Torsors under abelian schemes via Picard schemes

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Abstract – Given a torsor $P$ under a principally polarised abelian scheme $J$, projective over a noetherian base, we use the Picard scheme of $P$ to write down an explicit extension of $\mathbb{Z}$ by $J$ giving the class of $P$. As an application we give a version of Bhatt’s period-index result valid over an arbitrary noetherian base.

1. Introduction

Let $S$ be a noetherian scheme and $J$ an abelian $S$-scheme, polarised by an $S$-ample line bundle. Given a $J$-torsor $P$, projective over $S$, we can consider the Picard scheme $\text{Pic}_{P/S}$ of $P$ and its connected component of the identity $\text{Pic}^0_{P/S}$. In this note we find an explicit canonical extension $E_P$ of $\mathbb{Z}$ by $\text{Pic}^0_{P/S} \cong \text{Pic}^0_{J/S}$ with $E_P \subset \text{Pic}_{P/S}$ whose class is equal to the image of the class of $P$ under the map $H^1(S,J) \rightarrow H^1(S,\text{Pic}^0_{J/S})$ induced by the polarisation. See 4.1.2 for the precise statement. In particular, if the polarisation is principal, then this gives an explicit extension of $\mathbb{Z}$ by $J$ whose class is that of $P$.

Although this construction appears to be well known (cf. [1]), to the best of our knowledge no proof appears in the literature. To give an application, we generalise the period-index result of Bhatt [1] to an arbitrary base, see 4.2.1.

Conventions. Unless otherwise clear from the context, a product of schemes is viewed as a scheme over the rightmost factor, i.e. $X_1 \times X_2 \times \cdots X_n$ is considered as an $X_n$-scheme. This convention is necessary to avoid

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confusion about maps to the Picard scheme, such a map being given by
a line bundle on a product of two schemes viewed as family of schemes over
the right factor. It is for this reason that we consider right torsors in §3.
For a product $X_1 \times X_2 \times \cdots \times X_n$ and $(i_1, \ldots, i_m) \in [1, n]^m$ we write $p_{i_1i_2\cdots i_m}$ for the projection onto $X_{i_1} \times X_{i_2} \times \cdots \times X_{i_m}$.
We work locally on the big fppf site of a noetherian base scheme.

2. Reminders

Let $S$ be a noetherian scheme.

2.1 – Representability of the Picard functor

For a morphism of schemes $f : X \to S$, consider the functor on $S$-schemes

$$T \mapsto \text{Pic}_{X/S}(T) := \text{Pic}(X \times_S T)/\text{Pic}(T)$$

Theorem 2.1.1 (Grothendieck [4; 6, 9.4.8]). If $f$ projective and flat with
integral geometric fibres, then the sheafification of $\text{Pic}_{X/S}$ on the big étale site
of $S$ is representable by a separated $S$-scheme locally of finite type denoted
$\text{Pic}_{X/S}$.

Assume $f$ is as in the theorem. Then $\mathcal{L} \in \text{Pic}(X \times_S T)$ induces a
morphism $\lambda : T \to \text{Pic}_{X/S}$. We will need the following result.

Proposition 2.1.2. Assume $T \to S$ is an abelian scheme and $0 : S \to T$
its zero section. If $\mathcal{L}|_{X \times_S 0} \cong f^* \mathcal{M}$ for some $\mathcal{M} \in \text{Pic}(S)$, then $\lambda$ is a
homomorphism.

Proof. $\mathcal{L}|_{X \times_S 0} \cong f^* \mathcal{M}$ implies that the composition $\lambda \circ 0$ is the zero
section of $\text{Pic}_{X/S}$, i.e. $\lambda(0) = 0$. Now the claim follows from [10, Cor.
6.4]. □

2.1.3 (Poincaré bundle). Under the hypothesis of 2.1.1, assume that $f$
has a section. Then the functor $\text{Pic}_{X/S}$ is itself representable by $\text{Pic}_{X/S}$, and
therefore there is a universal line bundle $\mathcal{P}$ on $X \times_S \text{Pic}_{X/S}$, unique up to
tensor product by the pullback of a line bundle on $\text{Pic}_{X/S}$, called Poincaré
bundle. Explicitly, a line bundle $\mathcal{L}$ on $X \times_S T$ determines a morphism
$\lambda : T \to \text{Pic}_{X/S}$ and we have $(p_1 \times \lambda \circ p_2)^* \mathcal{P} \cong \mathcal{L} \otimes p_2^* \mathcal{M}$ for some $\mathcal{M} \in \text{Pic}(T)$, cf. [6, 9.4.3].
2.2 – Representability of Pic^0

The next theorem ensures the existence of the connected component of the identity of Pic.

