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Integrable derivations in the sense of Hasse-Schmidt for some binomial plane curves

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Abstract – We describe the module of integrable derivations in the sense of Hasse-Schmidt of the quotient of the polynomial ring in two variables over an ideal generated by the equation $x^n - y^q$.

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1. Introduction

Let k be a commutative ring and A a commutative k-algebra. We denote by $\mathrm{Der}_k(A)$ the A-module formed by the k-derivations of A. The Hasse-Schmidt derivations of length $m \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, also called higher derivations of order m (see [3]), were introduced by H. Hasse and F.K. Schmidt ([2]) and they have been used by several authors in different contexts (see [5], [7] or [8]). An important notion related with Hasse-Schmidt derivations is *integrability*. Let $m \in \overline{\mathbb{N}}$, we say that $\delta \in \mathrm{Der}_k(A)$ is m-integrable if there exists a Hasse-Schmidt derivation $(D_i)_{i\geq 0}$ of length m such that $\delta = D_1$. The set formed by the m-integrable k-derivations is a submodule of $\mathrm{Der}_k(A)$ for every m, which is denoted by $\mathrm{IDer}_k(A; m)$.

If k has characteristic 0 or A is 0-smooth over k, then any k-derivation is ∞-integrable ([3]), that means that $\mathrm{Der}_k(A) = \mathrm{IDer}_k(A; \infty)$. If we consider k a ring of positive characteristic and A any commutative k-algebra, the modules $\mathrm{IDer}_k(A; m)$ have better properties than $\mathrm{Der}_k(A)$ (see [4]). So exploring these modules seems interesting to better understand singularities in positive characteristic.

The aim of this paper is to describe the modules of m-integrable derivations, for $m \in \overline{\mathbb{N}}$, of the quotient of the polynomial ring in two variables over an ideal generated by an equation of type $x^n - y^q$.

This paper is organized as follows: In section 1 we recall the definition of Hasse-Schmidt derivations and give some known properties that will be useful in later sections. In section 2, we prove the relationship between integrable derivations of the quotient of a polynomial ring over $\langle h\rangle$ and over $\langle h^p\rangle$ where h is a polynomial, when k is a unique factorization domain. In section 3 we compute the modules of m-integrable k-derivations, where k is a reduced ring of characteristic $p > 0$, of the quotients of the polynomial ring in two variables over the ideal generated by the equation $x^n - y^q$ when n or q is not a multiple of p. Thanks to this, we can describe the integrable derivations of $k[x, y]/\langle x^n - y^q\rangle$ when n and q are both multiples of p and k is a unique factorization domain. In section 4 we compute the modules of m-integrable derivations in three examples taken from [1] assuming that k is a domain of positive characteristic.

2. Hasse-Schmidt derivations

In this section we recall the main definitions and properties of Hasse-Schmidt derivations. Most of the results presented in this section can be found in [3] and [6].

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Let k be a commutative ring and A a commutative k-algebra. We denote by $A[[T]]$ the ring of power series over A. For each integer $m \geq 1$, we will write $A[[T]]_m := A[[T]]/\langle T^{m+1}\rangle$ and $A[[T]]_\infty := A[[T]]$.

Definition 2.1. A *Hasse-Schmidt derivation* (over k) of A of length $m \geq 1$ (resp. of length ∞) is a sequence $D := (D_0, D_1, \ldots, D_m)$ (or resp. $D = (D_0, D_1, \ldots)$) of k-linear maps $D_i : A \to A$, satisfying the conditions:

$$D_0 = \mathrm{Id}_A, \quad D_i(xy) = \sum_{a+b=i} D_a(x)D_b(y)$$

for every $x, y \in A$ and for every integer i. We write $\mathrm{HS}_k(A; m)$ (resp. $\mathrm{HS}_k(A; \infty) = \mathrm{HS}_k(A)$) for the set of Hasse-Schmidt derivations (over k) of A of length m (resp. ∞).

Any Hasse-Schmidt derivation $D \in \mathrm{HS}_k(A; m)$ is determined by the k-algebra homomorphism

$$\varphi_D : \quad a \in A \quad \mapsto \quad \textstyle\sum_{i=0}^m D_i(a)T^i \in A[[T]]_m$$

satisfying $\varphi_D(x) \equiv x \mod T$. The morphism φ_D can be uniquely extended to a k-algebra automorphism $\widetilde{\varphi}_D : A[[T]]_m \to A[[T]]_m$ with $\widetilde{\varphi}_D(T) = T$. So, the set $\mathrm{HS}_k(A; m)$ has a canonical group structure. Namely, if $D, D' \in \mathrm{HS}_k(A; m)$, then $D \circ D' = D'' \in \mathrm{HS}_k(A; m)$ with $D''_n = \sum_{i+j=n} D_i \circ D'_j$ for $n \leq m$. Moreover, the component D_i of a Hasse-Schmidt derivation is a differential operator of order $\leq i$ vanishing at 1, in particular D_1 is a k-derivation. So, the map $(\mathrm{Id}, D_1) \in \mathrm{HS}_k(A; 1) \mapsto D_1 \in \mathrm{Der}_k(A)$ is a group isomorphism.

In this text we mainly use two operations on Hasse-Schmidt derivations: Let $D \in \mathrm{HS}_k(A; m)$ be a Hasse-Schmidt derivation of length $m \in \overline{\mathbb{N}}$.

(1) For every $a \in A$, the sequence $a \bullet D = (a^i D_i) \in \mathrm{HS}_k(A; m)$.

(2) For every integer $q \in \{1, \ldots, m\}$, we define the *truncation map* by $\tau_{mq}(D) = (\mathrm{Id}, D_1, \ldots, D_q) \in \mathrm{HS}_k(A; q)$.

Definition 2.2. Let $D \in \mathrm{HS}_k(A; m)$ where $m \in \overline{\mathbb{N}}$ and $n \geq m$. Let I be an ideal of A.

- We say that D is *I-logarithmic* if $D_i(I) \subseteq I$ for every i. The set of I-logarithmic Hasse-Schmidt derivations is denoted by $\mathrm{HS}_k(\log I; m)$, $\mathrm{HS}_k(\log I) := \mathrm{HS}_k(\log I; \infty)$ and $\mathrm{Der}_k(\log I) := \mathrm{HS}_k(\log I; 1)$.
- We say that D is *n-integrable* if there exists $E \in \mathrm{HS}_k(A, n)$ such that $\tau_{nm}(E) = D$. Any such E will be called an n-integral of D. If D is ∞-integrable we say that D is integrable. If $m = 1$, we write $\mathrm{IDer}_k(A; n)$ for the set of n-integrable derivations and $\mathrm{IDer}_k(A) := \mathrm{IDer}_k(A; \infty)$.
- If $D \in \mathrm{HS}_k(\log I; m)$, we say that D is *I-logarithmically n-integrable* if there exists $E \in \mathrm{HS}_k(\log I; n)$ such that E is an n-integral of D. We denote $\mathrm{IDer}_k(\log I; n)$ for the set of I-logarithmically n-integrable derivations (i.e. for $m = 1$) and $\mathrm{IDer}_k(\log I) := \mathrm{IDer}_k(\log I, \infty)$.

