Rend. Sem. Mat. Univ. Padova, DRAFT, 1–31

A necessary and sufficient condition for C^1-regularity of solutions of one-dimensional variational obstacle problems

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Abstract – In this paper we study the C^1-regularity of solutions of one-dimensional variational obstacle problems in $W^{1,1}$ when the obstacles are $C^{1,\sigma}$ and the Lagrangian is locally Hölder continuous and globally elliptic. In this framework, we prove that the solutions of one-dimensional variational obstacle problems are C^1 for all boundary data if and only if the value function is Lipschitz continuous at all boundary data.

Mathematics Subject Classification (2010). Primary: 49J05, 49N60; Secondary: 49J30, 49K30.

Keywords. One-dimensional variational obstacle problems, C^1-regularity, Value function, Conditional equa-continuity.

Contents

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1. Introduction

In this paper we are concerned with C^1-regularity of solutions of one-dimensional variational obstacle problems of type

$$\inf\left\{\mathcal{J}_L(u;[a,b]):=\int_a^b L(x,u(x),u'(x))dx : u\in\mathcal{A}_{f,g}(a,b,A,B)\right\} \tag{1}$$

with $L:\mathbb{R}^3\to[0,\infty[$ and $\mathcal{A}_{f,g}(a,b,A,B)\subset W^{1,1}([a,b])$ given by

$$\mathcal{A}_{f,g}(a,b,A,B):=\left\{u\in W^{1,1}([a,b]) : u(a)=A, u(b)=B, f\leqslant u\leqslant g\right\}, \tag{2}$$

where $(a,b,A,B)\in\mathbb{R}^4$ with $a<b$ and $f,g:\mathbb{R}\to\mathbb{R}$ with $f<g$, i.e., $f(x)<g(x)$ for all $x\in\mathbb{R}$, $f(a)\leqslant A\leqslant g(a)$ and $f(b)\leqslant B\leqslant g(b)$. Usually, the functions f and g are called obstacles. We prove that if the obstacles are $C^{1,\sigma}$ and if the Lagrangian L is locally Hölder continuous and globally elliptic, then the solutions of one-dimensional variational obstacle problems of type (1) are C^1-regular for all boundary data if and only if the value function associated with (1), i.e.,

$$(a,b,A,B)\mapsto\inf\left\{\mathcal{J}_L(u;[a,b]) : u\in\mathcal{A}_{f,g}(a,b,A,B)\right\},$$

is Lipschitz continuous at all boundary data (see Theorem 2.3). Without obstacles such a equivalence theorem was established by Sychev (see [Syc91]) and Mizel and Sychev (see [SM98] and see also [GST16]). The techniques used in the present paper are inspired from the works of Sychev and *al.*

The plan of the paper is as follows. In the next section we state the main result of the paper, i.e., Theorem 2.3. In Section 3, we give auxiliary results that we need for proving Theorem 2.3. (More precisely, we use Corollary 3.18, Theorem 3.2 and Lemma 3.21 for proving that C^1-regularity of solutions for all boundary data implies Lipschitz continuity of the value function at all boundary data, and we use Theorems 3.2 and 3.12 for proving the converse implication.) The proof of Theorem 2.3 is given in Section 4.

Notation. Troughout the paper, for any compact set $K\subset\mathbb{R}$, $C(K)$ denotes the class of continuous functions from K to $\mathbb{R}$ and $\|\cdot\|_{C(K)}$ is the uniform norm, i.e., for each $u\in C(K)$, $\|u\|_{C(K)}:=\sup\{|u(x)| : x\in K\}$.

2. Main result

Let $(a,b,A,B)\in\mathbb{R}^4$ with $a<b$ and $f,g:\mathbb{R}\to\mathbb{R}$ with $f<g$, , i.e., $f(x)<g(x)$ for all $x\in\mathbb{R}$, $f(a)\leqslant A\leqslant g(a)$ and $f(b)\leqslant B\leqslant g(b)$ and let $L:\mathbb{R}^3\to[0,\infty[$. In what follows, we consider the following two assumptions:

(H$_1$) L is locally Hölder continuous, i.e., for all compact $G \subset \mathbb{R}^3$ there exist $C = C(G) > 0$ and $\alpha = \alpha(G) > 0$ such that

$$|L(x_1, u_1, v_1) - L(x_2, u_2, v_2)| \leqslant C\left(|x_1 - x_2| + |u_1 - u_2| + |v_1 - v_2|\right)^{\alpha}$$

for all $(x_1, u_1, v_1), (x_2, u_2, v_2) \in G$;

(H$_2$) $L_{vv} \in C(\mathbb{R}^3)$ and there exists $\mu > 0$ such that $L_{vv} \geqslant \mu$ everywhere. (Then, we have $L_v \in C(\mathbb{R}^3)$ and

$$L(x, u, v_2) - L(x, u, v_1) - L_v(x, u, v_1)(v_2 - v_1) \geqslant \frac{\mu}{2}(v_2 - v_1)^2$$

for all $(x, u, v_1), (x, u, v_2) \in \mathbb{R}^3$.)

Let $V \subset \mathbb{R}^4$ be given by

$$V := \Big\{(a, b, A, B) \in \mathbb{R}^4 : a < b, f(a) \leqslant A \leqslant g(a), f(b) \leqslant B \leqslant g(b)\Big\}. \tag{3}$$

Let $\mathcal{V} : V \to [0, \infty]$ be the value function defined by

$$\mathcal{V}(a, A, b, B) := \inf\Big\{\mathcal{J}_L(u; [a, b]) : u \in \mathcal{A}_{f,g}(a, b, A, B)\Big\}. \tag{4}$$

Remark 2.1. If L is continuous and if $f, g \in C^1(\mathbb{R})$ then $\mathcal{V}(a, A, b, B) \in [0, \infty[$ for all $(a, b, A, B) \in V$. Indeed, let $(a, b, A, B) \in V$. (One has $\mathcal{V}(a, b, A, B) \geqslant 0$ because $L \geqslant 0$.) As $f, g \in C^1(\mathbb{R})$ we have $\overline{\mathcal{A}}_{f,g}(a, b, A, B) \neq \varnothing$ with

$$\overline{\mathcal{A}}_{f,g}(a, b, A, B) := \Big\{u \in C^1([a, b]) : u(a) = A, u(b) = B, f \leqslant u \leqslant g\Big\}. \tag{5}$$

But $\mathcal{J}_L(u; [a, b]) < \infty$ for all $u \in C^1([a, b])$ because L is continuous, hence $\mathcal{V}(a, A, b, B) < \infty$ since $\overline{\mathcal{A}}_{f,g}(a, b, A, B) \subset \mathcal{A}_{f,g}(a, b, A, B)$.

Let $\mathcal{S}_{f,g}(a, b, A, B)$ be the class of solutions of the variational obstacle problem (1), i.e.,

$$\mathcal{S}_{f,g}(a, b, A, B) := \Big\{u \in \mathcal{A}_{f,g}(a, b, A, B) : \mathcal{J}_L(u; [a, b]) = \mathcal{V}(a, b, A, B)\Big\}. \tag{6}$$

Remark 2.2. If L is continuous and satisfies (H$_2$) and if $f, g \in C^1(\mathbb{R})$ then $\mathcal{S}_{f,g}(a, b, A, B) \neq \varnothing$ for all $(a, b, A, B) \in V$, see Theorem 3.2.

Here is the main result of the paper which gives a necessary and sufficient condition for C^1-regularity of solutions of one-dimensional variational obstacle problems of type (1) when the obstacles f and g are $C^{1,\sigma}$ and the Lagrangian L is locally Hölder continuous and globally elliptic, i.e., L satisfies (H$_1$) and (H$_2$).

THEOREM 2.3. *Assume that* (H_1) *and* (H_2) *hold and* $f, g \in C^{1,\sigma}(\mathbb{R})$. *Then, the following two assertions are equivalent*:

(A_1) *for every* $(a,b,A,B) \in V$, $\mathcal{S}_{f,g}(a,b,A,B) \subset C^1([a,b])$;

(A_2) *for every* $(a,b,A,B) \in V$, $\mathcal{V}$ *is Lipschitz continuous at* (a,b,A,B).

REMARK 2.4. The assertion (A_1) implies the nonoccurence of the Lavrentiev gap phenomenon, i.e., for every $(a,b,A,B) \in V$, $\mathcal{V}(a,b,A,B) = \overline{\mathcal{V}}(a,b,A,B)$, where $\overline{\mathcal{V}} : V \to \mathbb{R}$ is defined by

$$\overline{\mathcal{V}}(a,b,A,B) := \inf\left\{\mathcal{J}_L(u;[a,b]) : u \in \overline{\mathcal{A}}_{f,g}(a,b,A,B)\right\}. \tag{7}$$

Indeed, given $(a,b,A,B) \in V$, since $\overline{\mathcal{V}}(a,b,A,B) \geqslant \mathcal{V}(a,b,A,B)$, it is sufficient to prove that $\overline{\mathcal{V}}(a,b,A,B) \leqslant \mathcal{V}(a,b,A,B)$. By Remark 2.2 (see Theorem 3.2) there exists $u \in \mathcal{S}_{f,g}(a,b,A,B)$, i.e., $u \in \mathcal{A}_{f,g}(a,b,A,B)$ and $\mathcal{J}_L(u;[a,b]) = \mathcal{V}(a,b,A,B)$. But $u \in C^1([a,b])$ by (A_1) hence $u \in \overline{\mathcal{A}}_{f,g}(a,b,A,B)$ and so $\mathcal{J}_L(u;[a,b]) \geqslant \overline{\mathcal{V}}(a,b,A,B)$, which gives the result. (For more details on the Lavrentiev gap phenomenon we refer the reader to [LAV26, MA34, CES83, BM92].)

3. Auxiliary results

3.1 – *An existence theorem for C^1-obstacles*

Given any $(a,b) \in \mathbb{R}^2$ such that $a < b$, let $W^{1,1}([a,b])$ be the class of $W^{1,1}$-Sobolev functions from $[a,b]$ to $\mathbb{R}$. (Note that $W^{1,1}([a,b]) = AC([a,b])$ where $AC([a,b])$ is the class of absolutely continuous functions from $[a,b]$ to $\mathbb{R}$, see [BGH98, Chapter 2] and the references therein.) We begin with the following semicontinuity theorem due to Tonelli (see [TON23]).

THEOREM 3.1 (Tonelli). *Assume that L is continuous and satisfies* (H_2). *Then, for each* $u_\infty \in W^{1,1}([a,b])$ *and each* $\{u_n\}_n \subset W^{1,1}([a,b])$ *such that* $\|u_n - u_\infty\|_{C([a,b])} \to 0$, *one has*

$$\varliminf_{n\to\infty} \mathcal{J}_L(u_n;[a,b]) \geqslant \mathcal{J}_L(u_\infty;[a,b]).$$

By using Tonelli's existence theory, we can establish the following existence result for C^1-obstacles.

THEOREM 3.2. *Let* $(a,b,A,B) \in V$ *with* V *given by* (3). *If L is continuous and satisfies* (H_2) *and if* $f, g \in C^1(\mathbb{R})$ *then* $\mathcal{S}_{f,g}(a,b,A,B) \neq \varnothing$.

To prove Theorem 3.2 we need the following two results: Lemma 3.3 below (whose proof can be found in [Man, Remark 2.3]) and Arzelà-Ascoli's theorem, i.e., Theorem 3.4 below (see for instance [McS57] for a proof).

