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On α-minimizing hypercones

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Abstract – In this paper we considerably extend the class of known α-minimizing hypercones using sub-calibration methods. Indeed, the improvement of previous results follows from a careful analysis of special cubic and quartic polynomials.

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1. Introduction

Let P_0 and P_1 be two distinct points in $\mathbb{R}\times\mathbb{R}_{\geq 0}$ and consider for $\alpha \geq 0$ the variational problem

$$\int y^\alpha \; d\mathscr{H}^1(x,y) \;\to\; \min$$

within the class

$$\mathscr{C} := \{\, \mathfrak{K}\colon [0,1] \to \mathbb{R}\times\mathbb{R}_{\geq 0} \;\text{ Lipschitz s.t. }\; \mathfrak{K}(0) = P_0, \mathfrak{K}(1) = P_1 \,\}.$$

Hence, with $\alpha = 0$ we are looking for the shortest curve joining P_0 and P_1, with $\alpha = \frac{1}{2}$ we gain a parametric version of the brachistochrone-problem, and the case $\alpha = 1$ leads to rotationally symmetric minimal surfaces in $\mathbb{R}^3$. On the other hand, the variational integral with $\alpha = 1$ appears when considering the potential energy of heavy chains.

Of course, the shortest path between P_0 and P_1 is a line, and the minimizing curve in the case $\alpha = \frac{1}{2}$ was named brachistochrone. However, the variational problem with $\alpha = 1$ may possess two distinct minimizers, namely a catenary and a Goldschmidt curve, which consists of three straight lines, cf. [11, ch. 8 sec. 4.3].

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In order to prove the minimality of the above mentioned curves it is sufficient to embed the corresponding curve into a field of extremals[1], i.e. into a foliation of extremal curves, cf. [10, ch. 6 sec. 2.3]. In fact, this can be directly justified by the divergence theorem. For this purpose let us look at the vector field

$$\xi(x,y) \coloneqq y^\alpha \cdot \nu(x,y),$$

where $\nu(x,y)$ are the normal fields orienting the curves from the foliation. Since all these curves are extremals, the vector field ξ is divergence-free. The conclusion then follows by applying the divergence theorem to the vector field ξ on the open set which is bounded by a critical curve and a comparison curve. In geometric measure theory setting, the critical curve is said being calibrated by ξ, and the vector field ξ is called *calibration*.[2]

In this paper we consider the higher dimensional variational problem and prove the minimizing property of special hypercones. Therefor we will construct suitable foliations. The crux hereby is to find an auxiliary function whose level sets are extremals.

First, we will weaken our considerations and look at "inner" and "outer" variations separately as in [5]. This gives simplified proofs and yields sub-solutions and sub-calibrations. The advantage of this weakened ansatz is that we can gain specific auxiliary functions. Moreover, we will show that a careful analysis of extremals as in [4] provides better results to our variational problem but loses the concrete representation of an auxiliary function.

1.1 – *The main result*

Let $m \in \{2,3,\dots\}$ and let $\mathcal{M}$ be an oriented Lipschitz-hypersurface in $\mathbb{R}^m \times \mathbb{R}_{\geq 0}$. Its *$\alpha$-energy* is given by

$$\mathcal{E}_\alpha(\mathcal{M}) \coloneqq \int_{\mathcal{M}} y^\alpha d\mathscr{H}^m(z), \tag{1}$$

where we use the notation $z \coloneqq (x,y) \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ and denote by $\mathscr{H}^m$ the m-dimensional Hausdorff measure. We show

[1] An argument which goes back to Weierstrass.

[2] Such method of conclusion is applicable even in a more general context and is well-known as Federer's differential form argument, cf. [9, 5.4.19].

THEOREM 1.1. *There exists an algebraic number $\alpha_m > \frac{2}{m}$ such that the cone*

$$\mathcal{C}_m^\alpha := \left\{ 0 \le y \le \sqrt{\frac{\alpha}{m-1}} \cdot |x| \right\}, \qquad \textit{with arbitrary } \alpha \ge \alpha_m,$$

is a local α-perimeter minimizer in $\mathbb{R}^m \times \mathbb{R}_{\ge 0}$.

REMARK 1.2. For α an integer, our result is equivalent to the area-minimizing property of the corresponding rotated cones in $\mathbb{R}^{m+\alpha+1}$. Indeed, with our lower bounds presented in rem. 1.5 we recover the area-minimizing property of all Lawson's cones, i.e. of the cones

$$C_{k,h} := \{ (x,y) \in \mathbb{R}^k \times \mathbb{R}^h \mid (h-1)|x|^2 = (k-1)|y|^2 \}$$

with $k, h \ge 2$ and $k+h \ge 9$ or $(k,h) \in \{(3,5),(5,3),(4,4)\}$, cf. [2, 14, 17], where k and h take over the parts of m and $\alpha+1$. For further reading on area-minimizing cones, see also [13] and the references contained therein.

REMARK 1.3. Following the minimal surfaces theory we will introduce the terminology of a *local α-perimeter minimizer* in the next section. Alternatively, we could say in theorem 1.1 that the hypercone

$$\mathcal{M}_m^\alpha := \partial \mathcal{C}_m^\alpha = \{ \sqrt{m-1} \cdot y = \sqrt{\alpha} \cdot |x| \}, \qquad \text{with arbitrary } \alpha \ge \alpha_m,$$

is α-minimizing in $\mathbb{R}^m \times \mathbb{R}_{\ge 0}$, where the boundary of $\mathcal{C}_m^\alpha$ is seen with respect to the induced topology.

REMARK 1.4. In our proof, we will specify polynomials $\boldsymbol{p_m}$ which characterize the corresponding α_m as the unique positive root. Moreover, we show $\alpha_m < \frac{12}{m}$, thus $\alpha_m \to 0$ with $m \to \infty$.

