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Mostow's Fibration for canonical embeddings of compact homogeneous *CR* manifolds

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Abstract – We define a class of compact homogeneous *CR* manifolds which are bases of Mostow fibrations having total spaces equal to their canonical complex realizations and Hermitian fibers. This is used to establish isomorphisms between their tangential Cauchy-Riemann cohomology groups and the corresponding Dolbeault cohomology groups of the embeddings.

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Contents

1. Introduction and preliminaries

The aim of this paper is to investigate relations between the cohomology groups of the tangential Cauchy Riemann complexes of $\mathfrak{n}$-reductive compact homogeneous *CR* mani-

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folds and the corresponding Dolbeault cohomology groups of their canonical embeddings. The class of $\mathfrak{n}$-reductive compact homogeneous *CR* manifolds was introduced in [1]: its objects are the minimal orbits, in homogeneous spaces of reductive complex groups, of their compact forms.

Results on the cohomology of the tangential *CR* complexes on general compact *CR* manifolds of arbitrary codimension were obtained in [14] (see also [8]), under suitable r-pseudoconcavity conditions, involving their *scalar Levi forms*, that were first introduced in [3, 25]. In this paper we will restrain to the homogeneous case.

The *CR* structure of a homogeneous *CR* manifold M_0 is efficiently described by considering its *CR algebra* at any point $p_0 \in M_0$: it is the pair $(\mathfrak{k}_0, \mathfrak{v})$ consisting of the real Lie algebra $\mathfrak{k}_0$ of its transitive group $\mathbf{K}_0$ of *CR*-automorphisms and of the subspace $\mathfrak{v} = d\pi^{-1}(T^{0,1}_{p_0}M_0)$ of the complexification $\mathfrak{k}$ of $\mathfrak{k}_0$ (see [22]). The *formal integrability* of the partial complex structure $T^{0,1}M_0$ of M_0 is equivalent to the fact that $\mathfrak{v}$ is a complex Lie subalgebra of $\mathfrak{k}$. The intersection $\mathfrak{v} \cap \bar{\mathfrak{v}}$ (conjugation is taken with respect to the real form $\mathfrak{k}_0$) is the complexification of the Lie algebra of the stabilizer of p_0 in $\mathbf{K}_0$ and the quotient $\mathfrak{v}/(\mathfrak{v} \cap \bar{\mathfrak{v}})$ represents the space $T^{0,1}_{p_0}M_0$ of *anti-holomorphic* complex tangent vectors at p_0.

We call $\mathfrak{n}$-reductive a homogeneous *CR* manifold for which $\mathfrak{v} = (\mathfrak{v} \cap \bar{\mathfrak{v}}) \oplus \mathfrak{n}(\mathfrak{v})$, i.e. for which $T^{0,1}_{p_0}M_0$ can be identified to the nilradical of $\mathfrak{v}$. It was shown in [1] that the intersection of any pair of Matsuki-dual orbits in a complex flag manifold M, with the *CR* structure inherited from M, is an $\mathfrak{n}$-reductive compact homogeneous *CR* manifold. Moreover, when M_0 is $\mathfrak{n}$-reductive, $\mathfrak{v}$ is the Lie algebra of a closed complex Lie subgroup $\mathbf{V}$ of $\mathbf{K}$ that contains the stabilizer of p_0 as its maximal compact subgroup, so that $M_0 = \mathbf{K}_0/\mathbf{V}_0 \hookrightarrow M_- = \mathbf{K}/\mathbf{V}$ is a generic *CR*-embedding. Vice versa, if M_- is a $\mathbf{K}$-homogeneous complex algebraic manifold, then a minimal $\mathbf{K}_0$-orbit M_0 in M_- is an $\mathfrak{n}$-reductive compact homogeneous *CR* manifold.

Since $\mathbf{K}_0$ is a maximal compact subgroup of a linear algebraic complex group $\mathbf{K}$, the quasi-projective manifold M_- can be viewed as a $\mathbf{K}_0$-equivariant fiber bundle on the basis M_0 (see [24]). We use this Mostow fibration of M_- onto M_0 to construct a nonnegative *smooth* exhaustion ϕ of M_-, with $\phi^{-1}(0) = M_0$, to relate the Dolbeault cohomology of M_- to the cohomology of the tangential *CR*-complex on M_0. This requires some precision on the structure of the fibers and forces us to introduce a further requirement on the *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$, namely to ask that, if $\mathfrak{w}$ is the largest complex subalgebra of $\mathfrak{k}$ with $\mathfrak{v} \subset \mathfrak{w} \subset (\mathfrak{v} + \bar{\mathfrak{v}})$, (see [22, Theorem 5.4]), then $\mathfrak{n}(\mathfrak{w})$ is the nilradical of a parabolic subalgebra of $\mathfrak{k}$. This condition is satisfied in many examples coming from Matsuki duality (cf. [21]) and can always be satisfied by *strengthening* the *CR* structure of an $\mathfrak{n}$-reductive M_0.

When we drop this extra assumption, we are still able to construct a *continuous* exhaustion, which, when M_0 is r-pseudoconcave, is still strictly r-pseudoconcave, allowing us to obtain results on the first $(r-1)$ tangential Cauchy Riemann and Dolbeault cohomology groups of M_0 and M_- (or up to $(r - \mathrm{hd}(\mathcal{F}) - 1)$ if we discuss cohomology with coefficients in a coherent sheaf $\mathcal{F}$).

Earlier versions of some results proved here were discussed in [19, 20].

The paper is organized as follows.

In §2 we discuss some basic facts on $\mathfrak{n}$-reductive *CR* manifolds. We skip from basic stuff on *CR* manifolds and *CR* algebras, for which we refer, e.g., to [14, 22], and only explain those special features which are necessary for the developments of the next sections.

Cartan and Mostow fibrations are related to the structure of negatively curved Riemannian symmetric space of the set of Hermitian symmetric matrices with determinant one. Hence we found convenient to discuss in §3, as a preliminary, some topics of the geometry of $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$.

In §4 we study decompositions of $\mathbf{K}$ with Hermitian fibers.

Example 3.7 shows that a $\mathbf{K}_0$-equivariant fibration of M_- with Hermitian fibers, as in [23], is not always possible. In §5 we describe the general structure of the fibers. To this aim, we consider a class of parabolic subalgebras associated to the pair $(\kappa_0, \mathfrak{v})$ and find a condition, that we call HNR from *horocyclic nilradical*, under which we get a Mostow fibration of M_- with Hermitian fibers.

In the final section §6 we apply these results to construct an exhaustion function which permits to relate some cohomology groups of the tangential *CR* complexes on M_0 to the corresponding cohomology groups of the Dolbeault complexes on M_- and analogous results for Čech cohomology with coefficients in a coherent sheaf. We conclude with the study of an example of a family of intersections of Matsuki-dual orbits and an application of §4 to obtain a pseudoconcavity result for which we do not require the validity of the HNR assumption.

2. Compact homogeneous *CR* manifolds and $\mathfrak{n}$-reductiveness

In this section we introduce the class of homogeneous *CR* manifold which is the object of this investigation. We found convenient to recall, in an initial short subsection, the definition of reductive Lie group, as it is not completely standard in the literature.

2.1 – *Reductive Lie groups*

We call *reductive* a Lie algebra κ whose radical is abelian: its commutator subalgebra $[\kappa, \kappa]$ is its semisimple ideal and its radical $\mathfrak{a}$ equals its center (see [7]).

Reductive κ's are characterized by having faithful semisimple representations.

An involution θ on a Lie algebra κ yields a direct sum decomposition

$$\kappa = \kappa_0 \oplus \mathfrak{p}_0, \quad \text{with} \quad \kappa_0 = \{X \in \kappa \mid \theta(X) = X\}, \quad \mathfrak{p}_0 = \{X \in \kappa \mid \theta(X) = -X\}.$$

A Lie group $\mathbf{K}$ is *reductive* (see [17]) if its Lie algebra κ is reductive and, moreover, there are an involution θ and an invariant bilinear form $\mathbf{b}$ on κ such that

(*i*) $\kappa_0 \perp \mathfrak{p}_0$ for $\mathbf{b}$;

(*ii*) $\mathbf{b} < 0$ on κ_0 and $\mathbf{b} > 0$ on $\mathfrak{p}_0$;

(*iii*) κ_0 is the Lie algebra of a compact subgroup $\mathbf{K}_0$ of $\mathbf{K}$ and

$$\mathbf{K}_0 \times \mathfrak{p}_0 \ni (x, X) \longrightarrow x \cdot \exp(X) \in \mathbf{K} \tag{2.1}$$

is a diffeomorphism onto;

(*iv*) every automorphism $\mathrm{Ad}(x)$ of the complexification $\kappa^{\mathbb{C}}$ of κ, with $x \in \mathbf{K}$, is inner, i.e. belongs to the analytic subgroup of the automorphis group of $\kappa^{\mathbb{C}}$ having Lie algebra $\mathrm{ad}(\kappa)$.

Then: θ is a *Cartan involution*, $\kappa = \kappa_0 \oplus \mathfrak{p}_0$ and (2.1) are *Cartan decompositions*, $\mathbf{K}_0$ is the associated *maximal compact subgroup*, $\mathbf{b}$ is the *invariant bilinear form*. The maximal compact subgroup $\mathbf{K}_0$ of $\mathbf{K}$ intersects all connected component of $\mathbf{K}$ (see [17, Proposition 7.19]). In particular, $\mathbf{K}$ has finitely many connected components.

2.2 – *Splittable Lie subalgebras*

Let κ be a reductive complex Lie algebra, and

$$\kappa = \mathfrak{z} \oplus \mathfrak{s}, \quad \text{with} \quad \mathfrak{z} = \{X \in \kappa \mid [X, \kappa] = \{0\}\}, \quad \mathfrak{s} = [\kappa, \kappa]$$

its decomposition into the direct sum of its center and its semisimple ideal. An element X of κ is *semisimple* if $\mathrm{ad}(X)$ is a semisimple derivation of κ, and *nilpotent* if $X \in \mathfrak{s}$ and $\mathrm{ad}(X)$ is nilpotent.

An equivalent formulation is obtained by considering a faithful matrix representation of κ in which the elements of $\mathfrak{z}$ are diagonal: then semisimple and nilpotent elements correspond to semisimple and nilpotent matrices, respectively.

Each $X \in \kappa$ admits a unique Jordan-Chevalley decomposition

$$X = X_s + X_n, \quad \text{with } X_s \text{ semisimple, } X_n \text{ nilpotent, and } [X_s, X_n] = 0.$$

A Lie subalgebra $\mathfrak{v}$ of κ is *splittable* if, for each $X \in \mathfrak{v}$, both X_s and X_n belong to $\mathfrak{v}$.

If $\mathfrak{v}$ is a Lie subalgebra of κ, the set

$$\mathfrak{n}_\kappa(\mathfrak{v}) = \{X \in \mathrm{rad}(\mathfrak{v}) \mid X \text{ is nilpotent}\}$$

is a nilpotent ideal of $\mathfrak{v}$, with

$$\mathrm{rad}_{\mathrm{n}}(\mathfrak{v}) = \mathrm{rad}(\mathfrak{v}) \cap [\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{n}_\kappa(\mathfrak{v}) \subset \mathrm{nil}(\mathfrak{v}),$$

where $\mathrm{nil}(\mathfrak{v})$ is the nilradical, i.e. the maximal nilpotent ideal of $\mathfrak{v}$, and $\mathrm{rad}_{\mathrm{n}}(\mathfrak{v})$ its nilpotent radical, i.e. the intersection of the kernels of all irreducible finite dimensional linear representations of $\mathfrak{v}$. Note that the nilpotent ideal $\mathfrak{n}_\kappa(\mathfrak{v})$, unlike $\mathrm{nil}(\mathfrak{v})$ and $\mathrm{rad}_{\mathrm{n}}(\mathfrak{v})$, depends on the inclusion $\mathfrak{v} \subset \kappa$ (cf. [6, §5.3]). We recall

Proposition 2.1 (see [6, §5.4]). *Every splittable Lie subalgebra* $\mathfrak{v}$ *admits a Levi-Chevalley decomposition*

$$\mathfrak{v} = \mathfrak{n}_\kappa(\mathfrak{v}) \oplus \mathfrak{v}_r, \tag{2.2}$$

with $\mathfrak{v}_r$ *reductive and uniquely determined modulo conjugation by elementary automorphisms of* $\mathfrak{v}$*, i.e. finite products of automorphisms of the form* $\exp(\mathrm{ad}(X))$*, with* $X \in \mathfrak{v}$ *and nilpotent.* □

2.3 – *Definition of* $\mathfrak{n}$*-reductive*

Let $\mathfrak{k}$ be the complexification of a compact Lie algebra $\mathfrak{k}_0$. Conjugation in $\mathfrak{k}$ will be understood with respect to its compact real form $\mathfrak{k}_0$. Note that all Lie subalgebras of a compact Lie algebra are compact and hence reductive.

Proposition 2.2. *For any complex Lie subalgebra* $\mathfrak{v}$ *of* $\mathfrak{k}$*, the intersection* $\mathfrak{v} \cap \bar{\mathfrak{v}}$ *is reductive and splittable. In particular,* $\mathfrak{v} \cap \bar{\mathfrak{v}} \cap \mathfrak{n}_{\mathfrak{k}}(\mathfrak{v}) = \{0\}$. *A splittable* $\mathfrak{v}$ *admits a Levi-Chevalley decomposition with a reductive Levi factor containing* $\mathfrak{v} \cap \bar{\mathfrak{v}}$.

