Rend. Sem. Mat. Univ. Padova 1xx (201x) **Rendiconti del Seminario Matematico della Università di Padova**

Lusin type theorems for Radon measures

Andrea Marchese

Abstract – We add to the literature the following observation. If μ is a singular measure on $\mathbb{R}^n$ which assigns measure zero to every porous set and $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function which is non-differentiable μ-a.e., then for every C^1 function $g : \mathbb{R}^n \to \mathbb{R}$ it holds

$$\mu\{x \in \mathbb{R}^n : f(x) = g(x)\} = 0.$$

In other words the Lusin type approximation property of Lipschitz functions with C^1 functions does not hold with respect to a general Radon measure.

Mathematics Subject Classification (2010). 26A16, 41A30.

Keywords. Lusin type approximation, Lipschitz function, Porous set.

1. Introduction

It is well known (see Theorem 3.1.16 of [7]) that given any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and $\varepsilon > 0$ there exists a C^1 function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathscr{L}^n\{x \in \mathbb{R}^n : f(x) \neq g(x)\} < \varepsilon,$$

where $\mathscr{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$. In this note we prove that in general it is not possible to replace the Lebesgue measure with a Radon measure μ. Indeed the following theorem shows that such approximation is not available whenever μ is a singular measure on $\mathbb{R}^n$ which assigns measure zero to every porous set (see §2 for the definition of porosity) and $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function which is non-differentiable μ-almost everywhere.

Andrea Marchese, Institut für Mathematik, Mathematisch-naturwissenschaftliche Fakultät, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich

E-mail: andrea.marchese@math.uzh.ch

Theorem 1.1. *Let μ be a singular measure on $\mathbb{R}^n$ which assigns measure zero to every porous set. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function which is non-differentiable μ-a.e. Then for every C^1 function g, it holds*

$$\mu(\{x \in \mathbb{R}^n : f(x) = g(x)\}) = 0.$$

The validity of Lusin type approximation properties in metric measure spaces has recently attracted some attention. For example, in [8], [13] and [14] the validity of Lusin type theorems for horizontal curves in Carnot Groups is studied and [4] extends a Lusin type theorem for gradients, originally established in [1], to a wide class of metric measure spaces which admit a differentiability structure. The forthcoming paper [9], provides a deeper investigation on the latter problem in the special metric measure space given by the Euclidean space $\mathbb{R}^n$ endowed with an arbitrary Radon measure μ, analyzing the possibility to prescribe not only the differential, but also some non-linear blowups at many points. The class of the "admissible" blowups is determined in terms of certain geometric properties of the measure μ, namely in terms of its decomposability bundle, introduced in [2]. Finally, let us mention the paper [12], where a result in the spirit of [1] was proved for maps from an infinite dimensional locally convex space, endowed with a Gaussian measure, to its Cameron-Martin space.

This paper is organized as follows. In §2 we recall two facts which are necessary to guarantee that the content of Theorem 1.1 is non-empty: firstly the existence of a singular measure μ on $\mathbb{R}^n$ which assigns measure zero to every porous set (Proposition 2.1) and secondly the existence of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is non-differentiable μ-almost everywhere (Proposition 2.4). These results are already present in the literature. Nevertheless, for the reader's convenience, we present here slightly simplified versions of the original proofs. In §3 we prove Theorem 1.1. In §4 we briefly discuss the possibility to extend and improve the main result of [1]: we observe that, in the one-dimensional case, the result is trivially valid with respect to any Radon measure and we show that, except for atomic measures, it is not possible to find any Lipschitz function having a unique non-differentiable blowup in a set of positive measure.

2. Notations and Prerequisites

2.1 – *Notations about sets and measures*

All the sets and functions considered in this note are tacitly assumed to be Borel measurable and measures are defined on the Borel σ-algebra. Moreover measures are positive, locally finite and *inner regular* (i.e. the measure of a set can be approximated from within by compact subsets). As usual, we say that a measure μ is *absolutely continuous* with respect to a measure ν (and we write $\mu \ll \nu$) if $\mu(E) = 0$ for every Borel set E such that $\nu(E) = 0$. We say that μ is *supported* on a set E if the complement set of E has measure zero and we say that μ is *singular* with respect to ν if there exists a Borel set E such that $\nu(E) = 0$ and μ is supported on E. When words like "nullset" and "singular measure" are used without further specification, they implicitly refer to the Lebesgue measure. Given a measure μ and a measurable function f we denote by $f\mu$ the measure satisfying

$$f\mu(A) := \int_A f \, \mathrm{d}\mu,$$

for every Borel set A. In particular, we write $\mu \llcorner A$ to indicate the measure $f\mu$, where $f = 1_A$ is the characteristic function of a Borel set A assuming values 0 and 1.