REMARK 1.5. First (integer) bounds can be found in [6], namely

$$\alpha_2 = 11,\ \alpha_3 = 6,\ \alpha_4 = \alpha_5 = \alpha_6 = 3,\ \alpha_7 = \cdots = \alpha_{11} = 2,\ \alpha_m = 1 \text{ for } m \ge 12.$$

Shortly thereafter, they were corrected in [7] to

$$\alpha_2 = 6, \quad \alpha_3 = 4, \quad \alpha_4 = 3, \quad \alpha_5 = \alpha_6 = 2, \quad \alpha_m = 1 \quad \text{for} \quad m \ge 7.$$

Our investigations show, that they can be improved to

$$\begin{array}{lll}
\alpha_2 \approx 5.881525129 & \alpha_7 \approx 0.963594772 & \alpha_{12} \approx 0.357996307 \\
\alpha_3 \approx 3.958758640 & \alpha_8 \approx 0.728989161 & \alpha_{13} \approx 0.317117533 \\
\alpha_4 \approx 2.829350458 & \alpha_9 \approx 0.581153278 & \ldots \\
\alpha_5 \approx 1.969224627 & \alpha_{10} \approx 0.481712568 & \alpha_{2017} \approx 0.001377480 \\
\alpha_6 \approx 1.352500103 & \alpha_{11} \approx 0.410855526 & \ldots
\end{array}$$

REMARK 1.6. For all $m = 2, 3, \dots$ we have $m+\alpha_m \geq 4+\sqrt{8}$, cf. remark 4.3, so, direct calculations yield that all hypercones $\mathcal{M}_m^\alpha$, with $\alpha \geq \alpha_m$, are (of course) $\mathcal{E}_\alpha$-stable, see also [8, p. 168].

REMARK 1.7. Although $\mathcal{M}_2^5$ is $\mathcal{E}_5$-stable, the corresponding cone $\mathcal{C}_2^5$ is not a (local) 5-perimeter minimizer in $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. Similarly, the hypercone $\mathcal{M}_6^1$ is $\mathcal{E}_1$-stable, but the cone $\mathcal{C}_6^1$ does not minimize the 1-perimeter in $\mathbb{R}^6 \times \mathbb{R}_{\geq 0}$, cf. [7]. Hence, the optimality question of our α_m's still remains open.

2. Notations and preliminary results

Let $\Omega \subseteq \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ be open (with respect to the induced topology) and let $\alpha > 0$. We say that $f \in BV^\alpha(\Omega)$ if $f \in L_1(\Omega)$ and the quantity

$$\int_\Omega y^\alpha |Df| \coloneqq \sup \left\{ \int_\Omega f(z) \operatorname{div}(\psi(z)) dz : \psi \in C_c^1(\Omega, \mathbb{R}^{m+1}), |\psi(z)| \leq y^\alpha \right\}$$

is finite. For a Lebesgue measurable set $E \subseteq \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ we call

$$\mathcal{P}_\alpha(E; \Omega) \coloneqq \int_\Omega y^\alpha |D\chi_E|$$

the *α-perimeter of E in Ω*. Furthermore, we call E an *α-Caccioppoli set in Ω* if E has a locally finite α-perimeter in Ω, i.e. $\chi_E \in BV_{loc}^\alpha(\Omega)$.

EXAMPLE 2.1. By the divergence theorem, if $E \subseteq \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ is an open set with regular boundary, then

$$\mathcal{P}_\alpha(E; \Omega) = \mathcal{E}_\alpha(\partial E \cap \Omega)$$

for all open sets Ω.

REMARK 2.2. Of course, several properties of the α-perimeter can be directly transferred from the known properties of the perimeter, cf. [12, 15].

REMARK 2.3. Note that there are α-Caccioppoli sets which are not Caccioppoli, i.e. do not possess a locally finite perimeter: In an arbitrary neighborhood of the origin consider the set

$$A \coloneqq \bigcup_{n=0}^\infty A_n,$$

where A_n is a triangle with vertices

$$\left(\tfrac{1}{2^{n+1}}, 0\right), \quad \left(\tfrac{1}{2^{n}}, 0\right) \text{ and } \left(\tfrac{3}{2^{n+2}}, \sqrt{\tfrac{1}{4(n+1)^2} - \tfrac{1}{2^{2n+4}}}\right).$$

Hereby, the A_n are chosen in such a way that

$$\left|\partial A_n \cap \left(\mathbb{R} \times \mathbb{R}_{>0}\right)\right| = \frac{1}{n+1}.$$

On the other hand, the α-perimeter of A is dominated by the convergent series

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha+1)(n+1)} \left(\frac{1}{4(n+1)^2} - \frac{1}{2^{2n+4}}\right)^{\alpha/2}.$$

Definition 2.4. Let E be an α-Caccioppoli set in Ω. We say that E is a *local α-perimeter minimizer in* Ω if in all bounded open sets $B \subseteq \Omega$ we have

$$\mathcal{P}_\alpha(E; B) \leq \mathcal{P}_\alpha(F; B) \quad \text{for all } F \text{ such that } F \triangle E \subset\subset B.$$

2.1 – *Under weakened conditions*

The following definitions and results are analogous to the observations in [5, sec. 1]. We only prove one proposition, which was not used in [5].

Definition 2.5. Let E be an α-Caccioppoli set in Ω. We say that E is a *local α-perimeter sub-minimizer in* Ω if in all bounded open sets $B \subseteq \Omega$ we have

$$\mathcal{P}_\alpha(E; B) \leq \mathcal{P}_\alpha(F; B) \quad \text{for all } F \subseteq E \text{ such that } E \backslash F \subset\subset B.$$

The connection with minimizers is given by

Proposition 2.6. *E is a local α-perimeter minimizer in Ω if and only if E as well as $\Omega \backslash E$ is a local α-perimeter sub-minimizer in Ω.*

The lower semicontinuity of the α-perimeter implies

Proposition 2.7. *Let $\{E_k\}_{k\in\mathbb{N}}$ and E be α-Caccioppoli sets in Ω with $E_k \subseteq E$ and suppose that E_k locally converge to E in Ω. If all E_k's are local α-perimeter sub-minimizers in Ω, then E is a local α-perimeter sub-minimizer in Ω as well.*

Furthermore, the existence of a so called sub-calibration ensures the sub-minimality.