Proof. We recall that $\mathfrak{v}$ is splittable if and only if its radical is splittable ([6, Ch.VII, §5, Théorème 2]). In this case, $\mathfrak{v}$ admits a Levi-Chevalley decomposition and all maximal reductive Lie subalgebras of $\mathfrak{v}$ can be taken as reductive Levi factors. The intersection $\mathfrak{v} \cap \bar{\mathfrak{v}}$ is reductive, being the complexification of the compact Lie algebra $\mathfrak{v} \cap \mathfrak{k}_0$. Then the reductive Levi factor in the Levi-Chevalley decomposition of $\mathfrak{v}$ can be taken to contain $\mathfrak{v} \cap \bar{\mathfrak{v}}$ (see e.g. [26]). □

Notation 2.1. In the following, for a complex Lie subalgebra $\mathfrak{v}$ of $\mathfrak{k}$, we shall use the notation

$$\mathfrak{L}_0(\mathfrak{v}) = \mathfrak{v} \cap \mathfrak{k}_0, \quad \mathfrak{L}(\mathfrak{v}) = \mathfrak{v} \cap \bar{\mathfrak{v}}.$$

Definition 2.1. Let $\mathbf{K}_0$ be a compact Lie group with Lie algebra $\mathfrak{k}_0$ and M_0 a $\mathbf{K}_0$-homogeneous *CR* manifold, with isotropy $\mathbf{V}_0$ and *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$ at a point $p_0 \in M_0$. We say that M_0, and its *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$, are $\mathfrak{n}$*-reductive* if

$$\mathfrak{v} = \mathfrak{n}_{\mathfrak{k}}(\mathfrak{v}) \oplus \mathfrak{L}(\mathfrak{v}),$$

i.e. if $\mathfrak{L}(\mathfrak{v}) = \mathfrak{v} \cap \bar{\mathfrak{v}}$ is a reductive complement of $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{v})$ in $\mathfrak{v}$.

Remark 2.3. If $(\mathfrak{k}_0, \mathfrak{v})$ is $\mathfrak{n}$-reductive, then $\mathfrak{v}$ is splittable. Indeed all elements of $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{v})$ are nilpotent and all elements of $\mathfrak{L}(\mathfrak{v})$ are splittable, because $\mathfrak{L}(\mathfrak{v})$ is the complexification of $\mathfrak{L}_0(\mathfrak{v})$, which is splittable because consists of semisimple elements. Then $\mathfrak{v}$ is splittable by [6, Ch,VII, §5, Théorème 1].

All submanifolds which are intersections of dual submanifold in the Matsuki duality, with the *CR* structure inherited by the embedding in the ambient flag manifold, are $\mathfrak{n}$-reductive (see [1, §1]). We exhibit here an example of a compact homogeneous *CR* manifold M_0 which is not $\mathfrak{n}$-reductive.

Example 2.4. Let $\mathbf{K}_0 = \mathbf{SU}(n)$, $n \geq 3$. Fix a complex symmetric nondegenerate $n \times n$ matrix S and consider the subgroup $\mathbf{V} = \{a \in \mathbf{SL}(n, \mathbb{C}) \mid a^t S a = S\}$ of $\mathbf{SL}(n, \mathbb{C})$, with Lie algebra $\mathfrak{v} = \{X \in \mathfrak{sl}(n, \mathbb{C}) \mid X^t S + S X = 0\}$. Set $\mathbf{V}_0 = \mathbf{V} \cap \mathbf{K}_0$ and $M_0 = \mathbf{K}_0/\mathbf{V}_0$. This is a $\mathbf{K}_0$-homogeneous *CR* manifold with *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$, where $\mathfrak{k}_0 \simeq \mathfrak{su}(n)$, $\mathfrak{v} \simeq \mathfrak{so}(n, \mathbb{C})$. If S and S^* are linearly independent, then $\mathfrak{v}$ is a semisimple Lie subalgebra of $\mathfrak{k}$ distinct from $\mathfrak{v} \cap \bar{\mathfrak{v}}$.

The *CR* manifolds of Definition 2.1 have canonical complex realizations:

Theorem 2.5 ([1, Theorem 4.3]). *Let M_0 be an $\mathfrak{n}$-reductive $\mathbf{K}_0$-homogeneous CR manifold, with CR algebra $(\kappa_0, \mathfrak{v})$ and isotropy $\mathbf{V}_0$ at some point $p_0 \in M_0$. Then there is a closed complex Lie subgroup $\mathbf{V}$ of the complexification $\mathbf{K}$ of $\mathbf{K}_0$ with $\mathbf{K}_0 \cap \mathbf{V} = \mathbf{V}_0$ and* $\mathrm{Lie}(\mathbf{V}) = \mathfrak{v}$ *such that the canonical map*

$$M_0 \simeq \mathbf{K}_0/\mathbf{V}_0 \longrightarrow M_- = \mathbf{K}/\mathbf{V} \tag{2.3}$$

is a generic CR embedding. □

Remark 2.6. Vice versa, if $M_- = \mathbf{K}/\mathbf{V}$ is the homogeneous complex manifold of the complexification $\mathbf{K}$ of $\mathbf{K}_0$, it is shown in [1, Prop.2.9] that any $\mathbf{K}_0$-orbit M_0 of minimal dimension in M_-, with the *CR* structure induced by the ambient space, is $\mathfrak{n}$-reductive.

3. Some remarks on $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$

Keep the notation of §2. As we explained in the introduction, we need to precise the structure of the fibers of the $\mathbf{K}_0$-equivariant Mostow fibration $M_- \to M_0$.

Mostow fibration ([23, 24]) extends to homogeneous spaces the Cartan decomposition of reductive Lie groups. Both are related to the fact that the positive definite $n \times n$ Hermitian symmetric matrices with determinant one are the points of a Riemannian symmetric space $\mathcal{M}_n$ with negative sectional curvature. We will discuss some topics on the geometry of $\mathcal{M}_n$ (see e.g. [11]).

Any compact Lie group $\mathbf{K}_0$ has, for some integer $n > 1$, a faithful linear representation in $\mathbf{SU}(n)$, which extends to a linear representation $\mathbf{K} \hookrightarrow \mathbf{SL}_n(\mathbb{C})$. Thus decompositions in $\mathbf{SL}_n(\mathbb{C})$ are preliminary to the general case.

The linear group $\mathbf{SL}_n(\mathbb{C})$ has the Cartan decomposition

$$\mathbf{SU}(n) \times \mathfrak{p}_0(n) \ni (x, X) \longrightarrow x \cdot \exp(X) \in \mathbf{SL}_n(\mathbb{C}),$$

where $\mathbf{SU}(n) = \{x \in \mathbf{SL}_n(\mathbb{C}) \mid x^* x = \mathrm{I}_n\}$ is its maximal compact subgroup consisting of $n \times n$ unitary matrices with determinant one, and $\mathfrak{p}_0(n)$ the subspace of the traceless Hermitian symmetric $n \times n$ matrices in $\mathfrak{sl}_n(\mathbb{C})$.

The quotient $\mathcal{M}_n = \mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$ is a symmetric space of the noncompact type and rank $(n-1)$, endowed with a Riemannian Riemanniansymmetricmetric with negative curvature. We can identify $\mathcal{M}_n$ with the set $\mathcal{P}_0(n)$ of positive definite Hermitian symmetric matrices in $\mathbf{SL}_n(\mathbb{C})$, which in turn is diffeomorphic to $\mathfrak{p}_0(n)$ via the exponential map. In this way $\mathcal{M}_n$ can be considered as an open subset of $\mathfrak{p}_0(n)$ and its tangent bundle $T\mathcal{M}_n$ is naturally diffeomorphic to the subbundle

$$T\mathcal{M}_n = \{(p, X) \in \mathcal{M}_n \times \mathfrak{p}(n) \mid p^{-1}X \in \mathfrak{p}_0(n)\}$$

of the trivial bundle $\mathcal{M}_n \times \mathfrak{p}(n)$, where we set $\mathfrak{p}(n) = \{X \in \mathbb{C}^{n\times n} \mid X^* = X\}$.

The special linear group $\mathbf{SL}_n(\mathbb{C})$ acts on $\mathcal{M}_n$ as a group of isometries, by

$$\mathbf{SL}_n(\mathbb{C}) \times \mathcal{M}_n \ni (z, p) \longrightarrow zpz^* \in \mathcal{M}_n,$$

and $\mathbf{SU}(n)$ is the stabilizer of the identity $e = \mathrm{I}_n$, that we choose as the base point.

The metric tensor on $\mathcal{M}_n$ is

$$(X, Y)_p = g_p(X, Y) = \mathrm{trace}(p^{-1}Xp^{-1}Y), \quad \forall p \in \mathcal{M}_n, \quad \forall X, Y \in T_p\mathcal{M}_n.$$

The curves

$$\mathbb{R} \ni t \to z \exp(tX)z^* \in \mathcal{M}_n, \quad \textit{for } X \in \mathfrak{p}_0(n),\ z \in \mathbf{SL}_n(\mathbb{C})$$

are the complete geodesics in $\mathcal{M}_n$ issued from $p = zz^*$ and

$$\mathrm{dist}(p_1, p_2) = \Big(\sum\nolimits_{i=1}^{n} |\log(\lambda_i(p_1^{-1}p_2)|^2\Big)^{1/2},$$

where $\lambda_i(p_1^{-1}p_2)$ are the eigenvalues of the matrix $p^{-1}p_2$, which are real and positive, the Riemanniandistance on $\mathcal{M}_n$.

3.1 – *Killing and Jacobi vector fields*

Since $\mathcal{M}_n$ is a Riemannian symmetric space of $\mathbf{SL}_n(\mathbb{C})$, the Lie algebra of its Killing vector fields is isomorphic to $\mathfrak{sl}_n(\mathbb{C})$. The correspondence is

$$\mathfrak{sl}_n(\mathbb{C}) \ni Z \longrightarrow \zeta_Z = \{p \to Zp + pZ^*\} \in \mathfrak{X}(\mathcal{M}_n).$$

For H in $\mathfrak{p}_0(n)$, the restriction to $[0, 1]$ of the geodesic $t \to \gamma_H(t) = \exp(tH)$ is the shortest path from $e = \gamma_H(0)$ to $h = \exp(H) = \gamma_H(1)$. We will denote by $\mathcal{J}(H)$ the space of Jacobi vector fields on γ_H and by $\mathcal{J}_0(H)$ its subspace consisting of those vanishing at $t = 0$. For each $Z \in \mathfrak{sl}_n(\mathbb{C})$, the restriction of ζ_{Z^*} to γ_H is a Jacobi vector field, that we denote by θ_Z:

$$\{\mathbb{R} \ni t \longrightarrow \theta_Z(t) = Z^* \exp(tH) + \exp(tH)Z\} \in \mathcal{J}(H).$$

To describe $\mathcal{J}(H)$ it is convenient to consider the commutator of H :

$$\begin{cases} \mathrm{C}(H) = \{Z \in \mathfrak{sl}_n(\mathbb{C}) \mid [Z, H] = 0\} = \mathrm{C}_u(H) \oplus \mathrm{C}_0(H), \quad \text{with} \\ \mathrm{C}_u(H) = \mathrm{C}(H) \cap \mathfrak{su}(n), \quad \mathrm{C}_0(H) = \mathrm{C}(H) \cap \mathfrak{p}_0(n). \end{cases}$$

Proposition 3.1. *The correspondence* $\theta : \mathfrak{sl}_n(\mathbb{C}) \ni Z \to \theta_Z \in \mathcal{J}(H)$ *is a linear map with kernel* $\mathrm{C}_u(H)$*. For each* $T \in \mathrm{C}_0(H)$*,* $J(t) = t \cdot \theta_T(t)$ *is a Jacobi vector field and*

$$\mathcal{J}(H) = \{\theta_Z + t\cdot\theta_T \mid Z \in \mathfrak{sl}_n(\mathbb{C}), \quad T \in \mathrm{C}_0(H)\}, \tag{3.1}$$

$$\mathcal{J}_0(H) = \{\theta_Y + t\cdot\theta_T \mid Y \in \mathfrak{su}(n), \quad T \in \mathrm{C}_0(H)\}. \tag{3.2}$$

Fix $Z \in \mathfrak{sl}_n(\mathbb{C})$ *and* $T \in \mathrm{C}_0(H)$*. Then*

$$J(t) = \theta_Z(t) + t\cdot\theta_T(t) = Z^* \exp(tH) + \exp(tH)Z + 2t \cdot T \cdot \exp(tH) \tag{3.3}$$

is the Jacobi vector field on γ_H satisfying the initial conditions:

$$\begin{cases} J(0) = Z + Z^*, \\ \dot{J}(0) = \frac{1}{2}[H, Z - Z^*] + 2T, \end{cases} \tag{3.4}$$

and we have

$$\begin{cases} \dot{J}(t) = \frac{1}{2}\theta_{[H,Z]+2T}(t), \\ \dfrac{D^k J(t)}{dt^k} = 2^{-k}\theta_{\mathrm{ad}_H^k(Z)}(t), \quad \textit{for } k \geq 2. \end{cases} \tag{3.5}$$

Proof. If $T \in \mathrm{C}_0(H)$, then θ_T is parallel and therefore also $t \cdot \theta_T$ is Jacobi on γ_H. To compute the covariant derivatives of the Jacobi vector field $J(t)$ defined in (3.3), we use the parallel transport $T_{\gamma_H(t)}\mathcal{M}_n \ni X \to \exp(sH/2)X\exp(sH/2) \in T_{\gamma_H(t+s)}\mathcal{M}_n$ along γ_H. Then

$$\begin{aligned} \dot{\theta}_Z(t) &= \left(\frac{d}{ds}\right)_{s=0} [\exp(-sH/2)\{Z^*\exp([t+s]H) + \exp([t+s]H)Z\}\exp(-sH/2)] \\ &= \tfrac{1}{2}[Z^*, H]\exp(tH) + \tfrac{1}{2}\exp(tH)[H, Z] = \tfrac{1}{2}\theta_{[H,Z]}(t). \end{aligned}$$

By iteration we obtain (3.5) and, in particular, (3.4).