Theorem 2.2.1. Under the hypotheses of 2.1.1.

(1) For all \(s \in S\) the connected component \(\text{Pic}^0_{X_s/s}\) of the identity element of \(\text{Pic}_{X_s/s}\) is a quasi-projective \(s\)-scheme.

(2) If for all \(s \in S\) \(\text{Pic}^0_{X_s/s}\) is a proper and smooth \(s\)-scheme of dimension independent of \(s\), then there is an open and closed subgroup scheme \(\text{Pic}^0_{X/S} \subseteq \text{Pic}_{X/S}\), proper over \(S\), whose fibre over \(s\) is \(\text{Pic}^0_{X_s/s}\), and whose formation commutes with arbitrary base change.

(3) In (2) \(\text{Pic}^0_{X/S}\) is projective over \(S\) if \(f\) is smooth.

(4) In (2) \(\text{Pic}^0_{X/S}\) is smooth over \(S\) if either \(S\) is reduced or \(f\) is a torsor under an abelian scheme.

This statement, except for smoothness in the non-reduced case in (4), can be found in [6, §9.5] (see also [4]). For smoothness in the case of an abelian scheme (from which the case of a torsor follows immediately) see [10, Prop. 6.7].

3. Picard schemes of \(J\)-torsors

Let \(J\) be an abelian \(S\)-scheme with \(S\)-ample polarisation \(\Theta \in \text{Pic}(J)\). We write

\[ \lambda : J \to \text{Pic}^0_{J/S} \]

for the isogeny induced by \(p^*_1 \Theta^{-1} \otimes \tau^* \Theta\), where \(\tau\) is the addition map on \(J\).

Let \(P\) be a \(J\)-torsor and let

\[ \tau_P : P \times_S J \to P \]

denote the \(J\)-action on \(P\). We usually write \(\tau\) for \(\tau_P\) when no confusion can occur. We assume throughout that \(P\) is projective over \(S\). For instance, this is true if \(P\) has finite order, see 3.1.1.

3.1 – Picard scheme of \(P\) and \(P \times_S J\)

Proposition 3.1.1. If \(P\) is a \(J\)-torsor of finite order, then \(P\) is projective over \(S\).
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Proof. Let $n$ be the order of $P$, and $J[n]$ the kernel of multiplication by $n$ on $J$. Then $J[n]$ is a finite flat $S$-group scheme of rank $n^{2 \dim S}$. Since $J$ is $n$-divisible for the flat topology, there is a $J[n]$-torsor $Q$ from which $P$ can be obtained. Then $P \times_S Q \cong J \times_S Q$ is projective over $Q$, and since $Q$ is finite faithfully flat over $S$ we can apply [3, II, 6.6.4] to deduce the result. □

Hence, if $P$ has finite order, then (by 2.1.1 and 2.2.1) $\text{Pic}_{P/S}$ and $\text{Pic}_{P/S}^0$ exist and the latter is a smooth and projective $S$-scheme. Similarly for $P \times_S J$ and $P \times_S P$.

**Proposition 3.1.2.** There is a canonical isomorphism $\mu : \text{Pic}_{P/S}^0 \cong \text{Pic}_{J/S}^0$. If $\sigma \in P(S)$ and $\iota : J \cong P$ is the isomorphism induced by $\sigma$, then the map $\iota^* : \text{Pic}_{P/S}^0 \to \text{Pic}_{J/S}^0$ is equal to $\mu$.

Proof. We claim that the canonical maps $\text{Pic}_{P/S}^0 \times_S \text{Pic}_{J/S}^0 \to \text{Pic}_{P \times_S J/S}^0$ and $\text{Pic}_{P/S}^0 \times_S \text{Pic}_{P/S}^0 \to \text{Pic}_{P \times_S P/S}^0$ are isomorphisms. Indeed, this is well known over a field\(^1\) and the general case follows from [3, IV, 17.9.5]. Moreover, the map $p_1 \times \tau : P \times_S J \to P \times_S P$ is an isomorphism of $P$-schemes. Thus, we have a canonical isomorphism

$$
\text{Pic}_{P/S}^0 \xrightarrow{\sim} \text{Pic}_{P \times_S P/S}^0 p_1^* \text{Pic}_{P/S}^0 (p_1 \times \tau)^* \text{Pic}_{P \times_S J/S}^0 \xrightarrow{\sim} \text{Pic}_{J/S}^0
$$

and we define $\mu$ to be this map.