DEFINITION 2.8. Let $E \subseteq \Omega$ be an α-Caccioppoli set in Ω with $\partial E \cap \Omega \in C^2$. We call a vector field $\xi \in C^1(\Omega, \mathbb{R}^{m+1})$ an *α-sub-calibration of E in Ω* if it fulfills

(i) $|\xi(z)| \leq y^\alpha$ for all $z \in \Omega$,

(ii) $\xi(z) = y^\alpha \cdot \nu_E(z)$ for all $z \in \partial E \cap \Omega$,

(iii) $\operatorname{div} \xi(z) \leq 0$ for all $z \in \Omega$,

where ν_E denotes the exterior unit normal vector field on ∂E.[3]

PROPOSITION 2.9. *If ξ is an α-sub-calibration of E in an open set $\mathcal{O} \subseteq \Omega$, then E is a local α-perimeter sub-minimizer in all Ω.*

Note that it suffices to find a sub-calibration on a subset of Ω which contains E since we only deal with inner deformations. Finally, we add

PROPOSITION 2.10. *If the cone $\mathcal{C}_m^\alpha$ is a local α-perimeter sub-minimizer in $\mathbb{R}^m \times \mathbb{R}_{>0} \backslash \{x = 0\}$, then $\mathcal{C}_m^\alpha$ is also a local α-perimeter sub-minimizer in the whole $\mathbb{R}^m \times \mathbb{R}_{\geq 0}$.*

PROOF. Firstly, we have for a bounded open set $\widetilde{B} \subset \mathbb{R}^m \times \mathbb{R}_{\geq 0}$:

$$\mathcal{P}_\alpha(\mathcal{C}_m^\alpha; \widetilde{B}) \leq \mathcal{P}_\alpha(F; \widetilde{B})$$

for all $F \subseteq \mathcal{C}_m^\alpha$ such that $\mathcal{C}_m^\alpha \backslash F \subset\subset \widetilde{B} \backslash \{x = 0 \vee y = 0\}$. Let now be $\widetilde{F} \subseteq \mathcal{C}_m^\alpha$ with $\mathcal{C}_m^\alpha \backslash \widetilde{F} \subset\subset \widetilde{B}$. For $\varepsilon > 0$ we consider the set

$$\widetilde{F}_\varepsilon := \widetilde{F} \cup \left(\mathcal{C}_m^\alpha \cap \{ |x| < \varepsilon \ \vee y < \varepsilon \}\right).$$

Hence,

$$\mathcal{C}_m^\alpha \backslash \widetilde{F}_\varepsilon \subset\subset \widetilde{B} \backslash \{x = 0 \vee y = 0\},$$

thus with the preliminary observation we have

$$\begin{aligned}\mathcal{P}_\alpha(\mathcal{C}_m^\alpha; \widetilde{B}) &\leq \mathcal{P}_\alpha(\widetilde{F}_\varepsilon; \widetilde{B}) \\ &\leq \mathcal{P}_\alpha(\widetilde{F}; \widetilde{B}) + c_1(m, \alpha, \widetilde{B}) \cdot \{\varepsilon^{m+\alpha} + \varepsilon^\alpha + \varepsilon^{m-1}\} \\ &\xrightarrow{\varepsilon \searrow 0} \mathcal{P}_\alpha(\widetilde{F}; \widetilde{B}).\end{aligned}$$

□

[3] Note that, in contrast to [4], our vector field has been weighted.

3. First proof of theorem 1.1

Arguing in this section as in [5] we give a first proof of theorem 1.1. Unfortunately, this does not lead to our best bounds, but gives the α_m's as constructible numbers. This study is based on the analysis of the cubic polynomial

$$\mathbf{Q}_{m,\alpha}(t) := (m-1)^4 t^3 - 3(m-1)^2 \alpha t^2 - 3(m-1)\alpha^2 t + \alpha^4.$$

LEMMA 3.1. *For all* $\alpha \geq \dfrac{2m^{3/2} + 3m - 1}{(m-1)^2}$, *we have*

$$\mathbf{Q}_{m,\alpha}(t) \geq 0 \quad \textit{for all } t \geq 0.$$

PROOF. For all admissible $m \in \{2, 3, \dots\}$ and $\alpha > 0$, the polynomial $\mathbf{Q}_{m,\alpha}(-t)$ has one sign change in the sequence of its coefficients

$$-(m-1)^4, \quad -3(m-1)^2\alpha, \quad 3(m-1)\alpha^2, \quad \alpha^4.$$

Hence, due to Descartes' rule of signs, $\mathbf{Q}_{m,\alpha}$ always has one negative root. On the other hand, $\mathbf{Q}_{m,\alpha}$ has none, a double or two distinct positive roots.

The number of real roots of the cubic polynomial $\mathbf{Q}_{m,\alpha}$ is determined by its discriminant

$$\vartheta = -27(m-1)^6 \alpha^6 \cdot \{(m-1)^2\alpha^2 - (6m-2)\alpha + 1 - 4m\}.$$

Summarizing, we have:

i) If $\vartheta > 0$, then $\mathbf{Q}_{m,\alpha}$ has one negative and two distinct positive roots.

ii) If $\vartheta = 0$, then $\mathbf{Q}_{m,\alpha}$ has one negative and a double positive root.

iii) If $\vartheta < 0$, then $\mathbf{Q}_{m,\alpha}$ has only one negative root.