Lemma 3.3. *Assume that L is continuous and satisfies* (H_2). *Then, for every $u \in W^{1,1}([a,b])$ such that $\mathcal{J}_L(u;[a,b]) \leqslant c$ and $\{(x,u(x)) : x \in [a,b]\} \subset K$, where $c > 0$ and K is a compact of $\mathbb{R}^2$, there exists $C = C(c,K,\mu) > 0$, where $\mu > 0$ is given by* (H_2), *such that $\|u'\|_{L^2([a,b])} \leqslant C$.*

Theorem 3.4 (Arzelà-Ascoli). *Let $\{u_n\}_n \subset C([a,b])$ be such that $\{u_n\}_n$ is uniformly bounded, i.e., $\sup_{n\geqslant 1}\|u_n\|_{C([a,b])} < \infty$, and equi-continuous, i.e., there exists an increasing function $\omega : [0,\infty[\to [0,\infty[$ with $\omega(0) = 0$ and $\lim_{t\to 0}\omega(t) = 0$ such that $|u_n(x) - u_n(y)| \leqslant \omega(|x-y|)$ for all $n \geqslant 1$ and all $x,y \in [a,b]$. Then, there exists $u_\infty \in C([a,b])$ such that (up to a subsequence) $\|u_n - u_\infty\|_{C([a,b])} \to 0$.*

Proof of Theorem 3.2. Fix $(a,b,A,B) \in V$. (We have $\mathcal{V}(a,A,b,B) \in [0,\infty[$, see Remark 2.1.) Let $\{u_n\}_n \subset \mathcal{A}_{f,g}(a,b,A,B)$ be such that

$$\begin{aligned}(8)\qquad \mathcal{V}(a,A,b,B) \leqslant \mathcal{J}_L(u_n;[a,b]) &< \mathcal{V}(a,A,b,B) + \frac{1}{n}\\ &\leqslant c := \max\big\{1, \mathcal{V}(a,A,b,B)+1\big\}\end{aligned}$$

for all $n \geqslant 1$. Then

$$(9)\qquad \forall n \geqslant 1 \quad \big\{(x,u_n(x)) : x \in [a,b]\big\} \subset K$$

with $K := \big\{(x,u) \in [a,b]\times\mathbb{R} : f(x) \leqslant u \leqslant g(x)\big\}$ and

$$(10)\qquad \lim_{n\to\infty}\mathcal{J}_L(u_n;[a,b]) = \mathcal{V}(a,A,b,B).$$

Taking (8) and (9) into account, by Lemma 3.3 we can assert that there exists $C > 0$ such that

$$(11)\qquad \forall n \geqslant 1 \quad \|u_n'\|_{L^2([a,b])} \leqslant C.$$

Hence, there exists $v_\infty \in L^2([a,b])$ such that (up to a subsequence)

$$(12)\qquad u_n' \rightharpoonup v_\infty \text{ in } L^2([a,b]).$$

On the other hand, as $\{u_n\}_n \subset W^{1,1}([a,b])$ one has

$$\begin{cases} \big|u_n(x)\big| - \big|u_n(a)\big| \leqslant \big|u_n(x) - u_n(a)\big| = \left|\displaystyle\int_a^x u_n'(t)dt\right| \leqslant \displaystyle\int_a^b \big|u_n'(t)\big|dt,\\ \big|u_n(x) - u_n(y)\big| = \left|\displaystyle\int_y^x u_n'(t)dt\right| \leqslant \displaystyle\int_y^x \big|u_n'(t)\big|dt,\end{cases}$$

for any $n \geqslant 1$ and any $x, y \in [a,b]$, and so

$$\begin{cases} \big|u_n(x)\big| \leqslant |A| + \left(\int_a^b \big|u_n'(t)\big|^2 dt\right)^{\frac{1}{2}} |b-a|^{\frac{1}{2}} \leqslant |A| + \|u_n'\|_{L^2([a,b])}|b-a|^{\frac{1}{2}}, \\ \big|u_n(x) - u_n(y)\big| \leqslant \left(\int_y^x \big|u_n'(t)\big|^2 dt\right)^{\frac{1}{2}} |x-y|^{\frac{1}{2}} \leqslant \|u_n'\|_{L^2([a,b])}|x-y|^{\frac{1}{2}}, \end{cases}$$

by using Cauchy-Schwarz's inequality and noticing that $u_n(a) = A$. Thus, taking (11) into account, we see that

$$\begin{cases} \forall n \geqslant 1 \quad \forall x \in [a,b] \quad \big|u_n(x)\big| \leqslant A + C|b-a|^{\frac{1}{2}}, \text{ and so } \sup\limits_{n\geqslant 1} \|u_n\|_{C([a,b])} < \infty, \\ \forall n \geqslant 1 \quad \forall x, y \in [a,b] \quad \big|u_n(x) - u_n(y)\big| \leqslant \omega(|x-y|) \text{ with } \omega(t) = Ct^{\frac{1}{2}}, \end{cases}$$

which show that $\{u_n\}_n$ is uniformly bounded and equi-continuous. From Theorem 3.4 it follows that there exists $u_\infty \in C([a,b])$ such that (up to a subsequence)

$$\|u_n - u_\infty\|_{C([a,b])} \to 0. \tag{13}$$

In particular $\{u_n\}_n$ pointwise converges to u_∞, and so $u_\infty(a) = A$ and $u_\infty(b) = B$ (resp. $f(x) \leqslant u_\infty(x) \leqslant g(x)$ for all $x \in [a,b]$) because $u_n(a) = A$ and $u_n(b) = B$ for all $n \geqslant 1$ (resp. $f(x) \leqslant u_n(x) \leqslant g(x)$ for all $n \geqslant 1$ and all $x \in [a,b]$). Combining (12) with (13) we deduce that v_∞ is the weak derivative of u_∞, and consequently $u_\infty \in W^{1,1}([a,b])$. Thus $u_\infty \in \mathcal{A}_{f,g}(a,b,A,B)$. Taking (10) into account, from Theorem 3.1 we conclude that

$$\mathcal{V}(a,b,A,B) = \lim_{n\to\infty} \mathcal{J}_L(u_n;[a,b]) \geqslant \mathcal{J}_L(u_\infty;[a,b]),$$

which implies that $u_\infty \in \mathcal{S}_{f,g}(a,b,A,B)$. ■

3.2 – *Regularity theorems for $C^{1,\sigma}$ -obstacles*

Recall first that every $u \in W^{1,1}([a,b])$ is uniformly continuous on $[a,b]$ and almost everywhere differentiable in $[a,b]$, i.e.,

$$\big|[a,b] \backslash \Omega_u\big| = 0 \text{ where } \Omega_u := \Big\{x \in [a,b] : u \text{ is differentiable at } x\Big\}.$$

(Note that $\Omega_u = \big\{x \in [a,b] : u'(x) \neq -\infty \text{ and } u'(x) \neq \infty\big\} = \big\{x \in [a,b] : |u'(x)| < \infty\big\}$.) The following notion of regularity comes from Tonelli.

Definition 3.5. We say that $u \in W^{1,1}([a,b])$ has *Tonelli's partial regularity* if

$$u' \in C\big([a,b];[-\infty,\infty]\big). \tag{14}$$

(In particular, (14) implies that:

- Ω_u is an open subset of $[a,b]$;
- $[a,b]\backslash\Omega_u = \Big\{x \in [a,b] : u'(x) = -\infty \text{ or } u'(x) = \infty\Big\} = \Big\{x \in [a,b] : |u'(x)| = \infty\Big\}$ is a closed subset of $[a,b]$.)

We denote the class of $u \in W^{1,1}([a,b])$ such that u has Tonelli's partial regularity by $W^{1,1}_{\mathrm{T}}([a,b])$.

For each $u \in W^{1,1}([a,b])$ and each $s,t \in [a,b]$ with $s<t$, we set

$$k_u(s,t) = \frac{u(s)-u(t)}{s-t}$$

and we define $u_{s,t} \in W^{1,1}([a,b])$ by

$$u_{s,t}(x) := \begin{cases} u(s) + k_u(s,t)(x-s) & \text{if } x \in]s,t[\\ u(x) & \text{if } x \in [a,b]\backslash]s,t[. \end{cases} \tag{15}$$

Then, for every $s,t \in [a,b]$ with $s<t$, one has

$$u'_{s,t}(x) = k_u(s,t) \text{ for all } x \in]s,t[.$$

Fix $\kappa,\lambda \in \mathbb{R}$ such that $\kappa<\lambda$ and $[a,b] \subset [\kappa,\lambda]$. Given $f,g \in C^1(\mathbb{R})$, let $C_0,\alpha_0>0$ be given by (H_1) with $G = [\kappa,\lambda]\times[-M_1,M_1]\times[-M_2,M_2]$, where:

$$\begin{cases} M_1 := \max\big\{\|f\|_{C([\kappa,\lambda])}, \|g\|_{C([\kappa,\lambda])}\big\} + M_2(\kappa-\lambda); \\ M_2 := \max\big\{\|f'\|_{C([\kappa,\lambda])}, \|g'\|_{C([\kappa,\lambda])}\big\}, \end{cases}$$

and let $\omega_{\kappa,\lambda} : [0,\infty[\times[0,\infty[\to[0,\infty[$ be given by

$$\omega_{\kappa,\lambda}(k,\varepsilon) := C_0\Big[\big(\omega_f^{\kappa,\lambda}(\varepsilon)+\omega_{f'}^{\kappa,\lambda}(\varepsilon)+k\varepsilon\big)^{\alpha_0}+\big(\omega_g^{\kappa,\lambda}(\varepsilon)+\omega_{g'}^{\kappa,\lambda}(\varepsilon)+k\varepsilon\big)^{\alpha_0}\Big], \tag{16}$$

where $\omega_f^{\kappa,\lambda}, \omega_{f'}^{\kappa,\lambda}, \omega_g^{\kappa,\lambda}, \omega_{g'}^{\kappa,\lambda} : [0,\infty[\to[0,\infty[$ are the moduli of continuity on $[\kappa,\lambda]$ of f, f', g and g' respectively.

Definition 3.6. Let K be a compact subset of $\mathbb{R}\times\mathbb{R}$ and let $c>0$. By $\mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$ we denote the class of $u \in W^{1,1}([a,b])$ with the following three properties:

- $\{(x,u(x)) : x \in [a,b]\} \subset K$;
- $\mathcal{J}_L(u;[a,b]) := \displaystyle\int_a^b L(x,u(x),u'(x))dx \leqslant c$;
- for every $s,t \in [a,b]$ with $s<t$, one has
$$\mathcal{J}_L(u;[a,b]) \leqslant \mathcal{J}_L(u_{s,t};[a,b]) + \omega_{\kappa,\lambda}\left(|k_u(s,t)|, |s-t|\right)|s-t|$$
with $\omega_{\kappa,\lambda} : [0,\infty[\times[0,\infty[\to [0,\infty[$ defined by (16).

The following regularity result is a particular case of [Man, Theorem 2.5] (for more details see [Man, Lemma 2.6, Remark 2.7, Theorem 2.8 and Corollary 2.9]).

Theorem 3.7. *Let K be a compact subset of $\mathbb{R}\times\mathbb{R}$ and let $c>0$. If* (H_1) *and* (H_2) *hold and if $f,g \in C^{1,\sigma}(\mathbb{R})$, then*
$$\mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c) \subset W^{1,1}_{\mathrm{T}}([a,b]),$$
i.e., every $u \in \mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$ has Tonelli's partial regularity.

Let us introduce another class of $W^{1,1}$-Sobolev functions.

Definition 3.8. Given $c>0$, let $\mathcal{L}_{f,g}(a,b,c) \subset W^{1,1}([a,b])$ denote the class of $u \in W^{1,1}([a,b])$ satisfying the second point of Definition 3.6 and the following two properties:

- $f \leqslant u \leqslant g$;
- for every $s,t \in [a,b]$ with $s<t$, one has
$$\mathcal{J}_L(u;[a,b]) \leqslant \mathcal{J}_L(v^u_{s,t};[a,b])$$
where $v^u_{s,t} \in W^{1,1}([a,b])$ is given by
$$v^u_{s,t}(x) = \begin{cases} u_{s,t}(x) & \text{if } f(x) \leqslant u_{s,t}(x) \leqslant g(x) \\ f(x) & \text{if } f(x) > u_{s,t}(x) \\ g(x) & \text{if } u_{s,t}(x) > g(x) \end{cases}$$
with $u_{s,t} \in W^{1,1}([a,b])$ defined by (15).

Remark 3.9. For each $u \in \mathcal{S}_{f,g}(a,b,A,B)$ we have $\mathcal{J}_L(u;[a,b]) = \mathcal{V}(a,b,A,B)$. So, given $c>0$, as $v^u_{s,t}(a) = u(a)$ and $v^u_{s,t}(b) = u(b)$, we see that
$$\mathcal{S}_{f,g}(a,b,A,B) \subset \mathcal{L}_{f,g}(a,b,c)$$
for all $(a,b,A,B) \in \mathbb{R}^4$ such that $\mathcal{V}(a,b,A,B) \leqslant c$.