2.2 – *Porosity*

We say that a set $E \subset \mathbb{R}^n$ is *porous at a point* x if there exist a constant $C(x) > 0$, a positive sequence $r_k \to 0$ and a sequence of points $y_k \in B(x, r_k)$ such that

$$B(y_k, C(x)r_k) \subset B(x, r_k) \quad \text{and} \quad E \cap B(y_k, C(x)r_k) = \emptyset, \tag{1}$$

where we denoted by $B(y, r)$ the open ball centered at y with radius r. We say that E is *porous* if it is porous at every point $x \in E$.

2.3 – *Existence of a singular measure which assigns measure zero to porous sets*

In order to guarantee that Theorem 1.1 actually applies to a non-trivial class of pairs (f, μ) we need to prove first of all the existence of a singular measure on $\mathbb{R}^n$ which assigns measure zero to every porous set. We prove the existence of such measure for $n = 1$. A proof of this fact can be found in [15]. In order to keep this note self-contained, we present here a slightly simpler proof. More precisely we exhibit a singular measure on $\mathbb{R}$ whose blowups assign positive measure to every open set. We easily get a measure on $\mathbb{R}^n$ with the same property, by taking the product of such measure on $\mathbb{R}$ with the Lebesgue measure $\mathscr{L}^{n-1}$ on $\mathbb{R}^{n-1}$. The fact that this product measure charges measure zero to every porous subset of $\mathbb{R}^n$ is a simple consequence of Lemma 2.3 below.

Proposition 2.1. *There exists a non-zero singular measure* ν *on* $\mathbb{R}^n$ *such that* $\nu(P) = 0$ *whenever* P *is porous.*

The construction uses the idea of *Riesz product measures*. On $[0, 1]$ we call i-th generation of dyadic intervals, $i = 0, 1, 2, \ldots$, all the intervals of the form

$$I = [a2^{-i}, (a + 1)2^{-i}], \text{ for } a = 0, \ldots, 2^i - 1.$$

We will make use of the following lemma, which is a well-known fact. Being unable to find a precise reference, we prefer to include its proof. Remember that a point

x is said to be a *Lebesgue continuity point* for a function f with respect to the measure μ if there holds

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|\,\mathrm{d}\mu \to 0, \quad \text{as } r\to 0.$$

LEMMA 2.2 ((Martingale Theorem)). *Let $(\mu_i)_{i\in\mathbb{N}}$ be a sequence of probability measures on $[0,1]$. Assume that $\mu_i = f_i\mathscr{L}^1$, where f_i is constant on the dyadic intervals of the i-th generation. Assume moreover that $\mu_j(I)=\mu_i(I)$ for every dyadic interval I of the i-th generation, for every $j>i$. Then μ_i weakly-* converges to a probability measure ν. Moreover the Radon-Nikodym derivative f of the absolutely continuous part of ν satisfies*

$$f=\lim_{i\to\infty} f_i, \quad \mathscr{L}^1-\text{a.e.}$$

PROOF. By the compactness theorem for measures (see Proposition 2.5 of [5]), there is a subsequence $(\mu_{i_h})_{h\in\mathbb{N}}$ weakly-* converging to a measure ν. Since the dyadic intervals generate the Borel σ-algebra, the hypotheses of the theorem guarantee that actually the whole sequence $(\mu_i)_{i\in\mathbb{N}}$ converges to ν.

To prove the second part of the theorem, denote ν_s the singular part of ν and let $S\subset[0,1]$ be a nullset such that $\nu_s([0,1]\setminus S)=0$. Fix a point $x\in[0,1]\setminus S$ with the following properties:

- x is a point of Lebesgue continuity for f with respect to $\mathscr{L}^1$;
- x is a continuity point for every f_n;
- $2^i\nu_s(I_i)\to 0$ as $i\to\infty$,

where we denoted by I_i the dyadic interval of the i-th generation containing x (the second property guarantees that such interval is unique). Notice that these three properties are satisfied by $\mathscr{L}^1$-almost every point in $[0,1]$ and in particular the third property follows from the Besicovitch Differentiation Theorem (see Theorem 2.10 of [5]). Observe that $\{I_i\}_{i\in\mathbb{N}}$ is a family of sets of *bounded eccentricity*, i.e. there exists $C>0$ such that each I_i is contained in a ball B, centered at x, with $\mathscr{L}^1(I_i)\geq C\mathscr{L}^1(B)$. Therefore the Lebesgue Theorem (see Theorem 7.10 in [11]) yields:

$$f_i(x)=\frac{\mu_i(I_i)}{\mathscr{L}^1(I_i)}=\frac{\mu(I_i)}{\mathscr{L}^1(I_i)}=\frac{\int_{I_i} f\,\mathrm{d}\mathscr{L}^1}{\mathscr{L}^1(I_i)}+\frac{\mu_s(I_i)}{\mathscr{L}^1(I_i)}\to f(x), \quad \text{as } i\to\infty.$$

