_2] & := & \theta_1 \circ \theta_2 - (-1)^{|\theta_1|.|\theta_2|} \theta_2 \circ \theta_1, \\ \mathcal{D}(\theta) & := & [d, \theta]. \end{array} \tag{1}$$

If $(\Lambda V, d)$ is a Sullivan minimal model of X, then (see §11 of [38]) that of $\mathrm{aut}_1(X)$ is given by :

$$\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}(\Lambda V); \mathcal{D}), \tag{2}$$

and the Quillen minimal model of $\mathrm{Baut}_1(X)$ is known to be DGLA-isomorphic to $\mathrm{Der}(\Lambda V)$. On the other hands, $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X))$ equals the length of the longest nontrivial bracket in $H_*(\mathrm{Der}(\Lambda V); \mathcal{D})$. Many computations of $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X))$ are done (see for example [33], [35], [24]).

Proof of Theorem 1.2. First, note that $\mathrm{aut}_1(X)$ is topologized with the open compact topology as a subspace of $\mathrm{map}(X, X)$. We know from (Proposition 2.2, [14]), that $\mathrm{aut}_1(X)$ has the homotopy type of a CW-complex and the H-homotopy type of a loop space. On the other hands, $\mathrm{aut}_1(X)$ is strictly a associative monoid and so admits a Dold-Lashof classifying space $\mathrm{Baut}_1(X)$. Moreover, we have ([11], Satz, 7.3)

$$\mathrm{aut}_1(X) \simeq \Omega \mathrm{Baut}_1(X). \tag{3}$$

$\mathrm{Baut}_1(X)$ is also simply connected, and hence rational homotopy machinery works well. In particular, $\mathrm{Baut}_1(X)$ admits a rationalization $\mathrm{Baut}_1(X)_{\mathbb{Q}}$.

On the other hands, it is well known that each grouplike structure on a connected space G induces on $\pi_*(G)$, a natural bilinear pairing $[-,-]$; the *Samelson product.* $(\pi_*(G) \otimes \mathbb{Q}, [-,-])$ is called the *Samelson Lie algebra* of G. If in addition, G is equipped with a multiplication, then (G, μ) admits a rationalization $(G_{\mathbb{Q}}, \mu_{\mathbb{Q}})$ which is also a grouplike.

The *rational homotopical nilpotency* $\mathrm{Hnil}_{\mathbb{Q}}(G)$ of $G = (G, \mu)$ is defined to be the homotopical nilpotency of $G_{\mathbb{Q}} = (G_{\mathbb{Q}}, \mu_{\mathbb{Q}})$. $\mathrm{Hnil}(\pi_*(G))$ denotes the usual nilpotency of the Samelson bracket $[-,-]$, while $\mathrm{WL}(X)$ denotes the longest Whitehead bracket in $\pi_*(X)$ and $\mathrm{WL}_{\mathbb{Q}}(X)$ that of its rationalization (i.e., $\mathrm{WL}_{\mathbb{Q}}(X) := \mathrm{WL}(X_{\mathbb{Q}})$).

All this invariants are well related in Proposition 2.3 of [14] which states that if G is a connected grouplike of CW loop space type (i.e., $G \simeq \Omega X$ for some simply connected CW-complex X), then

$$\mathrm{Hnil}_{\mathbb{Q}}(G) = \mathrm{Hnil}(\pi_*(G) \otimes \mathbb{Q}) = \mathrm{WL}_{\mathbb{Q}}(X). \tag{4}$$

Given a simply connected CW complex of finite type X, $\mathrm{cocat}_{\mathbb{Q}}(X)$ is defined in terms of Quillen models (see [34]) to be the least integer n (or infinity) such that the canonical projection $\pi_n : \mathbb{L}_W \to \mathbb{L}_W/\mathbb{L}_W^{\geq n+1}$ admits a homotopical retract. On of the main results of Sbaï is that

$$\mathrm{Hnil}(\pi_*(\Omega X) \otimes \mathbb{Q}) \leq \mathrm{cocat}_{\mathbb{Q}}(X), \quad \text{Theorem 11.5-[34]}. \tag{5}$$

