Torsors under abelian schemes via Picard schemes

Rémi Lodh (*)

Mathematics Subject Classification (2010). Primary: 14K30; Secondary: 14F22.

Keywords. Abelian scheme, torsor, Picard scheme, polarisation.

Abstract – Given a torsor $P$ under a principally polarised abelian scheme $J$, projective over a noetherian base, we use the Picard scheme of $P$ to write down an explicit extension of $\mathbb{Z}$ by $J$ giving the class of $P$. As an application we give a version of Bhatt’s period-index result valid over an arbitrary noetherian base.

1. Introduction

Let $S$ be a noetherian scheme and $J$ an abelian $S$-scheme, polarised by an $S$-ample line bundle. Given a $J$-torsor $P$, projective over $S$, we can consider the Picard scheme $\text{Pic}_{P/S}$ of $P$ and its connected component of the identity $\text{Pic}_{P/S}^0$. In this note we find an explicit canonical extension $E_P$ of $\mathbb{Z}$ by $\text{Pic}_{P/S}^0 \cong \text{Pic}_{J/S}^0$ with $E_P \subset \text{Pic}_{P/S}$ whose class is equal to the image of the class of $P$ under the map $H^1(S,J) \to H^1(S,\text{Pic}_{J/S}^0)$ induced by the polarisation. See 4.1.2 for the precise statement. In particular, if the polarisation is principal, then this gives an explicit extension of $\mathbb{Z}$ by $J$ whose class is that of $P$.

Although this construction appears to be well known (cf. [1]), to the best of our knowledge no proof appears in the literature. To give an application, we generalise the period-index result of Bhatt [1] to an arbitrary base, see 4.2.1.

Conventions. Unless otherwise clear from the context, a product of schemes is viewed as a scheme over the rightmost factor, i.e. $X_1 \times X_2 \times \cdots \times X_n$ is considered as an $X_n$-scheme. This convention is necessary to avoid
confusion about maps to the Picard scheme, such a map being given by a line bundle on a product of two schemes viewed as family of schemes over the right factor. It is for this reason that we consider right torsors in §3.

For a product \( X_1 \times X_2 \times \cdots X_n \) and \((i_1, \ldots, i_m) \in [1, n]^m\) we write \( p_{i_1 i_2 \cdots i_m} \) for the projection onto \( X_{i_1} \times X_{i_2} \times \cdots X_{i_m} \).

We work locally on the big fppf site of a noetherian base scheme.

2. Reminders

Let \( S \) be a noetherian scheme.

2.1 – Representability of the Picard functor

For a morphism of schemes \( f : X \to S \), consider the functor on \( S \)-schemes

\[
T \mapsto \text{Pic}_{X/S}(T) := \text{Pic}(X \times_S T)/\text{Pic}(T)
\]

**Theorem 2.1.1** (Grothendieck [4; 6, 9.4.8]). If \( f \) projective and flat with integral geometric fibres, then the sheafification of \( \text{Pic}_{X/S} \) on the big étale site of \( S \) is representable by a separated \( S \)-scheme locally of finite type denoted \( \text{Pic}_{X/S} \).

Assume \( f \) is as in the theorem. Then \( \mathcal{L} \in \text{Pic}(X \times_S T) \) induces a morphism \( \lambda : T \to \text{Pic}_{X/S} \). We will need the following result.

**Proposition 2.1.2.** Assume \( T \to S \) is an abelian scheme and \( 0 : S \to T \) its zero section. If \( \mathcal{L}|_{X \times_S 0} \cong f^* \mathcal{M} \) for some \( \mathcal{M} \in \text{Pic}(S) \), then \( \lambda \) is a homomorphism.