$\square$

The proof of Proposition 2.1 uses a blowup argument. Given a Radon measure ν on $\mathbb{R}$ and a point x we define the measure $\nu_{x,r}$ by

$$\nu_{x,r}(A):=\nu(x+rA), \text{ for every Borel set } A.$$

We denote by $\mathrm{Tan}(\nu, x)$ the set of the *blowups* of ν at x, i.e. all the possible weak-* limits of the form

$$\lim_{r_i \searrow 0} \kappa_i$$

where (we are interested only in the case in which the quotient is defined)

$$\kappa_i := \frac{\nu_{x,r_i} \llcorner B(0,1)}{\nu(B(x,r_i))}. \tag{2}$$

The following lemma gives a sufficient condition for a measure to assign measure zero to every porous set.

LEMMA 2.3. *Let ν be a locally finite measure on $\mathbb{R}^n$, such that for ν-a.e. x and for every $\eta \in \mathrm{Tan}(\nu, x) \neq \emptyset$, it holds $\eta(A) > 0$ for every open set A. Then $\nu(P) = 0$ for every porous set $P \subset \mathbb{R}$.*

PROOF. We assume by contradiction that ν satisfies the hypotheses of the lemma but there exists a porous set P with $\nu(P) > 0$. It is a general fact about tangent measures (see Remark 3.13 of [5]) that if E is a Borel set, then $\mathrm{Tan}(\nu \llcorner E, x) = \mathrm{Tan}(\nu, x)$ for ν-a.e $x \in E$. Then for ν-a.e. $x \in P$, every blowup η of $\nu \llcorner P$ at x is an element of $\mathrm{Tan}(\nu, x)$. In particular, by hypothesis, $\eta(B) > 0$ for every open ball $B \subset B(0,1)$. Instead we show that, for every $x \in P$, it is possible to find a blowup η of $\nu \llcorner P$ at the point x and a non-trivial ball $B \subset B(0,1)$ such that $\eta(B) = 0$.

Fix $x \in P$ and let $C := C(x)$, $(r_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ as in (1). Possibly passing to a subsequence, we may assume that $(y_k - x)/r_k$ converges to a point y in the closure of $B(0, 1 - C)$. This implies that, for every subsequence of $(r_k)_{k\in\mathbb{N}}$, such that the corresponding rescaled measures (defined in (2)) converge weakly-* to a measure $\eta \in \mathrm{Tan}(\nu \llcorner P, x)$, it holds $\eta(B(y, C/2)) = 0$. □

PROOF OF PROPOSITION 2.1. As we noticed in the discussion before Proposition 2.1, it is sufficient to prove the proposition for $n = 1$.

Consider the 1-periodic function $\varphi : \mathbb{R} \to \mathbb{R}$ which agrees with $2\chi_{[0,1/2]} - 1$ on $[0,1]$ and consider a non-increasing sequence of positive numbers a_i, $i = 0, 1, 2, \ldots$, such that $a_i < 1$, $a_i \searrow 0$ and $\sum_i a_i^2 = +\infty$. Further hypotheses on $(a_i)_{i\in\mathbb{N}}$ will be specified later. Define on $[0,1]$ the functions

$$\varphi_i(x) = a_i \varphi(2^i x), \quad \Phi_N = \sum_{i=0}^{N} \varphi_i, \quad \psi_i = 1 + \varphi_i, \quad \Psi_N = \prod_{i=0}^{N} \psi_i.$$

Consider now the measures $\mu_N = \Psi_N \mathscr{L}^1$. By the Martingale Theorem there exists a measure ν such that $\mu_N \overset{*}{\rightharpoonup} \nu$ as $N \to \infty$ and moreover $\Psi_N \to \frac{d\nu_{abs}}{dx}$ (the Radon-Nikodym derivative of the absolutely continuous part of ν) a.e. We will prove that ν is a singular measure and, for a suitable choice of (a_i), it satisfies $\nu(P) = 0$ for every porous set P.