The link between $\mathcal{L}_{f,g}(a,b,c)$ and $\mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$ is given by the following lemma. (For a proof, see [Man, Proof of Lemma 2.6].)

Lemma 3.10. *If $f, g \in C^1(\mathbb{R})$ then*

$$\mathcal{L}_{f,g}(a,b,c) \subset \mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$$

for all compact sets $K \subset \mathbb{R}\times\mathbb{R}$ such that $K \supset \{(x,u) \in [a,b]\times\mathbb{R} : f(x) \leqslant u \leqslant g(x)\}$ and all $c > 0$.

As a direct consequence of Remark 3.9 and Lemma 3.10 we have

Corollary 3.11. *Let $c > 0$. If* (H_1) *and* (H_2) *hold and if $f, g \in C^1(\mathbb{R})$, then*

$$\mathcal{S}_{f,g}(a,b,A,B) \subset \mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$$

for all $(a,b,A,B) \in \mathbb{R}^4$ and all compact sets $K \subset \mathbb{R}\times\mathbb{R}$ such that $\mathcal{V}(a,b,A,B) \leqslant c$ and $K \supset \{(x,u) \in [a,b]\times\mathbb{R} : f(x) \leqslant u \leqslant g(x)\}$.

By taking Theorem 3.7 into account, we obtain the following regularity result (see [Man, Corollary 2.9]).

Theorem 3.12. *Assume that* (H_1) *and* (H_2) *hold and $f, g \in C^{1,\sigma}(\mathbb{R})$. Then, for every $(a,b,A,B) \in \mathbb{R}^4$, we have*

$$\mathcal{S}_{f,g}(a,b,A,B) \subset W_T^{1,1}([a,b]),$$

i.e., every solution of the variational obstacle problem (1) *has Tonelli's partial regularity.*

3.3 – *Conditional equa-continuity*

The concept of conditional equa-continuity was introduced by Sychev in [Syc94].

Definition 3.13. We say that set $\mathcal{F} \subset W_T^{1,1}([a,b])$ has derivatives which are *conditionally equa-continuous* if for every $M > 0$ and every $\varepsilon > 0$ there exists $\delta(M,\varepsilon) > 0$ such that for all $u \in \mathcal{F}$, all $x_0 \in \Omega_u$ and all $x \in [a,b]$, one has

$$\Big(|u'(x_0)| < M \text{ and } |x - x_0| < \delta(M,\varepsilon)\Big) \Rightarrow \Big(x \in \Omega_u \text{ and } |u'(x) - u'(x_0)| \leqslant \varepsilon\Big).$$

The function $\delta :]0,\infty[\times]0,\infty[\to]0,\infty[$ is called *modulus of conditional equa-continuity* associated with the derivatives of $\mathcal{F}$.

The interest of Definition 3.13 comes from Lemma 3.14 below, which was proved by Sychev in [Syc94] (see [SM98, Lemmas 2.1 and 2.2] for a proof).

Lemma 3.14. *Let* $\mathcal{F} \subset W^{1,1}_T([a,b])$ *be such that its derivatives are conditionally equa-continuous and let* $\{u_n\}_n \subset \mathcal{F}$. *If* $\{u'_n\}_n$ *is uniformly integrable then there exists* $v_\infty \in C([a,b];[-\infty,\infty])$ *such that (up to a subsequence)* $\|u'_n - v_\infty\|_{C(K)} \to 0$ *for all compact set* $K \subset D_\infty$ *where* $D_\infty := \{x \in [a,b] : |v_0(x)| < \infty\}$ *and* $|[a,b]\backslash D_\infty| = 0$.

The following result is a particular case of [Man, Lemma 4.2].

Lemma 3.15. *Let* $\kappa, \lambda \in \mathbb{R}$ *be such that* $\kappa < \lambda$ *and* $[a,b] \subset [\kappa,\lambda]$. *Let* K *be a compact subset of* $\mathbb{R} \times \mathbb{R}$ *and let* $c > 0$. *If* (H_1) *and* (H_2) *hold and if* $f, g \in C^{1,\sigma}(\mathbb{R})$ *then the set* $\mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$ *(which is contained in* $W^{1,1}_T([a,b])$ *by Theorem 3.7) has derivatives which are conditionally equa-continuous whose modulus of conditional equa-continuity only depends on* κ, λ, $\omega_{\kappa,\lambda}$, L, K *and* c.

As a direct consequence of Lemma 3.15 and Corollary 3.11 we have

Lemma 3.16. *Let* $\kappa, \lambda \in \mathbb{R}$ *be such that* $\kappa < \lambda$, *let* $K \subset \mathbb{R} \times \mathbb{R}$ *be a compact set and let* $c > 0$. *If* (H_1) *and* (H_2) *hold and if* $f, g \in C^{1,\sigma}(\mathbb{R})$ *then, for every* $(a,b,A,B) \in \mathbb{R}^4$ *such that* $[a,b] \subset [\kappa,\lambda]$, $\big\{(x,u) \in [a,b] \times \mathbb{R} : f(x) \leqslant u \leqslant g(x)\big\} \subset K$ *and* $\mathcal{V}(a,b,A,B) \leqslant c$, *the set* $\mathcal{S}_{f,g}(a,b,A,B)$ *has derivatives which are conditionally equa-continuous whose modulus of conditional equa-continuity is the one associated with the derivatives of* $\mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$.

In what follows $(a,b,A,B) \in V$, where V is given by (3). By using Lemmas 3.14 and 3.16 we can prove the following proposition.

Proposition 3.17. *Assume that* (H_1) *and* (H_2) *hold and* $f, g \in C^{1,\sigma}(\mathbb{R})$, *and consider* $\{u_n\}_n \subset \mathcal{S}_{f,g}(a,b,A,B)$. *Then, there exists* $u_\infty \in \mathcal{S}_{f,g}(a,b,A,B)$ *such that (up to a subsequence)* $\|u_n - u_\infty\|_{C([a,b])} \to 0$ *and* $\|u'_n - u'_\infty\|_{C(K)} \to 0$ *for all compact set* $K \subset \Omega_{u_\infty}$.

Proof of Proposition 3.17. As $\{u_n\}_n \subset \mathcal{S}_{f,g}(a,b,A,B)$ we have

$$\forall n \geqslant 1 \quad \mathcal{J}_L(u_n;[a,b]) = \mathcal{V}(a,b,A,B). \tag{17}$$

Arguing as in the proof of Theorem 3.2, by using (17) instead of the right inequality in (8) and by noticing that (9) holds, from Lemma 3.3

we can assert that (11) is satisfied, which shows that $\{u_n'\}_n$ is uniformly integrable. Moreover, by Lemma 3.16, $\mathcal{S}_{f,g}(a,b,A,B)$ has derivatives which are conditionally equa-continuous. Hence, from Lemma 3.14 it follows that there exists $v_\infty \in C([a,b];[-\infty,\infty])$ such that (up to a subsequence)

$$\|u_n' - v_\infty\|_{C(K)} \to 0 \text{ for all compact set } K \subset D_\infty, \tag{18}$$

where $D_\infty := \{x \in [a,b] : |v_0(x)| < \infty\}$ and $|[a,b]\backslash D_\infty| = 0$. On the other hand, using the same method as in the proof of Theorem 3.2, by noticing that (17) implies (10), we can prove that there exists $u_\infty \in \mathcal{S}_{f,g}(a,b,A,B)$ such that (up to a subsequence)

$$\|u_n - u_\infty\|_{C([a,b])} \to 0. \tag{19}$$

As D_∞ and Ω_{u_∞} are open subsets of $[a,b]$, also is $D_\infty \cap \Omega_{u_\infty}$, and so for each $x \in D_\infty \cap \Omega_{u_\infty}$ there exists a compact set $K \subset D_\infty \cap \Omega_{u_\infty}$ such that $x \in K$. Hence, from (18) and (19) we can assert that $v_\infty(x) = u_\infty'(x)$ for all $x \in D_\infty \cap \Omega_{u_\infty}$. It follows that

$$\forall x \in \overline{D_\infty \cap \Omega_{u_\infty}} \quad v_\infty(x) = u_\infty'(x)$$

because $u_\infty' \in C([a,b];[-\infty,\infty])$ by Theorem 3.12, where $\overline{D_\infty \cap \Omega_{u_\infty}}$ denotes the closure of $D_\infty \cap \Omega_{u_\infty}$ in $[a,b]$. As $|[a,b]\backslash D_\infty \cap \Omega_{u_\infty}| = 0$ we have $\overline{D_\infty \cap \Omega_{u_\infty}} = [a,b]$, and the proof is complete. ■

As a consequence of Proposition 3.17 we have the following result that we will use in the proof of Theorem 2.3.

Corollary 3.18. *Assume that* (H_1) *and* (H_2) *hold and* $f,g \in C^{1,\sigma}(\mathbb{R})$. *Suppose also that* $\mathcal{S}_{f,g}(a,b,A,B) \subset C^1([a,b])$ *and let* $\{u_n\}_n \subset \mathcal{S}_{f,g}(a,b,A,B)$. *Then, there exists* $u_\infty \in \mathcal{S}_{f,g}(a,b,A,B)$ *such that (up to a subsequence)* $\|u_n - u_\infty\|_{C([a,b])} \to 0$ *and* $\|u_n' - u_\infty'\|_{C([a,b])} \to 0$.

Proof of Corollary 3.18. From Proposition 3.17 there exists $u_\infty \in \mathcal{S}_{f,g}(a,b,A,B)$ such that (up to a subsequence) $\|u_n' - u_\infty'\|_{C(K)} \to 0$ for all compact set $K \subset \Omega_{u_\infty}$. As $\mathcal{S}_{f,g}(a,b,A,B) \subset C^1([a,b])$ we have $u_\infty \in C^1([a,b])$, hence $\Omega_{u_\infty} = [a,b]$. It follows that we can take $K = [a,b]$ which gives $\|u_n' - u_\infty'\|_{C([a,b])} \to 0$, and the proof is complete. ■

3.4 – *Continuity of the value function*

The following lemma shows that the value function is continuous whenever the Lavrentiev gap phenomenon is absent.

Lemma 3.19. *If L is continuous and satisfies* (H_2) *and if* $f, g \in C^1(\mathbb{R})$ *then the value function* $\mathcal{V}$*, defined by* (2)-(3)-(4)*, is lower semicontinuous. If furthermore the Lavrentiev gap phenomenon is absent, i.e.,* $\mathcal{V}(a,b,A,B) = \overline{\mathcal{V}}(a,b,A,B)$ *for all* $(a,b,A,B) \in V$ *with* $\overline{\mathcal{V}}(a,b,A,B)$ *defined by* (3)-(7)-(5)*, then* $\mathcal{V}$ *is upper semicontinuous.*

Proof of Lemma 3.19. Let $(a,b,A,B) \in V$ and let $\{(a_n,b_n,A_n,B_n)\}_n \subset V$ be such that

$$\left\|(a_n,b_n,A_n,B_n) - (a,b,A,B)\right\|_\infty \to 0$$

with $\|(a_n,b_n,A_n,B_n)-(a,b,A,B)\|_\infty := \max\left\{|a_n-a|, |b_n-a|, |A_n-A|, |B_n-A|\right\}$. Then:

$$|a_n - a| \to 0; \tag{20}$$

$$|b_n - b| \to 0; \tag{21}$$

$$|A_n - A| \to 0; \tag{22}$$

$$|B_n - B| \to 0. \tag{23}$$

Thus:

(24) $\{a_n\}_n$ is bounded,
i.e., $\underline{a} \leqslant a_n \leqslant \overline{a}$ for all $n \geqslant 1$ and some $\underline{a}, \overline{a} \in \mathbb{R}$;

(25) $\{b_n\}_n$ is bounded,
i.e., $\underline{b} \leqslant b_n \leqslant \overline{b}$ for all $n \geqslant 1$ and some $\underline{b}, \overline{b} \in \mathbb{R}$;

(26) $\{A_n\}_n$ is bounded,
i.e., $\underline{A} \leqslant A_n \leqslant \overline{A}$ for all $n \geqslant 1$ and some $\underline{A}, \overline{A} \in \mathbb{R}$;

(27) $\{B_n\}_n$ is bounded,
i.e., $\underline{B} \leqslant B_n \leqslant \overline{B}$ for all $n \geqslant 1$ and some $\underline{B}, \overline{B} \in \mathbb{R}$,

where without loss of generality we can assume that $\overline{a} < \underline{b}$. (So we have $\underline{a} \leqslant a \leqslant \overline{a} < \underline{b} \leqslant b \leqslant \overline{b}$.)