**Proof.** \( \mathcal{L}|_{X \times_S 0} \cong f^* \mathcal{M} \) implies that the composition \( \lambda \circ 0 \) is the zero section of \( \text{Pic}_{X/S} \), i.e. \( \lambda(0) = 0 \). Now the claim follows from [10, Cor. 6.4]. \( \square \)

2.1.3 (Poincaré bundle). Under the hypothesis of 2.1.1, assume that \( f \) has a section. Then the functor \( \text{Pic}_{X/S} \) is itself representable by \( \text{Pic}_{X/S} \), and therefore there is a universal line bundle \( \mathcal{P} \) on \( X \times_S \text{Pic}_{X/S} \), unique up to tensor product by the pullback of a line bundle on \( \text{Pic}_{X/S} \), called Poincaré bundle. Explicitly, a line bundle \( \mathcal{L} \) on \( X \times_S T \) determines a morphism \( \lambda : T \to \text{Pic}_{X/S} \) and we have \((p_1 \times \lambda \circ p_2)^* \mathcal{P} \cong \mathcal{L} \otimes p_2^* \mathcal{M} \) for some \( \mathcal{M} \in \text{Pic}(T) \), cf. [6, 9.4.3].
2.2 – Representability of \( \text{Pic}^0 \)

The next theorem ensures the existence of the connected component of the identity of \( \text{Pic} \).

**Theorem 2.2.1.** Under the hypotheses of 2.1.1.

1. For all \( s \in S \) the connected component \( \text{Pic}^0_{X_s/s} \) of the identity element of \( \text{Pic}_{X_s/s} \) is a quasi-projective \( s \)-scheme.

2. If for all \( s \in S \) \( \text{Pic}^0_{X_s/s} \) is a proper and smooth \( s \)-scheme of dimension independent of \( s \), then there is an open and closed subgroup scheme \( \text{Pic}^0_{X/S} \subset \text{Pic}_{X/S} \), proper over \( S \), whose fibre over \( s \) is \( \text{Pic}^0_{X_s/s} \), and whose formation commutes with arbitrary base change.

3. In (2) \( \text{Pic}^0_{X/S} \) is projective over \( S \) if \( f \) is smooth.

4. In (2) \( \text{Pic}^0_{X/S} \) is smooth over \( S \) if either \( S \) is reduced or \( f \) is a torsor under an abelian scheme.

This statement, except for smoothness in the non-reduced case in (4), can be found in [6, §9.5] (see also [4]). For smoothness in the case of an abelian scheme (from which the case of a torsor follows immediately) see [10, Prop. 6.7].

3. Picard schemes of \( J \)-torsors

Let \( J \) be an abelian \( S \)-scheme with \( S \)-ample polarisation \( \Theta \in \text{Pic}(J) \). We write

\[
\lambda : J \to \text{Pic}^0_{J/S}
\]

for the isogeny induced by \( \tau^* \Theta - 1 \otimes \tau^* \Theta \), where \( \tau \) is the addition map on \( J \).

Let \( P \) be a \( J \)-torsor and let

\[
\tau_P : P \times_S J \to P
\]

denote the \( J \)-action on \( P \). We usually write \( \tau \) for \( \tau_P \) when no confusion can occur. We assume throughout that \( P \) is projective over \( S \). For instance, this is true if \( P \) has finite order, see 3.1.1.

3.1 – Picard scheme of \( P \) and \( P \times_S J \)

**Proposition 3.1.1.** If \( P \) is a \( J \)-torsor of finite order, then \( P \) is projective over \( S \).
Proof. Let \( n \) be the order of \( P \), and \( J[n] \) the kernel of multiplication by \( n \) on \( J \). Then \( J[n] \) is a finite flat \( S \)-group scheme of rank \( n^{2 \dim S} \). Since \( J \) is \( n \)-divisible for the flat topology, there is a \( J[n] \)-torsor \( Q \) from which \( P \) can be obtained. Then \( P \times_S Q \cong J \times_S Q \) is projective over \( Q \), and since \( Q \) is finite faithfully flat over \( S \) we can apply \([3, II, 6.6.4]\) to deduce the result. \( \square \)

Hence, if \( P \) has finite order, then (by 2.1.1 and 2.2.1) \( \text{Pic}_{P/S} \) and \( \text{Pic}_{P/S}^0 \) exist and the latter is a smooth and projective \( S \)-scheme. Similarly for \( P \times_S J \) and \( P \times_S P \).