ν **is singular.** To prove that ν is singular, according to Lemma 2.2 it is sufficient to prove that $\liminf_N \Psi_N = 0, \mathscr{L}^1$-a.e. Notice now that for $|t| < 1$ there holds

$$\log(1+t) \leq t - \frac{t^2}{8},$$

hence we have

$$\log(\Psi_N) = \sum_{i=0}^{N} \log(1+\varphi_i) \leq \sum_{i=0}^{N} \left(\varphi_i - \frac{\varphi_i^2}{8}\right) = \Phi_N - \sum_{i=0}^{N} \frac{a_i^2}{8}.$$

Since the random variable Φ_N has expected value $E(\Phi_N) = 0$ and variance $\sigma^2(\Phi_N) = \sum_{i=0}^{N} a_i^2$, then Chebyshev inequality (see 5.10.7 of [3]) gives

$$\mathscr{L}^1\left(\left\{x \in [0,1] : \Phi_N(x) > \sum_{i=0}^{N} \frac{a_i^2}{16}\right\}\right) \leq \frac{16^2}{\sum_{i=0}^{N} a_i^2},$$

and the right-hand side tends to zero as $N \to \infty$ because $\sum a_i^2 = +\infty$. Therefore we have

$$\liminf_N \Psi_N = \exp\left(\liminf_N \left(\Phi_N - \frac{\sum_{i=0}^{N} a_i^2}{8}\right)\right) = 0, \quad \mathscr{L}^1 - \text{a.e.}$$

$\nu(P) = 0$ **whenever** P **is a porous set.** Now we make the choice $a_0 = a_1 = 1/\sqrt{2}$ and for $i > 1$ $a_i := i^{-1/2}$. We want to show that for ν-a.e. point $x \in (0,1)$, every blowup of ν at x gives positive measure to every non-trivial interval $J \subset (-1,1)$. By Lemma 2.3, this guarantees that every porous set is ν-null.

Consider a point $x \in (0,1)$, a measure $\eta \in \mathrm{Tan}(\nu, x)$ and a sequence $r_j \searrow 0$ such that $\eta = \lim_j \kappa_j$, where κ_j is defined according to (2). We may further assume, possibly passing to a subsequence, that

$$r_j \leq \min\{2^{-j-2}, \mathrm{dist}(x, B(0,1)^c)\}. \tag{3}$$

For every $j \in \mathbb{N}$, there exist $i = i(j)$, and a dyadic interval $I_i(x)$, of the i-th generation, containing x, such that it also contains either $x + r_j$ or $x - r_j$, but no interval in the next generation has the same property. Note that $I_i(x)$ cannot contain both $x + r_j$ and $x - r_j$. In particular we have

$$r_j \leq |I_i(x)| \leq 4r_j,$$

hence, by (3), it holds $i(j) \geq j$. Denote by $I_i'(x)$ the adjacent dyadic interval of the same generation as $I_i(x)$, that together with $I_i(x)$ covers $(x - r_j, x + r_j)$. We claim that, eventually in j , the ratio

$$c_j(x) = \frac{\mu_{i-1}(I_i(x))}{\mu_{i-1}(I_i'(x))}$$

satisfies

$$e^{-8} \leq c_j(x) \leq e^8, \text{ for } \nu\text{−a.e. } x \in (0,1). \tag{4}$$

This would be sufficient to prove that $\eta(J) > 0$ for every non-trivial closed interval $J \subset (-1,1)$. Indeed we have

$$\eta(J) \geq \limsup_j \kappa_j(J) \geq \limsup_j \frac{\nu(I)}{\nu(I_i(x) \cup I_i'(x))}, \tag{5}$$

where $I = I(j)$ is the largest dyadic interval contained in $x + r_j J$. Note that I can be obtained, either from I_i or from I_i', by a number of subsequent subdivisions that depends only on J. Hence the fact that $\nu(I) = \mu_m(I)$ for every m sufficiently large and the bound (4) imply that the ratio in (5) is bounded from below by a positive constant.

To prove the claim (4), fix $x \in (0,1)$ and j and let $(\sigma_k(x))_{k\in\mathbb{N}}$ be the unique sequence made of 0's and 1's such that

$$\min\{I_i(x)\} = \sum_{k=0}^{i} 2^{-k}\sigma_k(x) \tag{6}$$

(see Figure 1), and analogously define $(\sigma_k'(x))_{k=1}^i$ replacing I_i with I_i' in (6).

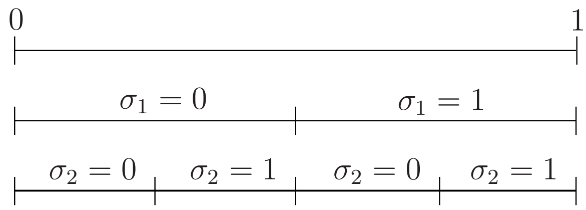

Figure 1. Values of the function $\sigma_k(x)$, for $k = 1, 2$.