Step 1: lower semicontinuity of the value function. We are going to prove that

$$\varliminf_{n\to\infty} \mathcal{V}(a_n,b_n,A_n,B_n) \geqslant \mathcal{V}(a,b,A,B). \tag{28}$$

Without loss of generality we can assume that $\varliminf_{n\to\infty} \mathcal{V}(a_n,b_n,A_n,B_n) < \infty$. Moreover, $\mathcal{V}(a_n,b_n,A_n,B_n) \geqslant 0$ for all $n \geqslant 1$ because $L \geqslant 0$, hence

$\underline{\lim}_{n\to\infty} \mathcal{V}(a_n, b_n, A_n, B_n) =: s \in [0, \infty[$. So (up to a subsequence) we have $\lim_{n\to\infty} \mathcal{V}(a_n, b_n, A_n, B_n) = s$, and consequently $\{\mathcal{V}(a_n, b_n, A_n, B_n)\}_n$ is bounded, i.e.,

$$0 \leqslant \mathcal{V}(a_n, b_n, A_n, B_n) \leqslant \bar{s} \text{ for all } n \geqslant 1 \text{ and some } \bar{s} \in \mathbb{R}. \tag{29}$$

By Theorem 3.2, for each $n \geqslant 1$, there exists $u_n \in \mathcal{S}_{f,g}(a_n, b_n, A_n, B_n)$, which means that:

$$\forall n \geqslant 1 \quad u_n \in \mathcal{A}_{f,g}(a_n, b_n, A_n, B_n); \tag{30}$$

$$\forall n \geqslant 1 \quad \mathcal{J}_L(u_n; [a_n, b_n]) = \mathcal{V}(a_n, b_n, A_n, B_n). \tag{31}$$

For each $n \geqslant 1$, we define $\widehat{u}_n : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in [a_n, b_n] \\ u_n(a_n) = A_n & \text{if } x \leqslant a_n \\ u_n(b_n) = B_n & \text{if } x \geqslant b_n. \end{cases} \tag{32}$$

Note that for every $n \geqslant 1$, $\widehat{u}_n \in W^{1,1}([c,d]) \subset C([c,d])$ for all $(c,d) \in \mathbb{R}^2$ with $c < d$. As L is continuous, we can consider $\theta_1, \theta_2 \in [0, \infty[$ given by:

$$\theta_1 := \sup\left\{L(x,u,0) : x \in [\underline{a}, \overline{b}] \text{ and } \underline{A} \leqslant u \leqslant \overline{A}\right\};$$
$$\theta_2 := \sup\left\{L(x,u,0) : x \in [\underline{a}, \overline{b}] \text{ and } \underline{B} \leqslant u \leqslant \overline{B}\right\}.$$

From (29) and (31) we see that $\mathcal{J}_L(u_n; [a_n, b_n]) \leqslant \bar{s} + 1$ for all $n \geqslant 1$. Moreover, taking (25), (26), (27) and (28) into account, for each $n \geqslant 1$ we have

$$\begin{aligned} \mathcal{J}_L(\widehat{u}_n; [\underline{a}, \overline{b}]) &= \int_{\underline{a}}^{a_n} L(x, A_n, 0)dx + \mathcal{J}_L(u_n; [a_n, b_n]) + \int_{b_n}^{\overline{b}} L(x, B_n, 0)dx \\ &\leqslant \theta_1(a_n - \underline{a}) + \bar{s} + 1 + \theta_2(\overline{b} - b_n) \\ &\leqslant \theta_1(\overline{a} - \underline{a}) + \bar{s} + 1 + \theta_2(\overline{b} - \underline{b}) =: c. \end{aligned}$$

Thus

$$\forall n \geqslant 1 \quad \mathcal{J}_L(\widehat{u}_n; [\underline{a}, \overline{b}]) \leqslant c. \tag{33}$$

For each $n \geqslant 1$, (recalling that L is positive and) using (31), we can assert that

$$\begin{aligned} \mathcal{J}_L(\widehat{u}_n; [a, b]) &\leqslant \int_a^{a_n} L(x, A_n, 0)dx + \mathcal{J}_L(u_n; [a_n, b_n]) + \int_{b_n}^{b} L(x, B_n, 0)dx \\ &\leqslant \theta_1(a_n - a) + \mathcal{V}(a_n, b_n, A_n, B_n) + \theta_2(b - b_n), \end{aligned}$$

and, according to (20) and (21), we deduce that

$$\lim_{n\to\infty} \mathcal{J}_L(\widehat{u}_n;[a,b]) \leqslant \lim_{n\to\infty} \mathcal{V}(a_n,b_n,A_n,B_n). \tag{34}$$

On the other hand, for any $n \geqslant 1$ we have

$$\Big\{(x,\widehat{u}_n(x)) : x \in [\underline{a},\overline{b}]\Big\} = X_n \cup Y_n \cup Z_n$$

with:

$$\begin{aligned} X_n &:= \Big\{(x,A_n) : x \in [\underline{a},a_n]\Big\};\\ Y_n &:= \Big\{(x,u_n(x)) : x \in [a_n,b_n]\Big\};\\ Z_n &:= \Big\{(x,B_n) : x \in [b_n,\overline{b}]\Big\}. \end{aligned}$$

As f and g are continuous, there exist $\underline{M},\overline{M} \in \mathbb{R}$ such that $\underline{M} \leqslant f(x)$ and $g(x) \leqslant \overline{M}$ for all $x \in [\underline{a},\overline{b}]$. But, for any $n \geqslant 1$, we have $[a_n,b_n] \subset [\underline{a},\overline{b}]$ and, taking (30) into account, $f(x) \leqslant u_n(x) \leqslant g(x)$ for all $x \in [a_n,b_n]$, hence $\underline{M} \leqslant u_n(x) \leqslant \overline{M}$ for all $x \in [a_n,b_n]$. So, according to (25), (26), (27) and (28), we see that:

$$\begin{aligned} X_n &\subset [\underline{a},\overline{a}] \times [\underline{A},\overline{A}] =: K_1;\\ Y_n &\subset [\underline{a},\overline{b}] \times [\underline{M},\overline{M}] =: K_2;\\ Z_n &\subset [\underline{b},\overline{b}] \times [\underline{B},\overline{B}] =: K_3. \end{aligned}$$

Consequently

$$\forall n \geqslant 1 \quad \Big\{(x,\widehat{u}_n(x)) : x \in [\underline{a},\overline{b}]\Big\} \subset K_1 \cup K_2 \cup K_3 =: K. \tag{35}$$

According to (33) and (35), from Lemma 3.3 we can assert that there exists $C > 0$ such that $\|\widehat{u}'_n\|_{L^2([\underline{a},\overline{b}])} \leqslant C$ for all $n \geqslant 1$. Hence, there exists $v_\infty \in L^2([\underline{a},\overline{b}])$ such that (up to a subsequence)

$$\widehat{u}'_n \rightharpoonup v_\infty \text{ in } L^2([\underline{a},\overline{b}]). \tag{36}$$

On the other hand, arguing as in the proof of Theorem 3.2 we see that:

$$\begin{aligned} \forall n \geqslant 1 \quad \forall x \in [\underline{a},\overline{b}] \quad |\widehat{u}_n(x)| &\leqslant |A_n| + \|\widehat{u}'_n\|_{L^2([\underline{a},\overline{b}])}|\overline{b}-\underline{a}|^{\frac{1}{2}}\\ &\leqslant \max\big\{|\underline{A}|,|\overline{A}|\big\} + C|\overline{b}-\underline{a}|^{\frac{1}{2}}; \end{aligned}$$

$$\forall n \geqslant 1 \quad \forall x,y \in [\underline{a},\overline{b}] \quad \big|\widehat{u}_n(x)-\widehat{u}_n(y)\big| \leqslant \omega(|x-y|) \text{ with } \omega(t) = Ct^{\frac{1}{2}}, \tag{37}$$

which shows that $\{\widehat{u}_n\}_n \subset C([\underline{a}, \overline{b}])$ is uniformly bounded and equicontinuous. By Theorem 3.4, there exists $u_\infty \in C([\underline{a}, \overline{b}])$ such that (up to a subsequence)

$$\|\widehat{u}_n - u_\infty\|_{C([\underline{a}, \overline{b}])} \to 0. \tag{38}$$

Combining (36) with (38) we deduce that v_∞ is the weak derivative of u_∞, and consequently $u_\infty \in W^{1,1}([\underline{a}, \overline{b}])$. In particular, we have:

$$\forall x \in [\underline{a}, \overline{b}] \quad |\widehat{u}_n(x) - u_\infty(x)| \to 0; \tag{39}$$

$$u_\infty \in W^{1,1}([a, b]) \text{ and } \|\widehat{u}_n - u_\infty\|_{C([a,b])} \to 0. \tag{40}$$

We claim that:

$$\forall x \in [a, b] \quad f(x) \leqslant u_\infty(x) \leqslant g(x); \tag{41}$$

$$u_\infty(a) = A \text{ and } u_\infty(b) = B. \tag{42}$$

Indeed, let $x \in]a, b[$. Then, there exists $\varepsilon > 0$ such that $a + \varepsilon < b - \varepsilon$ and $x \in [a + \varepsilon, b - \varepsilon]$. Moreover, by (20) and (21), there is $n_0 \geqslant 1$ such that $a_n \in]a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}[$ and $b_n \in]b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}[$ for all $n \geqslant n_0$. So, for any $n \geqslant n_0$, we have $x \in [a_n, b_n]$ and consequently $\widehat{u}_n(x) = u_n(x)$ by (32). But, by (30) we have $u_n \in \mathcal{A}_{f,g}(a_n, b_n, A_n, B_n)$, hence $f(x) \leqslant \widehat{u}_n(x) \leqslant g(x)$ for all $n \geqslant n_0$. Letting $n \to \infty$ and using (39), we deduce that $f(x) \leqslant u_\infty(x) \leqslant g(x)$. Thus $f(x) \leqslant u_\infty(x) \leqslant g(x)$ for all $x \in]a, b[$, which implies (41) because f, g and u_∞ are continuous.

Let us now prove that $u_\infty(a) = A$. Using (32) and (37) we see that

$$\begin{aligned} \big|u_\infty(a) - A\big| &\leqslant \big|u_\infty(a) - \widehat{u}_n(a)\big| + \big|\widehat{u}_n(a) - \widehat{u}_n(a_n)\big| + \big|\widehat{u}_n(a_n) - A\big| \\ &\leqslant \big|u_\infty(a) - \widehat{u}_n(a)\big| + \omega(|a - a_n|) + \big|A_n - A\big| \end{aligned}$$

for all $n \geqslant 1$. But $|u_\infty(a) - \widehat{u}_n(a)| \to 0$ by (39), $\omega(|a - a_n|) \to 0$ by (20) and $|A_n - A| \to 0$ by (22), hence $|u_\infty(a) - A| = 0$ by letting $n \to \infty$. By the same way we have $|u_\infty(b) - B| = 0$. So (42) is proved.

Thus $u_\infty \in \mathcal{A}_{f,g}(a, b, A, B)$. Consequently we have

$$\mathcal{V}(a, b, A, B) \leqslant \mathcal{J}_L(u_\infty; [a, b]). \tag{43}$$

Taking (34) and (40) into account, from Theorem 3.1 we deduce that

$$\mathcal{J}_L(u_\infty; [a, b]) \leqslant \varliminf_{n\to\infty} \mathcal{V}(a_n, b_n, A_n, B_n), \tag{44}$$

and (28) follows by combining (43) with (44).