For the last statement, it suffices to note that the map $\iota^*$ is equal to the composition $\text{Pic}_{P/S}^0 \xrightarrow{\iota^*} \text{Pic}_{P \times_S P/S}^0 (p_1 \times \tau)^* \text{Pic}_{P \times_S J/S}^0 \xrightarrow{\sigma^*} \text{Pic}_{J/S}^0$ and $\text{Pic}_{J/S}^0 \xrightarrow{\iota^*} \text{Pic}_{P \times_S J/S}^0 \xrightarrow{\sigma^*} \text{Pic}_{J/S}^0$ is the identity. □

\(^{1}\)[8, IV, Prop. 7]
3.2 – An exact sequence

Define a canonical map of sheaves

\( \psi : \text{Pic}_{P \times S/J} \to \text{Hom}_S(J, \text{Pic}_{J/S}^0) \)

as follows. To simplify the notation, we will write \( S \) to denote an arbitrary \( S \)-scheme. Let \( L \in \text{Pic}(P \times S J) \) and \( L_0 \in \text{Pic}(P) \) the restriction of \( L \) to \( P \times \{0\} \cong P \). Then \( L \otimes p_1^*L_0^{-1} \) defines a map \( \psi_L : J \to \text{Pic}_{P/S} \), which is unique up to tensor product by the pullback of a line bundle on \( J \). Moreover, since \( (L \otimes p_1^*L_0^{-1})_0 \cong \mathcal{O}_P \) we have \( \psi_L(0) = 0 \), thus the image of this map lies in \( \text{Pic}_{P/S}^0 \) and by 2.1.2 the induced morphism \( \psi_L : J \to \text{Pic}_{P/S}^0 \) is a homomorphism. Composing \( \psi_L \) with \( \mu \) we get an element \( \psi(L) \in \text{Hom}_S(J, \text{Pic}_{J/S}^0) \). It is clear from the definition that \( \psi \) is a homomorphism. Finally, note that if \( L = p_1^*M \), then \( L \otimes p_1^*L_0^{-1} \) is trivial, so \( \psi(L) = 0 \). Hence the image of the canonical map

\[ \text{Pic}_{P/S \times S} \text{Pic}_{J/S} \to \text{Pic}_{P \times S J/S} \]

lies in the kernel of \( \psi \).

**Proposition 3.2.1.** The sequence of sheaves on the big fppf site of \( S \)

\( 0 \to \text{Pic}_{P/S \times S} \text{Pic}_{J/S} \to \text{Pic}_{P \times S J/S} \xrightarrow{\psi} \text{Hom}_S(J, \text{Pic}_{J/S}^0) \to 0 \)

is exact.

**Proof.** This is local so we may assume \( P(S) \neq \emptyset \). Let \( L \in \text{Pic}(P \times S J) \). Let \( \psi_L : J \to \text{Pic}_{P/S}^0 \) be the map defined above. If this map is zero, then \( L \otimes p_1^*L_0^{-1} \) is isomorphic to the pullback of a line bundle on \( J \). Thus, the sequence is exact in the middle, and since it is easily seen to be left exact it remains to see that it is right exact. Let \( \mathcal{P} \) be a Poincaré bundle. By the universal property, given any homomorphism \( \alpha : J \to \text{Pic}_{P/S}^0 \) the pullback \( \mathcal{L} := (p_1 \times \alpha \circ p_2)^* \mathcal{P} \in \text{Pic}(P \times S J) \) is a line bundle inducing \( \alpha \) and \( \mathcal{L}_0 \) is the pullback of a line bundle on \( S \). Thus, \( \psi_L = \alpha \), as required. \( \square \)

3.3 – Fundamental map

Define a homomorphism of sheaves

\( \phi : \text{Pic}_{P/S} \to \text{Hom}_S(J, \text{Pic}_{J/S}^0) \)
as the composition

$$\text{Pic}_{P/S} \xrightarrow{\tau^*} \text{Pic}_{P \times S J} \xrightarrow{\psi} \text{Hom}_S(J, \text{Pic}^{0}_{J/S}).$$

That is, if $\mathcal{L} \in \text{Pic}(P)$, then the sheaf $\tau^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1}$ defines a morphism $\phi_{\mathcal{L}} : J \to \text{Pic}^{0}_{P/S}$ and we have $\phi(\mathcal{L}) := \mu \circ \phi_{\mathcal{L}}$.