Finally, we need to show that all J in $\mathcal{J}(H)$ have the form (3.3). Since ad_H is semisimple, $\mathfrak{sl}_n(\mathbb{C})$ decomposes into the direct sum of its image and its kernel. Hence $\mathfrak{p}_0(n) = [H, \mathfrak{su}(n)] \oplus \mathrm{C}_0(H)$, and this yields (3.1) and (3.2). □

For $X \in \mathfrak{p}_0(n)$, we will denote by J_X the geodesic on γ_H with

$$\begin{cases} J_X(0) = 0, \\ \dot{J}_X(0) = X, \end{cases} \tag{3.6}$$

while $\theta_X \in \mathcal{J}(H)$ satisfies $\theta_X(0) = 2X$, $\dot{\theta}_X(0) = 0$.

The nonconstant geodesics of a manifold with negative curvature have no conjugate points. Hence the map $\mathcal{J}_0(H) \ni J \to J(t) \in T_{\gamma_H(t)}\mathcal{M}_n$ is a linear isomorphism for all $t \neq 0$. Moreover, for every $J \in \mathcal{J}(H)$, the real map[1] $t \to \|J(t)\|$ is nonnegative and convex and therefore a nonzero $J(t) \in \mathcal{J}(H)$ vanishes for at most one value of $t \in \mathbb{R}$, corresponding to a minimum of $\|J(t)\|^2$ and thus to a solution of $(J(t)|\dot{J}(t)) = 0$.

Lemma 3.2. *If $J \in \mathcal{J}(H)$ is not parallel along γ_H and $(J(0)|\dot{J}(0)) = 0$, then*

$$\|J(0)\| < \|J(t)\| \textit{ for all } t \neq 0. \quad \square$$

Lemma 3.3. *The quadratic form*

$$\|J\|_H^2 = \int_0^1 (1-t)\left(\|\dot{J}(t)\|^2 + (J(t), \ddot{J}(t))\right) dt \tag{3.7}$$

[1] Here and in the following we drop the subscript indicating where norms and scalar products are computed, when we feel that this simplified notation does not lead to ambiguity.

is positive semidefinite on $\mathcal{J}(H)$ and

$$\|J\|_H^2 = 0 \Leftrightarrow J = \theta_T \;\; \textit{for a} \;\; T \in \mathrm{C}_0(H).$$

Proof. Let $J \in \mathcal{J}(H)$. Then $(\ddot{J}, J) = -(R(J, \dot\gamma_H)\dot\gamma_H | J) \geq 0$ for all t by the Jacobi equation, because $\mathcal{M}_n$ has negative sectional curvature. Hence $\|J\|_H^2 = 0$ if and only if $\dot{J}(t) = 0$ for all t. The statement follows because $\{\theta_T \mid T \in \mathrm{C}_0(H)\}$ is the space of the Jacobi vector fields that are parallel along γ_H. □

Lemma 3.4. *We have*

$$\|J(1)\|^2 = \|J(0)\|^2 + 2(J(0)|\dot{J}(0)) + 2\,\|J\|_H^2, \qquad \forall J \in \mathcal{J}(H). \tag{3.8}$$

Proof. We apply the integral form of the reminder in the first order Taylor's expansion to $f(t) = \|J(t)\|^2$. □

For further reference, we state an easy consequence of Lemma 3.4.

Lemma 3.5. *Let $Z \in \mathfrak{sl}_n(\mathbb{C})$, $X \in \mathfrak{p}_0(n)$, and* $\mathrm{trace}(X \cdot Z) = 0$. *Then*

$$\|\theta_Z(1) - J_X(1)\|^2 = \|Z + Z^*\|^2 + 2(H|[Z, Z^*]) + \|\theta_Z - J_X\|_H^2. \tag{3.9}$$

Proof. We apply (3.8) to $J = \theta_Z - J_X$.
Then

$$J(0) = Z + Z^*, \quad \dot{J}(0) = \tfrac{1}{2}[H, Z - Z^*] - X.$$

yields

$$\begin{aligned}\|\theta_Z(1) - J_X(1)\|^2 = \|J(1)\|^2 &= \|Z + Z^*\|^2 + 2(Z + Z^*|X + \tfrac{1}{2}[H, Z - Z^*]) + (J|J)_H \\ &= \|Z + Z^*\|^2 + ([H, Z - Z^*]|Z + Z^*) + (J|J)_H \\ &= \|Z + Z^*\|^2 + 2(H|[Z, Z^*]) + (J|J)_H.\end{aligned}$$

□

Let $J(t) = \theta_Z(t) + t\theta_T(t)$, with $Z \in \mathfrak{sl}_n(\mathbb{C})$ and $T \in \mathrm{C}_0(H)$. The two commuting Hermitian symmetric matrices H and T can be simultaneously diagonalized in an orthonormal basis of $\mathbb{C}^n$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of H, with multiplicities $n_1, \ldots, n_m$ and choose an orthonormal basis of $\mathbb{C}^n$ to get matrix representations

$$\begin{cases} H = \begin{pmatrix} \lambda_1 \mathrm{I}_{n_1} & & & \\ & \lambda_2 \mathrm{I}_{n_2} & & \\ & & \ddots & \\ & & & \lambda_m \mathrm{I}_{n_m} \end{pmatrix}, \quad T = \begin{pmatrix} \tau_1 & & & \\ & \tau_2 & & \\ & & \ddots & \\ & & & \tau_m \end{pmatrix}, \\ Z = \begin{pmatrix} z_{1,1} & z_{1,2} & \cdots & z_{1,m} \\ z_{2,1} & z_{2,2} & \cdots & z_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m,1} & z_{m,2} & \cdots & z_{m,m} \end{pmatrix}, \qquad \begin{array}{l} \text{with } \tau_i \in \mathbb{R}^{n_i \times n_i} \text{ diagonal,} \\ \text{and } z_{i,j} \in \mathbb{C}^{n_i \times n_j}. \end{array} \end{cases} \tag{3.10}$$

Let us extend the trace norm of $\mathfrak{p}_0(n)$ to a norm in $\mathfrak{sl}_n(\mathbb{C})$, by setting

$$|||A||| = \sqrt{\mathrm{trace}(AA^*)} \geq 0, \quad \forall A \in \mathfrak{sl}_n(\mathbb{C}).$$

Then

$$\begin{aligned}\|J(t)\|^2 &= \mathrm{trace}\left(Z^2 + Z^{*2} + 2e^{tH}Ze^{-tH}Z^* + 4t(Z+Z^*)T + 4t^2T^2\right)\\ &= \mathrm{trace}\left(2\,\mathrm{Re}\sum\nolimits_{i,j=1}^{m} z_{i,j}z_{j,i} + 2\sum\nolimits_{i,j=1}^{m} z_{i,j}z_{i,j}^{*}e^{t(\lambda_i-\lambda_j)} + 8t\,\mathrm{Re}\sum\nolimits_{i=1}^{m}\tau_i z_{i,i} + 4t^2\sum\nolimits_{i=1}^{m}\tau_i^2\right)\\ &= \sum\nolimits_{i\neq j}\left|\left|\left| z_{i,j}e^{t(\lambda_i-\lambda_j)/2} + z_{j,i}^{*}e^{t(\lambda_j-\lambda_i)/2}\right|\right|\right|^2 + \sum\nolimits_{i=1}^{m}|||\,2t\tau_i + z_{i,i} + \bar{z}_{i,i}\,|||^2.\end{aligned}$$

Set $Z(t) = \exp(tH/2)Z\exp(-tH/2) = (z_{i,j}(t))$, with $z_{i,j}(t) = z_{i,j}e^{t(\lambda_i-\lambda_j)/2} \in \mathbb{C}^{n_i\times n_j}$. We obtain the expression

$$\|J(t)\|^2 = \sum\nolimits_{i\neq j}|||z_{i,j}(t) + z_{j,i}^{*}(t)|||^2 + \sum\nolimits_{i=1}^{m}|||2t\tau_i + z_{i,i} + z_{i,i}^{*}|||^2. \tag{3.11}$$

If $J(t) = 0$, then each summand in (3.11) equals zero. For the terms in the first sum this amounts to the fact that $[H, Z(t)] = ((\lambda_i - \lambda_j)z_{i,j}(t))_{1\leq i,j\leq m}$ is Hermitian symmetric. Since $[H, Z(t)]$ and $[H, Z]$ are similar, we obtain:

Lemma 3.6. *Let $Z \in \mathfrak{sl}_n(\mathbb{C})$ and $H \in \mathfrak{p}_0(n)$. A necessary condition in order that there exists $T \in \mathrm{C}_0(H)$ such that the Jacobi vector field $J(t) = \theta_Z(t) + t\theta_T(t)$ on γ_H vanishes at some $t \in \mathbb{R}$ is that $[H, Z]$ is semisimple with real eigenvalues.* □

Example 3.7. We consider the matrices

$$H = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix} \in \mathfrak{sl}_3(\mathbb{R}), \quad Z = \begin{pmatrix}0 & a & 0\\ b & 0 & c\\ 0 & d & 0\end{pmatrix}, \quad Y = \begin{pmatrix}0 & \alpha & 0\\ -\bar{\alpha} & 0 & \beta\\ 0 & -\bar{\beta} & 0\end{pmatrix}.$$

We impose the conditions that Z be nilpotent and orthogonal to $X = [H, Y]$ and that $\theta_{Z+Y}(1) = 0$. This translates into the set of equations

$$\begin{cases} ab + cd = 0,\\ (\lambda_2-\lambda_1)(a\bar{\alpha} + b\alpha) + (\lambda_3-\lambda_2)(c\bar{\beta} + d\beta) = 0,\\ \alpha = (ae^{\lambda_1} + \bar{b}e^{\lambda_2})/(e^{\lambda_2} - e^{\lambda_1}),\\ \beta = (ce^{\lambda_2} + \bar{d}e^{\lambda_3})/(e^{\lambda_3} - e^{\lambda_2}).\end{cases}$$

By using the last two equation we reduce to the system

$$\begin{cases} ab + cd = 0,\\[1ex] \dfrac{\lambda_2-\lambda_1}{e^{\lambda_2}-e^{\lambda_1}}(|a|^2e^{\lambda_1} + ab(e^{\lambda_1}+e^{\lambda_2}) + |b|^2e^{\lambda_2})\\ \qquad\qquad + \dfrac{\lambda_3-\lambda_2}{e^{\lambda_3}-e^{\lambda_2}}(|c|^2e^{\lambda_2} + cd(e^{\lambda_2}+e^{\lambda_3}) + |d|^2e^{\lambda_3}) = 0\end{cases}$$

Assuming $ab \neq 0$ we obtain from the first equation $d = -ab/c$ and, as $\lambda_3 = -\lambda_1 - \lambda_2$, the system reduces to

$$(*)\qquad \begin{cases} \dfrac{\lambda_2 - \lambda_1}{e^{\lambda_2} - e^{\lambda_1}} \left(|a|^2 e^{\lambda_1} + ab(e^{\lambda_1} + e^{\lambda_2}) + |b|^2 e^{\lambda_2}\right) \\ \qquad + \dfrac{\lambda_1 + 2\lambda_2}{e^{\lambda_2} - e^{-\lambda_1 - \lambda_2}} \left(|c|^2 e^{\lambda_2} - ab(e^{\lambda_2} + e^{-\lambda_1-\lambda_2}) + \dfrac{|ab|^2}{c^2} e^{-\lambda_1 - \lambda_2}\right) = 0 \end{cases}$$

Let us restrict to the case where a, b, c are real. For any fixed a, b, c with $ab \neq 0$, the left hand side of $(*)$ is positive when $ab > 0$ and $|\lambda_1 + 2\lambda_2|$ is sufficiently small. Let us keep now λ_1 fixed and consider the left hand side of $(*)$ as a real valued function $f(\lambda_2)$ of the parameter λ_2. Then

$$\lim_{\lambda_2 \to +\infty} \lambda_2^{-1} f(\lambda_2) = |b|^2 + |c|^2 - ab.$$

If $ab > 0$, this is negative for $|a| \gg 1$. Then we can choose the parameters to satisfy $(*)$. In conclusion: *we can find H, Z, Y with $H \in \mathfrak{p}_0(3)$, $Z \in \mathfrak{sl}_3(\mathbb{C})$ nilpotent, and $Y \in \mathfrak{su}(3)$ with $X = [H, Y] \in \mathfrak{p}_0(3)$ trace-orthogonal to Z such that $\theta_{Z+Y}(0) \neq 0$ and $\theta_{Z+Y}(1) = 0$.*

Jacobi vector fields are used to compute the differential of the exponential map. In fact, for $H, X \in \mathfrak{p}_0(n)$, the covariant derivative $\frac{D}{dt}\exp(H + tX)|_{t=0}$ is the value at $t = 1$ of the Jacobian vector field $J_X \in \mathcal{J}_0(H)$. If $X = [H, Y] + T$, with $Y \in \mathfrak{su}(n)$ and $T \in \mathrm{C}_0(H)$, then

$$(3.12)\qquad \frac{D}{dt}\exp(H + tX)|_{t=0} = J_X(1) = [\exp(H), Y] + T\exp(H).$$

4. Decompositions with Hermitian fibers

4.1 – *Decomposition of* $\mathbf{SL}_n(\mathbb{C})$

Throughout this section, $\mathbf{V}$ is a closed complex Lie subgroup of $\mathbf{SL}_n(\mathbb{C})$, that admits a Levi-Chevalley decomposition $\mathbf{V} = \mathbf{V}_r \cdot \mathbf{V}_n$, with $\mathbf{V}_r$ algebraic reductive and $\mathbf{V}_n$ unipotent (cf. [31, Ch.I, §6.5]). We choose the embedding $\mathbf{V} \hookrightarrow \mathbf{SL}_n(\mathbb{C})$ in such a way that $\mathbf{V}_0 = \mathbf{V} \cap \mathbf{SU}(n)$ is a maximal compact sugbroup of $\mathbf{V}$ and a real form of $\mathbf{V}_r$ and set:

$$(4.1)\qquad \mathfrak{v} = \mathrm{Lie}(\mathbf{V}),\quad \mathfrak{v}_r = \mathrm{Lie}(\mathbf{V}_r),\quad \mathfrak{v}_n = \mathrm{Lie}(\mathbf{V}_n),\quad \mathfrak{v}_0 = (\mathfrak{v} \cap \mathfrak{su}(n)) = \mathrm{Lie}(\mathbf{V}_0),$$

$$(4.2)\qquad \mathfrak{m}_0 = (\mathfrak{v} + \mathfrak{v}^*)^\perp \cap \mathfrak{p}_0(n),\quad \mathfrak{v} = \mathfrak{v}_0 \oplus \mathfrak{v}',\ \text{ with }\ \mathfrak{v}' = (\mathfrak{v} \cap \mathfrak{p}_0(n)) \oplus \mathfrak{v}_n.$$

Remark 4.1. We have $(\mathfrak{v} + \mathfrak{v}^*) \cap \mathfrak{p}_0(n) = \{Z + Z^* \mid Z \in \mathfrak{v}\}$. Indeed, if $Z_1, Z_2 \in \mathfrak{v}$ and $Z_1 + Z_2^* \in \mathfrak{p}_0(n)$, then $Z = (Z_1 + Z_2)/2 \in \mathfrak{v}$ and $Z_1 + Z_2^* = Z + Z^*$. Hence the maps

$$(4.3)\qquad \begin{cases} \mathfrak{v}' \ni Z \to (Z + Z^*) \in (\mathfrak{v} + \mathfrak{v}^*) \cap \mathfrak{p}_0(n), \\ \mathfrak{v}' \oplus \mathfrak{m}_0 \ni (Z, X) \longleftrightarrow (Z^* + X + Z) \in \mathfrak{p}_0 \end{cases}$$

are $\mathbb{R}$-linear isomorphisms. Often we will write $Z \in \mathfrak{v}$ as a sum $Z = Z_0 + Z_n$, where it will be understood that $Z_0 \in (\mathfrak{v} \cap \mathfrak{p}_0(n))$ and $Z_n \in \mathfrak{v}_n$.