Proposition 3.1.2. There is a canonical isomorphism \( \mu : \text{Pic}^0_{P/S} \cong \text{Pic}^0_{J/S} \). If \( \sigma \in P(S) \) and \( \iota : J \cong P \) is the isomorphism induced by \( \sigma \), then the map \( \iota^* : \text{Pic}^0_{P/S} \to \text{Pic}^0_{J/S} \) is equal to \( \mu \).

Proof. We claim that the canonical maps
\[
\text{Pic}^0_{P/S} \times_S \text{Pic}^0_{J/S} \to \text{Pic}^0_{P \times_S J/S}
\]
and
\[
\text{Pic}^0_{P/S} \times_S \text{Pic}^0_{P/S} \to \text{Pic}^0_{P \times_S P/S}
\]
are isomorphisms. Indeed, this is well known over a field\(^1\) and the general case follows from \([3, IV, 17.9.5]\). Moreover, the map
\[
p_1 \times \tau : P \times_S J \to P \times_S P
\]
is an isomorphism of \( P \)-schemes. Thus, we have a canonical isomorphism
\[
\text{Pic}^0_{P/S} \xrightarrow{\sim} \text{Pic}^0_{P \times_S P/S} \xrightarrow{\sim} \text{Pic}^0_{P \times_S J/S} \xrightarrow{\sim} \text{Pic}^0_{J/S}
\]
and we define \( \mu \) to be this map.

For the last statement, it suffices to note that the map \( \iota^* \) is equal to the composition
\[
\text{Pic}^0_{P/S} \xrightarrow{\sim} \text{Pic}^0_{P \times_S P/S} \xrightarrow{(p_1 \times \tau)^*} \text{Pic}^0_{P \times_S J/S} \xrightarrow{\sigma^*} \text{Pic}^0_{J/S}
\]
and
\[
\text{Pic}^0_{J/S} \xrightarrow{\sim} \text{Pic}^0_{P \times_S J/S} \xrightarrow{\sigma^*} \text{Pic}^0_{J/S}
\]
is the identity. \( \square \)

\(^1\) \([8, IV, Prop. 7]\)
3.2 – An exact sequence

Define a canonical map of sheaves

\[ \psi : \text{Pic}_{P \times J/S} \rightarrow \text{Hom}_S(J, \text{Pic}_{J/S}^0) \]

as follows. To simplify the notation, we will write \( S \) to denote an arbitrary \( S \)-scheme. Let \( L \in \text{Pic}(P \times S J) \), and \( L_0 \in \text{Pic}(P) \) the restriction of \( L \) to \( P \times \{0\} \cong P \). Then \( L \otimes p_1^*L_0^{-1} \) defines a map \( \psi_L : J \rightarrow \text{Pic}_{P/S} \), which is unique up to tensor product by the pullback of a line bundle on \( J \). Moreover, since \( (L \otimes p_1^*L_0^{-1})_0 \cong O_P \) we have \( \psi_L(0) = 0 \), thus the image of this map lies in \( \text{Pic}_{P/S}^0 \) and by 2.1.2 the induced morphism \( \psi_L : J \rightarrow \text{Pic}_{P/S}^0 \) is a homomorphism. Composing \( \psi_L \) with \( \mu \) we get an element \( \psi(L) \in \text{Hom}_S(J, \text{Pic}_{J/S}^0) \). It is clear from the definition that \( \psi \) is a homomorphism. Finally, note that if \( L = p_1^*M \), then \( L \otimes p_1^*L_0^{-1} \) is trivial, so \( \psi(L) = 0 \). Hence the image of the canonical map

\[ \text{Pic}_{P/S} \times_S \text{Pic}_{J/S} \rightarrow \text{Pic}_{P \times S J/S} \]

lies in the kernel of \( \psi \).