Step 2: upper semicontinuity of the value function. Now, futhermore, we assume that the Lavrentiev gap phenomenon is absent. We have to prove that

$$\overline{\lim_{n\to\infty}}\, \mathcal{V}(a_n, b_n, A_n, B_n) \leqslant \mathcal{V}(a, b, A, B). \tag{45}$$

Fix any $\varepsilon > 0$. As $\mathcal{V}(a, b, A, B) = \overline{\mathcal{V}}(a, b, A, B)$ we can assert that there exists $u \in C^1([a, b])$ such that

$$\mathcal{J}_L(u; [a, b]) < \mathcal{V}(a, b, A, B) + \varepsilon. \tag{46}$$

Fix any $n \geqslant 1$. Let $u_n \in W^{1,1}([a_n, b_n])$ be defined by

$$u_n(x) := \begin{cases} u(x) & \text{if } x \in [a + \delta_n, b - \delta_n] \\ v_n(x) & \text{if } x \in [a_n, a + \delta_n] \text{ and } f(x) \leqslant v_n(x) \leqslant g(x) \\ f(x) & \text{if } x \in [a_n, a + \delta_n] \text{ and } f(x) > v_n(x) \\ g(x) & \text{if } x \in [a_n, a + \delta_n] \text{ and } v_n(x) > g(x) \\ w_n(x) & \text{if } x \in [b - \delta_n, b_n] \text{ and } f(x) \leqslant w_n(x) \leqslant g(x) \\ f(x) & \text{if } x \in [b - \delta_n, b_n] \text{ and } f(x) > w_n(x) \\ g(x) & \text{if } x \in [b - \delta_n, b_n] \text{ and } w_n(x) > g(x) \end{cases} \tag{47}$$

where $\delta_n := 2\|(a_n, b_n, A_n, B_n) - (a, b, A, B)\|_\infty$ and $v_n : [a_n, a + \delta_n] \to \mathbb{R}$ and $w_n : [b - \delta_n, b_n] \to \mathbb{R}$ are given by:

$$\begin{aligned} v_n(x) &:= u(a + \delta_n) + \frac{u(a + \delta_n) - A_n}{\delta_n + a - a_n}(x - a - \delta_n)); \\ w_n(x) &:= B_n + \frac{B_n - u(b - \delta_n)}{\delta_n + b_n - b}(x - b_n). \end{aligned}$$

(Note that by (20), (21), (22) and (23) we have $\delta_n \to 0$. So, without loss of generality we can assume that $a + \delta_n < b - \delta_n$ for all $n \geqslant 1$.) Then $u_n \in \mathcal{A}_{f,g}(a_n, b_n, A_n, B_n)$. Consequently

$$\forall n \geqslant 1 \quad \mathcal{V}(a_n, b_n, A_n, B_n) \leqslant \mathcal{J}_L(u_n; [a_n, b_n]). \tag{48}$$

On the other hand, for each $n \geqslant 1$, by definition of u_n, see (47), we have

$$
\begin{aligned}
(49) \qquad \mathcal{J}_L(u_n;[a_n,b_n]) &= \int_{a_n}^{a+\delta_n} L(x,u_n(x),u_n'(x))dx \\
&\quad + \int_{a+\delta_n}^{b-\delta_n} L(x,u_n(x),u_n'(x))dx \\
&\quad + \int_{b-\delta_n}^{b_n} L(x,u_n(x),u_n'(x))dx \\
&= \int_{a_n}^{a+\delta_n} L(x,u_n(x),u_n'(x))dx \\
&\quad + \int_{a}^{b} L(x,u(x),u'(x))1_{[a+\delta_n,b-\delta_n]}(x)dx \\
&\quad + \int_{b-\delta_n}^{b_n} L(x,u_n(x),u_n'(x))dx,
\end{aligned}
$$

where 1_I denotes the characteristic function of the set $I \subset \mathbb{R}$. Since $u \in C^1([a,b])$ and L is continuous (and positive), there exists $c_0 \geqslant 0$ such that $0 \leqslant L(x,u(x),u'(x)) \leqslant c_0$ for all $x \in [a,b]$. Hence $0 \leqslant L(x,u(x),u'(x))1_{[a+\delta_n,b-\delta_n]}(x) \leqslant c_0$ for all $n \geqslant 1$ and all $x \in [a,b]$. As $\{1_{[a+\delta_n,b-\delta_n]}\}_n$ pointwise converges to $1_{[a,b]}$, from Lebesgue's convergence theorem it follows that

$$
(50) \qquad \lim_{n\to\infty} \int_a^b L(x,u(x),u'(x))1_{[a+\delta_n,b-\delta_n]}(x)dx = \mathcal{J}_L(u;[a,b]).
$$

We claim that:

$$
(51) \qquad \lim_{n\to\infty} \int_{a_n}^{a+\delta_n} L(x,u_n(x),u_n'(x))dx = 0;
$$

$$
(52) \qquad \lim_{n\to\infty} \int_{b-\delta_n}^{b_n} L(x,u_n(x),u_n'(x))dx = 0.
$$

Indeed, let us prove (51). (The proof of (52) will follow by the same arguments.) For any $n \geqslant 1$ we set:

$V^n_{f,g} := \{x \in]a_n, a+\delta_n[: f(x) \leqslant v_n(x) \leqslant g(x)\}$;

$V^n_f := \{x \in]a_n, a+\delta_n[: f(x) > v_n(x)\}$;

$V^n_g := \{x \in]a_n, a+\delta_n[: v_n(x) > g(x)\}$.

Then, by definition of u_n, see (47), we have

$$
\begin{aligned}
(53) \qquad \mathcal{J}_L(u_n;[a_n,a+\delta_n]) &= \mathcal{J}_L(u_n;V^n_f) + \mathcal{J}_L(u_n;V^n_g) + \mathcal{J}_L(u_n;V^n_{f,g}) \\
&= \mathcal{J}_L(f;V^n_f) + \mathcal{J}_L(g;V^n_g) + \mathcal{J}_L(v_n;V^n_{f,g}).
\end{aligned}
$$

First of all, it is clear that:

$$\forall n \geqslant 1 \ \ \forall x \in V_f^n \ \ (x, f(x), f'(x)) \in [\underline{a}, \overline{b}] \times [-M_1, M_1] \times [-M_2, M_2]; \tag{54}$$

$$\forall n \geqslant 1 \ \ \forall x \in V_g^n \ \ (x, g(x), g'(x)) \in [\underline{a}, \overline{b}] \times [-M_1, M_1] \times [-M_2, M_2], \tag{55}$$

where:

$$\begin{cases} M_1 := \max\left\{\|f\|_{C([\underline{a},\overline{b}])}, \|g\|_{C([\underline{a},\overline{b}])}\right\} < \infty; \\ M_2 := \max\left\{\|f'\|_{C([\underline{a},\overline{b}])}, \|g'\|_{C([\underline{a},\overline{b}])}\right\} < \infty. \end{cases}$$

(Note that $M_1, M_2 \in [0, \infty[$ because $f, g \in C^1(\mathbb{R})$.) On the other hand, for any $n \geqslant 1$, as $\delta_n \geqslant 2|a_n - a|$ we have $\delta_n - |a_n - a| \geqslant \frac{1}{2}\delta_n$. Hence

$$\delta_n + a - a_n = \begin{cases} \delta_n + |a_n - a| \geqslant \delta_n \geqslant \frac{1}{2}\delta_n & \text{if } a \geqslant a_n \\ \delta_n - |a_n - a| \geqslant \frac{1}{2}\delta_n & \text{if } a \leqslant a_n. \end{cases}$$

Since $u(a) = A$, it follows that

$$\begin{aligned} \left|\frac{u(a+\delta_n) - A_n}{\delta_n + a - a_n}\right| &\leqslant \frac{2\,|u(a+\delta_n) - A_n|}{\delta_n} \\ &\leqslant \frac{2\,|u(a+\delta_n) - u(a)|}{\delta_n} + \frac{2\,|A - A_n|}{\delta_n}. \end{aligned}$$

But $\lim_{n\to\infty} \frac{2|u(a+\delta_n)-u(a)|}{\delta_n} = 2|u'(a)|$ because $u \in C^1([a,b])$ and, for every $n \geqslant 1$, $\frac{2|A-A_n|}{\delta_n} \leqslant 1$ because $\delta_n \geqslant 2|A_n - A|$, hence there exists $M_3 \geqslant 0$ such that

$$\forall n \geqslant 1 \quad \left|\frac{u(a+\delta_n) - A_n}{\delta_n + a - a_n}\right| \leqslant M_3.$$

Thus, we have

$$\forall n \geqslant 1 \ \ \forall x \in V_{f,g}^n \ \ (x, v_n(x), v_n'(x)) \in [\underline{a}, \overline{b}] \times [-M_1, M_1] \times [-M_3, M_3]. \tag{56}$$

Set $M_4 := \max\{M_2, M_3\}$. Since L is continuous (and positive), there exists $c_1 \geqslant 0$ such that

$$\forall (x, u, v) \in [\underline{a}, \overline{b}] \times [-M_1, M_1] \times [-M_4, M_4] \ \ 0 \leqslant L(x, u, v) \leqslant c_1,$$

and consequently, by using (54), (55) and (56), we see that:

$$\begin{aligned}
\forall n \geqslant 1 \quad 0 \leqslant \mathcal{J}_L(f; V_f^n) = \int_{V_f^n} L(x, f(x), f'(x))dx &\leqslant c_1 \left|V_f^n\right| \\
&\leqslant c_1(a + \delta_n - a_n); \\
\forall n \geqslant 1 \quad 0 \leqslant \mathcal{J}_L(g; V_{f,g}^n) = \int_{V_f^n} L(x, g(x), g'(x))dx &\leqslant c_1 \left|V_g^n\right| \\
&\leqslant c_1(a + \delta_n - a_n); \\
\forall n \geqslant 1 \quad \mathcal{J}_L(v_n; V_{f,g}^n) = \int_{V_f^n} L(x, v_n(x), v_n'(x))dx &\leqslant c_1 \left|V_{f,g}^n\right| \\
&\leqslant c_1(a + \delta_n - a_n),
\end{aligned}$$

which combined with (54) gives (51) because $\delta_n \to 0$ and $|a_n - a| \to 0$.

From (50), (50), (51) and (52) we deduce that

$$\lim_{n\to\infty} \mathcal{J}_L(u_n, [a_n, b_n]) = \mathcal{J}_L(u, [a, b]),$$

hence $\overline{\lim}_{n\to\infty} \mathcal{V}(a_n, b_n, A_n, B_n) \leqslant \mathcal{J}_L(u, [a, b])$ by (48). Using (46) we conclude that

$$\overline{\lim_{n\to\infty}} \mathcal{V}(a_n, b_n, A_n, B_n) < \mathcal{V}(a, b, A, B) + \varepsilon,$$

and (45) follows by letting $\varepsilon \to 0$. ■

Taking Remark 2.4 into account, as a direct consequence of Lemma 3.19 we have

Corollary 3.20. *Assume that L is continuous and satisfies* (H_2) *and* $f, g \in C^1(\mathbb{R})$. *If the condition* (A_1) *of Theorem 2.3 is verified, i.e., for every* $(a, b, A, B) \in V$, $\mathcal{S}_{f,g}(a, b, A, B) \subset C^1([a, b])$ *with* $\mathcal{S}_{f,g}(a, b, A, B)$ *defined by* (6), *then the value function* $\mathcal{V}$ *is continuous.*

By using Lemma 3.16, Theorem 3.1 and Corollary 3.20 we can establish the following result which will be used in the proof of Theorem 2.3. In what follows, for each $(a, b, A, B) \in V$, we set

$$\overline{\mathcal{S}}_{f,g}(a, b, A, B) := \left\{u \in \overline{\mathcal{A}}_{f,g}(a, b, A, B) : \mathcal{J}_L(u; [a, b]) = \overline{\mathcal{V}}(a, b, A, B)\right\}$$

(with $\overline{\mathcal{A}}_{f,g}(a, b, A, B)$ and $\overline{\mathcal{V}}(a, b, A, B)$ defined by (5) and (7) respectively).