We claim that the kernel of $\phi$ is exactly $\text{Pic}^{0}_{P/S}$. By [9, II.8, (i) p.74] this is true on the geometric fibres of $S$. Thus, $\ker(\phi) \subset \text{Pic}^{0}_{P/S}$, and if $\mathcal{L} \in \text{Pic}^{0}_{P/S}$ then $\phi_{\mathcal{L}}$ is zero on geometric fibres. But if a homomorphism of abelian schemes is zero on a geometric fibre then it must be zero, so $\text{Pic}^{0}_{P/S} \subset \ker(\phi)$. We have shown

**Proposition 3.3.1.** There is an exact sequence

$$0 \to \text{Pic}^{0}_{P/S} \to \text{Pic}_{P/S} \xrightarrow{\phi} \text{Hom}_S(J, \text{Pic}^{0}_{J/S})$$

The image of $\phi$ can be determined under certain assumptions, cf. [9, IV, §20] for the case $S$ is a point where it is shown that, up to tensor product with $\mathbb{Q}$, $\phi(\text{Pic}_{P/S})$ is equal to the subset of elements of $\text{End}_S(J) \cong \text{Hom}_S(J, \text{Pic}^{0}_{J/S})$ invariant under the Rosati involution.

4. The main result

We maintain the notation and assumptions of §3.

4.1 - Class of $P$

Define a $\text{Pic}^{0}_{P/S}$-torsor $\Pi$ by the formula

$$\Pi = \phi^{-1}(-\lambda)$$

where $\phi$ is the fundamental map (3).

**Theorem 4.1.1.** There is a canonical isomorphism of $\text{Pic}^{0}_{P/S}$-torsors $P' \cong \Pi$, where $P'$ is the $\text{Pic}^{0}_{P/S}$-torsor deduced from $P$ via the morphism $\mu^{-1} \circ \lambda : J \to \text{Pic}^{0}_{P/S}$.

**Proof.** To simplify the notation, if $f_i : J \to T_i$ are $S$-morphisms, we write $(f_1, f_2) := f_1 \circ p_1 \times f_2 \circ p_2 : J \times_S J \to T_1 \times_S T_2$, and similarly for more factors.

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2 Indeed, by [10, Prop. 6.1] it factors through a section, which must be the zero section since it is a homomorphism.
Consider the isomorphism
\[ p_1 \times \tau : P \times_S J \to P \times_S P \]
Let 1 = id and
\[ \sigma := (p_1 \times \tau) \circ (1, -1) \]
The line bundle \( \sigma^* p_1^* \Theta^{-1} \) defines a morphism \( \theta : P \to \text{Pic}_{P/S} \). We claim that its image lies in \( \Pi \), i.e.
\[ \phi \circ \theta = -\lambda \in \text{Hom}_S(J, \text{Pic}^0_{J/S}). \]
This is local on \( S \), so we may assume that \( P = J \). First note that
\[ \sigma^2 = 1 \]
hence
\[ \sigma^* p_2^* \Theta^{-1} = \sigma^* p_1^* \Theta^{-1} = (1, -1)^* \tau^* \Theta^{-1} \]  
(5)
Now \( \mathcal{M} = \tau^* \Theta^{-1} \otimes p_1^* \Theta \) induces the morphism \( -\lambda : J \to \text{Pic}^0_{J/S} \subset \text{Pic}_{J/S} \), hence \( \tau^* \Theta^{-1} = \mathcal{M} \otimes p_1^* \Theta^{-1} \) induces its translation by \( -\Theta \), i.e. \( -\lambda - \Theta : J \to \text{Pic}_{J/S} \). Since \( \phi \circ (\lambda - \Theta) = \phi(\lambda) + \phi(-\Theta) = \phi(-\Theta) = -\lambda \), this shows that \( \phi \circ \theta = -\lambda \).