By (4.3), the Euclidean subspace $\exp(\mathfrak{m}_0)$ is a natural candidate for the typical fiber F_0 of an $\mathbf{SU}(n)$-covariant fibration of $\mathbf{SL}_n(\mathbb{C})/\mathbf{V}$. As we will see, this is in fact the case for some important classes of $\mathbf{V}$'s.

Being algebraic, $\mathbf{V}$ admits the decomposition

$$\mathbf{V}_0 \times \mathfrak{v}' \ni (u, Z_0 + Z_n) \longleftrightarrow u \cdot \exp(Z_0) \cdot \exp(Z_n) \in \mathbf{V}, \tag{4.4}$$

which is a consequence of the Levi-Chevalley decomposition of $\mathbf{V}$ and of the polar Cartan decomposition of $\mathbf{V}_r$. Set

$$N = \{p \in \mathcal{M}_n \mid p = v^* v, \text{ for some } v \in \mathbf{V}\}. \tag{4.5}$$

Lemma 4.2. *The map $v \to v^* v$ defines, by passing to the quotients, an isomorphism*

$$\mathbf{V}/\mathbf{V}_0 \ni [v] \xrightarrow{\sim} v^* v \in N. \tag{4.6}$$

Proof. In fact the right action $v \cdot \zeta = v^* \cdot \zeta \cdot v$ of $\mathbf{V}$ on N is transitive and $\mathbf{V}_0$ is the stabilizer of $e = \mathrm{I}_n$. □

Lemma 4.3. *The map*

$$\mathfrak{v}' \ni (Z_0 + Z_n) \longrightarrow \exp(Z_n^*) \cdot \exp(Z_0) \cdot \exp(Z_n) \in N \tag{4.7}$$

is a diffeomorphism. In particular, N is diffeomorphic to a Euclidean space.

Proof. In fact, (4.7) is smooth and bijective and its inverse can be computed by using the diffeomorphisms $\mathbf{V}/\mathbf{V}_0 \simeq \mathfrak{v}'$ of (4.4), and (4.6). □

Lemma 4.4. *We can find a real $r > 0$ such that the map*

$$\lambda : \mathfrak{v}' \times \mathfrak{m}_0 \ni (Z_0 + Z_n, H) \longrightarrow \exp(Z_n^*) \exp(Z_0) \exp(H) \exp(Z_0) \exp(Z_n) \in \mathcal{M}_n \tag{4.8}$$

is a diffeomorphism of $\{\|H\| < r\}$ *onto* $\{p \in \mathcal{M}_n \mid \mathrm{dist}(p, N) < r\}$.

Proof. By (4.3), λ is a local diffeomorphism at all points where it has an injective differential. By using the isometries $p \to z^* {\cdot} p {\cdot} z$ of $\mathcal{M}_n$, we may reduce to points $(0, H)$, where, to compute the differential, we can use the Jacobi vector fields θ_Z and J_X on γ_H, that where defined in §3.1. Indeed, for $(Z, X) \in \mathfrak{v}' \times \mathfrak{m}_0$, $d\lambda(0, H)(Z, 0) = \theta_Z(1)$ and $d\lambda(0, H)(0, X) = J_X(1)$. Moreover, the maps $\mathfrak{v}' \ni Z \to \theta_Z(1) \in T_{\exp(H)}\mathcal{M}_n$ and $\mathfrak{m}_0 \ni X \to J_X(1) \in T_{\exp(H)}\mathcal{M}_n$ both are injective. Thus it suffices to verify that $\theta_Z(1) \neq J_X(1)$ when Z and X are not zero. By Lemma 3.5,

$$\|J_X(1) - \theta_Z(1)\|^2 \geq \|Z + Z^*\|^2 + 2(H|[Z, Z^*]), \quad \forall (Z, X) \in \mathfrak{v} \times \mathfrak{m}_0.$$

For $Z \in \mathfrak{v}'$, we have $\|Z\| = \|Z^*\| \leq \|Z + Z^*\|$. Thus

$$|(H|[Z, Z^*])| \leq \|H\| \cdot \|Z + Z^*\|^2.$$

This implies that, for some $r > 0$, (4.8) defines a local diffeomorphism, and hence a smooth covering, of $\mathfrak{v}' \times \{\|H\| < r\}$ onto $\{p \in \mathcal{M}_n \mid \mathrm{dist}(p, N) < r\}$. This is in fact a global diffeomorphism because both spaces are simply connected. □

Set

$$\mathbf{V}' = \{\exp(Z_0)\exp(Z_n) \mid Z_0 + Z_n \in \mathfrak{v}'\} \tag{4.9}$$

and consider the map

$$\mu : \mathbf{SU}(n) \times \mathfrak{m}_0 \times \mathbf{V}' \ni (u, X, v) \longrightarrow u \cdot \exp(X) \cdot v \in \mathbf{SL}_n(\mathbb{C}). \tag{4.10}$$

Proposition 4.5. *The map* (4.10) *is onto.*

There is a real $r > 0$ *for which* μ *is a diffeomorphism of* $\{\|X\| < r\}$ *onto the open manifold* $\{\zeta \in \mathbf{SL}_n(\mathbb{C}) \mid \operatorname{dist}(\zeta^*\zeta, N) < 2r\}$.

Proof. The set $N = \{z^*z \mid z \in \mathbf{V}\}$ is a properly embedded smooth submanifold of $\mathcal{M}_n$. Hence, for each $p \in \mathcal{M}_n$, there is a $z_p \in \mathbf{V}$ with

$$\operatorname{dist}(p, z_p^*z_p) = \operatorname{dist}(p, N).$$

The geodesic joining $z_p^*z_p$ to p has the form $[0,1] \ni t \to \gamma(t) = z_p^* \exp(tH) z_p$ for some $H \in \mathfrak{p}_0(n)$, and $\dot\gamma(0)$ is orthogonal to N at $z_p^*z_p$. The isometry $q \to z_p^{*\,-1} q\, z_p^{-1}$ maps N into itself, $z_p^*z_p$ to e and $\dot\gamma(0)$ to H. Thus $H \in T_e\mathcal{M}_n = \mathfrak{p}_0(n)$ belongs to $\mathfrak{m}_0$.

This shows that, if $\zeta \in \mathbf{SL}_n(\mathbb{C})$ and $z_p^*z_p$ is the nearest point in N to $p = \zeta^*\zeta$, then

$$p = \zeta^*\zeta = z_p^* \exp(H) z_p, \quad \text{for some } z_p \in \mathbf{V}' \text{ and } H \in \mathfrak{m}_0.$$

The matrix $u = \zeta \cdot z_p^{-1} \cdot \exp(-H/2)$ belongs to $\mathbf{SU}(n)$. Indeed

$$\begin{aligned} u^*u &= \exp(-H/2) \cdot [z_p^{-1}]^* \cdot \zeta^* \cdot \zeta \cdot z_p^{-1} \cdot \exp(-H/2) \\ &= \exp(-H/2) \cdot [z_p^{-1}]^* \cdot z_p^* \cdot \exp(H) \cdot z_p \cdot z_p^{-1} \cdot \exp(-H/2) = \mathrm{I}_n. \end{aligned}$$

Since $\zeta = u \cdot \exp(H/2) \cdot z_p$, this proves that (4.10) is onto.

The second part of the statement is then a consequence of Lemma 4.4. □

Corollary 4.6. *The map*

$$\mathbf{SU}(n) \times \mathfrak{m}_0 \ni (x, X) \longrightarrow \pi(x \cdot \exp(X)) \in \mathbf{SL}_n(\mathbb{C})/\mathbf{V}, \tag{4.11}$$

where $\pi : \mathbf{SL}_n(\mathbb{C}) \to \mathbf{SL}_n(\mathbb{C})/\mathbf{V}$ *is the projection onto the quotient, is onto. By passing to the quotient, it defines a surjective smooth map*

$$\mathbf{SU}(n) \times_{\mathbf{V}_0} \mathfrak{m}_0 \longrightarrow \mathbf{SL}_n(\mathbb{C})/\mathbf{V}, \tag{4.12}$$

where $\mathbf{SU}(n) \times_{\mathbf{V}_0} \mathfrak{m}_0$ *is the quotient of* $\mathbf{SU}(n) \times \mathfrak{m}_0$ *modulo the equivalence relation*

$$(x, X) \sim (x \cdot u, u^*Xu) \quad \textit{for} \quad x \in \mathbf{SU}(n), X \in \mathfrak{m}_0 \textit{ and } u \in \mathbf{V}_0.$$ □

4.2 – *Decomposition of* $\mathbf{K}$

Let $\mathbf{V}$ be a closed subgoup of the complexification $\mathbf{K}$ of a compact Lie group $\mathbf{K}_0$. We can assume that in turn $\mathbf{K}$ is a linear subgroup of $\mathbf{SL}_n(\mathbb{C})$, with $\mathbf{K}_0 = \mathbf{K} \cap \mathbf{SU}(n)$, and $\mathbf{V}_0 = \mathbf{V} \cap \mathbf{SU}(n)$ a maximal compact subgroup of $\mathbf{V}$. We obtain:

Proposition 4.7. *With* $\mathfrak{k}_0 = \mathfrak{m}_0 \cap \kappa$, *we have the commutative diagram with surjective arrows*

$$\begin{array}{ccc} \mathbf{K}_0 \times \mathfrak{k}_0 & \longrightarrow & \mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{k}_0 \\ & \searrow \quad \swarrow & \\ & \mathbf{K}/\mathbf{V}, & \end{array} \tag{4.13}$$

where the horizontal arrow is the projection onto the quotient, the left one is obtained by restricting (4.11)*, and the right one by passing to the quotient.*

We denoted by $\mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{k}_0$ the quotient of the product $\mathbf{K}_0 \times \mathfrak{k}_0$ by the equivalence relation $(x, X) \sim (x \cdot u, \mathrm{Ad}(u^{-1})(X))$ for $x \in \mathbf{K}_0$, $X \in \mathfrak{k}_0$ and $u \in \mathbf{V}_0$. The right arrow maps the equivalence class of (x, X) to $\pi(x \cdot \exp(X)) \in \mathbf{K}/\mathbf{V} \subset \mathbf{SL}_n(\mathbb{C})/\mathbf{V}$.

Proof. It is sufficient to follow the proof of Proposition 4.5 and check that, for $\zeta \in \mathbf{K}$, we obtain $X \in \mathfrak{k}_0$ and $x \in \mathbf{K}_0$.

In fact, in this case, $\zeta^*\zeta = z^* \exp(2X) z \in \mathbf{K} \cap \mathcal{P}_0(n)$, with $z \in \mathbf{V}$, implies that $\exp(2X) = z^{*-1}\zeta^*\zeta z^{-1} \in \exp(\mathfrak{m}_0) \cap \mathbf{K} = \exp(\mathfrak{k}_0)$. □

We have the analogous of Proposition 4.5.

Proposition 4.8. *The map*

$$\mathbf{K}_0 \times \mathfrak{k}_0 \times \mathbf{V}' \ni (u, X, v) \longrightarrow u \cdot \exp(X) \cdot v \in \mathbf{K} \tag{4.14}$$

is always surjective and there is $r_0 > 0$ *such that, for all* $0 < r \leq r_0$, *it is a diffeomorphism of* $\{\|X\| < r\}$ *onto a tubular neighborhood of* $M_0 = \mathbf{K}_0/\mathbf{V}_0$ *in* M_-. □

It is known that the right arrow in (4.13) *is* the Mostow fibration of $\mathbf{K}/\mathbf{V}$ when $\mathbf{V}$ is reductive (see e.g. [23, 29]). We give here a simple proof relying on the preparation done in §3.

Proposition 4.9. *If* $\mathbf{V}$ *is reductive, then the natural surjective map*

$$\mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{k}_0 \to M_- = \mathbf{K}/\mathbf{V} \tag{4.15}$$

is a diffeomorphism.