Lemma 3.21. *Assume that* (H_1) *and* (H_2) *hold and* $f, g \in C^{1,\sigma}(\mathbb{R})$. *Let* $(a, b, A, B) \in V$ *and let* $\{(a_n, b_n, A_n, B_n)\}_n \subset V$ *be such that* $\|(a_n, b_n, A_n, B_n) - (a, b, A, B)\|_\infty \to 0$ *and, for each* $n \geqslant 1$, *let* $u_n \in W^{1,1}([a_n, b_n])$ *be such that* $u_n \in \mathcal{S}_{f,g}(a_n, b_n, A_n, B_n)$, *i.e., for each* $n \geqslant 1$, $u_n \in \mathcal{A}_{f,g}(a_n, b_n, A_n, B_n)$ *and* $\mathcal{J}_L(u_n, [a_n, b_n]) = \mathcal{V}(a_n, b_n, A_n, B_n)$. *If* (A_1) *is satisfied then*

$$\forall n \geqslant 1 \quad u_n \in C^1([a_n, b_n]), \text{ and so } \forall n \geqslant 1 \quad u_n \in \overline{\mathcal{S}}_{f,g}(a_n, b_n, A_n, B_n), \tag{57}$$

and there exists $u_\infty \in W^{1,1}([a, b])$ *such that (up to a subsequence) one has*:

$$\|u_n - u_\infty\|_{C([a_n,b_n]\cap[a,b])} \to 0; \tag{58}$$

$$u_\infty \in \overline{\mathcal{S}}_{f,g}(a, b, A, B), \text{ and so, in particular, } u_\infty \in C^1([a, b]); \tag{59}$$

$$\|u'_n - u'_\infty\|_{C([a_n,b_n]\cap[a,b])} \to 0; \tag{60}$$

$$\|u'_n\|_{C([a_n,b_n])} \to \|u'_\infty\|_{C([a,b])}. \tag{61}$$

Proof of Lemma 3.21. First of all, since (A_1) is satisfied, it is clear that (57) holds.

Secondly, for each $n \geqslant 1$, let $\widehat{u}_n : \mathbb{R} \to \mathbb{R}$ be defined by (32). Since Lemma 3.21 assumes that the assertions (30) and (31) are satisfied, by arguing as in the proof of the step 1 of Lemma 3.19, we can assert that there exists $u_\infty \in W^{1,1}([\underline{a}, \overline{b}])$ such that (up to a subsequence)

$$\|\widehat{u}_n - u_\infty\|_{C([\underline{a},\overline{b}])} \to 0, \tag{62}$$

where $\underline{a} \leqslant a < b \leqslant \overline{b}$ and $\underline{a} \leqslant a_n < b_n \leqslant \overline{b}$ for all $n \geqslant 1$, and we can prove that (41) and (42) holds, which means that $u_\infty \in \mathcal{A}_{f,g}(a, b, A, B)$ and implies that

$$\mathcal{V}(a, b, A, B) \leqslant \mathcal{J}_L(u_\infty; [a, b]). \tag{63}$$

From (62) it is obvious that (58) holds. On the other hand, by the same arguments as in the proof of the step 1 of Lemma 3.19 we can also prove that (34) holds, which gives

$$\varlimsup_{n\to\infty} \mathcal{J}_L(\widehat{u}_n; [a, b]) \leqslant \mathcal{V}(a, b, A, B) \tag{64}$$

because $\mathcal{V}$ is continuous by Corollary 3.20. But $\|\widehat{u}_n - u_\infty\|_{C([a,b])} \to 0$ by (62), hence

$$\mathcal{J}_L(u_\infty; [a, b]) \leqslant \varliminf_{n\to\infty} \mathcal{J}_L(\widehat{u}_n; [a, b]) \tag{65}$$

by Theorem 3.1. From (63), (64) and (65) we deduce that $\mathcal{J}_L(u_\infty;[a,b]) = \mathcal{V}(a,b,A,B)$, hence $u_\infty \in \mathcal{S}_{f,g}(a,b,A,B)$ and (59) follows because (A_1) holds.

Thirdly, assume that (60) is false. Then, there exists $\varepsilon > 0$ such that (up to a subsequence) $\|u_n' - u_\infty'\|_{C([a_n,b_n]\cap[a,b])} > \varepsilon$ for all $n \geqslant 1$. So, there exists $\{x_n\}_n \subset \mathbb{R}$ with $x_n \in [a_n,b_n] \cap [a,b]$ such that

$$\forall n \geqslant 1 \quad |u_n'(x_n) - u_\infty'(x_n)| > \varepsilon. \tag{66}$$

As $\{x_n\}_n \subset [a,b]$ and $[a,b]$ is compact, there exists $x_\infty \in [a,b]$ such that (up to a subsequence)

$$x_n \to x_\infty. \tag{67}$$

As $u_\infty \in C^1([a,b])$ we can assert there exists $M > 0$ such that

$$\forall x \in [a,b] \quad |u_\infty'(x)| \leqslant M. \tag{68}$$

As (A_1) holds, by Corollary 3.20, the value function $\mathcal{V}$ is continuous, and so

$$\lim_{n\to\infty} \mathcal{V}(a_n,b_n,A_n,B_n) = \mathcal{V}(a,b,A,B) \tag{69}$$

because $\|(a_n,b_n,A_n,B_n) - (a,b,A,B)\|_\infty \to 0$. Setting $c := \mathcal{V}(a,b,A,B) + 1$, from (69) it follows that:

$\mathcal{V}(a,b,A,B) \leqslant c$;

$\forall n \geqslant n_0 \quad \mathcal{V}(a_n,b_n,A_n,B_n) \leqslant c$ with $n_0 \geqslant 1$ sufficiently large.

As $\|(a_n,b_n,A_n,B_n) - (a,b,A,B)\|_\infty \to 0$ and f,g are continuous, we can assert that there exist $\kappa,\lambda \in \mathbb{R}$ and a compact set $K \subset \mathbb{R}\times\mathbb{R}$ such that:

$[a,b] \subset [\kappa,\lambda]$;

$\forall n \geqslant 1 \quad [a_n,b_n] \subset [\kappa,\lambda]$;

$\big\{(x,u) \in [a,b]\times\mathbb{R} : f(x) \leqslant u \leqslant g(x)\big\} \subset K$;

$\forall n \geqslant 1 \quad \big\{(x,u) \in [a_n,b_n]\times\mathbb{R} : f(x) \leqslant u \leqslant g(x)\big\} \subset K$.

From Corollary 3.11 we deduce that:

$$\mathcal{S}_{f,g}(a,b,A,B) \subset \mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c); \tag{70}$$

$$\forall n \geqslant n_0 \quad \mathcal{S}_{f,g}(a_n,b_n,A_n,B_n) \subset \mathcal{L}_{\omega_{\kappa,\lambda}}(a_n,b_n,L,K,c). \tag{71}$$

Following Lemma 3.15, the sets $\mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$ and $\mathcal{L}_{\omega_{\kappa,\lambda}}(a_n,b_n,L,K,c)$ have derivatives which are conditionally equa-continuous with a same modulus of conditional equa-continuity which only depends on κ, λ, $\omega_{\kappa,\lambda}$, L, K and c. Let $\delta :]0,\infty[\times]0,\infty[\to]0,\infty[$ be a such modulus of conditional equa-continuity. Set $\overline{\delta} := \delta\left(\frac{\varepsilon}{4}+M,\frac{\varepsilon}{4}\right)$ with $M>0$ and $\varepsilon>0$ verifying (66) and (68) respectively. From (67) we see that there exists $n_1 \geqslant n_0$ such that

$$\forall n \geqslant n_1 \quad |x_n - x_\infty| \leqslant \frac{\overline{\delta}}{2}. \tag{72}$$

As moreover $x_\infty \in [a,b]$ and $x_n \in [a_n,b_n]\cap[a,b]$ for all $n \geqslant 1$, with $a_n \to a$ and $b_n \to b$, we can assert that there exist $n_2 \geqslant n_1$ and $y,z\in\mathbb{R}$ with $y<z$ such that:

$$[y,z]\subset[a,b]; \tag{73}$$

$$\forall n \geqslant n_2 \quad [y,z]\subset[a_n,b_n]; \tag{74}$$

$$[y,z]\subset\left[x_\infty-\frac{\overline{\delta}}{2}, x_\infty-\frac{\overline{\delta}}{2}\right]. \tag{75}$$

We claim that

$$\forall n \geqslant n_2 \quad \forall x\in[y,z] \quad \left|u_n'(x)-u_\infty'(x)\right| \geqslant \frac{\varepsilon}{4}. \tag{76}$$

Indeed, otherwise there exist $n \geqslant n_2$ and $x\in[y,z]$ such that

$$\left|u_n'(x)-u_\infty'(x)\right| < \frac{\varepsilon}{4}. \tag{77}$$

On the other hand, we have:

$$|x_n - x| \leqslant |x_n-x_\infty| + |x_\infty - x| < \overline{\delta} = \delta\left(\frac{\varepsilon}{4}+M,\frac{\varepsilon}{4}\right) \text{ by (72) and (75);}$$

$$\left|u_\infty'(x_n)\right| \leqslant M < \frac{\varepsilon}{4}+M \text{ by (68);}$$

$$\left|u_n'(x)\right| \leqslant \left|u_n'(x)-u_\infty'(x)\right| + \left|u_\infty'(x)\right| < \frac{\varepsilon}{4}+M \text{ by (77), (73) and (68).}$$

Moreover, since $u_\infty\in\mathcal{S}_{f,g}(a,b,A,B)$ and $u_n\in\mathcal{S}_{f,g}(a_n,b_n,A_n,B_n)$, from (70) and (71) we see that:

$$\begin{cases} u_\infty \in \mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c); \\ u_n \in \mathcal{L}_{\omega_{\kappa,\lambda}}(a_n,b_n,L,K,c). \end{cases}$$

Consequently, noticing that $u_\infty\in C^1([a,b])$ and $u_n\in C^1([a_n,b_n])$ and taking (74) and (73) into account, in view of Defintion 3.13, from the above we deduce that:

$$\left|u_\infty'(x)-u_\infty'(x_n)\right| \leqslant \frac{\varepsilon}{4}; \tag{78}$$

$$\left|u_n'(x_n)-u_n'(x)\right| \leqslant \frac{\varepsilon}{4}. \tag{79}$$

But, we have

$$\big|u_n'(x_n) - u_\infty'(x_n)\big| \leqslant \big|u_n'(x_n) - u_n'(x)\big| + \big|u_n'(x) - u_\infty'(x)\big| + \big|u_\infty'(x) - u_\infty'(x_n)\big|,$$

hence $|u_n'(x_n) - u_\infty'(x_n)| \leqslant \frac{3\varepsilon}{4}$ by using (77), (78) and (79), which contradicts (66), and (76) is proved. On the other hand, by (58) we have:

$$\lim_{n\to\infty} \big|u_n(y) - u_\infty(y)\big| = 0;$$
$$\lim_{n\to\infty} \big|u_n(z) - u_\infty(z)\big| = 0.$$

Hence

$$\lim_{n\to\infty} \big(\big|u_n(z) - u_\infty(z)\big| - \big|u_n(y) - u_\infty(y)\big|\big) = 0.$$

As $|u_n(z) - u_\infty(z)| - |u_n(y) - u_\infty(y)| = \int_y^z |u_n'(x) - u_\infty'(x)|dx$ for all $n \geqslant 1$, it follows that

$$\lim_{n\to\infty} \int_y^z \big|u_n'(x) - u_\infty'(x)\big|dx = 0. \tag{80}$$

But, from (76), we see that

$$\forall n \geqslant n_2 \quad \int_y^z \big|u_n'(x) - u_\infty'(x)\big|dx \geqslant \frac{\varepsilon}{4}(z-y) > 0,$$

and consequently, letting $n \to \infty$ and taking (80) into account, we obtain $0 \geqslant \frac{\varepsilon}{4}(z-y) > 0$ which is impossible. Thus (60) is proved.