Next, we show that \( \theta \) induces a morphism (hence isomorphism) of torsors. By [2, III, 1.4.6 (iii)] it suffices to show that the diagram
\[
\begin{array}{ccc}
P \times_S J & \mathbb{P} & P \\
\tau & \downarrow{\theta} & \downarrow{\theta} \\
\Pi \times_S \text{Pic}^0_{P/S} & \rightarrow & \text{Pic}_{P/S}
\end{array}
\]
commutes. This is again local on \( S \), so we may assume \( P = J \). Consider the diagram
\[
\begin{array}{ccc}
J \times_S J & \mathbb{P} & J \times_S J \\
(1, \tau) & \downarrow{(1, \theta, \lambda)} & \downarrow{(1, \theta)} \\
J \times_S \Pi \times_S \text{Pic}^0_{J/S} & \rightarrow & J \times_S \text{Pic}_{J/S}
\end{array}
\]
We must show that the pullback of the Poincaré bundle \( \mathcal{P} \) on \( J \times_S \text{Pic}_{J/S} \) clockwise has the same \( \equiv \)-class as the pullback counterclockwise in the diagram, where \( \equiv \) means isomorphism class up to tensor product with \( p_2^* \mathcal{M} \) for \( \mathcal{M} \in \text{Pic}(J \times_S J) \).
Before continuing with the proof we make a short digression. Define
\[ m = p_1 + p_2 + p_3 : J \times_S J \times_S J \to J \]
and
\[ m_{ij} = \tau \circ p_{ij} = p_i + p_j : J \times_S J \times_S J \to J \]
Fix any line bundle \( \mathcal{L} \) on \( J \) and consider the line bundle
\[ \mathcal{L}_1 := p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1} \otimes p_3^* \mathcal{L}^{-1} \otimes m_{12}^* \mathcal{L} \otimes m_{23}^* \mathcal{L} \]
on \( J \times_S J \times_S J \). It defines a morphism
\[ f_1 : \hat{J} \times_S J \times_S J \to \text{Pic}^0_{J/S} \]
where the hat denotes omission. Moreover, since \( \mathcal{L}_1|_{J \times_S (0,0)} \) is a pullback from \( S \), it follows from 2.1.2 that this is homomorphism of abelian schemes. Similarly, the bundle
\[ \mathcal{L}_2 := m^* \mathcal{L} \otimes m_{13}^* \mathcal{L}^{-1} \]
defines a homomorphism \( f_2 : \hat{J} \times_S J \times_S J \to \text{Pic}^0_{J/S} \). Now, by [9, II.6, Cor. 2 p.58] we know that for \( S \) a point we have \( \mathcal{L}_1 \cong \mathcal{L}_2 \). So the difference
\[ f_1 - f_2 : \hat{J} \times_S J \times_S J \to \text{Pic}^0_{J/S} \]
is a homomorphism of abelian schemes which is zero fibrewise. Hence it is zero. Applying this to \( \mathcal{L} = \Theta^{-1} \) we find
\[ m_{12}^* \Theta^{-1} \otimes p_1^* \Theta \otimes m_{13}^* \Theta^{-1} \equiv m^* \Theta^{-1} \]
Now, returning to the proof, we compute the pullback clockwise. By definition of \( \theta \) we have
\[ (1, \tau)^*(1, \theta)^* \mathcal{P} \equiv (1, \tau)^* \sigma_* p_2^* \Theta^{-1} \overset{(5)}{=} (1, \tau)^*(1, -1)^* \tau^* \Theta^{-1} \]
\[ = (1, -1, -1)^* \tau^* \Theta^{-1} \]
On the other hand, going counterclockwise we have by straightforward computation
\[ (1, +)^* \mathcal{P} \equiv p_{12}^* \mathcal{P} \otimes p_{13}^* \mathcal{P} \]
hence
\[ (1, \theta, \lambda)^*(1, +)^* \mathcal{P} \equiv (1, \theta, \lambda)^*(p_{12}^* \mathcal{P} \otimes p_{13}^* \mathcal{P}) = p_{12}^*(1, \theta)^* \mathcal{P} \otimes p_{13}^*(1, \lambda)^* \mathcal{P} \]
and using the definition of \( \theta \) we find
\[ (1, \theta, \lambda)^*(1, +)^* \mathcal{P} \equiv p_{12}^* (\sigma_* p_2^* \Theta^{-1}) \otimes p_{13}^*(1, \lambda)^* \mathcal{P} \]
\[ \overset{(5)}{=} p_{12}^*(1, -1)^* \tau^* \Theta^{-1} \otimes p_{13}^*(1, \lambda)^* \mathcal{P} \]
Now, since $\lambda$ is a homomorphism we have
\[(1, \lambda)^*P = (1, -1)^*(1, -\lambda)^*P \cong (1, -1)^*(p_1^*\Theta \otimes \tau^*\Theta^{-1})\]
modulo $p_2^*\text{Pic}(J)$. Substituting in the above we find
\[(1, \theta, \lambda)^*(1, +)^*P \equiv p_{12}^*(1, -1)^*\tau^*\Theta^{-1} \otimes p_{13}^*(1, -1)^*(p_1^*\Theta \otimes \tau^*\Theta^{-1})\]
\[= (1, -1, 1)^*(m_{12}^*\Theta^{-1} \otimes p_1^*\Theta \otimes m_{13}^*\Theta^{-1})\]
Finally, using (6) we get
\[(1, \theta, \lambda)^*(1, +)^*P \equiv (1, -1, -1)^*m^*\Theta^{-1}\]
as required. \[\square\]

