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A comparison of logarithmic overconvergent de Rham-Witt and log-crystalline cohomology for projective smooth varieties with normal crossing divisor

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Abstract – In this note we derive for a smooth projective variety X with normal crossing divisor Z an integral comparison between the log-crystalline cohomology of the associated log-scheme and the logarithmic overconvergent de Rham-Witt cohomology defined by Matsuue. This extends our previous result that in the absence of a divisor Z the crystalline cohomology and overconvergent de Rham-Witt cohomology are canonically isomorphic.

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1. Introduction

Let X be a smooth variety over a perfect field k of characteristic $p > 0$, let Z be a normal crossing divisor on X and $U = X \setminus Z$. Recently, Matsuue [M] constructed an (overconvergent) de Rham-Witt complex for the log-scheme (X, Z) and compared its hypercohomology with the rigid cohomology of U resp. the overconvergent de Rham-Witt cohomology of U, defined in [D-L-Z]. If X is in addition projective it is reasonable to compare the hypercohomology of the complexes $W^\dagger\Omega_X(\log Z)$ and $W\Omega_X(\log Z)$. We can extend the integral comparison between overconvergent and crystalline cohomology obtained in [L-Z] for the usual de Rham-Witt complexes in absence of a divisor, to the log-scheme (X, Z) as follows:

Theorem 1. *If X/k is smooth and projective with normal crossing divisor Z, then the canonical map, induced by the inclusion $W^\dagger\Omega_X(\log Z) \to W\Omega_X(\log Z)$*

$$\mathbb{H}^i(X, W^\dagger\Omega_X(\log Z)) \to \mathbb{H}^i(X, W\Omega_X(\log Z))$$

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is an isomorphism of $W(k)$-modules of finite type for all $i \geq 0$.

2. Proof of the Theorem

Let X be a smooth quasiprojective variety over a perfect field k of characteristic $p > 0$. Let $Z \subset X$ be a normal crossing divisor. We denote by $W\Omega_{X/k}(\log Z)$ resp. $W^\dagger\Omega_{X/k}(\log Z)$ the de Rham-Witt complex with log-poles along Z, resp. the overconvergent de Rham-Witt complex with log-poles along Z, as defined in [M]. The following Lemma is in analogy to Lemma (2.1) in [L-Z] but it makes only sense if X projective smooth over k.

Lemma 2. *Let X be projective and smooth over k. The following diagram is commutative for all $i \geq 0$:*

(1)
$$\begin{array}{ccc} \mathbb{H}^i(X, W^\dagger\Omega_{X/k}(\log Z)\otimes\mathbb{Q}) & \longrightarrow & \mathbb{H}^i(X, W\Omega_{X/k}(\log Z)\otimes\mathbb{Q}) \\ {\scriptstyle \iota_3\,\cong}\downarrow & & \downarrow{\scriptstyle \cong\,\iota_1} \\ H^i_{rig}(U) & \underset{\iota_2}{\longleftarrow} & H^i_{logcrys}((X,Z)/W(k))\otimes\mathbb{Q} \end{array}$$

Here the upper horizontal map is induced by the inclusion $W^\dagger\Omega_{X/k}(\log Z) \to W\Omega_{X/k}(\log Z)$. The isomorphism ($\iota_1$) is defined in [M] Theorem 7.2 and the isomorphism (ι_3) is defined in [M] Theorem 10.14. The isomorphism (ι_2) is shown by Shiho [S].

Proof. We reformulate the Lemma in such a way that it makes sense if X is only quasiprojective. The maps ι_1 and ι_3 are defined for quasiprojective X and are functorial with respect to open immersions of pairs (X, Z).

In order to define ι_2 in the quasiprojective case we replace H_{rig} by a different "overconvergent" cohomology. We define this cohomology for pairs (X, Z) as above, which may be inserted into a diagram

$$\begin{array}{ccc} (X,Z) & \xrightarrow{\ i\ } & (P,L) \\ \downarrow & & \downarrow \\ (\bar{X},\bar{Z}) & \xrightarrow{\ \bar{i}\ } & (\bar{P},\bar{L}), \end{array}$$

where the first vertical arrow is a closed immersion in a pair $(\bar{X}, \bar{Z})$ with $\bar{X}$ smooth and projective and $\bar{Z}$ a normal crossing divisor such that $Z = X \cap \bar{Z}$. The horizontal arrows are closed immersions into formal log-smooth schemes (P, L) resp. $(\bar{P}, \bar{L})$ over $W(k)$ (see [K]).