Obviously we have

$$\max\{c_j(x), c_j(x)^{-1}\} \leq \prod_{k=k_0+1}^{i} \frac{1+a_k}{1-a_k},$$

where k_0 is the last index smaller than $i = i(j)$ such that $\sigma_{k_0}(x) = \sigma_{k_0}'(x)$. Notice that if $k_0 < i-1$ and $I_i'(x)$ is the left neighborhood of $I_i(x)$, we have $\sigma_{k_0+1}(x) = 1$ and $\sigma_k(x) = 0$ for every $k = k_0+2, \ldots, i$; vice-versa if $I_i'(x)$ is the right neighborhood of $I_i(x)$, we have $\sigma_{k_0+1}(x) = 0$ and $\sigma_k(x) = 1$ for every $k = k_0+2, \ldots, i$.

For $\ell = 0, 1$, and for $j \geq 4$ denote

$$E_j^\ell = \{x \in (0,1) : \sigma_k(x) = \ell, \quad \text{for every } k \in [i - i^{1/2} + 2, i]\}.$$

Observe that, for j sufficiently large, the set of points x such that $c_j(x) \notin [e^{-8}, e^8]$ is contained in $E_j^0 \cup E_j^1$. Indeed assume by contradiction that $x \notin E_j^0 \cup E_j^1$, but either $c_j(x) > e^8$ or $c_j(x) < e^{-8}$. In both cases we have

$$\prod_{k=k_0+1}^{i} \frac{1 + a_k}{1 - a_k} > e^8. \tag{7}$$

Since $\log(1+t) < t$, for $t > 0$, then we have, for $k_0 \geq 3$

$$\prod_{k=k_0+1}^{i} \frac{1 + k^{-1/2}}{1 - k^{-1/2}} = \exp\left(\log\left(\prod_{k=k_0+1}^{i} \frac{1 + k^{-1/2}}{1 - k^{-1/2}}\right)\right) \leq \exp\left(4 \sum_{k=k_0+1}^{i} k^{-1/2}\right)$$

But it is easy to see that

$$4 \sum_{k=k_0+1}^{i} k^{-1/2} \leq 8, \tag{8}$$

whenever $k_0 \geq i - i^{1/2} \geq 2$, hence, for j sufficiently large, (7) implies that $x \in E_j^0 \cup E_j^1$.

Eventually we compute, for $i \geq 4$,

$$\nu(E_j^\ell) \leq \prod_{k=i-i^{1/2}+2}^{i} \frac{1 + k^{-1/2}}{2} \leq 2^{-i^{1/2}+2} \prod_{k=i-i^{1/2}+1}^{i} (1 + k^{-1/2})$$

$$\leq 2^{-i^{1/2}+2} \exp\left(\sum_{k=i-i^{1/2}+1}^{i} \ln(1 + k^{-1/2})\right) \leq e^2 2^{-i^{1/2}+2},$$

where in the last inequality we used (8) and the fact that $\log(1+t) < t$. Therefore

$$\nu\left(\bigcap_{h=4}^{\infty} \bigcup_{j=h}^{\infty} (E_j^0 \cup E_j^1)\right) = 0$$

and since, by the observation above, this set contains the set of points x such that $c_j(x) \notin [e^{-8}, e^8]$ frequently, the claim (4) is proved. $\square$

2.4 – *Existence of a Lipschitz function which is non differentiable a.e. with respect to a singular measure*

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called L-Lipschitz ($L > 0$) if

$$|f(y) - f(x)| \leq L|y - x|, \quad \text{for every } x, y \in \mathbb{R}^n.$$

We conclude this section of preliminary results proving that, given a singular measure μ on $\mathbb{R}$, there exists a 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is non-differentiable at μ-a.e. point. The original proof of a stronger statement is contained in [16]. The proof we present uses the Baire Theorem (see Theorem 2.2 of [10]).

Proposition 2.4. *Given a singular measure μ on $\mathbb{R}$, there exists a 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is non-differentiable at μ-a.e. point.*

Proof. Throughout the proof we will denote by X the complete metric space of real valued 1-Lipschitz functions on the real line, endowed with the supremum distance. We will actually prove a stronger statement: namely that the family of 1-Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ such that f is non-differentiable at μ-a.e. point. is *residual* (i.e. it contains the intersection of countably many open dense sets) and in particular, by the Baire Theorem, it is dense in X.