The statement of the lemma then follows since $-\vartheta$ has the same sign as the quadratic polynomial

$$\mathbf{q}_m(\alpha) := (m-1)^2\alpha^2 - (6m-2)\alpha + 1 - 4m,$$

whose sole positive root is

$$\alpha = \frac{2m^{3/2} + 3m - 1}{(m-1)^2}. \qquad \square$$

Proof of theorem 1.1 (with concrete bounds). We consider over $\mathbb{R}^m \times \mathbb{R}_{\geq 0}$ the function

$$F_{m,\alpha}(z) \coloneqq \frac{1}{4}\left\{\alpha^2|x|^4 - (m-1)^2 y^4\right\}.$$

It is

$$\nabla F_{m,\alpha}(z) = \big(\alpha^2|x|^2 x \;\;,\;\; -(m-1)^2 y^3\big).$$

Moreover, on $\{\, |\nabla F_{m,\alpha}| \neq 0 \,\}$ we have:

$$\nabla_x \; \frac{|x|^2}{|\nabla F_{m,\alpha}|} = \frac{2x}{|\nabla F_{m,\alpha}|} - \frac{3\alpha^4|x|^6 x}{|\nabla F_{m,\alpha}|^3}$$

and

$$\frac{\partial}{\partial y}\; \frac{1}{|\nabla F_{m,\alpha}|} = -\frac{3(m-1)^4 y^5}{|\nabla F_{m,\alpha}|^3}.$$

Hence,

$$\begin{aligned}
\operatorname{div}\left(-y^\alpha \frac{\nabla F_{m,\alpha}}{|\nabla F_{m,\alpha}|}\right) &= -\big\langle \nabla_x \;,\; \frac{y^\alpha \alpha^2 |x|^2 x}{|\nabla F_{m,\alpha}|}\big\rangle + \frac{\partial}{\partial y}\;\frac{(m-1)^2 y^{\alpha+3}}{|\nabla F_{m,\alpha}|} \\
&= -m\frac{y^\alpha\alpha^2|x|^2}{|\nabla F_{m,\alpha}|} - \big\langle x, \nabla_x\; \frac{y^\alpha\alpha^2|x|^2}{|\nabla F_{m,\alpha}|}\big\rangle \\
&\quad + \frac{(m-1)^2(\alpha+3)y^{\alpha+2}}{|\nabla F_{m,\alpha}|} + (m-1)^2 y^{\alpha+3}\frac{\partial}{\partial y}\;\frac{1}{|\nabla F_{m,\alpha}|} \\
&= |\nabla F_{m,\alpha}|^{-3}\{-(m-1)\alpha^6 y^\alpha |x|^8 - (m-1)^4(m+2)\alpha^2 y^{\alpha+6}|x|^2 \\
&\qquad + (m-1)^2(\alpha+3)\alpha^4 y^{\alpha+2}|x|^6 + (m-1)^6 \alpha y^{\alpha+8}\} \\
&= -|\nabla F_{m,\alpha}|^{-3}(m-1)\alpha y^\alpha |x|^6\, \mathbf{Q}_{m,\alpha}\left(\frac{y^2}{|x|^2}\right)\{\alpha|x|^2 - (m-1)y^2\}.
\end{aligned}$$

For $k \in \mathbb{N}$ consider the sets

$$E_k \coloneqq \left\{ z \in \mathbb{R}^m \times \mathbb{R}_{\geq 0} : F_{m,\alpha}(z) \geq \frac{1}{k} \right\} \subset \mathcal{C}_m^\alpha.$$

They all are α-Caccioppoli sets in $\mathbb{R}^m \times \mathbb{R}_{>0}\backslash\{x=0\}$ since

$$F_{m,\alpha} \in C^2\big(\big(\mathbb{R}^m \times \mathbb{R}_{>0}\backslash\{\, z = 0 \,\}\big)\backslash\mathcal{M}_m^\alpha\big),$$

whereby $\mathcal{M}_m^\alpha = \partial\mathcal{C}_m^\alpha = \{\, F_{m,\alpha} = 0 \,\}$. Furthermore, the E_k's locally converge to $\mathcal{C}_m^\alpha = \{\, F_{m,\alpha} \geq 0 \,\}$.

With lemma 3.1 we have

$$\mathbf{Q}_{m,\alpha}\left(\frac{y^2}{|x|^2}\right) \geq 0 \quad \text{for all} \quad x \neq 0, \ \ y \geq 0, \ \text{and for all } \alpha \geq \frac{2m^{3/2}+3m-1}{(m-1)^2}$$

consequently, due to the above computation of the divergence, the vector filed

$$\xi_+(z) := -y^\alpha \frac{\nabla F_{m,\alpha}(z)}{|\nabla F_{m,\alpha}(z)|}$$

is an α-sub-calibration for each E_k in $\{0 < \sqrt{m-1}\, y < \sqrt{\alpha}\,|x|\}$.

Hence, propositions 2.9, 2.7 and 2.10 ensure that $\mathcal{C}_m^\alpha$ is a local α-perimeter sub-minimizer in the whole $\mathbb{R}^m \times \mathbb{R}_{\geq 0}$.

In view of the characterization of α-perimeter minimizing sets, cf. proposition 2.6, the claim of theorem 1.1 follows for

$$\alpha \geq \frac{2m^{3/2}+3m-1}{(m-1)^2},$$

after proving the sub-minimality of the complement of $\mathcal{C}_m^\alpha$. We therefor argue as above considering the sets

$$D_k := \left\{ z \in \mathbb{R}^m \times \mathbb{R}_{\geq 0} : F_{m,\alpha}(z) \leq -\frac{1}{k} \right\}$$

and the vector field

$$\xi_-(z) := y^\alpha \frac{\nabla F_{m,\alpha}(z)}{|\nabla F_{m,\alpha}(z)|} \quad \text{on} \quad \{F_{m,\alpha} < 0\}.$$ □

Remark 3.2. All previous computations were carried out by hand.

Remark 3.3. For $m \geq 14$ we have $\frac{2m^{3/2}+3m-1}{(m-1)^2} > \frac{12}{m}$ and $\frac{12}{m}$ is an upper bound for our best α_m's.

Remark 3.4. Improvements of these bounds can be achieved by an alternative auxiliary function. As seen in the proof, such a function F should fulfill the following conditions

1. $F \in C^2\big((\mathbb{R}^m \times \mathbb{R}_{>0} \backslash \{\, x = 0 \,\}) \backslash \mathcal{M}_m^\alpha\big) \cap C^0(\mathbb{R}^m \times \mathbb{R}_{\geq 0})$,
2. $\{\, F \geq 0 \,\} = \mathcal{C}_m^\alpha, \quad \{\, F = 0 \,\} = \partial \mathcal{C}_m^\alpha = \mathcal{M}_m^\alpha$,
3. $F \cdot \operatorname{div}\left(-y^\alpha \frac{\nabla F}{|\nabla F|}\right) \leq 0 \quad \text{in } \{\, \nabla F \neq 0 \,\}$.