We denote by (P_K, L_K) resp. $(\bar{P}_K, \bar{L}_K)$ the associated rigid log-varieties. We denote by $j : U \to X$ and $\bar{j} : U \to \bar{X}$ the open immersions. Then the rigid cohomology is given by

(2)
$$H^{\cdot}_{rig}(U) = \mathbb{H}^{\cdot}(\bar{j}^\dagger\Omega^{\cdot}_{]\bar{X}[^{log}_{\bar{P}_K}}).$$

Indeed, we note that for an exact closed immersion i (resp. $\bar{i}$) we have by definition $]X[^{log}_{P_K} =]X[_{P_K}$ (resp. $]\bar{X}[^{log}_{\bar{P}_K} =]\bar{X}[_{\bar{P}_K}$) (see [S] pp. 56-58). Therefore for $\bar{i}$ exact the right hand side of (2) is rigid cohomology. If we only assume that $\bar{\imath}$ is a closed immersion then by [K] 4.10 there exists a factorisation $\bar{\imath} = f \circ \check{\imath}$, where $\check{\imath}$ is an exact closed immersion into a formal log-smooth scheme $(\bar{P}', \bar{L}')$ and where f is formally log-étale. By definition $]\bar{X}[^{log}_{\bar{P}_K} =]\bar{X}[_{\bar{P}'_K}$ and therefore f induces a map $]\bar{X}[^{log}_{\bar{P}_K} \to]\bar{X}[_{\bar{P}_K}$. We can apply the claim in ([S], p. 114) to conclude that

$$H^{\cdot}_{rig}(U) = \mathbb{H}^{\cdot}(\bar{j}^{\dagger}\Omega^{\cdot}_{]\bar{X}[^{log}_{\bar{P}_K}}) = \mathbb{H}^{\cdot}(\bar{j}^{\dagger}\Omega^{\cdot}_{]\bar{X}[_{\bar{P}_K}}).$$

By restricting strict tubular neighbourhoods of $]U[_{\bar{P}_K}$ in $]\bar{X}[_{\bar{P}_K}$ to strict tubular neighbourhoods of $]U[_{P_K}$ in $]X[_{P_K}$ we obtain a canonical map

$$H^{\cdot}_{rig}(U) \to \mathbb{H}^{\cdot}(j^{\dagger}\omega^{\cdot}_{]X[^{log}_{P_K}}) = \mathbb{H}^{\cdot}(j^{\dagger}\Omega^{\cdot}_{]X[^{log}_{P_K}}) =: H^{\cdot}_{rig}(U, X) \tag{3}$$

If X is proper then the map (3) is an isomorphism: both are the rigid cohomology of U!

We have a natural map $\mathbb{H}^{\cdot}(\omega^{\cdot}_{]X[^{log}_{P_K}}) \to \mathbb{H}^{\cdot}(j^{\dagger}\omega^{\cdot}_{]X[^{log}_{P_K}})$ where this time the source of the arrow is log-analytic cohomology, which coincides with log-convergent ([S], Corollary 2.3.9) which coincides with log-crystalline cohomology ([S], Theorem 3.1.1). If we assume that X is proper and smooth (in our case this implies that X is projective) then the above maps are all isomorphic ([S], Theorem 2.4.4), yielding the isomorphism labelled ι_2 in the diagram (1).

The cohomology groups $H^{\cdot}_{rig}(U, X)$ defined in terms of an embedding $(X, Z) \to (P, L)$ are in fact independent of this embedding by the usual argument. Now we may write a diagram which makes sense for quasiprojective X.

$$\begin{array}{ccc} \mathbb{H}^i(X, W^{\dagger}\Omega_{X/k}(\log Z) \otimes \mathbb{Q}) & \longrightarrow & \mathbb{H}^i(X, W\Omega_{X/k}(\log Z) \otimes \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^{\cdot}_{rig}(U, X) & \longleftarrow & H^i_{\log-crys}((X, Z)/W(k)) \otimes \mathbb{Q}. \end{array} \tag{4}$$

Here the vertical map on the left hand side is obtained by composing the corresponding map of (1) with (3). We remark that the cohomology group $H^{\cdot}_{rig}(U, X)$ may be defined even in the case where the embedding $(X, Z) \to (P, L)$ exists only locally. This is shown by simplicial methods in the proof of [S] Theorem 2.4.4. Therefore the last diagram makes sense even if we can't choose $(X, Z) \to (P, L)$ globally. We note that all maps in the diagram are functoriel with respect to open immersions. Since (3) is an isomorphism for X projective and smooth the commutativity of the diagram (4) would imply the commutativity of (1).