**Proposition 3.2.1.** The sequence of sheaves on the big fppf site of \( S \)

\[ 0 \rightarrow \text{Pic}_{P/S} \times_S \text{Pic}_{J/S} \rightarrow \text{Pic}_{P \times S J/S} \xrightarrow{\psi} \text{Hom}_S(J, \text{Pic}_{J/S}^0) \rightarrow 0 \]

is exact.

**Proof.** This is local so we may assume \( P(S) \neq \emptyset \). Let \( L \in \text{Pic}(P \times S J) \). Let \( \psi_L : J \rightarrow \text{Pic}_{P/S}^0 \) be the map defined above. If this map is zero, then \( L \otimes p_1^*L_0^{-1} \) is isomorphic to the pullback of a line bundle on \( J \). Thus, the sequence is exact in the middle, and since it is easily seen to be left exact it remains to see that it is right exact. Let \( P \) be a Poincaré bundle. By the universal property, given any homomorphism \( \alpha : J \rightarrow \text{Pic}_{P/S}^0 \) the pullback \( L := (p_1 \times \alpha \circ p_2)^*P \in \text{Pic}(P \times S J) \) is a line bundle inducing \( \alpha \) and \( L_0 \) is the pullback of a line bundle on \( S \). Thus, \( \psi_L = \alpha \), as required. \( \square \)

3.3 – Fundamental map

Define a homomorphism of sheaves

\[ \phi : \text{Pic}_{P/S} \rightarrow \text{Hom}_S(J, \text{Pic}_{J/S}^0) \]
as the composition

$$\text{Pic}_{P/S} \xrightarrow{\tau^*} \text{Pic}_{P \times _S J} \xrightarrow{\psi} \text{Hom}_S(J, \text{Pic}^0_{J/S}).$$

That is, if $L \in \text{Pic}(P)$, then the sheaf $\tau^* L \otimes p_1^* L^{-1}$ defines a morphism $\phi_L : J \to \text{Pic}^0_{P/S}$ and we have $\phi(L) := \mu \circ \phi_L$.

We claim that the kernel of $\phi$ is exactly $\text{Pic}^0_{P/S}$. By [9, II.8, (i) p.74] this is true on the geometric fibres of $S$. Thus, $\ker(\phi) \subset \text{Pic}^0_{P/S}$, and if $L \in \text{Pic}^0_{P/S}$ then $\phi_L$ is zero on geometric fibres. But if a homomorphism of abelian schemes is zero on a geometric fibre then it must be zero,\(^2\) so $\text{Pic}^0_{P/S} \subset \ker(\phi)$. We have shown

**Proposition 3.3.1.** There is an exact sequence

$$0 \to \text{Pic}^0_{P/S} \to \text{Pic}_{P/S} \xrightarrow{\phi} \text{Hom}_S(J, \text{Pic}^0_{J/S})$$

The image of $\phi$ can be determined under certain assumptions, cf. [9, IV.20] for the case $S$ is a point where it is shown that, up to tensor product with $\mathbb{Q}$, $\phi(\text{Pic}_{P/S})$ is equal to the subset of elements of $\text{End}_S(J) \cong \text{Hom}_S(J, \text{Pic}^0_{J/S})$ invariant under the Rosati involution.

4. The main result

We maintain the notation and assumptions of §3.

4.1 – Class of $P$

Define a $\text{Pic}^0_{P/S}$-torsor $\Pi$ by the formula

$$\Pi = \phi^{-1}(-\lambda)$$

where $\phi$ is the fundamental map (3).

**Theorem 4.1.1.** There is a canonical isomorphism of $\text{Pic}^0_{P/S}$-torsors $P' \cong \Pi$, where $P'$ is the $\text{Pic}^0_{P/S}$-torsor deduced from $P$ via the morphism $\mu^{-1} \circ \lambda : J \to \text{Pic}^0_{P/S}$.