Remark 3.5. In fact, corresponding auxiliary functions can be found in papers concerning the minimizing property of Lawson's cones, namely

- in [16]: $F(x,y) = \big(|x|^2 - |y|^2\big)\big(|x|^2 + |y|^2\big)$, for $k = h = 4$.
- in [3]:

$$\begin{aligned} F(x,y) = &\big((h-1)|x|^2 - (k-1)|y|^2\big) \\ &\cdot \big((5k-h-4)(h-1)|x|^2 - (5h-k-4)(k-1)|y|^2\big), \end{aligned}$$

  for $k+4 < 5h$ and $(k,h) \neq (3,5)$, and for $h+4 < 5k$ and $(k,h) \neq (5,3)$.

- in [1]:

$$F(x,y) = \big((h-1)|x|^2 - (k-1)|y|^2\big) \cdot \begin{cases} \big((h-1)|x|^2\big)^{\beta}, & \text{in "}\{F>0\}\text{"}, \\ \big((k-1)|y|^2\big)^{\beta}, & \text{in "}\{F<0\}\text{"}, \end{cases}$$

  where β was chosen in a way, that such an argumentation was admissible for all Lawson's cones.

- in [5]: $F(x,y) = \frac{1}{4}\big(|x|^2 - |y|^2\big)\big(|x|^2 + |y|^2\big)$, for $k = h \geq 4$.

Note that

- in [3, 1] computer algebra systems were used to perform the symbolic manipulations.
- the argumentation using sub-calibration method from [5] is applicable to the function

$$F(x,y) = \frac{1}{4}\big((h-1)|x|^2 - (k-1)|y|^2\big)\big((h-1)|x|^2 + (k-1)|y|^2\big)$$

  and yields the minimality of all Lawson's cones with

$$\begin{aligned} (k,h) \notin \{&(2,7),(2,8),(2,9),(2,10),(2,11),(3,5), \\ &(5,3),(7,2),(8,2),(9,2),(10,2),(11,2)\}. \end{aligned}$$

  However, we have already performed such computations above and the exceptional cases correspond to the given bounds in lemma 3.1 for integer values, where k and h take over the parts of m and $\alpha + 1$.

Remark 3.6. With the aid of a suitable parametrization Davini detected the existence of an auxiliary function which was applicable to all Lawson's cones. All his computations he carried out by hand, cf. [4].

4. Second proof of theorem 1.1 with better bounds

Since the hypercones $\mathcal{M}_m^\alpha = \partial\mathcal{C}_m^\alpha$ are invariant under the action of $SO(m)$ on the first m components, we will look for a foliation consisting of extremal hypersurfaces with the same type of symmetry. In fact, recalling (1), a dimension reduction and the special parametrization[4]

$$\begin{cases} |x| &= \mathrm{e}^{v(t)} \cdot \cos t, \\ y &= \mathrm{e}^{v(t)} \cdot \sin t, \end{cases} \tag{2}$$

with $v \in C^2(0, \frac{\pi}{2})$ yields as Euler-Lagrange equation

$$\ddot{v} = \left(1 + \dot{v}^2\right) \cdot \left\{m + \alpha + \frac{m - \alpha - 1 - (m + \alpha - 1)\cos(2t)}{\sin(2t)} \cdot \dot{v}\right\}, \tag{3}$$

cf. [4], where m and α take over the parts of k and $h - 1$.

Hence, with $w := \dot{v}$ the initial problem reduces to a question about the behavior of solutions of the following ordinary differential equation of first order:

$$\dot{w} = \left(1 + w^2\right) \cdot \left\{m + \alpha + \frac{m - \alpha - 1 - (m + \alpha - 1)\cos(2t)}{\sin(2t)} \cdot w\right\}. \tag{4}$$

The existence of a solution follows, for example, from the existence of an upper and a lower solution of (4). Arguing as Davini we will directly give an upper solution and the difficult part is in finding the conditions on m and α under which a suitable lower solution exists. Note that we push the argumentation from [4] to the extreme, since $\alpha > 0$ is real valued and not necessarily an integer. Our study is based on the analysis of the quartic polynomial

$$\boldsymbol{P_{m,\alpha}}(\gamma) := a_4\gamma^4 + a_3\gamma^3 + a_2\gamma^2 + a_1\gamma + a_0,$$

[4] Note that the simplification in [4] towards the argumentation as in [2] comes from such a parametrization.

with

$$\begin{aligned}
a_4 &= (m+\alpha)^3,\\
a_3 &= -(m+\alpha)^2(m+\alpha+1),\\
a_2 &= (m+\alpha)(2m+6\alpha-4m\alpha-1),\\
a_1 &= 4m^2\alpha+4\alpha^2 m-4\alpha^2-5\alpha-m+1,\\
a_0 &= -8(m-1)\alpha.
\end{aligned}$$

LEMMA 4.1. *There exists an algebraic number $\alpha_m > \frac{2}{m}$ such that for all $\alpha \geq \alpha_m$ we can find a value $\boldsymbol{\gamma_{m,\alpha}} \in (0, 1-\frac{1}{m+\alpha})$ with*

$$\boldsymbol{P_{m,\alpha}}(\boldsymbol{\gamma_{m,\alpha}}) \geq 0.$$

PROOF. Note that

$$\boldsymbol{P_{m,\alpha}}(0) = -8(m-1)\alpha < 0$$

and

$$\boldsymbol{P_{m,\alpha}}(1-\tfrac{1}{m+\alpha}) = -\frac{8(m-1)\alpha}{m+\alpha} < 0.$$