Proof. In this case $\mathbf{V}$, being algebraic and self-adjoint, has the Cartan decomposition $\mathbf{V} = \mathbf{V}_0 \times \exp(\mathfrak{v}')$, with $\mathfrak{v}' = \mathfrak{v} \cap \mathfrak{p}_0(n)$. By Lemma 3.2, the map

$$\lambda_\kappa : \mathfrak{v}' \times \mathfrak{k}_0 \ni (Z, H) \longrightarrow \exp(Z^*) \cdot \exp(H) \cdot \exp(Z) \in \mathbf{K} \cap \mathcal{P}_0(n)$$

is surjective. Moreover, it is a local diffeomorphism at every point of $\mathfrak{v}' \times \mathfrak{k}_0$. In fact, we can reduce to prove this fact at points $(0, H)$, where the differential at (Z, X) is $J(1)$ for $J(t) = \theta_Z + J_X \in \mathcal{J}(H)$. Then $\|J(1)\| \geq \|J(0)\| = 2\|Z\| > 0$ for $Z \neq 0$, while $J_X(1) \neq 0$ if $X \neq 0$. Since $\kappa \cap \mathfrak{p}_0(n) = \mathfrak{v}' \oplus \mathfrak{k}_0$, this proves that $d\lambda_\kappa(0, H)$ is a linear isomorphism. Thus, being a connected covering of a simply connected space, λ_κ is a global diffeomorphism.

Hence, for every $\zeta \in \mathbf{K}$, there is a unique pair $(Z, H) \in \mathfrak{v}' \times \mathfrak{k}_0$ such that

$$\zeta^* \cdot \zeta = \exp(Z^*) \cdot \exp(H) \cdot \exp(Z);$$

then $u = \zeta \cdot \exp(-Z) \cdot \exp(-\frac{1}{2}H) \in \mathbf{K}_0$ and we obtain the direct product decomposition

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{k}_0) \cdot \exp(\mathfrak{v}'), \tag{4.16}$$

from which the statement follows. □

The complex $\mathbf{K}$-homogeneous M_- of Proposition 4.9 corresponds to an M_- which is the Stein complexification of a totally real $\mathbf{K}_0$-homogeneous compact M_0. An M_0 having a positive *CR* dimension corresponds to a $\mathbf{V}$ having a nontrivial unipotent radical.

Before investigating cases where, even though $\mathfrak{v}_n \neq 0$, (4.15) is nevertheless a diffeomorphism, we observe that, when we know that decomposition (4.10) is unique, we can extract some extra information from the minimal distance characterization of $z_p^* z_p$ in the proof of Proposition 4.5. For instance, as a corollary of Proposition 4.5, we obtain the following

Proposition 4.10. *For $h \in \mathcal{P}_0(n)$, denote by $D_\ell(h)$ the minor determinant of the first ℓ rows and columns of h. Set $D_0(h) = 1$ and let $0 < \lambda_1(h) \leq \cdots \leq \lambda_n(h)$ be the eigenvalues of h. Then*

$$\operatorname{dist}(h, e) = \sum\nolimits_{\ell=1}^{n} |\log(\lambda_\ell(h))|^2 \geq \sum\nolimits_{\ell=1}^{n} |\log(D_\ell(h)/D_{\ell-1}(h))|^2. \tag{4.17}$$

If h is not diagonal, we have strict inequality.

Proof. We take $\mathbf{V}$ equal to the group of unipotent upper triangular matrices in $\mathbf{GL}_n(\mathbb{C})$. The element $\delta = e^\Delta \in N_h = \{z^* h z \mid z \in \mathbf{V}\}$, with $\Delta \in \mathfrak{p}_0$, at minimal distance from e satisfies $\operatorname{trace}([Z + Z^*]\Delta) = 0$ for all nilpotent upper triangular Z and hence is diagonal. The unique diagonal $\delta = z^* h z$ in N_h is the one obtained by the Gram-Schmidt orthogonalization procedure. The proof is complete. □

The orbit of a point $p \in \mathcal{M}_n$ by the group of unipotent upper triangular matrices of $\mathbf{SL}_n(\mathbb{C})$ is an example of a *horocycle* of maximal dimension in a symmetric space of noncompact type. We will generalize this situation while outlining a class of subroups $\mathbf{V}$ for which $F_0 = \exp(\mathfrak{k}_0)$ can be taken as the fiber of the $\mathbf{K}_0$-covariant fibration.

Following [32, p.17], we call *horocyclic* in κ the nilpotent subalgebras which are nilradicals of parabolic subalgebras of κ.

Lemma 4.11. *Let* $\mathfrak{q}$ *be a parabolic subalgebra of* $\mathfrak{sl}_n(\mathbb{C})$, *with nilradical* $\mathfrak{q}_n$. *Assume that* $\mathfrak{q} \cap \mathfrak{q}^*$ *is a reductive Levi factor of* $\mathfrak{q}$. *Let* $H \in \mathfrak{q} \cap \mathfrak{p}_0(n)$. *Then, for* $Z_0 \in \mathfrak{q} \cap \mathfrak{q}^*$, $T \in \mathrm{C}_0(H) \cap \mathfrak{q}$ *and* $Z_n \in \mathfrak{q}_n$ *the Jacobi vector fields* $J_1 = \theta_{Z_0} + t\theta_T$ *and* $J_2 = \theta_{Z_n}$ *are orthogonal at all points of* γ_H.

Proof. We show, separately, that θ_{Z_0} and θ_T are both orthogonal to θ_{Z_n} at all points of γ_H. We have

$$\begin{aligned}(\theta_T(t)|\theta_{Z_n}(t)) &= \mathrm{trace}(2Te^{-tH}(e^{tH}Z_n^* + Z_n e^{tH})) = 2\mathrm{trace}(TZ_n + TZ_n^*) = 0,\\ (\theta_{Z_0}(t)|\theta_{Z_n}(t)) &= \mathrm{trace}((e^{-tH}Z_0^* + Z_0 e^{-tH})(e^{tH}Z_n + Z_n^* e^{tH}))\\ &= \mathrm{trace}(Z_0^*(e^{tH}Z_n e^{-tH}) + Z_0^* Z_n^* + Z_0 Z_n + (e^{tH}Z_0 e^{-tH})Z_n^*) = 0\end{aligned}$$

because $\mathfrak{q} \cap \mathfrak{q}^*$ and $\mathfrak{q}_n$ are orthogonal for the trace form of the canonical representation of $\mathfrak{sl}_n(\mathbb{C})$. Indeed, The expression in the last line is twice the sum of the real parts of the product of Z_0 and Z_n and of $e^{-tH}Z_0^* e^{tH} \in \mathfrak{q} \cap \mathfrak{q}^*$ and Z_n. □

Proposition 4.12. *If* $\mathfrak{v}_n$ *is horocyclic in* κ, *then*

$$\mathbf{V}' \times \mathfrak{k}_0 \ni (v, H) \longrightarrow v^* \exp(H)\, v \in \mathcal{M}(\mathbf{K}) = \mathcal{P}_0(n) \cap \mathbf{K} \tag{4.18}$$

is a diffeomorphism.

Proof. In fact, we can find a parabolic $\mathfrak{q}$ in $\mathfrak{sl}_n(\mathbb{C})$ such that $\mathfrak{q} \cap \mathfrak{q}^*$ is its reductive Levi factor and $\mathfrak{v}_n = \mathfrak{q}_n \cap \kappa$. Then we can reduce to proving the proposition in the case where $\mathbf{K} = \mathbf{SL}_n(\mathbb{C})$ and $\mathfrak{k}_0 = \mathfrak{m}_0$. We want to show that (4.8) is a local diffeomorphism. To this aim, with the notation of §3.1, it suffices to prove that, for $Z \in \mathfrak{v}$ and $H, X \in \mathfrak{m}_0$, we have $\theta_Z(1) \neq J_X(1)$ when $Z + X \neq 0$. We split Z into the sum $Z = Z_0 + Z_n$, with $Z_0 \in \mathfrak{v} \cap \mathfrak{p}_0(n)$ and $Z_n \in \mathfrak{v}_n$. Then the fact that $\theta_{Z_0}(1) + J_X(1) \neq 0$ if $Z_0 + X \neq 0$ follows from Lemma 3.2 because of Lemma 4.11. Hence (4.8) is a connected covering of a simply connected manifold and thus a global diffeomorphism. □

Proposition 4.12 can be slightly generalized. It was shown in [22, p.251] that there is a unique maximal complex Lie subalgebra $\mathfrak{w}$ of κ with $\mathfrak{v} \subseteq \mathfrak{w} \subseteq \mathfrak{v} + \bar{\mathfrak{v}}$. The *CR*-algebra $(\kappa_0, \mathfrak{v})$ and the corresponding $\mathbf{K}_0$-homogeneous *CR* manifold M_0 are called *weakly nondegenerate* when $\mathfrak{w} = \mathfrak{v}$. If this is not the case, M_0 turns out to be the total space of a complex *CR*-bundle with nontrivial fibers over a weakly nondegenerate $\mathbf{K}_0$-homogeneous *CR* manifold M_0', having *CR* algebra $(\kappa_0, \mathfrak{w})$.

Proposition 4.13. *Let* $\mathfrak{w}$ *be the largest complex Lie algebra with* $\mathfrak{v} \subseteq \mathfrak{w} \subseteq \mathfrak{v} + \bar{\mathfrak{v}}$. *If* $\mathfrak{w}_n = \mathfrak{n}(\mathfrak{w})$ *is horocyclic in* κ, *then* (4.18) *is a diffeomorphism.*

Proof. As above, we reduce the proof to the case where $\mathbf{K} = \mathbf{SL}_n(\mathbb{C})$. The proof follows the same pattern of the proof of Proposition 4.12. We denote by $\mathfrak{q}$ a parabolic Lie subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ with $\mathfrak{q}_n = \mathfrak{w}_n$ and use the notation of §3.1. We need to prove that, for $Z \in \mathfrak{v}' = (\mathfrak{v} \cap \mathfrak{p}_0(n)) \oplus \mathfrak{v}_n$ and $X, H \in \mathfrak{m}_0$, we have $\theta_Z(1) + J_X(1) \neq 0$ if $Z + X \neq 0$.

To this aim it is convenient to split Z into a sum $Z = U + W$, with $U \in \mathfrak{v}' \cap \mathfrak{w} \cap \overline{\mathfrak{w}}$ and $W \in \mathfrak{q}_n$. Let us consider first $J = \theta_U + J_X$. We note that $\dot{J}(0) = X + \frac{1}{2}[X, U - U^*]$ is orthogonal to $J(0) = U + U^*$. Indeed $(X|U + U^*) = 0$ because $\mathfrak{w} + \overline{\mathfrak{w}} = \mathfrak{v} + \overline{\mathfrak{v}}$ and, since $[U, U^*] \in \mathfrak{w} \cap \mathfrak{p}_0(n)$,

$$(U + U^*|[H, U - U^*]) = \operatorname{trace}([H, U - U^*](U + U^*)) = 2\operatorname{trace}(H \cdot [U, U^*]) = 0.$$

By Lemma 3.2, this implies that $J(1) \neq 0$ if $Z + X \neq 0$. Finally, we note that $\theta_W(0)$ and $\dot{\theta}_W(0)$ are orthogonal to both $J(0)$ and $\dot{J}(0)$ to conclude, using again Lemma 3.2, that $J_Z(1) + J_X(1) = J(1) + J_W(1) \neq 0$ when $X + Z = (X + U) + W \neq 0$.

This shows that (4.18), being a connected smooth covering of a simply connected manifold, is a global diffeomorphism. □

By using the argument in the proof of Proposition 4.9, we conclude:

Theorem 4.14. *Let* $\mathfrak{w}$ *be the largest complex Lie algebra with* $\mathfrak{v} \subseteq \mathfrak{w} \subseteq \mathfrak{v} + \overline{\mathfrak{v}}$. *If* $\mathfrak{w}_n = \mathfrak{n}(\mathfrak{w})$ *is horocyclic in* $\mathfrak{k}$, *then* (4.15) *is a global diffeomorphism and therefore we obtain the* $\mathbf{K}_0$*-equivariant Mostow fibration of* M_- *over* M_0

$$\begin{array}{ccc} \mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{f}_0 & \longrightarrow & M_- \\ & \searrow \quad \swarrow & \\ & M_0 & \end{array} \tag{4.19}$$

with Hermitian fiber. □

We keep the notation of §2.3 and denote by $\mathfrak{w}$ the largest Lie subalgebra of $\mathfrak{k}$ with

$$\mathfrak{v} \subseteq \mathfrak{w} \subseteq \mathfrak{v} + \overline{\mathfrak{v}}. \tag{4.20}$$

Definition 4.1. We say that $(\mathfrak{k}_0, \mathfrak{v})$ is HNR if $\mathfrak{w}_n = \mathfrak{n}(\mathfrak{w})$ is horocyclic.

For further reference, we reformulate the result obtained so far in the following form.