**Proof.** To simplify the notation, if $f_1 : J \to T_1$ are $S$-morphisms, we write $(f_1, f_2) := f_1 \circ p_1 \times f_2 \circ p_2 : J \times_S J \to T_1 \times_S T_2$, and similarly for more factors.

\(^2\) Indeed, by [10, Prop. 6.1] it factors through a section, which must be the zero section since it is a homomorphism.
Consider the isomorphism
\[ p_1 \times \tau : P \times_S J \to P \times_S P \]
Let 1 = id and
\[ \sigma := (p_1 \times \tau) \circ (1, -1) \]
The line bundle \( \sigma^* p_1^* \Theta^{-1} \) defines a morphism \( \theta : P \to \text{Pic}_{P/S} \). We claim that its image lies in \( \Pi \), i.e.
\[ \phi \circ \theta = -\lambda \in \text{Hom}_S(J, \text{Pic}^0_{J/S}). \]
This is local on \( S \), so we may assume that \( P = J \). First note that
\[ \sigma^2 = 1 \]

hence
\[ (5) \quad \sigma^* p_1^* \Theta^{-1} = \sigma^* p_1^* \Theta^{-1} = (1, -1)^* \tau^* \Theta^{-1} \]

Now \( \mathcal{M} = \tau^* \Theta^{-1} \otimes p_1^* \Theta \) induces the morphism \( -\lambda : J \to \text{Pic}^0_{J/S} \subset \text{Pic}_{J/S} \), hence \( \tau^* \Theta^{-1} = \mathcal{M} \otimes p_1^* \Theta^{-1} \) induces its translation by \( -\Theta \), i.e., \( -\lambda - \Theta : J \to \text{Pic}_{J/S} \). Since \( \phi \circ (\lambda - \Theta) = \phi(\lambda) + \phi(-\Theta) = \phi(-\Theta) = -\lambda \), this shows that \( \phi \circ \theta = -\lambda \).