Further, for all admissible $m \in \{2,3,\dots\}$ and $\alpha > \frac{2}{m}$ the coefficients of $\boldsymbol{P_{m,\alpha}}$ fulfill:

$$\begin{aligned}
a_4 &= (m+\alpha)^3 > 0,\\
a_3 &= -(m+\alpha)^2(m+\alpha+1) < 0,\\
a_1 &= 5\alpha(\tfrac{m^2}{4}-1)+4\alpha^2(m-1)+m(\tfrac{11}{4}m\alpha-1)+1 > 0,\\
a_0 &= -8(m-1)\alpha < 0,
\end{aligned}$$

consequently, $\boldsymbol{P_{m,\alpha}}(-\gamma)$ has, regardless of the value a_2, always one sign change in the sequence of its coefficients $a_4,\ -a_3,\ a_2,\ -a_1,\ a_0$. Hence, due to Descartes' rule of signs, $\boldsymbol{P_{m,\alpha}}$ always has one negative root. Moreover, we have

$$\boldsymbol{P_{m,\alpha}}(\gamma + 1 - \tfrac{1}{m+\alpha}) = \widetilde{a}_4\gamma^4 + \widetilde{a}_3\gamma^3 + \widetilde{a}_2\gamma^2 + \widetilde{a}_1\gamma + \widetilde{a}_0,$$

$$\begin{aligned} \text{with}\quad \widetilde{a}_4 &= (m+\alpha)^3 > 0,\\ \widetilde{a}_3 &= (m+\alpha)^2(3m+3\alpha-5) > 0,\\ \widetilde{a}_2 &= (m+\alpha)\{(m-2)(3m-4) + \tfrac{2\alpha}{m}(m^2-3m+\tfrac{3}{2}\alpha m)\} > 0,\\ \widetilde{a}_0 &= -\tfrac{8(m-1)\alpha}{m+\alpha} < 0, \end{aligned}$$

thus, regardless of the value $\widetilde{a}_1$, we always have one sign change in the sequence of coefficients of the polynomial $\boldsymbol{P_{m,\alpha}}(\gamma + 1 - \frac{1}{m+\alpha})$. In other words, $\boldsymbol{P_{m,\alpha}}$ always has one root in $(1 - \frac{1}{m+\alpha}, \infty)$.

All in all, $\boldsymbol{P_{m,\alpha}}$ has none, a double or two distinct roots in the interval $(0, 1 - \frac{1}{m+\alpha})$. To determine the nature of roots of the quartic equation

$$\boldsymbol{P_{m,\alpha}}(\gamma) = 0 \tag{5}$$

we convert it by the change of variable $\gamma = u + \frac{m+\alpha+1}{4(m+\alpha)}$ to the depressed quartic

$$u^4 + pu^2 + qu + r = 0, \tag{5*}$$

with coefficients

$$\begin{aligned} p &= -\tfrac{1}{8(m+\alpha)^2}\{3m^2 - 10m + 11 + 3\alpha^2 + 2(19m-21)\alpha\} < 0,\\ q &= -\tfrac{1}{8(m+\alpha)^3}\{\alpha^3 + \alpha^2(11-13m) - \alpha(m-1)(13m+23)\\ &\qquad\qquad + (m-3)(m-1)^2\},\\ r &= -\tfrac{1}{256(m+\alpha)^4}\{3\alpha^4 + 172\alpha^3 - 1630\alpha^2 + 204\alpha + 3m^4 - 180\alpha m^3 - 20m^3\\ &\qquad -366\alpha^2m^2 + 1796\alpha m^2 + 34m^2 - 180\alpha^3 m + 1988\alpha^2 m\\ &\qquad -1788\alpha m + 12m - 45\}, \end{aligned}$$

and consider its resolvent cubic, namely

$$\zeta^3 + 2p\zeta^2 + (p^2 - 4r)\zeta - q^2 = 0. \tag{5**}$$

We have $p < 0$ and $p^2 - 4r > 0$ as

$$\begin{aligned} 16(m+\alpha)^4(p^2-4r) &= 3\alpha^4 + 4(3m-5)\alpha^3 + \left(274m^2 - 316m + 50\right)\alpha^2\\ &\quad + 4(m-1)(3m^2+52m+45)\alpha\\ &\quad + (m-1)^2(3m^2-14m+19). \end{aligned}$$

Consequently, (5**) has no negative roots, since there is no sign change in the sequence of the coefficients $-1,\ 2p,\ 4r-p^2,\ -q^2$. On the other hand, (5**) has one or three positive roots depending on the sign of its discriminant

$$\theta = 4p^2(p^2-4r)^2 - 4(p^2-4r)^3 - 36p(p^2-4r)q^2 + 32p^3q^2 - 27q^4.$$

In view of the foregoing, it follows:

i) If $\theta > 0$, then $\boldsymbol{P_{m,\alpha}}$ has two distinct roots in $(0, 1-\frac{1}{m+\alpha})$.

ii) If $\theta = 0$, then $\boldsymbol{P_{m,\alpha}}$ has one double root in $(0, 1-\frac{1}{m+\alpha})$.

iii) If $\theta < 0$, then $\boldsymbol{P_{m,\alpha}}$ has no roots in $(0, 1-\frac{1}{m+\alpha})$.

So, the statement of the lemma follows for such values of m and α for which $\theta = \theta_m(\alpha) \geq 0$. We have:

$$\begin{aligned}
\frac{(m+\alpha)^{12}}{16\alpha(m-1)}\cdot\theta_m(\alpha) = \\
&16(m-1)^2\alpha^8 \\
&-4(m-1)(8m^2+3)\alpha^7 \\
&-(16m^4-256m^3+584m^2-496m+153)\alpha^6 \\
&+2(32m^5-224m^4+1238m^3-2738m^2+2545m-852)\alpha^5 \\
&-(m-1)(16m^5+48m^4-1712m^3+6672m^2-4321m-641)\alpha^4 \\
&-2(16m^7-208m^6+250m^5+2302m^4-3214m^3-588m^2 \\
&\qquad\qquad +1566m-123)\alpha^3 \\
&+(16m^8-192m^7+984m^6-2864m^5+1001m^4 \\
&\qquad\qquad +4184m^3-3870m^2+794m-52)\alpha^2 \\
&-2(m-1)(22m^6-148m^5+363m^4-381m^3+185m^2-60m+2)\alpha \\
&-(m-2)^3(m-1)^2m =: \boldsymbol{p_m}(\alpha).
\end{aligned}$$