Remark 2.3. The set $\mathrm{IDer}_k(A; n)$ is a submodule of $\mathrm{Der}_k(A)$ for every $n \in \overline{\mathbb{N}}$ thanks to the group structure of $\mathrm{HS}_k(A; n)$ and operation 1 defined before. Namely, if $\delta_1, \delta_2 \in \mathrm{IDer}_k(A; n)$ and $a_1, a_2 \in A$ then, an n-integral of $a_1\delta_1 + a_2\delta_2 \in \mathrm{IDer}_k(A; n)$ is $(a_1 \bullet D^1) \circ (a_2 \bullet D^2) \in \mathrm{HS}_k(A; n)$ where $D^i \in \mathrm{HS}_k(A; n)$ is an n-integral of δ_i for $i \in \{1, 2\}$.

Definition 2.4. The k-algebra A has a *leap* at $s > 1$ if the inclusion $\mathrm{IDer}_k(A; s-1) \supsetneq \mathrm{IDer}_k(A; s)$ is proper. The set of leaps of A over k is denoted by $\mathrm{Leaps}_k(A)$.

We recall that a k-algebra A is *0-smooth* over k if it has the following property: for any k-algebra C, any ideal N of C satisfying $N^2 = 0$, and any k-algebra homomorphism $u : A \to C/N$, there exists a lifting $v : A \to C$ of u to C, as a k-algebra homomorphism.

Theorem 2.5 ([3], Th. 27.1). *If the ring A is 0-smooth over a ring k, then a Hasse-Schmidt derivation of length $m < \infty$ over k can be extended to a Hasse-Schmidt derivation of length ∞.*

The following result is a straightforward consequence of Theorem 2.5.

Corollary 2.6. *Consider $R = k[x_1, \ldots, x_d]$ the polynomial ring over k. Any Hasse-Schmidt derivation of R (over k) of length $m \geq 1$ is integrable.*

Let us consider a k-algebra A, $I \subseteq A$ an ideal and $m \in \overline{\mathbb{N}}$. We denote by $\Pi : \mathrm{IDer}_k(\log I; m) \to \mathrm{IDer}_k(A/I; m)$ the map given by:

$$\delta \in \mathrm{IDer}_k(\log I; m) \mapsto \Pi(\delta) = \overline{\delta} \in \mathrm{IDer}_k(A/I; m),\ \overline{\delta}(x+I) = \delta(x) + I \quad \forall x \in A \tag{1}$$

The following proposition is clear thanks to Corollary 2.1.9 of [6].

Proposition 2.7. *Consider $R = k[x_1, \ldots, x_d]$ and $I \subseteq R$ an ideal. Then, the following short sequence of R-modules is exact:*

$$0 \to I(\mathrm{Der}_k(R)) \to \mathrm{IDer}_k(\log I; m) \xrightarrow{\Pi} \mathrm{IDer}_k(R/I; m) \to 0.$$

Corollary 2.8. *Let $I \subset R = k[x_1, \ldots, x_d]$ be an ideal and $A = R/I$. Then, $s \in \mathrm{Leaps}_k(A)$ if and only if the inclusion $\mathrm{IDer}_k(\log I; s-1) \supsetneq \mathrm{IDer}_k(\log I; s)$ is proper.*

Proof. The corollary comes from applying the *five lemma* to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(\mathrm{Der}_k(R)) & \longrightarrow & \mathrm{IDer}_k(\log I; m) & \xrightarrow{\Pi} & \mathrm{IDer}_k(A; m) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\text{incl.}} & & \downarrow{\text{incl.}} & & \\ 0 & \longrightarrow & I(\mathrm{Der}_k(R)) & \longrightarrow & \mathrm{IDer}_k(\log I; m-1) & \xrightarrow{\Pi} & \mathrm{Der}_k(A; m-1) & \longrightarrow & 0. \end{array}$$

□

Let us recall two results.

Proposition 2.9 ([6], Prop. 2.2.4). *Let $f \in R$, $I = \langle f \rangle$, and $J^0 = \langle \partial_{x_1}(f), \ldots, \partial_{x_d}(f) \rangle$ the gradient ideal. If $\delta : R \to R$ is an I-logarithmic k-derivation with $\delta \in J^0\,\mathrm{Der}_k(R)$, then δ admits an I-logarithmic integral $D \in \mathrm{HS}_k(\log I)$ with $D_i(f) = 0$ for every $i > 1$. In particular, if $\delta(f) = 0$, the integral D can be taken with $\varphi_D(f) = f$.*

Lemma 2.10 ([8], Corollary 1.1). *Let k be a ring of characteristic $p > 0$, A a k-algebra and $h \in A$. Consider $D \in \mathrm{HS}_k(A; m)$ with $m \in \overline{\mathbb{N}}$ and $\tau \geq 0$. Then, for every $i \leq m$,*

$$D_i\left(h^{p^\tau}\right) = \begin{cases} 0 & \text{if} \quad p^\tau \not| i \\ D_{i/p^\tau}(h)^{p^\tau} & \text{if} \quad p^\tau | i. \end{cases}$$

3. I^p-logarithmic derivations

In this section let us consider $R = k[x_1, \ldots, x_d]$ the polynomial ring in d variables over a unique factorization domain (UFD) k of characteristic $p > 0$ and $h \in R$ a polynomial. In this section we give some easy results about the relationship between the modules of i-integrable derivations of $R/\langle h \rangle$ and $R/\langle h^{p^\tau} \rangle$. This will allow us to calculate the leaps and generators of the modules of i-integrable derivations of the curve $h = x^n - y^q$ when n and q are multiples of p (Corollaries 4.2 and 4.3). In addition, this relationship will be used in future references.

Thanks to Proposition 2.7, it is enough to study this relationship between $\langle h \rangle$-logarithmically n-integrable derivations and $\langle h^p \rangle$-logarithmically n-integrable derivations. We start with a general result.

Lemma 3.1. *Let A be a k-algebra, $n \in \overline{\mathbb{N}}$ and an integer $m \leq n$. Consider $g \in A$ and $D \in \mathrm{HS}_k(A; n)$. Suppose that $D_j(g) \in \langle g \rangle$ for every integer $j \in \{0, 1, \ldots, m-1\}$. Then, for every integer $r \geq 1$, we have $D_m(g^r) \in r g^{r-1} D_m(g) + \langle g^r \rangle$.*

Proof. It is easy to prove that $D_j(g^r) \in \langle g^r \rangle$ for every integer $j \in \{0, \ldots, m-1\}$ and every integer $r \geq 1$. Thanks to this, we can prove that $D_m(g^r)$ belongs to $r g^{r-1}(g) D_m(g) + \langle g^r \rangle$ by induction on $r \geq 1$. □

From now on, k will be a UFD of characteristic $p > 0$ and $R = k[x_1, \ldots, x_d]$.

Proposition 3.2. *If $f, g \in R$ are coprime then, for every $n \in \overline{\mathbb{N}}$, we have* $\mathrm{HS}_k(\log fg; n) = \mathrm{HS}_k(\log f; n) \cap \mathrm{HS}_k(\log g; n)$.

Proof. The inclusion $\supseteq$ is clear. Let us consider $D \in \mathrm{HS}_k(\log fg; n)$. To conclude the result we prove that $D_i(f) \in \langle f\rangle$ and $D_i(g) \in \langle g\rangle$ by induction on $i \geq 0$. $\square$

As a straightforward consequence of the previous proposition, we have the following result.