Theorem 4.15. *If* $(\mathfrak{k}_0, \mathfrak{v})$ *is* HNR, *then we have the direct product decomposition*

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \mathbf{V}'. \tag{4.21}$$ □

Example 4.16. *Minimal orbit of* $\mathbf{SU}(2, 2)$ *in* $\mathcal{F}_{1,2}(\mathbb{C}^4)$.
We fix in $\mathbb{C}^4$ the Hermitian form associated to the matrix

$$\begin{pmatrix} \mathrm{I}_2 & \\ & -\mathrm{I}_2 \end{pmatrix}.$$

We let the corresponding group $\mathbf{SU}(2, 2)$ operate on the flag manifold $\mathcal{F}_{1,2}(\mathbb{C}^4)$, consisting of the pairs (ℓ_1, ℓ_2) of a line ℓ_1 and a 2-plane ℓ_2 with $0 \in \ell_1 \subset \ell_2 \subset \mathbb{C}^4$. The minimal orbit is

$$M_0 = \{(\ell_1, \ell_2) \mid \ell_1 \subset \ell_2 = \ell_2^{\perp}\},$$

where the orthogonal is taken with respect to the fixed Hermitian form. It is the total space of a $\mathbb{CP}^1$-bundle over a smooth real manifold and in particular is Levi-flat of *CR* dimension 1. With $\mathbf{K}_0 = \mathbf{S}(\mathbf{U}(2)\times\mathbf{U}(2))$, $\mathbf{K} = \mathbf{S}(\mathbf{GL}_2(\mathbb{C})\times\mathbf{GL}_2(\mathbb{C}))$, the stabilizer

$$\mathbf{V} = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \middle| \, a \in \mathbf{ST}_2^+(\mathbb{C}) \right\}$$

of the base point $\mathrm{p}_0 = (\langle e_1 + e_3\rangle, \langle e_1 + e_3, e_2 + e_4\rangle)$ (here $\mathbf{T}_2^+(\mathbb{C})$ is the group of upper triangular 2×2 complex matrices with non vanishing determinant and $\mathbf{ST}_2^+(\mathbb{C})$ its normal subgroup consisting of those having determinant 1) has Lie algebra

$$\mathfrak{v} = \left\{ \begin{pmatrix} \lambda & \alpha & & \\ 0 & -\lambda & & \\ & & \lambda & \alpha \\ & & 0 & -\lambda \end{pmatrix} \middle| \, \lambda, \alpha \in \mathbb{C} \right\}.$$

Clearly $\mathfrak{v}_n$ is not horocyclic. We note that

$$\mathfrak{w} = \mathfrak{v} + \bar{\mathfrak{v}} = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \middle| \, X \in \mathfrak{sl}_2(\mathbb{C}) \right\} = \mathfrak{v}'$$

is a complex Lie algebra. Thus, although $\mathbf{V}$ is not HNR, nevertheless we have a Mostow fibration with Hermitian fibers by Theorem 4.14.

Remark 4.17. Example 3.7 shows that (4.18) is not, in general, a diffeomorphism when $(\mathfrak{k}_0, \mathfrak{v})$ is not HNR.

5. Mostow fibration in general and the HNR condition

5.1 – *The set* $\mathfrak{P}_0(\mathfrak{v})$

To better understand the notion introduced in Definition 4.1 and to characterize the fiber of the Mostow fibration of M_- on M_0 in general, it is convenient to rehearse some notions that were introduced in [1, §3]. We simply assume, at the beginning, that $\mathfrak{k}$ is any reductive Lie algebra over $\mathbb{C}$.

For a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{k}$, let us denote by $\mathfrak{n}(\mathfrak{a})$ the ideal consisting of the $\mathrm{ad}_{\mathfrak{k}}$-nilpotent elements of its radical. Starting from any splittable Lie subalgebra $\mathfrak{v}$ of $\mathfrak{k}$ we construct a sequence $\{\mathfrak{v}_{(h)}\}$ of Lie subalgebras by setting recursively

$$(5.1)\qquad \begin{cases} \mathfrak{v}_{(0)} = \mathfrak{v}, \\ \mathfrak{v}_{(h+1)} = \mathbf{N}_{\mathfrak{k}}(\mathfrak{n}(\mathfrak{v}_{(h)})) = \{Z \in \mathfrak{k} \mid [Z, \mathfrak{n}(\mathfrak{v}_{(h)})] \subset \mathfrak{n}(\mathfrak{v}_{(h)})\}, \quad \forall h \geq 0. \end{cases}$$

Each $\mathfrak{v}_{(h)}$, with $h \geq 1$, is the normalizer in $\mathfrak{k}$ of the ideal of $\mathrm{ad}_{\mathfrak{k}}$-nilpotent elements of the radical of $\mathfrak{v}_{(h-1)}$. It was shown in [1] that $\mathfrak{v}_{(h)} \subsetneqq \mathfrak{v}_{(h+1)}$ and $\mathfrak{n}(\mathfrak{v}_{(h)}) \subsetneqq \mathfrak{n}(\mathfrak{v}_{(h+1)})$ for all $h \geq 0$, and that the union $\mathfrak{e} = \bigcup_{h\geq 0}\mathfrak{v}_{(h)}$ is a parabolic subalgebra of $\mathfrak{k}$, with $\mathfrak{v} \subset \mathfrak{e}$ and $\mathfrak{n}(\mathfrak{v}) = \mathfrak{v}_n \subset \mathfrak{n}(\mathfrak{e})$. We call $\mathfrak{e}$ the *parabolic regularization of* $\mathfrak{v}$. Hence

$$(5.2)\qquad \mathfrak{P}(\mathfrak{v}) = \{\mathfrak{q} \mid \mathfrak{q} \text{ is parabolic in } \mathfrak{k} \text{ and } \mathfrak{v} \subset \mathfrak{q},\ \mathfrak{n}(\mathfrak{v}) \subset \mathfrak{n}(\mathfrak{q})\}$$

is nonempty. Let us prove a general simple lemma on parabolic Lie subalgebras.

Lemma 5.1. *If* $\mathfrak{q}_1, \mathfrak{q}_2$ *are parabolic Lie subalgebras of* κ*, then the Lie subalgebra* $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2 + \mathfrak{n}(\mathfrak{q}_1)$ *is parabolic in* κ*.*

Proof. We know (see e.g. [6, Ch.VIII,Prop.10]) that $\mathfrak{q}_1 \cap \mathfrak{q}_2$ contains a Cartan subalgebra $\mathfrak{h}$ of κ. If $\mathcal{R}$ is the corresponding set of roots, then each $\mathfrak{q}_i$ ($i = 1, 2$) decomposes into a direct sum

$$\mathfrak{q}_i = \mathfrak{h} \oplus \sum\nolimits_{\substack{\alpha\in\mathcal{R},\\ \alpha(A_i)\geq 0}} \kappa_\alpha,$$

where $A_1, A_2 \in \mathfrak{h}_{\mathbb{R}}$ and, for each $\alpha \in \mathcal{R}$, $\kappa_\alpha = \{Z \in \kappa \mid [A, Z] = \alpha(A)Z,\ \forall A \in \mathfrak{h}_{\mathbb{R}}\}$ is the root space of α.

Take $\epsilon > 0$ so small that $\epsilon \cdot |\alpha(A_2)| < \alpha(A_1)$ if $\alpha(A_1) > 0$. Then

$$\mathfrak{q} = \mathfrak{h} \oplus \sum\nolimits_{\substack{\alpha\in\mathcal{R},\\ \alpha(A_1+\epsilon A_2)>0}} \kappa_\alpha,$$

is parabolic. In fact, if $\mathfrak{L}(\mathfrak{q}_i)$ are the $\mathfrak{h}$-invariant reductive summands of $\mathfrak{q}_i$ and $\mathfrak{n}(\mathfrak{q}_i)$ the ideals of nilpotent elements of their radicals, we have

$$\mathfrak{q} = (\mathfrak{L}(\mathfrak{q}_1) \cap \mathfrak{L}(\mathfrak{q}_2)) \oplus (\mathfrak{L}(\mathfrak{q}_1) \cap \mathfrak{n}(\mathfrak{q}_2)) \oplus \mathfrak{n}(\mathfrak{q}_1). \qquad \square$$

¿From now on we assume that κ is the complexification of its compact real form κ_0. Conjugation in κ will be understood with respect to κ_0 . Using parabolic regularization and Lemma 5.1 we obtain

Proposition 5.2. *If* $(\kappa_0, \mathfrak{v})$ *is* $\mathfrak{n}$*-reductive, then* $\mathfrak{P}(\mathfrak{v})$ *contains a* $\mathfrak{q}$ *having a conjugation-invariant reductive Levi subalgebra.*

Proof. We can take $\mathfrak{q} = (\mathfrak{e} \cap \bar{\mathfrak{e}}) + \mathfrak{n}(\mathfrak{e})$, for the parabolic regularization $\mathfrak{e}$ of $\mathfrak{v}$. $\square$

This shows that, for an $\mathfrak{n}$-reductive $(\kappa_0, \mathfrak{v})$, the set

$$\mathfrak{P}_0(\mathfrak{v}) = \{\mathfrak{q} \in \mathfrak{P}(\mathfrak{v}) \mid \mathfrak{q} = (\mathfrak{q} \cap \bar{\mathfrak{q}}) \oplus \mathfrak{n}(\mathfrak{q})\} \tag{5.3}$$

is nonempty. For $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$ we will use $\mathfrak{L}(\mathfrak{q}) = \mathfrak{q} \cap \bar{\mathfrak{q}}$. The parabolic regularizazion produces a *small* $\mathfrak{e}$ and a corresponding smaller $(\mathfrak{e} \cap \bar{\mathfrak{e}}) \oplus \mathfrak{n}(\mathfrak{e})$ in $\mathfrak{P}_0(\mathfrak{v})$. We are however more interested in the *maximal* elements of $\mathfrak{P}(\mathfrak{v})$. To explain the meaning of maximality, we prove (cf. [1, Proposition 20])

Proposition 5.3. *If* $(\kappa_0, \mathfrak{v})$ *is* $\mathfrak{n}$*-reductive and* $\mathfrak{q}$ *any maximal element of* $\mathfrak{P}_0(\mathfrak{v})$*, then*

$$\mathfrak{q} = \mathrm{Lie}\,(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q})) \quad \textit{and} \quad \mathfrak{n}(\mathfrak{q}) = \sum\nolimits_h \mathrm{ad}^h(\mathfrak{L}(\mathfrak{q}))(\mathfrak{n}(\mathfrak{v})). \tag{5.4}$$

Proof. Let $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$ and denote by $\mathfrak{z}$ the center of $\mathfrak{L}(\mathfrak{q})$. Being invariant under conjugation, it is the complexification of the Lie subalgebra $\mathfrak{z}_0$ of a maximal torus $\mathfrak{t}_0$ of κ_0. Set

$\mathfrak{z}_{\mathbb{R}} = i\mathfrak{z}_0$. Following the construction of Konstant in [18], we consider the set $\mathcal{Z}$ consisting of the nonzero elements ν of the dual $\mathfrak{z}_{\mathbb{R}}^*$ for which

$$\kappa_\nu = \{X \in \kappa \mid [Z, X] = \nu(Z)X, \ \forall Z \in \mathfrak{z}_{\mathbb{R}}\} \neq \{0\}.$$

This set $\mathcal{Z}$ shares many properties of the root system of a semisimple Lie algebra. With the scalar product defined on $\mathfrak{z}_{\mathbb{R}}$ by the restriction of the trace form of a faithful linear representation of κ and the corrisponding dual scalar product on $\mathfrak{z}_{\mathbb{R}}^*$, we have

(*i*) $\quad \nu \in \mathcal{Z} \Longrightarrow -\nu \in \mathcal{Z}, \ \text{and} \ \overline{\kappa}_\nu = \kappa_{-\nu},$

(*ii*) $\quad \nu_1, \nu_2, \nu_1 + \nu_2 \in \mathcal{Z} \Longrightarrow [\kappa_{\nu_1}, \kappa_{\nu_2}] = \kappa_{\nu_1+\nu_2},$

(*iii*) $\quad \nu_1, \nu_2 \in \mathcal{Z}$ and $(\nu_1|\nu_2) > 0 \Longrightarrow \nu_1 - \nu_2 \in \mathcal{Z},$

(*iv*) $\quad \forall \nu \in \mathcal{Z}, \ \kappa_\nu$ is an irreducible $\mathfrak{L}(\mathfrak{q})$-module,

(*v*) $\quad \mathfrak{n}(\mathfrak{q}) = \sum_{\nu>0} \kappa_\nu,$ for some lexicographic order in $\mathcal{Z}$,

(*vi*) $\quad \exists$ a basis $\{\mu_1, \ldots, \mu_\ell\} \subset \mathcal{Z}$ of positive simple roots of $\mathfrak{z}_{\mathbb{R}}^*$.

The Lie subalgebra $\text{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q}))$ is contained in $\mathfrak{q}$ and is a direct sum

$$\text{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q})) = \mathfrak{L}(\mathfrak{q}) \oplus \sum_{\nu\in\mathcal{E}} \kappa_\nu,$$

for a subset $\mathcal{E}$ of $\mathcal{Z}^+ = \{\nu > 0\}$. Assume that there is a positive simple root μ_i which does not belong to $\mathcal{E}$. Since μ_i is simple, $\mathfrak{q}' = \mathfrak{q} \oplus \kappa_{-\mu_i}$ is still a parabolic Lie subalgebra. Let us show that it is an element of $\mathfrak{P}_0(\mathfrak{v})$. We have

$$\mathfrak{q}' = \mathfrak{L}(\mathfrak{q}') \oplus \mathfrak{n}(\mathfrak{q}'), \ \text{ with } \ \mathfrak{L}(\mathfrak{q}') = \mathfrak{L}(\mathfrak{q}) \oplus \kappa_{\mu_i} \oplus \kappa_{-\mu_i} \ \text{ and } \ \mathfrak{n}(\mathfrak{q}') = \sum_{\nu\in(\mathcal{Z}^+\setminus\{\mu_i\})} \kappa_\nu\,.$$

Note that $\mathfrak{L}(\mathfrak{q}') = \mathfrak{q}' \cap \bar{\mathfrak{q}}'$. An element $X \in \mathfrak{n}(\mathfrak{v})$ can be written in a unique way as a sum $X = \sum_{\nu\in\mathcal{E}} X_\nu$ with $X_\nu \in \kappa_\nu$. Then $X \in \mathfrak{n}(\mathfrak{q}')$, because $\mathcal{E} \subset \mathcal{Z}^+ \setminus \{\mu_i\}$. This shows that $\mathfrak{n}(\mathfrak{v}) \subset \mathfrak{n}(\mathfrak{q}')$, i.e that $\mathfrak{q}' \in \mathfrak{P}_0(\mathfrak{v})$. Thus, if $\mathfrak{q}$ is maximal in $\mathfrak{P}_0(\mathfrak{v})$, then $\text{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q}))$ contains all κ_{μ_i} for $i = 1, \ldots, \ell$ and thus is equal to $\mathfrak{q}$, because (*ii*) and the fact that every positive root is a sum o simple positive roots yield that $\text{Lie}(\sum_{i=1}^{\ell} \kappa_{\mu_i}) = \mathfrak{n}(\mathfrak{q})$. Finally, it follows from the discussion above that $\mathfrak{n}(\mathfrak{q})$ is the $\text{ad}(\mathfrak{L}(\mathfrak{q}))$-module generated by $\mathfrak{n}(\mathfrak{v})$. □