Note that the polynomial $\boldsymbol{p_m}$ has three changes of sign in its sequence of coefficients if $m = 2,\dots,6$ and five changes if $m \geq 7$, so that Descartes' rule of signs is not applicable to show that $\boldsymbol{p_m}$ has only one positive root. To prove the latter we will now apply Sturm's theorem. For that purpose we consider the canonical Sturm chain

$$\boldsymbol{p_{m}}_{,0}(\alpha),\ \boldsymbol{p_{m}}_{,1}(\alpha),\dots,\ \boldsymbol{p_{m}}_{,8}(\alpha)$$

and count the number of sign changes in these sequences for $\alpha = 0$ and $\alpha \to \infty$:

		$\alpha = 0$	$\alpha \to \infty$
sign of	$\boldsymbol{p_{m,0}}(\alpha)$	0 $m = 2$ $-$ $m \geq 3$	$+$
	$\boldsymbol{p_{m,1}}(\alpha)$	$-$	$+$
	$\boldsymbol{p_{m,2}}(\alpha)$	$+$	$+$
	$\boldsymbol{p_{m,3}}(\alpha)$	$+$	$-$ $m = 2, \ldots, 28$ $+$ $m \geq 29$
	$\boldsymbol{p_{m,4}}(\alpha)$	$-$ $m = 2$ $+$ $m \geq 3$	$-$
	$\boldsymbol{p_{m,5}}(\alpha)$	$-$ $m = 2, 3$ $+$ $m = 4, 5$ $-$ $m \geq 6$	$-$ $m = 2, \ldots, 4$ $+$ $m = 5, \ldots, 10$ $-$ $m \geq 11$
	$\boldsymbol{p_{m,6}}(\alpha)$	$+$ $m = 2$ $-$ $m \geq 3$	$+$ $m = 2, \ldots, 22$ $-$ $m \geq 23$
	$\boldsymbol{p_{m,7}}(\alpha)$	$+$ $m = 2, \ldots, 6$ $-$ $m \geq 7$	$+$
	$\boldsymbol{p_{m,8}}(\alpha)$	$+$	$+$
sign changes		3	2

Hence, due to Sturm's theorem, the polynomial $\boldsymbol{p_m}$ has always $3 - 2 = 1$ positive root which we denote by α_m.

Moreover we have

$$\begin{aligned} m^8 \cdot \boldsymbol{p_m}\left(\frac{2}{m}\right) = &-25m^{14} - 80m^{13} + 1611m^{12} - 5114m^{11} - 2544m^{10} \\ &- 19620m^9 + 65904m^8 - 135888m^7 + 228832m^6 \\ &- 215760m^5 + 111152m^4 - 18688m^3 - 7232m^2 \\ &- 6656m + 4096 \quad < 0 \end{aligned}$$

and

$$\begin{aligned} m^8 \cdot \boldsymbol{p_m}\left(\frac{12}{m}\right) = &\, 1775m^{14} - 23560m^{13} + 74111m^{12} + 324326m^{11} \\ &- 1065244m^{10} - 8010880m^9 + 62969424m^8 \\ &- 283180848m^7 + 790863552m^6 - 674075520m^5 \\ &- 1637169408m^4 + 2203656192m^3 + 5992869888m^2 \\ &- 13329432576m + 6879707136 \quad > 0 \quad \text{for all } m \geq 2, \end{aligned}$$

thus,

$$\frac{2}{m} < \alpha_m < \frac{12}{m}\,. \qquad \square$$

REMARK 4.2. The lengthy symbolic manipulations were completed here with the aid of the *Wolfram Language* on a *Raspberry Pi 2, Model B*. The following computations will again be carried out by hand:

PROOF OF THEOREM 1.1. Denoting the right-hand side of (4) by $H_{m,\alpha}(t,w)$ we see that

$$g_{m,\alpha}(t) \coloneqq (m+\alpha)\cdot\frac{\sin(2t)}{(m+\alpha-1)\cos(2t)-(m-\alpha-1)}$$

fulfills

$$H_{m,\alpha}(t, g_{m,\alpha}(t)) = 0 \quad \text{on } (0, \boldsymbol{t_{m,\alpha}}) \cup (\boldsymbol{t_{m,\alpha}}, \tfrac{\pi}{2}),$$

where

$$\boldsymbol{t_{m,\alpha}} \coloneqq \frac{1}{2}\arccos\left(\frac{m-\alpha-1}{m+\alpha-1}\right) = \arctan\sqrt{\frac{\alpha}{m-1}}\,.$$

Since ${g_{m,\alpha}}'(t) \geq 0$, the function $g_{m,\alpha}$ is an upper solution of (4). As we are interested in a solution of (4), which has the same growth properties as $g_{m,\alpha}$, it is natural to ask for a lower solution of the form $\gamma\cdot g_{m,\alpha}$ with $\gamma \in (0,1)$, i.e., we should have

$$\text{(6)} \quad \gamma\cdot {g_{m,\alpha}}'(t) \leq H_{m,\alpha}(t, \gamma\cdot g_{m,\alpha}(t)) \qquad \text{for all } t \in (0, \boldsymbol{t_{m,\alpha}}) \cup (\boldsymbol{t_{m,\alpha}}, \tfrac{\pi}{2}).$$

For $t \neq \boldsymbol{t_{m,\alpha}}$ this is equivalent to

$$(6^*) \qquad a \cdot \cos^2(2t) - 2b \cdot \cos(2t) + c \geq 0,$$

with

$$a = (1-\gamma)\big((m+\alpha-1)^2 - \gamma^2(m+\alpha)^2\big),$$
$$b = (m-\alpha-1)\big(m+\alpha-1-\gamma(m+\alpha)\big),$$
$$c = (1-\gamma)\gamma^2(m+\alpha)^2 - 2\gamma(m+\alpha-1) + (1-\gamma)(m-\alpha-1)^2.$$