Corollary 3.3. *Let $f_1, \ldots, f_m \in R$. If f_i, f_j are coprime whenever $i \neq j$ then, for every $n \in \overline{\mathbb{N}}$ we have* $\mathrm{HS}_k(\log\langle f_1 \cdots f_m\rangle; n) = \bigcap_i \mathrm{HS}_k(\log f_i; n)$ *and* $\mathrm{IDer}_k(\log\langle f_1 \cdots f_m\rangle; n) \subseteq \bigcap_i \mathrm{IDer}_k(\log f_i; n)$.

Remark 3.4. In general, equality for the modules of n-integrable derivations in Corollary 3.3 does not hold. For example: Let $k = \mathbb{F}_2$ and $f = y^2$ and $g = x^2 - y$ two polynomials of $k[x, y]$. Then $\partial_x \in \mathrm{IDer}_k(\log f; 4) \cap \mathrm{IDer}_k(\log g; 4)$. However $\partial_x \notin \mathrm{IDer}_k(\log fg; 4)$.

Lemma 3.5. *Let $f \in R$ be an irreducible polynomial, $a \geq 1$ and $n \in \overline{\mathbb{N}}$. Consider $D \in \mathrm{HS}_k(R; n)$. Suppose that $D_i(f^a)^p \in \langle f^{ap}\rangle$ for every $i \leq n$. Then, $D \in \mathrm{HS}_k(\log f^a; n)$.*

Proof. We write $a = sp^\alpha$ where $\alpha = \mathrm{val}_p(a) \geq 0$ and $s \geq 1$. By Lemma 2.10,

$$D_i\left(f^{sp^\alpha}\right) = \begin{cases} 0 & \text{if} \quad p^\alpha \not| i \\ D_{i/p^\alpha}(f^s)^{p^\alpha} & \text{if} \quad p^\alpha | i. \end{cases}$$

Hence, we can focus on the case $n \geq p^\alpha$ and $i = jp^\alpha \leq n$. It's enough to show that $D_j(f) \in \langle f\rangle$ because, if this is true, we have that $D_j(f^s) \in \langle f^s\rangle$ by Lemma 3.1 and we would have the result.

Since $i = jp^\alpha \leq n$, by hypothesis

$$D_j\left(f^s\right)^{p^{\alpha+1}} = D_{jp^\alpha}\left(f^{sp^\alpha}\right)^p \in \left\langle f^{sp^{\alpha+1}}\right\rangle. \tag{2}$$

When $j = 1$, $D_1\left(f^s\right) = sf^{s-1}D_1(f)$. Substituting in the previous expression, we have that

$$D_1\left(f^s\right)^{p^{\alpha+1}} = sf^{(s-1)p^{\alpha+1}}D_1(f)^{p^{\alpha+1}} \in \left\langle f^{sp^{\alpha+1}}\right\rangle. \tag{3}$$

Since R is UFD and $f, s \neq 0$, $D_1(f)^{p^{\alpha+1}} \in \left\langle f^{p^{\alpha+1}}\right\rangle \subseteq \langle f\rangle$ and hence $D_1(f) \in \langle f\rangle$. Let us assume that $D_l(f) \in \langle f\rangle$ for every $l < j$ with $jp^\alpha \leq n$. Thanks to the hypothesis, we can use Lemma 3.1, and we have $D_j\left(f^s\right) = sf^{s-1}D_j(f) + Ff^s$ for some $F \in R$. Substituting this expression in (2), we obtain that $sf^{(s-1)p^{\alpha+1}}D_j(f)^{p^{\alpha+1}} \in \langle f^{sp^{\alpha+1}}\rangle$. Observe that it is the same condition that (3), so we can deduce that $D_j(f) \in \langle f\rangle$ and the lemma is proved. $\square$

Proposition 3.6. *Let k be a UFD of characteristic $p > 0$ and $R = k[x_1, \ldots, x_d]$ the polynomial ring over k. Let h be a polynomial of R. For every $n \in \overline{\mathbb{N}}$, we have* $\mathrm{IDer}_k(\log h; n) = \mathrm{IDer}_k\left(\log h^p, np\right)$.

Proof. Let $D_1 \in \mathrm{IDer}_k(\log h; n)$ and $D \in \mathrm{HS}_k(\log h; n)$ an n-integral. If $n < \infty$, from Corollary 2.6, D is np-integrable, so let D' be an np-integral of D. If $n = \infty$, we put $D' = D$ (observe that $D'_1 = D_1$). We have to see that $D'_i(h^p) \in \langle h^p\rangle$ for every $i \leq np$. By Lemma 2.10, if i is not a multiple of p, then $D_i(h^p) = 0$. If i is multiple of p, then $i = jp$ where $j \in \{1, \ldots, n\}$ and $D'_i(h^p) = D'_j(h)^p = D_j(h)^p \in \langle h^p\rangle$. Therefore $\mathrm{IDer}_k(\log h; n) \subseteq \mathrm{IDer}_k(\log h^p; n)$.

Let $D_1 \in \mathrm{IDer}_k\left(\log h^p; np\right)$ and $D \in \mathrm{HS}_k\left(\log h^p; np\right)$ an np-integral of D_1. Let $h = h_1^{a_1} \cdots h_m^{a_m}$ be the factorization of h in irreducible factors where $h_i \neq h_j$ if $i \neq j$. Then, $h_1^{a_1 p}, \ldots, h_m^{a_m p}$ are coprime. By Corollary 3.3, $D \in \mathrm{HS}_k(\log h^p; np) = \cap_i \mathrm{HS}_k\left(\log h_i^{a_i p}; np\right)$. Hence, $D_j(h_i^{a_i})^p = D_{jp}(h_i^{a_i p}) \in \langle h_i^{a_i p}\rangle$ for every integer $j \leq n$. By Lemma 3.5, $\tau_{np,n}(D) \in \mathrm{HS}_k(\log h_i^{a_i}; n)$ for every integer $i \in \{1, \ldots, m\}$. So, $\tau_{np,n}(D) \in \mathrm{HS}_k(\log h; n)$. This concludes the proof of the lemma. $\square$

Corollary 3.7. *Under the hypotheses of Proposition 3.6, for every $\tau \geq 0$ and $n \in \overline{\mathbb{N}}$,* $\mathrm{IDer}_k(\log h; n) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right)$.

Proposition 3.8. *Let k be a UFD of characteristic $p > 0$, $R = k[x_1, \ldots, x_d]$ the polynomial ring over k, $h \in R$ and $\tau \geq 1$. We denote $A := R/\left\langle h^{p^\tau}\right\rangle$ and $A' = R/\langle h\rangle$. Then,*

$$\mathrm{Leaps}_k(A) = \begin{cases} \{np^\tau \mid n \in \mathrm{Leaps}_k(A')\} & \textit{if } \mathrm{Der}_k\left(\log h\right) = \mathrm{Der}_k(R) \\ \{np^\tau \mid n \in \mathrm{Leaps}_k(A')\} \cup \{p^\tau\} & \textit{if } \mathrm{Der}_k\left(\log h\right) \neq \mathrm{Der}_k(R). \end{cases}$$

PROOF. By Corollary 2.8, A has a leap at $s > 1$ if and only if the inclusion $\mathrm{IDer}_k\left(\log h^{p^\tau}; s-1\right) \supsetneq \mathrm{IDer}_k\left(\log h^{p^\tau}; s\right)$ is proper. First of all, we will prove the next two equalities:

1. For $s < p^\tau$, $\mathrm{IDer}_k\left(\log h^{p^\tau}; s\right) = \mathrm{Der}_k(R)$.

Let $D_1 \in \mathrm{Der}_k(R) = \mathrm{IDer}_k(R)$ and $D \in \mathrm{HS}_k(R; s)$ an s-integral. By Lemma 2.10, $D \in \mathrm{HS}_k(\log p^\tau; s)$. So, $D_1 \in \mathrm{IDer}_k(\log p^\tau; s)$. Since the other inclusion always holds, we have the equality and $s \notin \mathrm{Leaps}_k(A)$.