Hence to prove the Lemma it suffices to show that (4) is commutative. By the Mayer-Vietoris sequence it is enough to prove the commutativity for $X = \operatorname{Spec} A$ affine and $Z = \{t_1 \cdot \ldots \cdot t_r = 0\}$ for a regular system of parameters $t_1, \ldots, t_r \in A$.

In this case we denote by $\tilde{A}$ a lifting of A to a smooth $W(k)$-algebra. The cohomology group $\mathbb{H}^{\cdot}(j^{\dagger}\omega^{\cdot}_{]X[^{log}_{P_K}})$ is the cohomology of the complex $\Omega_{(\hat{\tilde{A}})^{\dagger}_{\tilde{t}_1 \cdot \ldots \cdot \tilde{t}_r}}$.

But then the commutativity of (4) follows because we have a commutative diagram of complexes

$$\begin{tikzcd} \Omega_{\tilde{A}^\dagger}(\log \tilde{Z}) \arrow[r] \arrow[d] & \Omega_{\hat{\tilde{A}}}(\log \tilde{Z}) \arrow[dl] \\ \Omega_{(\hat{\tilde{A}})^\dagger_{\tilde{t}_1\cdots\tilde{t}_r}} . & \end{tikzcd}$$

□

Proposition 3. *Let X/k be smooth, Z a normal crossing divisor on X. We have quasi-isomorphisms of complexes*

$$W^\dagger\Omega^{\cdot}_{X/K}(\log Z)/p^n \cong W\Omega^{\cdot}_{X/k}(\log Z)/p^n \cong W_n\Omega^{\cdot}_{X/k}(\log Z)$$

Proof. In the absence of Z this was shown in [L-Z], Proposition 2.2. It suffices to show the Proposition locally in the Zariski-topology. So let $X = \operatorname{Spec} B$ be smooth and Z given by an equation $t_1 \cdots t_r = 0$ for $t_1, \ldots, t_r \in B$ such that $\operatorname{Spec} B/(t_i)$ is smooth for all i. Using Steenbrink's weight filtration [St] which can be defined on $W^\dagger\Omega_{X/k}(\log Z)$, resp. $W\Omega_{X/k}(\log Z)$ resp. $W_n\Omega^{\cdot}_{X/k}(\log Z)$ as in [M], §10.2, one applies [M] Lemmas 8.4 and 10.8 to get three residue isomorphisms

$$Res : \mathrm{Gr}_j\, W^\dagger\Omega^{\cdot}_{B/k}(\log Z) \xrightarrow{\sim} \bigoplus_{\substack{c|J|=j \\ J=(\alpha_1,\ldots,\alpha_j)}} W^\dagger\Omega^{\cdot-j}_{B/(t_{\alpha_1},\ldots,t_{\alpha_j})},$$

$$Res : \mathrm{Gr}_j\, W\Omega^{\cdot}_{B/k}(\log Z) \xrightarrow{\sim} \bigoplus_{\substack{c|J|=j \\ J=(\alpha_1,\ldots,\alpha_j)}} W\Omega^{\cdot-j}_{B/(t_{\alpha_1},\ldots,t_{\alpha_j})}$$

$$Res : \mathrm{Gr}_j\, W_n\Omega^{\cdot}_{B/k}(\log Z) \xrightarrow{\sim} \bigoplus_{\substack{c|J|=j \\ J=(\alpha_1,\ldots,\alpha_j)}} W_n\Omega^{\cdot-j}_{B/(t_{\alpha_1},\ldots,t_{\alpha_j})}$$

on the graded quotients of the weight filtrations.