To finish, let us prove (61). We are going to prove that

$$\forall \varepsilon > 0 \;\; \exists N \geqslant 1 \;\; \forall n \geqslant N \;\; \big|\|u_n'\|_{C([a_n,b_n])} - \|u_\infty'\|_{C([a,b])}\big| \leqslant \varepsilon.$$

Let $\varepsilon > 0$. First of all, as $u_\infty' \in C^1([a,b])$ we have $\|u_\infty'\|_{C([a,b])} < C$ for some $C > 0$. Hence

$$\forall n \geqslant 1 \;\; \|u_\infty'\|_{C([a_n,b_n]\cap[a,b])} < C. \tag{81}$$

But by (60) we have

$$\big|\|u_n'\|_{C([a_n,b_n]\cap[a,b])} - \|u_\infty'\|_{C([a_n,b_n]\cap[a,b])}\big| \to 0, \tag{82}$$

and so we can assert that there exists $N_1 \geqslant 1$ such that

$$\forall n \geqslant N_1 \;\; \|u_n'\|_{C([a_n,b_n]\cap[a,b])} < C + 1. \tag{83}$$

As previously, following Lemma 3.15, the sets $\mathcal{L}_{\omega_{\kappa,\lambda}}(a,b,L,K,c)$ and $\mathcal{L}_{\omega_{\kappa,\lambda}}(a_n,b_n,L,K,c)$ have derivatives which are conditionally equa-continuous with a same modulus of conditional equa-continuity which only depends on κ, λ, $\omega_{\kappa,\lambda}$, L, K and c. Let $\delta :]0,\infty[\times]0,\infty[\to]0,\infty[$ be a such modulus of conditional equa-continuity. Set $\widehat{\delta} := \delta\left(C+1,\frac{\varepsilon}{3}\right)$. As $a_n \to a$ and $b_n \to b$ there exists $N_2 \geqslant 1$ such that

$$\forall n \geqslant N_2 \quad \begin{cases} |a_n - a| < \widehat{\delta} \\ |b_n - b| < \widehat{\delta}. \end{cases}$$

Setting $N_3 := \max\{N_1, N_2\}$ and taking (81) and (83) into account, in view of Definition 3.13, we deduce that:

$$\forall n \geqslant N_3 \quad \begin{cases} \left|\|u'_\infty\|_{C([a,b])} - \|u'_\infty\|_{C([a_n,b_n]\cap[a,b])}\right| \leqslant \frac{\varepsilon}{3} \\ \left|\|u'_n\|_{C([a_n,b_n])} - \|u'_n\|_{C([a_n,b_n]\cap[a,b])}\right| \leqslant \frac{\varepsilon}{3}. \end{cases} \tag{84}$$

On the other hand, by (82) there exists $N_4 \geqslant 1$ such that

$$\forall n \geqslant N_4 \quad \left|\|u'_n\|_{C([a_n,b_n]\cap[a,b])} - \|u'_\infty\|_{C([a_n,b_n]\cap[a,b])}\right| \leqslant \frac{\varepsilon}{3}. \tag{85}$$

Setting $N := \max\{N_3, N_4\}$, from (84) and (85) we conclude that for every $n \geqslant N$,

$$\begin{aligned} \left|\|u'_n\|_{C([a_n,b_n])} - \|u'_\infty\|_{C([a,b])}\right| &\leqslant \left|\|u'_n\|_{C([a_n,b_n])} - \|u'_n\|_{C([a_n,b_n]\cap[a,b])}\right| \\ &\quad + \left|\|u'_n\|_{C([a_n,b_n]\cap[a,b])} - \|u'_\infty\|_{C([a_n,b_n]\cap[a,b])}\right| \\ &\quad + \left|\|u'_\infty\|_{C([a_n,b_n]\cap[a,b])} - \|u'_\infty\|_{C([a,b])}\right| \\ &\leqslant \varepsilon. \end{aligned}$$

Thus (61) is proved and the the proof of Lemma 3.21 is complete. ■

4. Proof of the main result

In this section we prove Theorem 2.3.

Proof of Theorem 2.3. The proof is divided into two implications.

$(\mathbf{A_1}) \Rightarrow (\mathbf{A_2})$**.** Fix $(a,b,A,B) \in V$. We are going to prove that $\mathcal{V}$ is Lipschitz continuous at (a,b,A,B). First of all, there exists $M > 0$ such that

$$\forall u \in \mathcal{S}_{f,g}(a,b,A,B) \quad \|u'\|_{C([a,b])} \leqslant M. \tag{86}$$

Indeed, otherwise there exists $\{u_n\}_n \subset \mathcal{S}_{f,g}(a,b,A,B)$ such that

$$\|u'_n\|_{C([a,b])} \to \infty. \tag{87}$$

But, by (A_1), $\mathcal{S}_{f,g}(a,b,A,B) \subset C^1([a,b])$, and so, by using Corollary 3.18, we can assert that there exists $u_\infty \in C^1([a,b])$ such that (up to a subsequence) $\|u_n' - u_\infty'\|_{C([a,b])} \to 0$. Hence $\|u_n'\|_{C([a,b])} \to \|u_\infty'\|_{C([a,b])} < \infty$ which contradicts (87). Thus (86) holds.

Let $\{(a_n, b_n, A_n, B_n)\}_n \subset V$ be such that

$$\|(a_n, b_n, A_n, B_n) - (a, b, A, B)\|_\infty \to 0.$$

By Theorem 3.2, for each $n \geqslant 1$ there exists $u_n \in \mathcal{S}_{f,g}(a_n, b_n, A_n, B_n)$. As (A_1) is satisfied (we have $u_n \in C^1([a_n, b_n])$ for all $n \geqslant 1$ and) from Lemma 3.21 we can assert that there exists $u_\infty \in \mathcal{S}_{f,g}(a,b,A,B)$ such that $\|u_n'\|_{C([a_n,b_n])} \to \|u_\infty'\|_{C([a,b])}$. Hence

$$\|u_\infty'\|_{C([a,b])} \leqslant M \tag{88}$$

by (86) and consequently

$$\forall n \geqslant N_1 \quad \|u_n'\|_{C([a_n,b_n])} \leqslant M + 1 \tag{89}$$

for some $N_1 \geqslant 1$. Noticing that f, g are continuous, we can assert that there exists a compact set $K := [\underline{a}, \overline{b}] \times [-M_1, M_1] \subset \mathbb{R} \times \mathbb{R}$, which does not depend on n, such that:

$$\big\{(x,u) \in [a,b] \times \mathbb{R} : f(x) \leqslant u \leqslant g(x)\big\} \subset K;$$
$$\forall n \geqslant N_2 \quad \big\{(x,u) \in [a_n,b_n] \times \mathbb{R} : f(x) \leqslant u \leqslant g(x)\big\} \subset K \text{ for some } N_2 \geqslant 1.$$

For each $n \geqslant 1$, set $\delta_n := 2\|(a_n, b_n, A_n, B_n) - (a, b, A, B)\|_\infty$. Then, there exists $N_3 \geqslant N_2$ such that for every $n \geqslant N_3$, $a_n \leqslant a + \delta_n < b_n$, $a_n < b - \delta_n \leqslant b_n$, $a \leqslant a_n + \delta_n < b$, $a < b_n - \delta_n \leqslant b$, $a_n + \delta_n < b_n$ and $a_n < b_n - \delta_n$.

For any $n \geqslant N_3$ we consider $\widehat{u}_{\infty,n} \in W^{1,1}([a_n, b_n])$ defined by

$$\widehat{u}_{\infty,n}(x) := \begin{cases} u_\infty(x) & \text{if } x \in [a_n + \delta_n, b_n - \delta_n] \\ v_{\infty,n}(x) & \text{if } x \in [a_n, a_n + \delta_n] \text{ and } f(x) \leqslant v_{\infty,n}(x) \leqslant g(x) \\ f(x) & \text{if } x \in [a_n, a_n + \delta_n] \text{ and } f(x) > v_{\infty,n}(x) \\ g(x) & \text{if } x \in [a_n, a_n + \delta_n] \text{ and } v_n(x) > g(x) \\ w_{\infty,n}(x) & \text{if } x \in [b_n - \delta_n, b_n] \text{ and } f(x) \leqslant w_{\infty,n}(x) \leqslant g(x) \\ f(x) & \text{if } x \in [b_n - \delta_n, b_n] \text{ and } f(x) > w_{\infty,n}(x) \\ g(x) & \text{if } x \in [b_n - \delta_n, b_n] \text{ and } w_n(x) > g(x) \end{cases}$$

where $v_{\infty,n} : [a_n, a_n + \delta_n] \to \mathbb{R}$ and $w_{\infty,n} : [b_n - \delta_n, b_n] \to \mathbb{R}$ are given by:

$$v_{\infty,n}(x) := u_\infty(a_n + \delta_n) + \frac{u_\infty(a_n + \delta_n) - A_n}{\delta_n}(x - a_n - \delta_n));$$
$$w_{\infty,n}(x) := B_n + \frac{B_n - u_\infty(b_n - \delta_n)}{\delta_n}(x - b_n).$$

Then $\widehat{u}_{\infty,n} \in \mathcal{A}_{f,g}(a_n, b_n, A_n, B_n)$. Consequently

$$\forall n \geqslant N_3 \quad \mathcal{V}(a_n, b_n, A_n, B_n) \leqslant \mathcal{J}_L(\widehat{u}_{\infty,n}; [a_n, b_n]). \tag{90}$$

For any $n \geqslant N_3$ we set:

$V^n_{f,g} := \big\{x \in]a_n, a_n + \delta_n[: f(x) \leqslant v_{\infty,n}(x) \leqslant g(x)\big\}$;
$V^n_f := \big\{x \in]a_n, a_n + \delta_n[: f(x) > v_{\infty,n}(x)\big\}$;
$V^n_g := \big\{x \in]a_n, a_n + \delta_n[: v_{\infty,n}(x) > g(x)\big\}$.

First of all, it is clear that:

$$\forall n \geqslant N_3 \quad \forall x \in V^n_f \quad (x, f(x), f'(x)) \in K \times [-M_2, M_2]; \tag{91}$$

$$\forall n \geqslant N_3 \quad \forall x \in V^n_g \quad (x, g(x), g'(x)) \in K \times [-M_2, M_2], \tag{92}$$

where $M_2 := \max\big\{\|f'\|_{C([\underline{a},\bar{b}])}, \|g'\|_{C([\underline{a},\bar{b}])}\big\}$. On the other hand, since $u_\infty(a) = A$, for any $n \geqslant N_3$, one has

$$\left|\frac{u_\infty(a_n + \delta_n) - A_n}{\delta_n}\right| \leqslant \frac{|u_\infty(a_n + \delta_n) - u_\infty(a)|}{\delta_n} + \frac{|A - A_n|}{\delta_n}.$$

As $u_\infty \in C^1([a, b])$, by using Lagrange's finite-increment theorem, we can assert that there exists $x_n \in]a, a_n + \delta_n[$ such that $\frac{|u_\infty(a_n+\delta_n)-u_\infty(a)|}{\delta_n} = |u'_\infty(x_n)|$, and so $\frac{|u_\infty(a_n+\delta_n)-u_\infty(a)|}{\delta_n} \leqslant M$ by (88). Moreover, for every $n \geqslant 1$, $\frac{|A-A_n|}{\delta_n} \leqslant \frac{1}{2}$ because $\delta_n \geqslant 2|A_n - A|$, hence

$$\forall n \geqslant N_3 \quad \left|\frac{u_\infty(a_n + \delta_n) - A_n}{\delta_n}\right| \leqslant M_3$$

with $M_3 := M + \frac{1}{2}$. Thus, we have

$$\forall n \geqslant N_3 \quad \forall x \in V^n_{f,g} \quad (x, v_{\infty,n}(x), v'_{\infty,n}(x)) \in K \times [-M_3, M_3]. \tag{93}$$

In the same way, setting:

$W^n_{f,g} := \big\{x \in]b_n - \delta_n, b_n[: f(x) \leqslant w_{\infty,n}(x) \leqslant g(x)\big\}$;
$W^n_f := \big\{x \in]b_n - \delta_n, b_n[: f(x) > w_{\infty,n}(x)\big\}$;
$W^n_g := \big\{x \in]b_n - \delta_n, b_n[: w_{\infty,n}(x) > g(x)\big\}$,

we see that:

$$\forall n \geqslant N_3 \quad \forall x \in W^n_f \quad (x, f(x), f'(x)) \in K \times [-M_2, M_2]; \tag{94}$$

$$\forall n \geqslant N_3 \quad \forall x \in W^n_g \quad (x, g(x), g'(x)) \in K \times [-M_2, M_2], \tag{95}$$

$$\forall n \geqslant N_3 \quad \forall x \in W^n_{f,g} \quad (x, w_{\infty,n}(x), v'_{\infty,n}(x)) \in K \times [-M_3, M_3]. \tag{96}$$