Fix any compact nullset E and define inductively an infinitesimal sequence of positive numbers $(\varepsilon_i)_{i\in\mathbb{N}}$ and a sequence of open sets $(E_i)_{i\in\mathbb{N}}$, with the following properties

- $E \subset E_{i+1} \subset E_i$;
- E_i is a finite union of disjoint open intervals;
- $\mathscr{L}^1(E_i) \leq \varepsilon_i$;
- $\varepsilon_{i+1} \leq \alpha_i \varepsilon_i$, where

$$\alpha_i := \min\{\mathscr{L}^1(I) : I \text{ is a connected component of } E_i\}.$$

Define the following subsets of X (we will write c.c. for "connected component"):

$$U_i = \{g \in X : g(b) - g(a) > (b - a) - \varepsilon_{i+1}, \text{ whenever } (a, b) \text{ is a c.c. of } E_i\},$$

$$V_i = \{g \in X : g(b) - g(a) < \varepsilon_{i+1} - (b - a), \text{ whenever } (a, b) \text{ is a c.c. of } E_i\},$$

$$A_j = \bigcup_{i\geq j} U_i, \quad B_j = \bigcup_{i\geq j} V_i.$$

Obviously U_i and V_i are open sets for every i, and therefore A_j and B_j are also open, for every j. Moreover, U_i and V_i are $2\varepsilon_i$-nets in X, by which we mean that for every element $\phi \in X$ there is an element $\phi_i \in U_i$ (respectively V_i) such that $\|\phi - \phi_i\|_\infty \leq 2\varepsilon_i$. To prove this fact, consider for every function $\phi \in X$ the function

$$\phi_i(x) = \phi\bigg(x - \int_{-\infty}^{x} \chi_{E_i}(t)\,\mathrm{d}t\bigg) + \int_{-\infty}^{x} \chi_{E_i}(t)\,\mathrm{d}t,$$

which has the following properties: $\phi_i'(x) = \phi'(x - \int_{-\infty}^{x} \chi_{E_i}(t)\,\mathrm{d}t)$ for a.e. $x \notin E_i$ and $\phi_i'(x) = 1$ if $x \in E_i$. This is clearly an element of U_i and $\|\phi - \phi_i\|_\infty \leq 2\varepsilon_i$. The proof that V_i is a $2\varepsilon_i$-net is analogous.

As a consequence, A_j and B_j are dense for every j. Finally,

$$A = \left(\bigcap_{j=1}^{\infty} A_j\right) \cap \left(\bigcap_{j=1}^{\infty} B_j\right)$$

is a residual set in X (in particular it is dense).

Next we prove that every function $f \in A$ is not differentiable at any point of E. More precisely, we claim that

$$f'_+(x) := \limsup_{|h| \searrow 0} \frac{f(x+h) - f(x)}{h} = 1$$

and

$$f'_-(x) := \liminf_{|h| \searrow 0} \frac{f(x+h) - f(x)}{h} = -1$$

for every $x \in E$. Fix $\varepsilon > 0$ and take $i \in \mathbb{N}$ such that $3\varepsilon_i < \varepsilon$, and $f \in U_i$. Let $I = (a, b)$ be the connected component of E_i containing x. Take a point $y \in I$ such that

$$\operatorname{dist}(x, y) \geq \frac{\mathscr{L}^1(I)}{3}.$$

Let I' be the open interval with end points x and y. Since on (a, b) we have $f' \leq 1$ a.e. and $f(b) - f(a) \geq b - a - \varepsilon_{i+1}$, then we also have

$$\int_{I'} f'(t)\,\mathrm{d}t \geq |x - y| - \varepsilon_{i+1}.$$

Therefore we conclude:

$$\frac{f(y) - f(x)}{y - x} \geq \frac{|y - x| - \varepsilon_{i+1}}{|y - x|} \geq 1 - \frac{3\varepsilon_{i+1}}{\mathscr{L}^1(I)} \geq 1 - \frac{3\varepsilon_{i+1}}{\alpha_i} \geq 1 - 3\varepsilon_i \geq 1 - \varepsilon.$$

Similarly we can prove that $f'_-(x) = -1$ for every $x \in E$.

Eventually we consider a sequence of compact nullsets $E^k \subset \mathbb{R}$ such that $\mu(\mathbb{R} \setminus \bigcup_k E^k) = 0$. Since for every k the set A^k of 1-Lipschitz functions which are non-differentiable at all points of E^k is a residual set, then also the intersection of the sets A^k is residual and it is contained in the set of all 1-Lipschitz functions which are non-differentiable at μ-a.e. point. $\square$

3. Proof of Theorem 1.1

By Proposition 2.1 and Proposition 2.4 we deduce that the class of pairs (μ, f) satisfying the assumption of Theorem 1.1 is non empty, at least for $n = 1$. To prove the same fact for $n \geq 2$, one should replace our Proposition 2.4 with Theorem 1.14 of [6].

To prove Theorem 1.1 assume by contradiction that there exists a C^1 function g such that, denoting

$$A := \{x \in \mathbb{R}^n : g(x) = f(x)\},$$

there holds $\mu(A) > 0$. We can assume that f is 1-Lipschitz and g is globally L-Lipschitz for some $L > 0$.

Denote $h := f - g$. Observe that h is $(1 + L)$-Lipschitz and $h \equiv 0$ on A. We claim that Dh exists and it is equal to 0 at μ-a.e. point of A, which is a contradiction because it implies that f is differentiable on a set of positive measure μ.