Analogously, we obtain

Proposition 5.4. *If* $\mathfrak{q}$ *is any maximal element of* $\mathfrak{P}(\mathfrak{v})$*, then*

$$\mathfrak{q} = \text{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q})), \tag{5.5}$$

for any reductive Levi factor $\mathfrak{L}(\mathfrak{q})$ *of* $\mathfrak{q}$*, and* $\mathfrak{n}(\mathfrak{q})$ *is the* $\text{ad}(\mathfrak{L}(\mathfrak{q}))$*-module generated by* $\mathfrak{n}(\mathfrak{v})$. □

5.2 – *A remark on the* HNR *condition*

Assume that $(\kappa_0, \mathfrak{v})$ is $\mathfrak{n}$-reductive and let $\mathbf{Q}$ be the parabolic subgroup of $\mathbf{K}$ corresponding to a $\mathfrak{q}$ in $\mathfrak{P}_0(\mathfrak{v})$. Let $\mathbf{Q}_n$ be the unipotent radical of $\mathbf{Q}$ and set $\mathbf{V}' = \mathbf{V} \cdot \mathbf{Q}_n$. Then $\mathbf{V}' \cap \overline{\mathbf{V}}' = \mathbf{V} \cap \overline{\mathbf{V}}$ and therefore the minimal $\mathbf{K}_0$ orbits in $M_- = \mathbf{K}/\mathbf{V}$ and $M'_- = \mathbf{K}/\mathbf{V}'$ are diffeomorphic as $\mathbf{K}_0$-homogeneous manifolds: the *CR* algebras $(\kappa_0, \mathfrak{v})$ and $(\kappa_0, \mathfrak{v} + \mathfrak{q}_n)$ define two *CR* structures on the same $M_0 = \mathbf{K}_0/\mathbf{V}_0$, the latter being *stronger* than the first. These are the *CR* structures inherited from the embeddings $M_0 \hookrightarrow M_-$ and $M_0 \hookrightarrow M'_-$. Note that M'_- is the basis of a complex fiber bundle $M_- \to M'_-$, with Stein fibers biholomorphic to $\mathbb{C}^k$ for some nonnegative integer k (cf. [1, Thm.30]). The choice of a maximal $\mathfrak{q}$ in $\mathfrak{P}_0(\mathfrak{v})$ leads to a *minimal* $\mathfrak{v} + \mathfrak{q}_n$, while a minimal $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$ to a *maximal* $\mathfrak{v} + \mathfrak{q}_n$, defining, when $(\kappa_0, \mathfrak{v})$ is not HNR, a maximal $\mathbf{K}_0$-homogeneous *CR* structure on M_0 which is HNR and stronger than the original one.

Example 5.5. *Minimal orbit of* $\mathbf{SU}(2,3)$ *in* $\mathcal{F}_{1,3}(\mathbb{C}^5)$.

We denote by $\mathcal{F}_{1,3}(\mathbb{C}^5)$ the flag manifold consisting of the pairs (ℓ_1, ℓ_3) of a line ℓ_1 and a 3-plane ℓ_3 of $\mathbb{C}^5$ with $0 \in \ell_1 \subset \ell_3$. We fix the Hermitian symmetric form of signature $(2,3)$ in $\mathbb{C}^n$, corresponding to the matrix

$$\begin{pmatrix} \mathrm{I}_2 & \\ & -\mathrm{I}_3 \end{pmatrix},$$

and consider the minimal orbit for the action of the real Lie group $\mathbf{SU}(2,3)$ in $\mathcal{F}_{1,3}(\mathbb{C}^5)$:

$$M_0 = \{(\ell_1, \ell_3) \in \mathcal{F}_{1,3}(\mathbb{C}^5) \mid \ell_1 \subset \ell_3^{\perp} \subset \ell_3\}.$$

Fix on M_0 the base point $\mathrm{p}_0 = (\langle e_1 + e_3\rangle, \langle e_1 + e_3,\ e_2 + e_5,\ e_5\rangle)$. Its stabilizer in $\mathbf{K}$ is

$$\mathbf{V} = \left\{ \begin{pmatrix} \lambda_1 & z_1 & & & \\ 0 & \lambda_2 & & & \\ & & \lambda_1 & 0 & z_1 \\ & & 0 & \lambda_3 & z_2 \\ & & 0 & 0 & \lambda_2 \end{pmatrix} \middle| \lambda_i, z_i \in \mathbb{C},\ \ \lambda_1^2 \cdot \lambda_2^2 \cdot \lambda_3 = 1 \right\},$$

with Lie algebra

$$\mathfrak{v} = \left\{ \begin{pmatrix} \lambda_1 & z_1 & & & \\ 0 & \lambda_2 & & & \\ & & \lambda_1 & 0 & z_1 \\ & & 0 & \lambda_3 & z_2 \\ & & 0 & 0 & \lambda_2 \end{pmatrix} \middle| \lambda_i, z_i \in \mathbb{C},\ \ 2\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \right\}.$$

The normalizer of $\mathfrak{v}_n$ in κ is the parabolic

$$\mathfrak{q} = \left\{ \begin{pmatrix} \lambda_1 & z_1 & & & \\ 0 & \lambda_2 & & & \\ & & \lambda_3 & \alpha_1 & z_2 \\ & & \alpha_2 & \lambda_4 & z_3 \\ & & 0 & 0 & \lambda_5 \end{pmatrix} \middle| \lambda_i, z_i, \alpha_i \in \mathbb{C},\ \ \sum\nolimits_{i=1}^{5} \lambda_i = 0 \right\},$$

which is also a maximal element in $\mathfrak{P}_0(\mathfrak{v})$ and hence $(\mathfrak{s}(\mathfrak{u}(2)\times\mathfrak{u}(3)),\mathfrak{v})$ is not HNR.

The Lie algebra

$$\tilde{\mathfrak{v}} = \mathfrak{v} + \mathfrak{q}_n = \left\{ \begin{pmatrix} \lambda_1 & z_1 & & & \\ 0 & \lambda_2 & & & \\ & & \lambda_1 & 0 & z_2 \\ & & 0 & \lambda_3 & z_3 \\ & & 0 & 0 & \lambda_2 \end{pmatrix} \middle| \lambda_i, z_i \in \mathbb{C},\quad 2(\lambda_1+\lambda_2)+\lambda_3 = 0 \right\}$$

is the Lie algebra of the stabilizer $\tilde{\mathbf{V}}$ in $\mathbf{K} = \mathbf{S}(\mathbf{GL}_2(\mathbb{C})\times\mathbf{GL}_3(\mathbb{C}))$ of $p_0' \in \mathcal{F}_{1,2,4}(\mathbb{C}^5)$ for $p_0' = (\langle e_1+e_3\rangle, \langle e_1+e_3, e_4\rangle, \langle e_1, e_3, e_4, e_2+e_5\rangle)$. This corresponds to the intersection of the $\mathbf{SU}(2,3)$-orbit

$$M_+' = \{(\ell_1,\ell_2,\ell_4)\in\mathcal{F}_{1,2,4}(\mathbb{C}^5) \mid \ell_1 = \ell_2\cap\ell_2^\perp,\ \dim(\ell_4\cap\ell_4^\perp) = 1\}$$

with its Matsuki dual $\mathbf{K}$-orbit M_-'. With $L_2 = \langle e_1, e_2\rangle$ and $L_3 = \langle e_3, e_4, e_5\rangle$, we have

$$M_-' = \left\{ \begin{matrix} (\ell_1,\ell_2,\ell_4) \\ \in \mathcal{F}_{1,2,4}(\mathbb{C}^5) \end{matrix} \middle| \begin{matrix} \dim(\ell_1\cap L_2) = 0,\ \dim(\ell_1\cap L_3) = 0,\ \dim(\ell_2\cap L_2) = 0, \\ \dim(\ell_2\cap L_3) = 1,\ \dim(\ell_4\cap L_2) = 1,\ \dim\ell_4\cap L_3 = 2 \end{matrix} \right\}.$$

This shows that, in this case, the strengthening of the *CR* structure on M_0 corresponds to considering the compact intersection with its Matsuki dual of an intermediate orbit in some complex flag manifold of the same complex semisimple Lie group (in this case of $\mathbf{SL}_5(\mathbb{C})$).

Proposition 5.6. *Assume that* $(\mathfrak{k}_0,\mathfrak{v})$ *is* $\mathfrak{n}$*-reductive. Then, if* $\mathfrak{w}$ *is a complex Lie subalgebra of* $\mathfrak{k}$ *with* $\mathfrak{v}\subseteq\mathfrak{w}\subseteq\mathfrak{v}\oplus\bar{\mathfrak{v}}$*, then also* $(\mathfrak{k}_0,\mathfrak{w})$ *is* $\mathfrak{n}$*-reductive.*

Proof. The reductive Lie group $\mathfrak{k}$ has an invariant nondegenerate bilinear form β, which is real and negative definite on $\mathfrak{k}_0$. We observe that, if the pair $(\mathfrak{k}_0,\mathfrak{v})$ is $\mathfrak{n}$-reductive, then $\mathfrak{v}_n = \mathfrak{v}\cap\mathfrak{v}^\perp$, where $\mathfrak{v}^\perp = \{Z\in\mathfrak{k} \mid \beta(Z,Z') = 0,\ \forall Z'\in\mathfrak{v}\}$, and that $\mathfrak{v}+\bar{\mathfrak{v}}$ has the direct sum decomposition

$$\mathfrak{v}+\bar{\mathfrak{v}} = \mathfrak{v}\oplus\bar{\mathfrak{v}}_n.$$

If $\mathfrak{w}$ is a complex Lie subalgebra with $\mathfrak{v}\subseteq\mathfrak{w}\subseteq\mathfrak{v}+\bar{\mathfrak{v}}$, then $\mathfrak{w} = \mathfrak{v}\oplus(\mathfrak{w}\cap\bar{\mathfrak{v}}_n)$. Since β defines a duality pairing between $\mathfrak{v}_n$ and $\bar{\mathfrak{v}}_n$, we obtain the decomposition

$$\mathfrak{w} = (\mathfrak{w}\cap\bar{\mathfrak{w}})\oplus\mathfrak{w}_n,\quad \text{with}\quad \mathfrak{w}_n = \mathfrak{v}_n\cap(\mathfrak{w}\cap\bar{\mathfrak{v}}_n)^\perp,\quad \mathfrak{w}\cap\bar{\mathfrak{w}} = (\mathfrak{v}\cap\bar{\mathfrak{v}})\oplus(\mathfrak{v}_n\cap\bar{\mathfrak{w}})\oplus(\bar{\mathfrak{v}}_n\cap\mathfrak{w}),$$

showing that also $(\mathfrak{k}_0,\mathfrak{w})$ is $\mathfrak{n}$-reductive. □

Remark 5.7. If $(\mathfrak{k}_0,\mathfrak{v})$ is $\mathfrak{n}$-reductive, then $\mathfrak{v}$ is the Lie algebra of an algebraic Lie subgroup $\mathbf{V}$ of $\mathbf{K}$. This is the content of [1, Thm.26]. In particular, all Lie subalgebras $\mathfrak{w}$ with $\mathfrak{v}\subseteq\mathfrak{w}\subseteq\mathfrak{v}+\bar{\mathfrak{v}}$ are $\mathrm{Lie}(\mathbf{W})$ for an algebraic Lie subgroup $\mathbf{W}$ of $\mathbf{K}$.

Example 5.8. *Minimal orbit of* $\mathbf{SU}(2,3)$ *in* $\mathcal{F}_{1,2}(\mathbb{C}^5)$.

We partly use the notation of Example 5.5. Denote by M_0 the minimal orbit of $\mathbf{SU}(2,3)$ in the flag $\mathcal{F}_{1,2}(\mathbb{C}^5)$ of nested lines and 2-planes.

$$M_0 = \{(\ell_1, \ell_2 \in \mathcal{F}_{1,2}(\mathbb{C}^5) \mid \ell_2 \subset \ell_2^{\perp}\}$$

is a *CR* manifold of type $(3,4)$. It is the total space of a $\mathbb{CP}^1$-bundle on the *CR* manifold M_0' of isotropic 2-planes in the Grassmannian $\mathcal{G}r_2(\mathbb{C}^4)$, which has type $(2,4)$. The stabilizer $\mathbf{V}$ of the base point $p_0 = (\langle e_1 + e_3\rangle, \langle e_1 + e_3, e_2 + e_4\rangle)$, has Lie algebra

$$\mathfrak{v} = \left\{ \begin{pmatrix} \lambda_1 & z_1 & & & \\ 0 & \lambda_2 & & & \\ & & \lambda_1 & z_1 & z_2 \\ & & 0 & \lambda_2 & z_3 \\ & & 0 & 0 & \lambda_3 \end{pmatrix} \middle| \begin{matrix} \lambda_i, z_i \in \mathbb{C} \\ 2\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{matrix} \right\}.$$

The largest $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$ has

$$\mathfrak{q}_n = \left\{ \begin{pmatrix} 0 & z_1 & & & \\ 0 & 0 & & & \\ & & 0 & z_2 & z_3 \\ & & 0 & 0 & z_4 \\ & & 0 & 0 & 0 \end{pmatrix} \middle| z_i \in \mathbb{C} \right\}$$

and hence $(\mathfrak{s}(\mathfrak{su}(2) \times \mathfrak{su}(3)), \mathfrak{v})$ is not HNR. We note however that

$$\mathfrak{w} = \left\{ \begin{pmatrix} \lambda_1 & \zeta_1 & & & \\ \zeta_2 & \lambda_2 & & & \\ & & \lambda_1 & \zeta_1 & z_1 \\ & & \zeta_2 & \lambda_2 & z_2 \\ & & 0 & 0 & \lambda_3 \end{pmatrix} \middle| \begin{matrix} \lambda_i, \zeta_i, z_i \in \mathbb{C} \\ 2\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{matrix} \right\} \subset \mathfrak{v} + \bar{\mathfrak{v}}$$

has a horocyclic $\mathfrak{w}_n$. The orthogonal $\mathfrak{m}_0$ of $\mathfrak{v} + \bar{\mathfrak{v}}$ in $\mathfrak{s}(\mathfrak{p}(2) \times \mathfrak{p}(3))$ is

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} X & & \\ & -X & \\ & & 0 \end{pmatrix} \middle| X \in \mathfrak{p}(2) \right\}$$

and, according to Theorem 4.14 it can be used to describe the typical fiber of the Mostow fibration $M_- \to M_0$ in this case.