Set $M_4 := \max\{M, M_2, M_3\}$. Then, M_4 does not depend on n. Taking (88), (91), (92), (93) (94), (95) and (96) into account, we deduce that:

$$\forall x \in [a,b] \ \ (x, u_\infty(x), u'_\infty(x)) \in K \times [-M_4, M_4]; \tag{97}$$

$$\forall n \geqslant N_3 \ \ \forall x \in [a_n,b_n] \ \ (x, \widehat{u}_{\infty,n}(x), \widehat{u}'_{\infty,n}(x)) \in K \times [-M_4, M_4]. \tag{98}$$

Set $C_1 := \sup\big\{L(x,u,v) : (x,u,v) \in K \times [-M_4, M_4]\big\} \in [0,\infty[$. From the above, we see that C_1 does not depend on n. Taking (90) into account, it follows that for every $n \geqslant N_3$ we have

$$\begin{aligned}
\mathcal{V}(a_n,b_n,A_n,B_n) - \mathcal{V}(a,b,A,B) &\leqslant \mathcal{J}_L(u_{\infty,n},[a_n,b_n]) - \mathcal{J}_L(u_\infty,[a,b]) \\
&\leqslant \int_{a_n}^{b_n} L(x, \widehat{u}_{\infty,n}(x), \widehat{u}'_{\infty,n}(x))dx \\
&\quad - \int_a^b L(x, u_\infty(x), u'_\infty(x))dx \\
&= \int_{a_n}^{a_n+\delta_n} L(x, \widehat{u}_{\infty,n}(x), \widehat{u}'_{\infty,n}(x))dx \\
&\quad + \int_{b_n-\delta_n}^{b_n} L(x, \widehat{u}_{\infty,n}(x), \widehat{u}'_{\infty,n}(x))dx \\
&\quad - \int_a^{a+\delta_n} L(x, u_\infty(x), u'_\infty(x))dx \\
&\quad - \int_{b-\delta_n}^{b} L(x, u_\infty(x), u'_\infty(x))dx,
\end{aligned}$$

and so, recalling that L is positive,

$$\begin{aligned}
\mathcal{V}(a_n,b_n,A_n,B_n) - \mathcal{V}(a,b,A,B) &\leqslant \int_{a_n}^{a_n+\delta_n} L(x, \widehat{u}_{\infty,n}(x), \widehat{u}'_{\infty,n}(x))dx \\
&\quad + \int_{b_n-\delta_n}^{b_n} L(x, \widehat{u}_{\infty,n}(x), \widehat{u}'_{\infty,n}(x))dx \\
&\leqslant 2C_1\delta_n.
\end{aligned}$$

As $\delta_n = 2\|(a_n,b_n,A_n,B_n) - (a,b,A,B)\|_\infty$ we obtain

$$\mathcal{V}(a_n,b_n,A_n,B_n) - \mathcal{V}(a,b,A,B) \leqslant \widehat{C}_1 \|(a_n,b_n,A_n,B_n) - (a,b,A,B)\|_\infty. \tag{99}$$

For all $n \geqslant N_3$ with $\widehat{C}_1 := 4C_1$. Let us now consider, for any $n \geqslant N_4 :=$

$\max\{N_1, N_3\}$, $\widehat{u}_n \in W^{1,1}([a,b])$ defined by

$$\widehat{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in [a+\delta_n, b-\delta_n] \\ v_n(x) & \text{if } x \in [a, a+\delta_n] \text{ and } f(x) \leqslant v_n(x) \leqslant g(x) \\ f(x) & \text{if } x \in [a, a+\delta_n] \text{ and } f(x) > v_n(x) \\ g(x) & \text{if } x \in [a, a+\delta_n] \text{ and } v_n(x) > g(x) \\ w_n(x) & \text{if } x \in [b-\delta_n, b] \text{ and } f(x) \leqslant w_n(x) \leqslant g(x) \\ f(x) & \text{if } x \in [b-\delta_n, b] \text{ and } f(x) > w_n(x) \\ g(x) & \text{if } x \in [b-\delta_n, b] \text{ and } w_n(x) > g(x) \end{cases}$$

where $v_n : [a, a+\delta_n] \to \mathbb{R}$ and $w_n : [b-\delta_n, b] \to \mathbb{R}$ are given by:

$$v_n(x) := u_n(a+\delta_n) + \frac{u_n(a+\delta_n) - A}{\delta_n}(x - a - \delta_n);$$
$$w_n(x) := B + \frac{B - u_n(b-\delta_n)}{\delta_n}(x - b).$$

Then $\widehat{u}_n \in \mathcal{A}_{f,g}(a,b,A,B)$. Consequently

$$\forall n \geqslant N_4 \quad \mathcal{V}(a,b,A,B) \leqslant \mathcal{J}_L(\widehat{u}_n; [a,b]). \tag{100}$$

Similarly as in the above, by using (89) instead of (88), we can assert that:

$$\forall n \geqslant N_4 \quad \forall x \in [a_n, b_n] \quad (x, u_n(x), u_n'(x)) \in K \times [-M_4, M_4]; \tag{101}$$
$$\forall n \geqslant N_4 \quad \forall x \in [a_n, b_n] \quad (x, \widehat{u}_n(x), \widehat{u}_n'(x)) \in K \times [-M_4, M_4]. \tag{102}$$

with $M_4 := \max\{M_1, M_2, M_3'\}$ with $M_3' := M + \frac{3}{2}$. Set $C_2 := \sup\big\{L(x,u,v) : (x,u,v) \in K \times [-M_4, M_4]\big\} \in [0, \infty[$. By taking (100) into account, in the same manner as in the above, we obtain

$$\mathcal{V}(a,b,A,B) - \mathcal{V}(a_n,b_n,A_n,B_n) \leqslant \widehat{C}_2 \|(a_n,b_n,A_n,B_n) - (a,b,A,B)\|_\infty. \tag{103}$$

for all $n \geqslant N_4$ with $\widehat{C}_2 := 4C_2$. Set $N := \max\{N_3, N_4\}$ and $C := \max\{\widehat{C}_1, \widehat{C}_2\}$. Then, C does not depend on n, and combining (99) with (103) we conclude that

$$\Big|\mathcal{V}(a_n,b_n,A_n,B_n) - \mathcal{V}(a,b,A,B)\Big| \leqslant C\|(a_n,b_n,A_n,B_n) - (a,b,A,B)\|_\infty,$$

for all $n \geqslant N$, which proves that $\mathcal{V}$ is Lipschitz continuous at (a,b,A,B).

$(\mathbf{A_2}) \Rightarrow (\mathbf{A_1})$**.** Assume that (A_1) is false. Then, there exists $(a,b,A,B) \in V$ such that $\mathcal{S}_{f,g}(a,b,A,B) \not\subset C^1([a,b])$. So, taking Theorems 3.2 and 3.12 into account, there exists $u \in \mathcal{S}_{f,g}(a,b,A,B)$ such that $u \in W_T^{1,1}([a,b])$

and $u \notin C^1([a,b])$. Thus $[a,b]\backslash\Omega_u \neq \emptyset$, which means that without loss of generality we can assert that there exists $a_0, b_0 \in [a,b]$ such that $[a_0, b_0[\subset \Omega_u$ and $|u'(b_0)| = \infty$. Set $A_0 = u(a_0)$ and $B_0 = u(b_0)$. Then $(a_0, b_0, A_0, B_0) \in V$ and $u \in \mathcal{S}_{f,g}(a_0, b_0, A_0, B_0)$. Moreover, since $f < g$, without loss of generality we can assume that $f(b_0) < B_0$. As L satisfies (H_2) there exists $M > 0$ such that for every $(x,u,v) \in \mathbb{R}^3$,

$$|v| \geqslant M \Rightarrow L(x,u,v) \geqslant \frac{\mu v^2}{4}, \tag{104}$$

where $\mu > 0$ is given by (H_2). Let $\{B_n\}_n \subset \mathbb{R}$ be such that $B_n \to B_0$ and $(a_0, b_0, A_0, B_n) \in V$ with $B_n < B_0$ for all $n \geqslant 1$. Fix any $n \geqslant 1$. Let us consider $x_n \in [a_0, b_0]$ such that $u(x_n) = B_n$ and $u(x) \geqslant B_n$ for all $x \in [x_n, b_0]$. Then $x_n \to b_0$. Let us define $u_n : [a_0, b_0] \to \mathbb{R}$ by

$$u_n(x) := \begin{cases} u(x) & \text{if } x \in [a_0, x_n] \\ B_n & \text{if } x \in [x_n, b_0]. \end{cases}$$

Then $u_n \in \mathcal{A}_{f,g}(a_0, b_0, A_0, B_n)$. From the above it follows that for every $n \geqslant 1$,

$$\begin{aligned} \mathcal{V}(a_0, b_0, A_0, B_0) - \mathcal{V}(a_0, b_0, A_0, B_n) &\geqslant \mathcal{J}_L(u, [a_0, b_0]) - \mathcal{J}_L(u_n, [a_0, b_0]) \\ &\geqslant \int_{x_n}^{b_0} L(x, u(x), u'(x))dx \\ &\quad - \int_{x_n}^{b_0} L(x, u_n(x), u_n'(x))dx \\ &\geqslant \int_{x_n}^{b_0} L(x, u(x), u'(x))dx \\ &\quad - \int_{x_n}^{b_0} L(x, B_n, 0)dx \\ &\geqslant \int_{x_n}^{b_0} L(x, u(x), u'(x))dx \\ &\quad -C(b_0 - x_n), \end{aligned}$$

where $C := \sup\{L(x,u,0) : (x,u) \in [a_0, b_0] \times [A_0, B_0]\} \in [0, \infty[$. But u has Tonelli's regularity and $|u'(b_0)| = \infty$, hence $|u'(x)| \to \infty$ as $x \to b_0$ and so, because $x_n \to b_0$, there exists $N_1 \geqslant 1$ such that for each $n \geqslant N_1$, $|u'(x)| \geqslant M$ for all $x \in [x_n, b_0]$. Taking (104) into account and using Jensen's inequality, we deduce that for every $n \geqslant N_1$,

$$\begin{aligned}
\mathcal{V}(a_0,b_0,A_0,B_0)-\mathcal{V}(a_0,b_0,A_0,B_n) &\geqslant \frac{\mu}{4}\int_{x_n}^{b_0}\big(u'(x)\big)^2\,dx - C(b_0-x_n)\\
&\geqslant (b_0-x_n)\left(\frac{1}{(b_0-x_n)}\int_{x_n}^{b_0}u'(x)dx\right)^2\\
&\quad -C(b_0-x_n)\\
&=(b_0-x_n)\frac{\mu}{4}\left(\frac{(u(b_0)-u(x_n))}{b_0-x_n}\right)^2\\
&\quad -C(b_0-x_n)\\
&=|B_0-B_n|\left[\frac{\mu}{4}\frac{|B_0-B_n|}{b_0-x_n}-C\frac{b_0-x_n}{|B_0-B_n|}\right].
\end{aligned}$$

Thus

$$\frac{\mathcal{V}(a_0,b_0,A_0,B_0)-\mathcal{V}(a_0,b_0,A_0,B_n)}{|B_0-B_n|}\geqslant \frac{\mu}{4}\frac{|B_0-B_n|}{b_0-x_n}-C\frac{b_0-x_n}{|B_0-B_n|}. \tag{105}$$

for all $n \geqslant N_1$. As $|u'(b_0)| = \infty$ and $x_n \to b_0$ we have $\frac{|B_0-B_n|}{b_0-x_n} \to \infty$ and $\frac{b_0-x_n}{|B_0-B_n|} \to 0$. Letting $n \to \infty$ in (105) we conclude that

$$\lim_{n\to\infty}\frac{\mathcal{V}(a_0,b_0,A_0,B_0)-\mathcal{V}(a_0,b_0,A_0,B_n)}{|B_0-B_n|}=\infty,$$

which implies that $\mathcal{V}$ is not Lipschitz continuous at (a_0,b_0,A_0,B_0), and so contradicts the assumption (A_2). ■

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