2. Let s be an integer such that $np^\tau < s < (n+1)p^\tau$ for some $n \geq 1$. Then $\mathrm{IDer}_k\left(\log h^{p^\tau}; s\right) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right)$.

Let $D_1 \in \mathrm{IDer}_k(\log h^{p^\tau}; np^\tau)$ and $D \in \mathrm{HS}_k(\log h^{p^\tau}; np^\tau)$ an np^τ-integral of D_1. By Corollary 2.6, we can consider $D' \in \mathrm{HS}_k(R; s)$ a s-integral of D. By Lemma 2.10, $D'_i(h^{p^\tau}) = 0$ for every integer $i \in \{np^\tau + 1, \ldots, s\}$. So, $D_1 \in \mathrm{IDer}_k(\log h^{p^\tau}; s)$. Since the other inclusion always holds, we have the equality and $s \notin \mathrm{Leaps}_k(A)$.

Thanks to these two equalities we know that the leaps are given at $s = np^\tau$ for some $n \geq 1$. Let us suppose that $s = p^\tau$. By Corollary 3.7 and the point 1., $\mathrm{Der}_k(R) = \mathrm{IDer}_k\left(\log h^{p^\tau}; s-1\right) \supseteq \mathrm{IDer}_k\left(\log h^{p^\tau}; p^\tau = s\right) = \mathrm{Der}_k(\log h)$. Hence, A has a leap at p^τ if and only if $\mathrm{Der}_k(\log h) \neq \mathrm{Der}_k(R)$.

Now, let us consider $s = np^\tau$ for $n \geq 2$. By Corollary 3.7 and the point 2.

$$\begin{aligned} \mathrm{IDer}_k(\log h; n-1) = & \ \mathrm{IDer}_k\left(\log h^{p^\tau}; (n-1)p^\tau\right) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau - 1\right) \\ \supseteq & \ \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right) = \mathrm{IDer}_k(\log h; n). \end{aligned}$$

Then, A has a leap at $s = np^\tau$ if and only if n is a leap of $R/\langle h\rangle$ and we have proved the result. □

4. Integrable derivations for $x^n - y^q$

Let $q, n \geq 1$ be integers, $R = k[x, y]$ the polynomial ring in two variables over a reduced ring k of characteristic $p > 0$ and $h = x^n - y^q \in R$. In this section we will study the modules of i-integrable derivations of $A := R/\langle h\rangle$ for every $i \in \overline{\mathbb{N}}$.

Let $\alpha := \mathrm{val}_p(n)$ be the p-adic valuation of n and $s = n/p^\alpha$. We will denote by m the remainder of the division of q by p and $\beta := \mathrm{val}_p(q - m)$. We write

$$\gamma := \min\{i | ip^\alpha \geq q-1\} = \lceil (q-1)/p^\alpha \rceil$$

where $\lceil * \rceil$ is the least integer greater than or equal to $*$ and, for $\delta \in \mathrm{Der}_k(\log h)$, we denote $\overline{\delta} = \Pi(\delta)$ (see (1)).

PROPOSITION 4.1. *Under the above notations, we have the following properties.*

- *If $n, q \not\equiv 0 \mod p$, then $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2}\rangle$ where $\delta_1 = qx\partial_x + ny\partial_y$ and $\delta_2 = qy^{q-1}\partial_x + nx^{n-1}\partial_y$.*
- *If $n \equiv 0 \mod p$ and $q = 1$, then $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\partial_x}\rangle$.*
- *If $\alpha, m \geq 1$ and $q \geq 2$, then*

$$\mathrm{IDer}_k(A; i) = \begin{cases} \begin{cases} \langle \overline{\partial_x}\rangle & 1 \leq i < p^\alpha \\ \langle \overline{x\partial_x}, \overline{y^\gamma \partial_x}\rangle & p^\alpha \leq i < p^{\alpha+\beta} \\ \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x}\right\rangle & i \geq p^{\alpha+\beta} \text{ or } i = \infty \end{cases} & \text{if } s = 1,\ \alpha \leq \beta,\ m = 1 \\ \begin{cases} \langle \overline{\partial_x}\rangle & 1 \leq i < p^\alpha \\ \langle \overline{x\partial_x}, \overline{y^\gamma \partial_x}\rangle & i \geq p^\alpha \text{ or } i = \infty \end{cases} & \text{otherwise.} \end{cases}$$

PROOF. Let $\delta = u\partial_x + v\partial_y$ be a k-derivation of R. To prove this result it is enough to show which derivations are h-logarithmically i-integrable for $i \in \overline{\mathbb{N}}$ (Proposition 2.7).

- *Case $n, q \not\equiv 0 \mod p$.*

We have to find the pairs (u, v) such that $\delta(h) = nux^{n-1} - qvy^{q-1} \in \langle h\rangle$. It easy to see that $\mathrm{Der}_k(\log h) = \langle \delta_1, \delta_2\rangle$ where $\delta_1 = qx\partial_x + ny\partial_y$ and $\delta_2 = qy^{q-1}\partial_x + nx^{n-1}\partial_y$. Note that h is a quasi-homogenous polynomial with respect to the weights $w(x) = q$ and $w(y) = n$. By Theorem 1.2. of [8], the Euler vector field, δ_1, is h-logarithmically ∞-integrable. On the other hand, the gradient of h is $J^0 = \langle x^{n-1}, y^{q-1}\rangle$, so $\delta_2 \in J^0\,\mathrm{Der}_k(R)$ and from Proposition 2.9 we know that δ_2 is h-logarithmically ∞-integrable too. So, $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2}\rangle$.

- *Case* $n \equiv 0 \mod p$ *and* $q = 1$.

The condition for δ to be h-logarithmic is that $v \in \langle h \rangle$, so $\mathrm{Der}_k(\log h) = \langle \partial_x, h\partial_y \rangle$. In this case $J^0 = \langle 1 \rangle$, hence any $\langle h \rangle$-logarithmic derivation is integrable (Prop. 2.9). Then, $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\partial}_x \rangle$.

- *Case* $\alpha, m \geq 1$ *and* $q \geq 2$.