Consider the canonical map
\[Z \to \text{Hom}(J, \text{Pic}^0_J/S) : 1 \mapsto \lambda\]
Pulling back (4) by this map we obtain an exact sequence
\[(E_P) \quad 0 \to \text{Pic}^0_{P/S} \to \text{Pic}_{P/S} \times \text{Hom}_{S}(J, \text{Pic}^0_J/S)Z \to Z \to 0\]

**Corollary 4.1.2.** The class of this extension in $H^1(S, \text{Pic}^0_J/S)$ is equal to the class of the image of $P$ under the map $H^1(S, J) \to H^1(S, \text{Pic}^0_J/S)$ induced by $\lambda$.

**Proof.** The class of this extension is equal to the image of 1 under the coboundary map $Z \to H^1(S, \text{Pic}^0_J/S)$ arising from $(E_P)$. By [2, III, 3.5.5] this class is the negative of that of the torsor $\phi^{-1}(\lambda)$, i.e. $\phi^{-1}(-\lambda)$. \[\square\]

**4.2 – An application**

As an application we apply Bhatt’s argument to obtain a generalisation of his result [1] to an arbitrary base. \(^3\) To lighten the notation write $J^1 := \text{Pic}^0_{J/S}$. Let $\alpha \in \text{Br}(J^1)$ denote the image of $P$ under the canonical maps $H^1(S, J) = H^1(S, \text{Pic}^0_{J/S}) \to H^1(S, \text{Pic}_{J^1/S}) \to H^2(J^1, \mathbb{G}_m)$. \(^4\)

\(^3\) Note our notation differs slightly from [1], e.g. the Picard scheme is denoted $\text{Pic}_X/S$ there.

\(^4\) The last map is defined using the identification $H^1(S, \text{Pic}_{J^1/S}) = \ker(F^1 \text{Br}(J^1) \to \text{Br}(S))$, where $F^1$ is the first step in the filtration coming from the Leray spectral sequence for $J^1 \to S$ and $0 : S \to J^1$ is the zero section.
Proposition 4.2.1. Assume \( \text{Br}(S) = 0 \) and that the order \( p \) of the image of the class of \( P \) in \( H^1(S, \text{Pic}^0_{P/S}) \) is finite. Then for every \( j \in J^t \) there is an open neighbourhood \( U \subset J^t \) and a finite étale morphism \( T \to U \) of degree \( p^{\dim S_J} \) splitting \( \alpha \).

Proof. The key result is the existence [7; 1, Thm. A.1.1] of a twisted Fourier-Mukai transform \( F : D(P) \to D(J^t, -\alpha) \), where \( D(J^t, -\alpha) \) is a derived category of \(-\alpha\)-twisted sheaves.

Now, by 4.1.2 there is an element \( \mathcal{L} \in H^0(S, \text{Pic}_{P/S}) \) such that \( \phi(\mathcal{L}) = p \cdot \lambda \). Since \( \text{Br}(S) = 0 \), \( \mathcal{L} \) is representable by a line bundle. By Mukai theory \( F := \mathcal{F}(\mathcal{L}) \) is an \(-\alpha\)-twisted vector bundle of rank \( \chi(\mathcal{L}_s) = \chi(\Theta^{op}) = p^{\dim S_J} \). Then \( A := \text{End}(\mathcal{F}) \) is an Azumaya algebra of rank \( p^{2 \dim S_J} \) representing the class of \(-\alpha\). So the opposite algebra of \( A \) represents \( \alpha \).

By [5, 5.7], in an open neighbourhood of any point \( j \in J^t \) there is a finite étale morphism of degree \( p^{\dim S_J} \) trivialising \( \alpha \). \( \square \)

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References

[1] B. Bhatt, A period-index result, appendix to [7].