To prove the claim, consider the set $P \subset A$ of points where either Dh does not exist or $Dh \neq 0$. In particular, for every $x \in P$, there exist a constant $C(x) > 0$ and a sequence of points $y_k \to x$ such that

$$|h(y_k)| > C(x)|y_k - x|, \quad \text{for every } k \in \mathbb{N}.$$

Then for every $x \in P$ and for every $k \in \mathbb{N}$ we would have $h \neq 0$ on the open ball B_k centered at y_k with radius $(L + 1)^{-1}C(x)|y_k - x|$. Since $P \subset A$ this implies that the set P is porous. Hence $\mu(P) = 0$.

4. Lusin type theorem for gradients

4.1 – *Prescribing derivatives*

In this section we discuss the possibility to extend and possibly to improve the result of [1], when we replace the Lebesgue measure with any Radon measure. We will consider only the one-dimensional setting. The higher dimensional case and further results are discussed in [9]. The first trivial observation is that the result of [1] is valid with respect to any Radon measure.

Theorem 4.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function and μ be a Radon measure on $\mathbb{R}$. Then for every $\varepsilon > 0$ there exist a set $E \subset \mathbb{R}$ with $\mu(\mathbb{R} \setminus E) < \varepsilon$ and a C^1 function $f : \mathbb{R} \to \mathbb{R}$ such that $f' = g$ on the set E.*

Proof. Fix $\varepsilon > 0$. By the standard Lusin theorem (see Theorem 2.24 of [11]), there exist a set $E \subset \mathbb{R}$ with $\mu(\mathbb{R} \setminus E) < \varepsilon$ and a bounded and continuous function $h : \mathbb{R} \to \mathbb{R}$ such that $h = g$ on the set E. Denote

$$f(x) := \int_0^x h(t)dt.$$

Clearly f is C^1 and it holds $f' = h = g$ on the set E. □

4.2 – *Prescribing non-differentiable blowups*

Next we want to investigate the possibility to modify Theorem 4.1, prescribing some non-linear local behavior at many points. Clearly the corresponding f cannot be a C^1 function, but we can require it to be Lipschitz. When μ is the Lebesgue measure, the Rademacher theorem is an obstruction to our aim. Indeed it guarantees that the local behavior of any Lipschitz function f at most of the points must be linear. Since by Proposition 2.4 we know that for a singular measure μ one can find Lipschitz functions which are μ-almost everywhere non-differentiable, we wonder if it is possible to find a Lipschitz function with an arbitrarily prescribed non-differentiability local behavior.

We denote by $\mathrm{Tan}(f, x)$ the set of the *blowups* of f at x, i.e. all the possible limits (with respect to the uniform convergence) of the form

$$\lim_{r_i \searrow 0} f_{x,r_i},$$

where $f_{x,r}(y) = r^{-1}(f(x + ry) - f(x))$. Given a pair (a, b) in $\mathbb{R}^2$ we say that a function $f : \mathbb{R} \to \mathbb{R}$ is (a, b)-*differentiable* at the point x_0 if the two limits

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}, \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

exist and they are equal to a and b respectively.

Note that, for a Lipschitz function f, this is the only unique blowup of f at x_0 that one can prescribe, if "prescribing a unique blowup" is intended in the sense of finding a function g_{x_0} such that $Tan(f, x_0) = \{g_{x_0}\}$. Observe that this requirement forces g_{x_0} to be positively homogeneous, because every rescaling of a blowup is also a blowup. The following proposition shows that, unless μ is atomic (i.e. there exists a countable set N such that $\mu(\mathbb{R} \setminus N) = 0$), it is not possible to prescribe a non-linear unique blowup at many points.

Proposition 4.2. *Let μ be a Radon measure on $\mathbb{R}$ and let $a, b : \mathbb{R} \to \mathbb{R}$ be bounded Borel functions, such that $a(x) \neq b(x)$ μ-a.e. Then the following property (P) holds if and only if μ is an atomic measure.*

(P) For every $\delta > 0$ there exist a set $E \subset \mathbb{R}$ with $\mu(\mathbb{R} \setminus E) < \delta$ and a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in E$, f is $(a(x), b(x))$-differentiable.

Proof. If μ is an *atomic measure* then it is very easy to prove the validity of property (P) constructing, for every δ, an appropriate piecewise affine function f.