Note that $n = sp^\alpha$. In order for δ to be h-logarithmic, $v \in \langle h \rangle$ so $\mathrm{Der}_k(\log h) = \langle \partial_x, h\partial_y \rangle$. Since $h\partial_y$ is the zero derivation on A, we can focus on the h-logarithmically integrability of $\delta = u\partial_x$ with $u \in R$. Let $u_x \in k[x, y]$ and $u_y \in k[y]$ such that

$$u = u_x(x, y)x + u_y(y) \Rightarrow \delta = u\partial_x = u_x x\partial_x + u_y\partial_x.$$

Since h is a quasi-homogeneous polynomial with respect to the weights $w(x) = q$ and $w(y) = sp^\alpha$, the Euler vector field, $\chi = qx\partial_x$, is h-logarithmically integrable, and hence also $u_x x\partial_x$ are. Since $\mathrm{IDer}_k(\log h; i)$ is an R-module for every i,

$$\delta \in \mathrm{IDer}_k(\log h; i) \Leftrightarrow u_y\partial_x \in \mathrm{IDer}_k(\log h; i).$$

Let us consider $\delta = u\partial_x$ where $u \in k[y]$. Let $\varphi : R \to R[[T]]$ be a k-algebra homomorphism:

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[[T]] \\ & x & \longmapsto & x + uT + u_2T^2 + \cdots \\ & y & \longmapsto & y \quad + \quad v_2T^2 + \cdots . \end{array}$$

To show that δ is i-integrable it is enough to prove that there exist u_j, v_j for every integer $j \in \{2 \ldots, i\}$ such that $\varphi(h) \in \langle h \rangle \mod T^{i+1}$, or, equivalently, the coefficients of T^j in $\varphi(h)$ belong to $\langle h \rangle$ for every $j \leq i$. We will denote by T_j the coefficient of T^j in the equation

$$\varphi(h) = \left(x^{p^\alpha} + u^{p^\alpha}T^{p^\alpha} + u_2^{p^\alpha}T^{2p^\alpha} + \cdots\right)^s - \left(y + v_2T^2 + v_3T^3 + \cdots\right)^q. \tag{4}$$

Suppose that there exists an integer i such that $i \in \{2 \ldots, p^\alpha\}$. Then, $T_2 = -qy^{q-1}v_2$ has to belong to $\langle h \rangle$. Hence, $v_2 \in \langle h \rangle$, so we can put $v_2 = 0$. Let us assume that $v_l = 0$ for every integer $l \in \{2 \ldots, p^\alpha\}$. In this case, $T_i = -qy^{q-1}v_i$ and, as the same before, we can put $v_i = 0$. Then, for every integer $i < p^\alpha$, we have

$$\mathrm{Der}_k(A) = \mathrm{IDer}_k(A; i) = \langle \overline{\partial_x} \rangle$$

and we can write the equation (4) as:

$$\left(x^{p^\alpha} + u^{p^\alpha}T^{p^\alpha} + u_2^{p^\alpha}T^{2p^\alpha} + \cdots\right)^s - \left(y + v_{p^\alpha}T^{p^\alpha} + v_{p^\alpha+1}T^{p^\alpha+1} + \cdots\right)^q \in \langle h \rangle. \tag{5}$$

Now, we have to see that there are $u_{p^\alpha}, v_{p^\alpha} \in R$ such that

$$T_{p^\alpha} = sx^{p^\alpha(s-1)}u^{p^\alpha} - qy^{q-1}v_{p^\alpha} \in \langle h \rangle. \tag{6}$$

Since $u \in k[y]$, the previous expression implies that $u^{p^\alpha} \in \langle y^{q-1} \rangle$. Therefore, if we write $u = \sum_{i \geq 0} u_i y^i$ with $u_i \in k$, then $u_i^{p^\alpha} = 0$ for every integer i such that $ip^\alpha < q - 1$, so $u_i = 0$ because k is reduced. Hence, we can write $u = w(y)y^\gamma$ where $\gamma = \min\{i | ip^\alpha \geq q - 1\}$ and $w(y) \in k[y]$. Substituting the expression of u on (6), we can deduce that

$$sx^{p^\alpha(s-1)}w^{p^\alpha}y^{\gamma p^\alpha - (q-1)} - qv_{p^\alpha} \in \langle h \rangle \;\Rightarrow\; v_{p^\alpha} \in (s/q)x^{p^\alpha(s-1)}w^{p^\alpha}y^{\gamma p^\alpha - (q-1)} + \langle h \rangle. \tag{7}$$

Therefore, A has a leap at p^α and

$$\mathrm{IDer}_k(A; p^\alpha) = \langle \overline{x\partial_x}, \overline{y^\gamma \partial_x} \rangle \text{ where } \gamma = \min\{i |\; ip^\alpha \geq q - 1\}.$$

Let us write $q = tp^\beta + m$. Note that the only case where $\gamma p^\alpha = q - 1$ is $q = tp^\beta + 1$ and $\alpha \leq \beta$. Let us focus on this case when $s = 1$.

- *Case* $q = tp^\beta + 1$, $\alpha \leq \beta$ *and* $s = 1$. Observe that $t \neq 0$ because $q \geq 2$. It is easy to see that $\gamma = tp^{\beta-\alpha}$. We will study the integrability of $w(y)y^\gamma\partial_x$ in this particular case.

  Substituting the values of q and s in the equation (5) and (7) we obtain:

$$\left(x^{p^\alpha} + u^{p^\alpha} T^{p^\alpha} + u_2^{p^\alpha} T^{2p^\alpha} + \cdots\right) - \left(y^{p^\beta} + v_{p^\alpha}^{p^\beta} T^{p^{\alpha+\beta}} + v_{p^\alpha+1}^{p^\beta} T^{(p^\alpha+1)p^\beta} + \cdots\right)^t \left(y + v_{p^\alpha} T^{p^\alpha} + \cdots\right) \in \langle h \rangle$$

and

$$v_{p^\alpha} = cw^{p^\alpha} + Fh$$

for $c = s/q$ and some $F \in k[x, y]$. Let us consider an integer i such that $p^\alpha < i < p^{\alpha+\beta}$. If $i = jp^\alpha$ for some $j \geq 2$, then $T_i = u_j^{p^\alpha} - y^{tp^\beta} v_i$. Otherwise, $T_i = -y^{tp^\beta} v_i$. So, $wy^\gamma \partial_x$ is h-logarithmically i-integrable for every $i < p^{\alpha+\beta}$ (it's enough to put $u_j = v_i = 0$ so that $T_i \in \langle h \rangle$). Now,

$$T_{p^{\alpha+\beta}} = u_{p^\beta}^{p^\alpha} - ty^{(t-1)p^\beta+1} v_{p^\alpha}^{p^\beta} - y^{tp^\beta} v_{p^{\alpha+\beta}}$$

has to belong to $\langle h \rangle$. So, substituting the value of v_{p^α}, we have that

$$u_{p^\beta}^{p^\alpha} - ctw^{p^{\alpha+\beta}} y^{(t-1)p^\beta+1} - y^{tp^\beta} v_{p^{\alpha+\beta}} = G\left(x^{p^\alpha} - y^{tp^\beta+1}\right)$$

for some $G \in k[x, y]$. The coefficient of y^j with $j = (t-1)p^\beta + 1$ in this equality is $tcw_0^{p^\alpha} = 0$ where w_0 is the independent term of w. Since R is reduced, $w_0 = 0$. Hence, $y^\gamma \partial_x$ is not $p^{\alpha+\beta}$-integrable. However, if $w = w'y$ with $w' \in k[y]$, the previous equation is

$$u_{p^\beta}^{p^\alpha} - ctw'^{p^{\alpha+\beta}} y^{q+p^\beta(p^\alpha-1)} - y^{tp^\beta} v_{p^{\alpha+\beta}} = G\left(x^{p^\alpha} - y^{tp^\beta+1}\right).$$

Then, there exists a solution, for instance $u_{p^\beta} = 0$ and $v_{p^{\alpha+\beta}} = -ctw'^{p^{\alpha+\beta}} y^{p^\beta(p^\alpha-1)+1}$. In conclusion, in this case A has a leap at $p^{\alpha+\beta}$ and

$$\mathrm{IDer}_k\left(A; p^{\alpha+\beta}\right) = \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x} \right\rangle.$$

Until now we saw that, for every $q \geq 2$, $\mathrm{IDer}_k(A; p^\alpha) = \langle \overline{x\partial_x}, \overline{y^\gamma \partial_x} \rangle$ where $\gamma = \min\{i \mid ip^\alpha \geq q-1\}$ and moreover, when $q = tp^\beta + 1$, $1 \leq \alpha \leq \beta$ and $s = 1$, $y^\gamma \partial_x$ is not h-logarithmically integrable but $\mathrm{IDer}_k\left(A; p^{\alpha+\beta}\right) = \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x} \right\rangle$.