Note that (6*) is valid on $(0, \frac{\pi}{2})$ as long as $\gamma \in (0, 1 - \frac{1}{m+\alpha})$. The latter is equivalent to $a > 0$. Hence, the left hand side of (6*) is bounded below by

$$c - \frac{b^2}{a}.$$

In other words, to find an adequate lower solution, it suffices to find conditions on m and α under which a $\gamma \in (0, 1 - \frac{1}{m+\alpha})$ exists with

$$c - \frac{b^2}{a} \geq 0$$

$$\overset{\gamma\in(0,1)}{\underset{m\geq 2,\ \alpha > \frac{2}{m}}{\Longleftrightarrow}} \qquad \boldsymbol{P_{m,\alpha}}(\gamma) \geq 0,$$

and lemma 4.1 yields the desired conclusion. Consequently, we gain for $\boldsymbol{\gamma_{m,\alpha}}$:

$$\boldsymbol{\gamma_{m,\alpha}} \cdot {g_{m,\alpha}}'(t) \leq H_{m,\alpha}(t, \boldsymbol{\gamma_{m,\alpha}} \cdot g_{m,\alpha}(t)) \qquad \text{on } (0, \boldsymbol{t_{m,\alpha}}) \cup (\boldsymbol{t_{m,\alpha}}, \tfrac{\pi}{2}),$$

i.e., the function $\boldsymbol{\gamma_{m,\alpha}} \cdot g_{m,\alpha}$ is a lower solution of (4), so that we can proceed as in [4]: Due to results from classical ordinary differential equations theory it follows the existence of a C^1-solution $\boldsymbol{w_{m,\alpha}}$ of (4) on $(0, \boldsymbol{t_{m,\alpha}}) \cup (\boldsymbol{t_{m,\alpha}}, \frac{\pi}{2})$. Moreover, $\boldsymbol{w_{m,\alpha}}$ satisfies

$$0 < \boldsymbol{\gamma_{m,\alpha}} \cdot g_{m,\alpha}(t) \leq \boldsymbol{w_{m,\alpha}}(t) \leq g_{m,\alpha}(t) \quad \text{on } (0, \boldsymbol{t_{m,\alpha}})$$

and

$$0 > \boldsymbol{\gamma_{m,\alpha}} \cdot g_{m,\alpha}(t) \geq \boldsymbol{w_{m,\alpha}}(t) \geq g_{m,\alpha}(t) \quad \text{on } (\boldsymbol{t_{m,\alpha}}, \tfrac{\pi}{2}),$$

as well as

$$\lim_{t \nearrow \boldsymbol{t_{m,\alpha}}} \boldsymbol{w_{m,\alpha}}(t) = +\infty, \qquad \lim_{t \searrow \boldsymbol{t_{m,\alpha}}} \boldsymbol{w_{m,\alpha}}(t) = -\infty,$$

$$\lim_{t \searrow 0} \boldsymbol{w_{m,\alpha}}(t) = 0 = \lim_{t \nearrow \frac{\pi}{2}} \boldsymbol{w_{m,\alpha}}(t).$$

Let us denote by $\boldsymbol{v_{m,\alpha}}$ the antiderivative of $\boldsymbol{w_{m,\alpha}}$ with

$$\lim_{t\searrow 0} \boldsymbol{v_{m,\alpha}}(t) = 0 \quad \text{and} \quad \lim_{t\nearrow \frac{\pi}{2}} \boldsymbol{v_{m,\alpha}}(t) = 0.$$

Reconstructing the auxiliary function from its level curves which are parametrized by

$$\begin{cases} |x| & = \lambda \cdot \mathrm{e}^{\boldsymbol{v_{m,\alpha}}(t)} \cdot \cos t, \\ y & = \lambda \cdot \mathrm{e}^{\boldsymbol{v_{m,\alpha}}(t)} \cdot \sin t, \end{cases}$$

with $\lambda > 0$ and $t \in (0, \boldsymbol{t_{m,\alpha}}) \cup (\boldsymbol{t_{m,\alpha}}, \frac{\pi}{2})$, we gain

$$\boldsymbol{F_{m,\alpha}}(x,y) := \begin{cases} \sqrt{|x|^2+y^2} \cdot \mathrm{e}^{-\boldsymbol{v_{m,\alpha}}(\arctan \frac{y}{|x|})}, & 0 < \arctan \frac{y}{|x|} < \boldsymbol{t_{m,\alpha}}, \\ -\sqrt{|x|^2+y^2} \cdot \mathrm{e}^{-\boldsymbol{v_{m,\alpha}}(\arctan \frac{y}{|x|})}, & \boldsymbol{t_{m,\alpha}} < \arctan \frac{y}{|x|} < \frac{\pi}{2}. \end{cases}$$

Note that, since $\boldsymbol{v_{m,\alpha}}$ satisfies (3), we obtain

$$\operatorname{div}\left(-y^{\alpha} \frac{\nabla \boldsymbol{F_{m,\alpha}}}{|\nabla \boldsymbol{F_{m,\alpha}}|}\right) = 0, \quad \text{on } \big(\mathbb{R}^m \times \mathbb{R}_{>0} \backslash \{\, x = 0 \,\}\big) \backslash \mathcal{M}_m^{\alpha}.$$

We than conclude as in our first proof above because $\boldsymbol{F_{m,\alpha}}$ has the desired properties, cf. remark 3.4. □

Remark 4.3. The crucial ingredient in our argumentation was to find conditions on $m \geq 2$ and $\alpha > 0$ under which a $\gamma \in (0,1)$ exists such that (6*) is fulfilled on $(0, \boldsymbol{t_{m,\alpha}}) \cup (\boldsymbol{t_{m,\alpha}}, \frac{\pi}{2})$. For $t \to \boldsymbol{t_{m,\alpha}}$ the inequality (6*) is equivalent to

$$(1-\gamma)\gamma \geq \frac{2(m+\alpha-1)}{(m+\alpha)^2}.$$

The last inequality has solutions in $(0,1)$ as long as $m+\alpha \geq 4+\sqrt{8}$. Hence,

$$\max\{\, 4 - m + \sqrt{8}, 0 \,\}$$

are lower bounds for the optimal α_m's. With our values we have already reached the lower bounds quite close, so, for $m = 4$ we have

$$\alpha_4 - \sqrt{8} < \frac{1}{1000}.$$

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