Now assume property (P) holds for the functions a and b. Then it also holds if we replace a and b with

$$a_1 := a - (a + b)/2, \quad b_1 := b - (a + b)/2.$$

Indeed, given $\delta > 0$ one can apply Theorem 4.1 to the function $g := -(a+b)/2$ with parameter $\varepsilon := \delta/2$, thus obtaining a set E_0 and a function f_0. Then by property (P) for a and b, there exist a set E_1 and a Lipschitz function f_1 such that $\mu(\mathbb{R} \setminus E_1) < \delta/2$ for every $x \in E_1$, f_1 is $(a(x), b(x))$-differentiable. Hence the function $f := f_0 + f_1$ and the set $E := E_0 \cup E_1$ yield property (P) for the fixed parameter δ and the functions a_1 and b_1.
Now we have that $a_1(x)$ and $b_1(x)$ have different sign (non zero) for μ-almost every point x. Note that this implies that μ-almost every $x \in E$ is a strict local maximum or minimum for f. We claim that there are at most countably many such points, which implies that μ is an atomic measure. To prove the claim, for every $i \in \mathbb{N}$ we denote by A_i the set of points x in E such that $f(x)$ is the unique minimum of f in the interval $(x - 1/i, x + 1/i)$. By construction, the set A_i is discrete for every $i \in \mathbb{N}$, hence the union of the sets A_i (which contains μ-a.e. point of E) is at most countable. $\square$

Even if for a general measure it is not possible to prescribe any form of non-differentiable first order approximation, it might be possible to prescribe the existence of a non-linear blow up, at many points x. Such problem is treated in [9]. In particular we prove the following perhaps surprising result. For every singular measure μ on the line, the generical (in the sense of Baire categories) 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ has the following property: for μ-a.e. point x, the set $Tan(f, x)$ of all blowups of f at x coincides with the set of all 1-Lipschitz functions with value 0 at the origin. In other words the generical Lipschitz function attains every possible blowup at μ a.e. point.

Acknowledgments

The author wishes to thank Bernd Kirchheim and David Preiss for valuable discussions and the referee for carefully checking the paper and for several remarks. The author is supported by the ERC-grant "Regularity of area-minimizing currents" (306247).

References

[1] Alberti, Giovanni. A Lusin type theorem for gradients. *J. Funct. Anal.* 100 (1991), no. 1, 110–118.

[2] Alberti, Giovanni; Marchese, Andrea. Differentiability of Lipschitz functions with respect to measures in the Euclidean Space. *Geom. Funct. Anal.* 26 (2016) 1–66.

[3] Ash, Robert B. *Real analysis and probability.* Probability and Mathematical Statistics, No. 11. Academic Press, New York-London, 1972.

[4] David, Guy C. Lusin-type theorems for Cheeger derivatives on metric measure spaces. *Anal. Geom. Metr. Spaces* 3 (2015), 296–312.

[5] De Lellis, Camillo. *Rectifiable sets, densities and tangent measures.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[6] De Philippis, Guido; Rindler Filip. On the structure of $\mathscr{A}$-free measures and applications. To appear in *Ann. of Math. (2).* Preprint availble at https://arxiv.org/abs/1601.06543.

[7] Federer, Herbert. *Geometric measure theory.* Grundlehren der mathematischen Wissenschaften, 153. Springer, Berlin-New York 1969. Reprinted in the series Classics in Mathematics. Springer, Berlin-Heidelberg 1996.

[8] Hajasz, Piotr; Mirra, Jacob. The Lusin theorem and horizontal graphs in the Heisenberg group. *Anal. Geom. Metr. Spaces* 1 (2013), 295–301.

[9] Marchese, Andrea; Schioppa Andrea. Lipschitz functions with prescribed blowups at many points. *In preparation.*

[10] Rudin, Walter. *Functional Analysis. Second edition.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

[11] Rudin, Walter. *Real and Complex Analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill Science/Engineering/Math, 1986.

[12] Shaposhnikov, Alexander V. A Luzin-type theorem for vector fields on the Wiener space. (Russian) *Dokl. Akad. Nauk* 434 (2010), no. 6, 744–748. Translation in *Dokl. Math.* 82 (2010), no. 2, 803–807.

[13] Speight, Gareth. Lusin Approximation and Horizontal Curves in Carnot Groups. To appear in *Revista Matematica Iberoamericana*. Preprint available at http://arxiv.org/abs/1412.2531.

[14] Speight, Gareth; Le Donne, Enrico Lusin Approximation for Horizontal Curves in Step 2 Carnot Groups. Preprint available at http://arxiv.org/abs/1602.02607.

[15] Tkadlec, Josef. Construction of a finite Borel measure with σ-porous sets as null sets. *Real Anal. Exchange* 12 (1986/1987), no. 1, 349–353.

[16] Zahorski, Zygmunt. Sur l'ensemble des points de non-dérivabilité d'une fonction continue. *Bull. Soc. Math. France* 74 (1946), 147–178.

Received submission date; revised revision date