Let us rewrite $\gamma := \gamma + 1$ in the latter case. We will see that $y^\gamma \partial_x$ is integrable on A for every $q \geq 2$. Consider the k-algebra homomorphism

$$\begin{array}{rccl} \varphi: & A & \longrightarrow & A[[T]] \\ & x & \longmapsto & x + y^\gamma T \\ & y & \longmapsto & y + v_1 T^{p^\alpha} + v_2 T^{2p^\alpha} + \cdots \end{array}$$

where $v_i = C_i x^{p^\alpha(s-\sigma)} y^{i\gamma p^\alpha - (\tau+1)q+1}$ for $i = \tau s + \sigma$ with $\tau \geq 0$ and $\sigma \in \{1, \ldots, s\}$,

$$C_i = \frac{1}{q}\left[\binom{s}{i} - \sum_{j \in I_i} D_j\right] \text{ where } \binom{s}{i} = 0 \text{ if } i > s,$$

$$I_i = \left\{ j = (j_0, j_1, \ldots, j_{i-1}) \mid j_k \geq 0 \ \forall k = 0, \ldots, i-1, |j| = q, \sum_{k=1}^{i-1} kj_k = i \right\}$$

and, for every $j = (j_0, j_1, \ldots, j_l)$ with $l \geq 1$,

$$D_j = \binom{q}{j} C_1^{j_1} \cdots C_l^{j_l} \text{ with } \binom{q}{j} = \frac{q!}{j_0! \cdots j_l!}.$$

We have to prove that φ is well defined. First we see that $i\gamma p^\alpha - (\tau+1)q + 1 \geq 0$, i.e. $(\tau s + \sigma)\gamma p^\alpha - \tau q \geq q - 1$.

- When $\gamma p^\alpha > q - 1$, then $\gamma p^\alpha \geq q$, but q is not multiple of p, so $\gamma p^\alpha \geq q + 1$ and therefore

  $$(\tau s + \sigma)\gamma p^\alpha - \tau q \geq (\tau s + \sigma)(q+1) - \tau q = (\tau(s-1) + \sigma)q + \tau s + \sigma \geq q - 1$$

  because $s - 1 \geq 0$ and $\sigma \geq 1$.

- Let us consider $\gamma p^\alpha = q-1$. As we have seen before, the previous equality only holds if $q = tp^\beta + 1$ and $\alpha \leq \beta$. If $s = 1$, then we have considered $\gamma + 1$, so we are in the first point. Therefore, we have just considered $s \geq 2$. In this case, we have to prove that $(\tau s+\sigma)\gamma p^\alpha - \tau q = (\tau s+\sigma)(q-1) - \tau q \geq q-1$. Then
$$(\tau s+\sigma)(q-1) - \tau q \geq (2\tau+\sigma)(q-1) - \tau q = (\tau+\sigma)q - (2\tau+\sigma).$$
So,
$$(\tau+\sigma)q - (2\tau+\sigma) \geq q-1 \Leftrightarrow (\tau+\sigma-1)q \geq 2\tau+\sigma-1$$
and this is true because $q \geq 2$ and $\tau+\sigma-1 \geq 0$. Note that if $\tau+\sigma-1 = 0$ then $\tau = 0$ and $\sigma = 1$, so $2\tau+\sigma-1 = 0$ too.

Now, we have to show that $\varphi(h) = 0$ in $A[[T]]$. The equation is:
$$\varphi(h) = \left(x^{p^\alpha} + y^{\gamma p^\alpha} T^{p^\alpha}\right)^s - \left(y + v_1 T^{p^\alpha} + v_2 T^{2p^\alpha} + \cdots\right)^q.$$
Since all degrees of the monomial which appeared in this equation are multiple of p^α, let us denote T_i to the coefficient of degree ip^α. Then
$$T_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \widetilde{T}_i$$
where $\widetilde{T}_i$ is the coefficient of T^{ip^α} from $\left(y + v_1 T^{p^\alpha} + v_2 T^{2p^\alpha} + \cdots\right)^q$. This coefficient can be found on
$$\left(y + v_1 T^{p^\alpha} + \cdots + v_i T^{ip^\alpha}\right)^q = \sum_{|j|=q} \binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} T^{p^\alpha(j_1+\cdots+ij_i)}.$$
We just have to consider all j such that $j_1 + \cdots + ij_i = i$. Observe that there exists only one j holding this equation such that $j_i \neq 0$, This j is $(q-1, 0, \ldots, 0, 1)$ where 1 is in the position i. So, we can identify the set of all these j with $I_i \cup (q-1, 0, \ldots, 0, 1)$. Let us calculate a term of $\widetilde{T}_i$. Fixed j, we have
$$\binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} = \binom{q}{j} C_1^{j_1} \ldots C_i^{j_i} x^{ap^\alpha} y^b = D_j x^{ap^\alpha} y^b$$
where
$$a = \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}(s-\sigma) \geq 0 \quad \text{and} \quad b = j_0 + \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\left(\gamma p^\alpha(\tau s+\sigma) - (\tau+1)q + 1\right) \geq 0.$$
We are going to study these exponents:
$$a = s \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma} - \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\sigma = s(q-j_0) - \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\sigma.$$
On the other side, we have
$$\sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}(\tau s+\sigma) = s \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\tau + \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\sigma = ls + r$$
where $i = ls + r$ (remember: $l \geq 0$ and $1 \leq r \leq s$). Then, if we denote $J = \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\tau$ and we substitute on a, we have
$$a = s(q-j_0) - ((l-J)s + r) = s(q-j_0-l+J) - r \geq 0.$$
If $q - j_0 - l + J < 1$, then $a < 0$ so $q - j_0 - l + J \geq 1$ and we can write
$$a = (q-j_0-l+J-1)s + s - r.$$
Observe that $s - r \geq 0$ because $1 \leq r \leq s$. Now,
$$\begin{aligned} b = \;& j_0 + \gamma p^\alpha \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}(\tau s+\sigma) - q \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma}\tau - q \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma} + \sum_{1\leq \tau s+\sigma \leq i} j_{\tau s+\sigma} \\ = \;& \gamma p^\alpha i - qJ - q(q-j_0) + (q-j_0) + j_0 = i\gamma p^\alpha - q(J+q-j_0-1). \end{aligned}$$

So,

$$\binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} = D_j x^{(q-j_0-l+J-1)sp^\alpha+(s-r)p^\alpha} y^{i\gamma p^\alpha - q(J+q-j_0-1)}.$$

Since $x^{sp^\alpha} = y^q$ in A,

$$\binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} = D_j x^{(s-r)p^\alpha} y^{i\gamma p^\alpha + q(q-j_0-l+J-1)-q(J+q-j_0-1)} = D_j x^{(s-r)p^\alpha} y^{i\gamma p^\alpha - lq}.$$