Thus, from (3), (4) and (5), we deduce that

$$\begin{array}{rcl} \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X)) & \geq & \mathrm{Hnil}(\pi_*(\Omega \mathrm{Baut}_1(X)) \otimes \mathbb{Q}) \\ & = & \mathrm{Hnil}(\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q}) \\ & = & \mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) \end{array}. \tag{6}$$

□

Remark 2.1 (Proofs of Propositions 1.3 and 1.4).

- Following Theorem II.5 of [34], the equality holds in (5) when $\mathrm{Baut}_1(X)$ is coformal. Thus (6) gives $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) = \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ and this achieves the proof of Proposition 1.3.
- By Proposition III.1.6.2 and Corollaire III.1.7 of [34], we have $\mathrm{cocat}(\mathrm{Baut}_1(X)) = 2$, when $\mathrm{Baut}_1(X)$ is formal with a finite cocategory. Theorem 1.2 achieves the proof of Proposition 1.4.

3. Related results and open questions

The first open problem with what our results intersect nicely is that of the formality or coformality of $\mathrm{Baut}_1(X)$. When $\mathrm{Baut}_1(X)$ is of finite rational cocategory, then Proposition 1.3 states that the situation where $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) \neq \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ is an obstruction of the coformality of $\mathrm{Baut}_1(X)$, while the inequality $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) > 2$ can be considered (thanks to Proposition 1.4) as an obstruction of the formality of $\mathrm{Baut}_1(X)$. This agrees nicely with a Smith's result (Theorem 4.1, [36]) wherein the formality of $\mathrm{Baut}_1(X)$, viewed as an universal cover of $\mathrm{Baut}(X)$, is well and deeply studied.

Note also that H-spaces are formal, and that still yet opened a more general question, that if $\mathrm{Baut}_1(X)$ is a rational H-space (see Problem 3.2, [36]). Smith pointed out in Problem 3.2, [36], that $\mathrm{Baut}_1(G)$ is rarely H-space when G is a topological group. Indeed, we have

PROPOSITION 3.1. *If G is a topological group such that $\mathrm{Baut}_1(G)$ is a rational H-space with a finite rational cocategory, then*

$$\mathrm{Hnil}(G) \leq 2.$$

PROOF. Since $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(G))$ is finite, then (following Proposition 1.4, Examples 2.5 and 2.7 in [14] and Proposition 7 [33]) $\mathrm{Baut}_1(G)$ may be H-space only if $\mathrm{Hnil}(G) \leq 2$ (i.e. G homotopically trivial or abelian or its inner automorphism group is abelian). □

Class of 2-nilpotent groups have generated many interest among geometers (see [21]). For more detailed results on groups of rational homotopicall nilpotency 2, we refer the reader to both [19] and [20]).

Another basic and famous open problem on self homotopy equivalences wherewith our results intersect, is that asked some 50 years ago about the *realizability* of $\mathrm{aut}(X)$ (see [2], [25]). It was asked :

CONJECTURE 3.2 (Realizability Open Question). *For a given group G, is there any CW-complex X such that $\mathrm{aut}(X) \cong G$?.*

However, deeply studied in a lot of surveys ([1], [10], [25], [26], [32]), this longstanding open problem continues to give rise to many of research interests. A rational and light version was proposed by Arkowitz and Lupton in [3].