Next, we show that \( \theta \) induces a morphism (hence isomorphism) of torsors. By [2, III, 1.4.6 (iii)] it suffices to show that the diagram
\[
\begin{array}{ccc}
P \times_S J & \overset{\tau}{\longrightarrow} & P \\
\downarrow{(1, \mu^{-1} \circ \lambda)} & & \downarrow{\theta} \\
\Pi \times_S \text{Pic}^0_{P/S} & \overset{+}{\longrightarrow} & \text{Pic}_{P/S}
\end{array}
\]
commutes. This is again local on \( S \), so we may assume \( P = J \). Consider the diagram
\[
\begin{array}{ccc}
J \times_S J & \overset{(1, \tau)}{\longrightarrow} & J \times_S J \\
\downarrow{(1, \theta, \lambda)} & & \downarrow{(1, \theta)} \\
J \times_S \Pi \times_S \text{Pic}^0_{J/S} & \overset{(1,+)}{\longrightarrow} & J \times_S \text{Pic}_{J/S}
\end{array}
\]
We must show that the pullback of the Poincaré bundle \( \mathcal{P} \) on \( J \times_S \text{Pic}_{J/S} \) clockwise has the same \( \equiv \)-class as the pullback counterclockwise in the diagram, where \( \equiv \) means isomorphism class up to tensor product with \( p_{23}^* \mathcal{M} \) for \( \mathcal{M} \in \text{Pic}(J \times_S J) \).
Before continuing with the proof we make a short digression. Define
\[ m = p_1 + p_2 + p_3 : J \times_S J \times_S J \to J \]
and
\[ m_{ij} = \tau \circ p_{ij} = p_i + p_j : J \times_S J \times_S J \to J \]
Fix any line bundle \( \mathcal{L} \) on \( J \) and consider the line bundle
\[ \mathcal{L}_1 := p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1} \otimes p_3^* \mathcal{L}^{-1} \otimes m_{12}^* \mathcal{L} \otimes m_{23}^* \mathcal{L} \]
on \( J \times_S J \times_S J \). It defines a morphism
\[ f_1 : \hat{J} \times_S J \times_S J \to \text{Pic}^0_{J/S} \]
where the hat denotes omission. Moreover, since \( \mathcal{L}_1 \mid_{J \times_S (0,0)} \) is a pullback from \( S \), it follows from 2.1.2 that this is homomorphism of abelian schemes. Similarly, the bundle
\[ \mathcal{L}_2 := m^* \mathcal{L} \otimes m_{13}^* \mathcal{L}^{-1} \]
defines a homomorphism \( f_2 : \hat{J} \times_S J \times_S J \to \text{Pic}^0_{J/S} \). Now, by [9, II.6, Cor. 2 p.58] we know that for \( S \) a point we have \( \mathcal{L}_1 \cong \mathcal{L}_2 \). So the difference \( f_1 - f_2 : \hat{J} \times_S J \times_S J \to \text{Pic}^0_{J/S} \) is a homomorphism of abelian schemes which is zero fibrewise. Hence it is zero. Applying this to \( \mathcal{L} = \Theta^{-1} \) we find
\[ m_{12}^* \Theta^{-1} \otimes p_1^* \Theta \otimes m_{13}^* \Theta^{-1} \equiv m^* \Theta^{-1} \tag{6} \]
Now, returning to the proof, we compute the pullback clockwise. By definition of \( \theta \) we have
\[ (1, \tau)^*(1, \theta)^* \mathcal{P} \equiv (1, \tau)^* \sigma_* p_2^* \Theta^{-1} \overset{(5)}{=} (1, \tau)^*(1, -1)^* \tau^* \Theta^{-1} \]
\[ = (1, -1, -1)^* m^* \Theta^{-1} \]
On the other hand, going counterclockwise we have by straightforward computation
\[ (1, +)^* \mathcal{P} \equiv p_{12}^* \mathcal{P} \otimes p_{13}^* \mathcal{P} \]
hence
\[ (1, \theta, \lambda)^*(1, +)^* \mathcal{P} \equiv (1, \theta, \lambda)^*(p_{12}^* \mathcal{P} \otimes p_{13}^* \mathcal{P}) = p_{12}^*(1, \theta)^* \mathcal{P} \otimes p_{13}^*(1, \lambda)^* \mathcal{P} \]
and using the definition of \( \theta \) we find
\[ (1, \theta, \lambda)^*(1, +)^* \mathcal{P} \equiv p_{12}^*(\sigma_* p_2^* \Theta^{-1}) \otimes p_{13}^*(1, \lambda)^* \mathcal{P} \]
\[ \overset{(5)}{=} p_{12}^*(1, -1)^* \tau^* \Theta^{-1} \otimes p_{13}^*(1, \lambda)^* \mathcal{P} \]
Now, since $\lambda$ is a homomorphism we have
\[(1, \lambda)^*P = (1, -1)^*(1, -\lambda)^*P \cong (1, -1)^*(p_1^*\Theta \otimes \tau^*\Theta^{-1})\]
modulo $p_2^*\text{Pic}(J)$. Substituting in the above we find
\[
(1, \theta, \lambda)^*(1, +)^*P \equiv p_{12}^*(1, -1)^*\tau^*\Theta^{-1} \otimes p_{13}^*(1, -1)^*(p_1^*\Theta \otimes \tau^*\Theta^{-1})
\]
\[
= (1, -1)^*(p_{12}^*\tau^*\Theta^{-1} \otimes p_{13}^*(p_1^*\Theta \otimes \tau^*\Theta^{-1}))
\]
\[
= (1, -1, -1)^*(m_{12}^*\Theta^{-1} \otimes p_1^*\Theta \otimes m_{13}^*\Theta^{-1})
\]
Finally, using (6) we get
\[
(1, \theta, \lambda)^*(1, +)^*P \equiv (1, -1, -1)^*m^*\Theta^{-1}
\]
as required. \(\square\)