Hence,

$$\begin{aligned}
\widetilde{T}_i &= \sum_{\substack{|j|=q \\ j_1+\ldots+ij_i=i}} D_j x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} = \left(\sum_{j\in I_i} D_j + D_{(q-1,0,\ldots,0,1)}\right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} \\
&= \left(\sum_{j\in I_i} D_j + qC_i\right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} = \left(\sum_{j\in I_i} D_j + q(1/q)\left[\binom{s}{i} - \sum_{j\in I_i} D_j\right]\right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} \\
&= \binom{s}{i} x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq}.
\end{aligned}$$

So,

$$T_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \binom{s}{i} x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq}$$

If $i > s$, then $\binom{s}{i} = 0$, and hence $T_i = 0$. If $i \le s$, then $i = 0 \cdot s + i$, i.e. $l = 0$ and $r = i$, then

$$T_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} = 0$$

so, the morphism φ is well defined and the proposition is proved. □

As a straightforward consequence of Proposition 4.1 and Corollaries 3.7 and 3.8, we have the following results.

Corollary 4.2. *Under the hypotheses of Proposition* 4.1*, assume that k is a UFD of characteristic $p > 0$, $m = 0$, $\alpha \ge 1$ and $\beta = \mathrm{val}_p(q) \ge 1$. We write $\tau = \min\{\alpha, \beta\}$, $n' = n/p^\tau$ and q/p^τ. Then,* $\mathrm{Der}_k(A) = \langle \overline{\partial}_x, \overline{\partial}_y \rangle$ *and for every $i \ge 0$,* $\mathrm{IDer}_k(A; p^{\tau+i}) = \{\overline{\delta} \mid \delta \in \mathrm{IDer}_k(\log\langle x^{n'} - y^{q'} \rangle, p^i)\}$.

Corollary 4.3. *Under the hypotheses of Proposition* 4.1*, we have the following properties.*

- *If $n, q \not\equiv 0 \mod p$ then,* $\mathrm{Leaps}_k(A) = \emptyset$.
- *If $n \equiv 0 \mod p$ and $q = 1$ then,* $\mathrm{Leaps}_k(A) = \emptyset$.
- *If $\alpha, m \ge 1$ and $q \ge 2$, then*

$$\mathrm{Leaps}_k(A) = \begin{cases} \{p^\alpha, p^{\alpha+\beta}\} & \text{if } s = 1,\ \alpha \le \beta,\ m = 1 \\ \{p^\alpha\} & \text{otherwise.} \end{cases}$$

Moreover, if k is a UFD, $m = 0$, $\alpha, \beta \ge 1$ and we denote $\tau = \min\{\alpha, \beta\}$, $n' = n/p^\tau$ and $q' = q/p^\tau$, we have that $\mathrm{Leaps}_k(A) = \{p^\tau\} \cup \{ip^\tau \mid i \in \mathrm{Leaps}_k(B)\}$ *where* $B = k[x,y]/\left\langle x^{n'} - y^{q'} \right\rangle$.

Examples 4.4. Let us consider k a reduced ring of characteristic $p = 3$ and $h = x^3 - y^4 \in k[x,y]$, then $\gamma = 1$. According with Proposition 4.1 and the previous corollary, $\mathrm{Leaps}_k(k[x,y]/\langle h \rangle) = \{3, 9\}$ and

$$\mathrm{IDer}_k(A; i) = \begin{cases} \langle \overline{\partial_x} \rangle & 1 \le i < 3 \\ \langle \overline{x\partial_x}, \overline{y\partial_x} \rangle & 3 \le i < 9 \\ \langle \overline{x\partial_x}, \overline{y^2\partial_x} \rangle & i \ge 9. \end{cases}$$

Now, if we consider $h = x^3 - y^5$, then $\gamma = 2$, $\mathrm{Leaps}_k(k[x,y]/\langle h \rangle) = \{3\}$ and

$$\mathrm{IDer}_k(A; i) = \begin{cases} \langle \overline{\partial_x} \rangle & 1 \le i < 3 \\ \langle \overline{x\partial_x}, \overline{y^2\partial_x} \rangle & i \ge 3. \end{cases}$$

Remark 4.5. Note that if k is not reduced, Proposition 4.1 is not true. For example, if $k = \mathbb{F}_3[t]/\langle t^3\rangle$ and $h = x^3 - y^5$, then $\overline{t\partial_x} \in \mathrm{IDer}_k(A)$ with the integral

$$\begin{array}{ccc} A & \to & A[[T]] \\ x & \mapsto & x + tT \\ y & \mapsto & y. \end{array}$$

Corollary 4.6. *Under the hypotheses of Proposition* 4.1*, assume that* $\alpha, m \geq 1$ *and* $q \geq 2$*. We denote* $B_i := \mathrm{Ann}_A\left(\mathrm{IDer}_k(A; i-1)/\,\mathrm{IDer}_k(A; i)\right)$ *for* $i > 1$*. Then,*

$$B_i = \begin{cases} \langle x, y^\gamma\rangle & \text{if } i = p^\alpha \\ \langle y\rangle & \text{if } i = p^{\alpha+\beta},\ s = 1,\ \alpha \leq \beta \text{ and } m = 1. \end{cases}$$

Moreover, $B_i \supseteq J^0 = \langle y^{q-1}\rangle$ *where* J^0 *is the gradient ideal of* h *defined in Proposition* 2.9.

Proof. Let us start with $i = p^\alpha$. From Proposition 4.1, we can deduce that

$$\mathrm{IDer}_k\,(A; p^\alpha - 1)\,/\,\mathrm{IDer}_k\,(A; p^\alpha) = \langle \partial_x\rangle/\langle x\partial_x, y^\gamma\partial_x\rangle$$

where $\partial_x \in \mathrm{Der}_k(A)$. By definition, $a \in B_i$ if $a\partial_x = 0 \mod \langle x\partial_x, y^\gamma\partial_x\rangle$, i.e. if there exist $F, G \in A$ such that $a\partial_x = Fx\partial_x + Gy^\gamma\partial_x$. Applying this derivation to x, we have that $a \in \langle x, y^\gamma\rangle$.

Now, when $\alpha \leq \beta$, $s = m = 1$ and $i = p^{\alpha+\beta}$, from Proposition 4.1,

$$\mathrm{IDer}_k\left(A; p^{\alpha+\beta} - 1\right)/\,\mathrm{IDer}_k\left(A; p^{\alpha+\beta}\right) = \langle x\partial_x, y^\gamma\partial_x\rangle/\langle x\partial_x, y^{\gamma+1}\partial_x\rangle = \langle y^\gamma\partial_x/y^{\gamma+1}\partial_x\rangle.$$

In this case, $a \in B_{p^{\alpha+\beta}}$ if and only if $ay^\gamma\partial_x \in \langle y^{\gamma+1}\partial_x\rangle$, i.e. if $(a - Fy)y^\gamma\partial_x = 0$ for some $F \in A$. This implies that $a \in \langle y\rangle$ and we have proved the corollary. □

5. Other examples

We are going to calculate the integrable derivations of the quotient of a polynomial ring over some non-binomial equations. These examples have been taken from the article [1].

Example 1.