CONJECTURE 3.3 (Realizability Open Question, Rational version). *Given a finite group G, is there a rational 1-connected CW-complex X such that $\mathrm{aut}(X) \cong G$?*

A complete and positive answer was given in [9]. Our Theorem 1.2 combined whith that of Costoya-Viruel (Theorem 1.1-[9]) leads to the following :

PROPOSITION 3.4. *If G is a finite group, with a classifying space BG of a finite rational cocategory, then G is of finite rational homotopical nilpotency. Moreover we have*

$$\mathrm{Hnil}_{\mathbb{Q}}(G) \leq \mathrm{cocat}_{\mathbb{Q}}(BG).$$

Note that the problem of the realization has not been asked only of self homotopy equivalences but also for their classifying spaces. In fact, Schlessinger asked (see [13], page 519) :

CONJECTURE 3.5 (Realizability Open Question, $\mathrm{Baut}_1(X)$ version). *Which simply connected spaces Y have the rational homotopy type of some $\mathrm{Baut}_1(X)$?*

The affirmed cases known until now (see [27], [36], [28], [40]) converge to the fact that $\mathrm{Baut}_1(X)$ may have the rational homotopy type of a finite product of some Eilenberg-Mac Lane spaces.

Finally, and before closing this overview by posing the Theriault's question in terms of fibrations $p : X \to B$, let us recall that $\mathrm{aut}(p)$ denotes conventionally the monoid of all fibrewise self homotopy equivalences $f : X \to X$ satisfying $p \circ f = p$. $\mathrm{aut}_1(p)$ denotes the identity component. One can recover the precedent case of $\mathrm{aut}_1(X)$ by taking $B = *$.

Conjecture 3.6 (Theriault Open Question, Fibrations version). *Let $p : X \to B$ be a fibration of connected CW-complexes with a connected fibre F. Is it true that* $\mathrm{Hnil}(\mathrm{aut}_1(p))$ *is finite whenever* $\mathrm{cocat}(\mathrm{Baut}_1(p))$ *is?*

Theriault's question can also be asked for $\mathrm{aut}_{\#}(X)$ (resp. $\mathrm{aut}_{\star}(X)$); the set of self homotopy equivalences that induce the identity on $\pi_*(X)$ (resp. $H^*(X;\mathbb{Q})$). This fibrewise setting is interesting, because one can control not only the fibre F but also the fibration. For example, whenever X or B is finite, we have:

- $\mathrm{aut}(p)$ has the homotopy type of a CW-complex and the H-homotopy type of a loop-space (Proposition 2.2, [14]);
- $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(p)) = \mathrm{nil}(\pi_*(\mathrm{aut}_1(p)) \otimes \mathbb{Q}) \leq \mathrm{card}\{n \mid \pi_n(F) \otimes \mathbb{Q} = 0\}$ (Proposition 2.2-[14] and Theorem 5.2-[14]);
- Sullivan minimal model of $\mathrm{aut}_1(p)$ is given in Theorem 1-[14] by
$$\pi_*(\mathrm{aut}_1(p)) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}_{\Lambda V}(\Lambda V \otimes \Lambda W)).$$
Here $\Lambda V \to \Lambda V \otimes \Lambda W$ is the Koszul-Sullivan model of the fibration $p : X \to B$ and $\mathrm{Der}_{\Lambda V}(\Lambda V \otimes \Lambda W)$ denotes the space of derivation on $\Lambda V \otimes \Lambda W$ that vanish on ΛV;
- $\mathcal{D}\mathrm{er}_{\Lambda V}(\Lambda V \otimes \Lambda W)$ is the Quillen model for $\mathrm{Baut}_1(p)$ (see Theorem 1-[6]);
- nilpotency and localization of $\mathrm{aut}_{\#}(X)$ are well studied respectively in [8] and in [28];
- some interesting computations or bounds of Hnil are given for $\mathrm{aut}_1(p)$ and for $\mathrm{aut}_{\#}(X)$, respectively in [24] and in [15];
- some interesting interpretations of $\mathrm{Hnil}_{\mathbb{Q}}$ of both $\mathrm{Baut}_1(X), \mathrm{Baut}_{\#}(X)$ and $\mathrm{Baut}_{\star}(X)$ are discussed in the section 4 of [33];
- the open problem of the realizability of $\mathrm{aut}_{\#}(X)$ is partially resolved in [3] and in [31].

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4. References

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