Consider the canonical map
\[
Z \to \text{Hom}_S(J, \text{Pic}^0_J/S) : 1 \mapsto \lambda
\]
Pulling back (4) by this map we obtain an exact sequence
\[
(E_P) \quad 0 \to \text{Pic}_P^0 \to \text{Pic}_P/S \times_{\text{Hom}_S(J, \text{Pic}^0_J)} Z \to Z \to 0
\]

**Corollary 4.1.2.** The class of this extension in $H^1(S, \text{Pic}^0_P/S)$ is equal to the class of the image of $P$ under the map $H^1(S, J) \to H^1(S, \text{Pic}^0_P/S)$ induced by $\lambda$.

**Proof.** The class of this extension is equal to the image of 1 under the coboundary map $Z \to H^1(S, \text{Pic}^0_P/S)$ arising from $(E_P)$. By [2, III, 3.5.5] this class is the negative of that of the torsor $\phi^{-1}(\lambda)$, i.e. $\phi^{-1}(-\lambda)$. \(\square\)

### 4.2 - An application

As an application we apply Bhatt’s argument to obtain a generalisation of his result [1] to an arbitrary base.\(^3\) To lighten the notation write $J^t := \text{Pic}^0_J/S$. Let $\alpha \in \text{Br}(J^t)$ denote the image of $P$ under the canonical maps $H^1(S, J) = H^1(S, \text{Pic}^0_J/S) \to H^1(S, \text{Pic}_{J^t}/S) \to H^2(J^t, \mathbb{G}_m).$\(^4\)

\(^3\)Note our notation differs slightly from [1], e.g. the Picard scheme is denoted $\text{Pic}_{X/S}$ there.

\(^4\)The last map is defined using the identification $H^1(S, \text{Pic}_{J^t}/S) = \ker(F^1\text{Br}(J^t) \to \text{Br}(S))$, where $F^1$ is the first step in the filtration coming from the Leray spectral sequence for $J^t \to S$ and $0 : S \to J^t$ is the zero section.
Proposition 4.2.1. Assume $\text{Br}(S) = 0$ and that the order $p$ of the image of the class of $P$ in $H^1(S, \text{Pic}^0_{P/S})$ is finite. Then for every $j \in J^i$ there is an open neighbourhood $U \subset J^i$ and a finite étale morphism $T \to U$ of degree $p^{\dim S} J \cdot \chi(\Theta_s)$ splitting $\alpha$. \footnote{Here $\chi(\Theta_s) = \sum_i (-1)^i h^i(J_s, \Theta)$ is the usual locally constant function on $S$ [9, II.5, Cor. p.50].}

Proof. The key result is the existence [7; 1, Thm. A.1.1] of a twisted Fourier-Mukai transform $F : D(P) \to D(J^i, -\alpha)$, where $D(J^i, -\alpha)$ is a derived category of $-\alpha$-twisted sheaves. Let $\chi = \chi(\Theta_s)$.

Now, by 4.1.2 there is an element $L \in H^0(S, \text{Pic}_{P/S})$ such that $\phi(L) = p \cdot \lambda$. Since $\text{Br}(S) = 0$, $L$ is representable by a line bundle. By Mukai theory $F := F(L)$ is an $-\alpha$-twisted vector bundle of rank $\chi(L_s) = \chi(\Theta_s^{\otimes p}) = p^{\dim S} J \chi$. Then $A := \text{End}(F)$ is an Azumaya algebra of rank $(p^{\dim S} J \chi)^2$ representing the class of $-\alpha$. So the opposite algebra of $A$ represents $\alpha$. By [5, 5.7], in an open neighbourhood of any point $j \in J^i$ there is a finite étale morphism of degree $p^{\dim S} J \chi$ trivialising $\alpha$. \qed

Acknowledgement. I thank the referee for helpful remarks.

References

[1] B. Bhatt, A period-index result, appendix to [7].