Since all $B/(t_{\alpha_1}, \ldots, t_{\alpha_j})$ are smooth, the entries in the overconvergent and usual de Rham-Witt complexes are p-torsion free. We have exact sequences, using the weight filtration of complexes

$$0 \to P_j W^\dagger\Omega_{X/k}(\log Z) \to P_{j+1}W^\dagger\Omega_{X/k}(\log Z) \to \mathrm{Gr}_j\, W^\dagger\Omega_{X/k}(\log Z) \to 0$$

$$0 \to P_j W\Omega_{X/k}(\log Z) \to P_{j+1}W\Omega_{X/k}(\log Z) \to \mathrm{Gr}_j\, W\Omega_{X/k}(\log Z) \to 0.$$

By the residue isomorphism these exact sequences of complexes remain exact after tensoring with $\mathbb{Z}/p^n$. Moreover, by the residue isomorphism and [L-Z] Prop. 2.2 we see that $\mathrm{Gr}_j\, W^\dagger\Omega_{X/k}(\log Z)/p^n$ and $\mathrm{Gr}_j\, W\Omega_{X/k}(\log Z)/p^n$ are quasi-isomorphic complexes. By induction this shows that $W^\dagger\Omega_{X/k}(\log Z)/p^n$ and $W\Omega_{X/k}(\log Z)/p^n$ are quasi-isomorphic too. Since $W\Omega^{\cdot}_{C/k}/p^n$ is quasi-isomorphic to $W_n\Omega^{\cdot}_{C/k}$ for any smooth algebra C/k by [I] 3.17.3 we can apply the residue isomorphism and induction on the weight filtration again to obtain a quasi-isomorphism

$$W\Omega_{X/k}(\log Z)/p^n \cong W_n\Omega_{X/k}(\log Z).$$

This proves Proposition 3. □

We can now finish the proof of the Theorem. Using $W^\dagger\Omega_{X/k}(\log Z)$ instead of $W^\dagger\Omega_{X/k}$ and $W\Omega_{X/k}(\log Z)$ instead of $W\Omega_{X/k}$ and noting that by Proposition 3:

$$\begin{aligned}\varprojlim \mathbb{H}^i\left(X, W^\dagger\Omega_{X/k}(\log Z)[/p^n]\right) &= \varprojlim \mathbb{H}^i\left(X, W_n\Omega_{X/k}(\log Z)\right)\\ &= \mathbb{H}^i\left(X, W\Omega_{X/k}(\log Z)\right),\end{aligned}$$

where the last equality holds because all $\mathbb{H}^i(X, W_n\Omega_{X/k}(\log Z))$ are of finite length as X is proper, one can apply the arguments in [L-Z] on page 1392, because

$$\mathbb{H}^i\left(X, W\Omega_{X/k}(\log Z)\right) \cong H^i_{\log -crys}\left((X,Z)/W(k)\right)$$

is a $W(k)$-module of finite type. This finishes the proof of the Theorem.

Remark: Let $Z_1, \ldots, Z_d$ be irreducible components of the normal crossing divisor Z; let $Z_J = Z_{j_1} \cap \cdots \cap Z_{j_l}$ for $J = \{j_1, \ldots, j_l\} \subset \{1, \ldots, j\}$ and let $Z^{(l)} = \coprod_{|J|=l} Z_J$ for all $l \geq 0$.

Then the above residue isomorphisms which also hold globally ([M], §8.2) give rise to p-adic weight spectral sequences

$$E_1^{-l,h+l} = \mathbb{H}^{h-l}_{crys}\left(Z^{(l)}/W(k)\right)(-l) \Longrightarrow H^h_{\log -crys}\left((X,Z)/W(k)\right)$$

and

$$\check{E}_1^{-l,h+l} = \mathbb{H}^{h-l}\left(W^\dagger\Omega^\cdot_{Z^{(l)}/k}\right)(-l) \Longrightarrow \mathbb{H}^h\left(W^\dagger\Omega^\cdot_X(\log Z)\right).$$

By ([M], Theorem 9.1) and ([N], Theorem 5.2) the first spectral sequence degenerates at E_2 after tensoring with K. As it is not known whether the p-adic weight spectral sequence degenerates before tensoring with K, we cannot deduce the main Theorem from the result in the case without log structure as shown in [L-Z] by using the weight spectral sequence.

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