Let k be a domain of characteristic $p > 0$ and $h = x^p + tx^{p+1} \in R = k[x]$ with $t \in k$. Let $A = R/\langle h\rangle$. The module $\mathrm{Der}_k(\log h)$ is generated by $(1+tx)\partial_x$. From Example (2.1.2) of [6], we have that $(1+tx)\partial_x$ is h-logarithmically $(p-1)$-integrable. So, let us consider $E \in \mathrm{HS}_k(\log h; p-1)$ an integral of $u(1+tx)\partial_x$ where $u \in R$. From Corollary 2.6, there exists $D \in \mathrm{HS}_k(R)$ an integral of E. In order for D to be h-logarithmic,

$$D_p(x^p + tx^{p+1}) = D_1(x)^p + t(xD_1(x)^p + D_p(x)x^p) = u^p(1+tx)^{p+1} + tD_p(x)x^p \in \langle h\rangle.$$

So, $u \in \langle x\rangle$ and $\mathrm{IDer}_k(\log h; p) = \langle x(1+tx)\partial_x\rangle$. Observe that this generator is ∞-integrable, for example $x \in A \mapsto x + x(1+tx)T \in A[[T]]$ is an integral. In conclusion, $\mathrm{Leaps}_k(A) = \{p\}$ and

$$\mathrm{IDer}_k(A; i) = \begin{cases} \langle\overline{(1+tx)\partial_x}\rangle & \text{if } i \leq p-1 \\ \langle\overline{x(1+tx)\partial_x}\rangle & \text{if } i \geq p. \end{cases}$$

Example 2.

Let k be a domain of characteristic $p = 2$ and $h = x^4 + y^6 + y^7 \in R = k[x,y]$. Let $A = R/\langle h\rangle$. In this case, the module of h-logarithmic derivations is generated by ∂_x and $h\partial_y$. Since $h\partial_y$ is h-logarithmically ∞-integrable, we can focus on the h-logarithmically integrability of $u\partial_x$ where $u \in k[x,y]$. Let $\varphi : R \to R[[T]]$ be a k-algebra homomorphism

$$\begin{array}{rccl} \varphi: & R & \longrightarrow & R[[T]] \\ & x & \longmapsto & x + uT + u_2T^2 + \cdots \\ & y & \longmapsto & y \quad + \quad v_2T^2 + \cdots. \end{array}$$

We want to see that there exist $u_i, v_i \in R$ for $i \geq 2$ such that φ is h-logarithmic. The coefficient of T^i for $i = 2, 3$ in $\varphi(h)$ is $y^6 v_i$. In order for φ to be h-logarithmic, $v_i \in \langle h \rangle$, so we can put $v_i = 0$. In fact, we can put $v_i = 0$ for every integer i such that $i \not\equiv 0 \mod 4$. Thanks to this, we can write:

$$\varphi(h) = (x + uT + u_2T^2 + \cdots)^4 + (y + v_4T^4 + v_8T^8 + \cdots)^6(1 + y + v_4T^4 + v_8T^8 + \cdots). \tag{8}$$

The coefficient of T^4 in (8) is $T_4 := u^4 + y^6 v_4$ and it has to belong to $\langle h \rangle$. Hence, $u \in \langle x, y^2 \rangle$ and $\mathrm{IDer}_k(\log h; 4) = \langle x\partial_x, y^2\partial_x, h\partial_y \rangle$. It's easy to prove the following lemma through the calculation of a term in the equation (8):

Lemma 5.1. *Suppose that $u_j = 0$ for every $j \geq 2$ and $v_{4n} \in \langle y^2 \rangle$ for every $n < i$, then there exists $v_{4i} \in \langle y^2 \rangle$ such that the coefficient of T^{4i} in (8) belongs to $\langle h \rangle$.*

Using this lemma repeatedly we deduce that $y^2\partial_x$ and $xy\partial_x$ are h-logarithmically integrable since a possible solution so that T_4 is h-logarithmic is $v_4 = y^2$ and $v_4 = (1 + y)y^4$ respectively. Therefore, we need to see the h-logarithmically integrability of $ux\partial_x$ where $u \in k[x]$. In this case, $v_4 \in (1 + y)u^4 + \langle h \rangle$. Calculating the coefficient of T^8 in (8), we obtain $T_8 := u_2^4 + y^6 v_8 + v_4^2(1 + y)y^4$. In order for T_8 to be in $\langle h \rangle$, $u \in \langle x \rangle$. Hence, $v_4 \in \langle x^4, h \rangle$. We deduce that $x^2\partial_x$ is h-logarithmically integrable by the following lemma:

Lemma 5.2. *Suppose that $u_j = 0$ for every $j \geq 2$ and $v_{4n} \in \langle x^4 \rangle$ for every $n < i$, then there exists $v_{4i} \in \langle x^4 \rangle$ such that the coefficient of T^{4i} in (8) belongs to $\langle h \rangle$.*

In conclusion, $\mathrm{Leaps}_k(A) = \{4, 8\}$ and

$$\mathrm{IDer}_k(A; i) = \begin{cases} \langle \overline{\partial_x} \rangle & \text{if } 1 \leq i < 4 \\ \langle \overline{x\partial_x}, \overline{y^2\partial_x} \rangle & \text{if } 4 \leq i < 8 \\ \langle \overline{x^2\partial_x}, \overline{xy\partial_x}, \overline{y^2\partial_x} \rangle & \text{if } i \geq 8. \end{cases}$$

Example 3.

Let k be a domain of characteristic $p = 3$ and $h = x^3 + y^5 + x^2y^2 \in R = k[x, y]$. Let $A = R/\langle h \rangle$. The module of h-logarithmic derivations is generated by $\delta_1 := x^2\partial_x + y^3\partial_y$ and $\delta_2 := 2y^2\partial_x + (x + y^2)\partial_y$. These two derivations are h-logarithmically integrable. To verify this claim, let us consider $\varphi : R \to R[[T]]$ a k-algebra homomorphism

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[[T]] \\ & x & \longmapsto & x + u_1T + u_2T^2 + \cdots \\ & y & \longmapsto & y + v_1T + v_2T^2 + \cdots . \end{array}$$

As in the previous example, we want to prove that there exist $u_i, v_i \in R$ for $i \geq 2$ such that $\varphi(h) \in \langle h \rangle$ where u_1 and v_1 are determined by δ_1 or δ_2. By calculating a generic term of $\varphi(h)$, we can show the following lemmas:

Lemma 5.3. *Let $u_1 = x^2$ and $v_1 = y^3$. Suppose that $v_j = 0$ for every $j \geq 2$ and $u_n \in \langle x^2 \rangle$ for every $n < i$. Then, there exists $u_i \in \langle x^2 \rangle$ such that the coefficient of T^i in $\varphi(h)$ belongs to $\langle h \rangle$.*

Lemma 5.4. *Let $u_1 = 2y^2$ and $v_1 = x + y^2$. Suppose $u_n \in \langle xy, y^3 \rangle$ and $v_n \in \langle y^2 \rangle$ for every $2 \leq n < i$. Then, there exist $u_i \in \langle xy, y^3 \rangle$ and $v_i \in \langle y^2 \rangle$ such that the coefficient of T^i in $\varphi(h)$ belongs to $\langle h \rangle$.*

Using Lemma 5.3 for the integrability of δ_1 and Lemma 5.4 for the integrability of δ_2, we can deduce that $\mathrm{Leaps}_k(A) = \emptyset$ and $\mathrm{IDer}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2} \rangle$.

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