5.3 – *Decomposition of unipotent Lie groups*

A *unipotent* Lie group is a connected and simply connected Lie group $\mathbf{N}$ having a nilpotent Lie algebra $\mathfrak{n}$. Then the exponential map $\exp : \mathfrak{n} \to \mathbf{N}$ is an algebraic diffeomorphism and each Lie subalgebra $\mathfrak{e}$ of $\mathfrak{n}$ is the Lie algebra of an analytic closed subgroup $\mathbf{E}$ of $\mathbf{N}$.

Proposition 5.9. *Let* $\mathbf{N}$ *be a unipotent Lie group and* $\mathbf{S}$ *a group of automorphisms of its Lie algebra* $\mathfrak{n}$, *which acts on* $\mathfrak{n}$ *in a completely reducible way. If* $\mathbf{E}$ *a Lie subgroup of* $\mathbf{N}$ *with an* $\mathbf{S}$*-invariant Lie algebra* $\mathfrak{e}$, *then we can find an* $\mathbf{S}$*-invariant linear complement* $\mathfrak{l}$ *of* $\mathfrak{e}$ *in* $\mathfrak{n}$ *such that*

$$\mathfrak{l} \times \mathbf{E} \ni (X, x) \longrightarrow \exp(X) \cdot x \in \mathbf{N} \tag{5.6}$$

is a diffeomorphism onto.

Proof. We argue by recurrence on the sum of the dimension n of $\mathfrak{n}$ and the codimension k of $\mathfrak{e}$ in $\mathfrak{n}$. The statement is indeed trivial when $n = 1$, or $k = 0$. If $k = 1$, then $\mathfrak{e}$ is an ideal in $\mathfrak{n}$ and has a 1-dimensional $\mathbf{S}$-invariant complement $\mathfrak{l}$ in $\mathfrak{n}$. Since $\mathfrak{l}$ is a Lie subalgebra, using e.g. [30, Lemma 3.18.5] we conclude that (5.6) is a diffeomorphism in this case.

Assume now that $k > 1$ and that the statement has already been proved for subalgebras $\mathfrak{e}$ of codimension lesser than k or nilpotent Lie algebras $\mathfrak{n}$ of dimension lesser than n. Since $\mathfrak{n}$ is nilpotent, its center $\mathfrak{c}$ has positive dimension and is $\mathbf{S}$-invariant. If $\mathfrak{c} \cap \mathfrak{e} \neq \{0\}$, then $\mathbf{A} = \exp(\mathfrak{c} \cap \mathfrak{e})$ is a nontrivial normal subgroup of $\mathbf{N}$. Since $\dim(\mathbf{N}/\mathbf{A}) < n$ and $\mathbf{S}$ acts in a completely reducible way on $\mathfrak{n}/(\mathfrak{c} \cap \mathfrak{e})$, by the recursive assumption we can find an $\mathbf{S}$-invariant linear complement $\mathfrak{l}$ of $\mathfrak{e}$ in $\mathfrak{n}$ such that, for its projection $\mathfrak{l}'$ in $\mathfrak{n}/(\mathfrak{c} \cap \mathfrak{e})$, the map

$$f' : \mathfrak{l}' \times (\mathbf{E}/\mathbf{A}) \ni (X', x') \longrightarrow \exp(X') \cdot x' \in \mathbf{N}/\mathbf{A}$$

is a diffeomorphism. This implies that (5.6) is also a diffeomorphism. In fact, if $\zeta \in \mathbf{N}$, by the surjectivity of f' there is a pair $(X, y) \in \mathfrak{l} \times \mathbf{E}$ such that $\exp(X) \cdot y = \zeta \cdot a$, for some $a \in \mathbf{A}$. This shows that $\zeta = \exp(X) \cdot (y \cdot a^{-1})$ and therefore (5.6) is onto. If $\zeta = \exp(X_1) \cdot (x_1) = \exp(X_2) \cdot (x_2) \cdot a$, with $X_1, X_2 \in \mathfrak{l}$, $x_1, x_2 \in \mathbf{E}$ and $a \in \mathbf{A}$, then $X_1 = X_2 = X$ because the projection $\mathfrak{l} \to \mathfrak{l}'$ is a linear isomorphism. Moreover, the correspondence $\zeta \to X$ is C^∞-smooth, because f'^{-1} is smooth. Then $\zeta \to x = \exp(-X) \cdot \zeta \in \mathbf{E}$ is also smooth, and $\zeta \to (X, \exp(-X)\zeta)$ yields a smooth inverse of (5.6).

If $\mathfrak{c} \cap \mathfrak{e} = \{0\}$, then by the recurrence assumption, we can take an $\mathbf{S}$-invariant linear complement $\mathfrak{l}$ of $\mathfrak{e}$ in $\mathfrak{n}$ containing $\mathfrak{c}$ and such that

$$f' : (\mathfrak{l}/\mathfrak{c}) \times ((\mathbf{E} \cdot \mathbf{C})/\mathbf{C}) \ni (X', x') \longrightarrow \exp(X') \cdot x' \in \mathbf{N}/\mathbf{C}.$$

is a diffeomorphism. We claim that, with this choice, (5.6) is a diffeomorphism. Indeed, $(\mathbf{E} \cdot \mathbf{C})/\mathbf{C} \simeq \mathbf{E}$ and therefore for $\zeta \in \mathbf{N}$ there is a unique $x \in \mathbf{E}$, with $x = \phi(\zeta)$ for a smooth function $\phi : \mathbf{N} \to \mathbf{E}$, such that, for some $Z \in \mathfrak{c}$ and $Y \in \mathfrak{l}$,

$$\zeta \cdot \exp(Z) = \exp(Y) \cdot x \Rightarrow \zeta = \exp(Y - Z) \cdot x.$$

The exponential is a diffeomorphism of $\mathfrak{n}$ onto $\mathbf{N}$. If we denote by $\log : \mathbf{N} \to \mathfrak{n}$ its inverse, we obtain $X = Y - Z = \log(\zeta \cdot x^{-1}) \in \mathfrak{l}$ and

$$\mathbf{N} \ni \zeta \to \left(\log(\zeta \cdot [\phi(\zeta)]^{-1}), \phi(\zeta)\right) \in \mathfrak{l} \times \mathbf{E}$$

is a smooth inverse of (5.6). This completes the proof. □

With the notation of the previous section, we will apply Proposition 5.9 to the case where $\mathbf{N} = \mathbf{Q}_n$ and $\mathfrak{n} = \mathfrak{q}_n$, for a minimal $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{w})$, while $\mathfrak{e} = \mathfrak{v}_n$ and $\mathbf{S} = \mathrm{Ad}(\mathbf{V}_0)$. Since $\mathbf{V}_0$ is compact, its adjoint action on $\mathfrak{q}_n$ is completely reducible.

5.4 – *Structure of the typical fiber*

The quotient $\mathbf{K}/\mathbf{Q}$ of $\mathbf{K}$ by a parabolic subgroup $\mathbf{Q}$ is compact and thus a homogeneous space of its compact form $\mathbf{K}_0$. Thus

$$\mathbf{K} = \mathbf{K}_0 \cdot \mathbf{Q}. \tag{5.7}$$

Set $\kappa = \mathrm{Lie}(\mathbf{K})$, $\mathfrak{q} = \mathrm{Lie}(\mathbf{Q})$, and choose $\mathbf{K}_0$ to contain a maximal compact subgroup of $\mathbf{Q}$. Then $\mathbf{Q}$ has a Levi-Chevalley decomposition $\mathbf{Q} = \mathbf{L}(\mathbf{Q}) \cdot \mathbf{Q}_n$, whose reductive factor $\mathbf{L}(\mathbf{Q})$ has Lie algebra $\mathfrak{L}(\mathfrak{q}) = \mathfrak{q} \cap \bar{\mathfrak{q}}$. The conjugation is taken with respect to the real compact form κ_0 and $\mathbf{Q}_n$ is the unipotent factor of $\mathbf{Q}$, with Lie algebra $\mathfrak{q}_n$. We consider the Cartan decomposition $\kappa = \kappa_0 \oplus \mathfrak{p}_0$, with $\mathfrak{p}_0 = i \cdot \kappa_0$. Using the Cartan decomposition of $\mathbf{L}(\mathbf{Q})$, we obtain the direct product decomposition

$$\mathbf{Q} = \mathbf{L}(\mathbf{Q}) \cdot \exp(\mathfrak{n}(\mathfrak{q})) = \mathbf{L}_0(\mathbf{Q}) \cdot \exp(\mathfrak{p}_0 \cap \mathfrak{q}) \cdot \exp(\mathfrak{n}(\mathfrak{q})). \tag{5.8}$$

We keep the notation of the previous sections, with $\mathfrak{w}$ the maximal complex Lie subalgebra with $\mathfrak{v} \subseteq \mathfrak{w} \subseteq \mathfrak{v} + \bar{\mathfrak{v}}$ and take $\mathfrak{q}$ in $\mathfrak{P}_0(\mathfrak{w})$. Then $\mathfrak{e} = \mathfrak{v} + \mathfrak{q}_n$ is a Lie subalgebra of κ and the pair $(\kappa_0, \mathfrak{e})$ has the HNR property. Set

$$\mathfrak{f}_0 = \mathfrak{p}_0 \cap (\mathfrak{v} + \mathfrak{q}_n)^{\perp}. \tag{5.9}$$

By (4.21), we obtain the direct product decomposition

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{v}_n + \mathfrak{q}_n) \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0). \tag{5.10}$$

We use Proposition 5.9 to decompose $\exp(\mathfrak{v}_n + \mathfrak{q}_n)$: we can find an $\mathrm{Ad}(\mathbf{V}_0)$-invariant linear subspace $\mathfrak{l}$ of $(\mathfrak{v}_n + \mathfrak{q}_n)$ such that $\mathfrak{v}_n + \mathfrak{q}_n = \mathfrak{l} \oplus \mathfrak{v}_n$ and

$$\mathfrak{l} \oplus \mathfrak{v}_n \ni (X, Y) \longrightarrow \exp(X) \cdot \exp(Y) \in \mathbf{V}_n \cdot \mathbf{Q}_n = \exp(\mathfrak{v}_n + \mathfrak{q}_n) \tag{5.11}$$

is a diffeomorphism. We obtained:

THEOREM 5.10. *Let* $\mathfrak{f}_0$ *and* $\mathfrak{l}$ *be defined by* (5.9) *and* (5.11). *Then we have a direct product decomposition*

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{l}) \cdot \mathbf{V}', \tag{5.12}$$

where $\mathbf{V}' = \exp(\mathfrak{v}_n) \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0)$.

Then $\mathbf{F}_0 = \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{l})$, *with the adjoint action of* $\mathbf{V}_0$, *is the typical fiber of the Mostow fibration:*

$$M_- \simeq \mathbf{K}/\mathbf{V} \simeq \mathbf{K}_0 \times_{\mathbf{V}_0} \mathbf{F}_0. \tag{5.13}$$ □

LEMMA 5.11. *If* $\mathbf{N}$ *is a unipotent subgoup of* $\mathbf{K}$, *then, for every* $p \in \mathcal{P}_0(n)$, *the map*

$$\mathbf{N} \ni z \longrightarrow z^* p z \in N_p = \{z^* p z \mid z \in \mathbf{N}\} \tag{5.14}$$

is a diffeomorphism.

Proof. In fact the stabilizer $\mathbf{Stab}(p)$ of p for the right action

$$\mathbf{K} \times \mathcal{P}_0(\kappa) \ni (z, x) \to z^* \cdot x \cdot z \in \mathcal{P}_0(\kappa)$$

of $\mathbf{K}$ on $\mathcal{P}_0(\kappa)$ is a compact group and hence has trivial intersection with $\mathbf{N}$. Thus (5.14) is a diffeomorphism with the image, being the restriction to $\mathbf{N} \simeq \mathbf{N}/\{e_{\mathbf{K}}\}$ of the diffeomorphism $\mathbf{K}/\mathbf{Stab}(p) \to \mathcal{P}_0(\kappa)$. □

Corollary 5.12. *Fix* $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{w})$ *and let* $\mathfrak{k}_0$ *and* $\mathfrak{l}$ *be the corresponding subspaces of* κ *of Theorem 5.10. Then the elements* $X \in \mathfrak{k}_0$ *and* $Z \in \mathfrak{l}$ *of the decomposition*

$$\zeta = u \cdot \exp(X) \cdot \exp(Z) \cdot v, \quad \textit{with} \quad u \in \mathbf{K}_0,\ v \in \exp(\mathfrak{v}_n) \cdot \exp(v \cap \mathfrak{p}_0)$$

are obtained in the following way:

(a) $[0, 1] \ni t \to \exp(2tX)$ *is the geodesic in* $\mathcal{P}_0(\kappa)$ *joining* $e_{\mathbf{K}}$ *to the unique point* p_0 *of* $\tilde{N}_{\zeta^*\cdot\zeta} = \{z^* \cdot \zeta^* \cdot \zeta \cdot z \mid z \in \mathbf{V} \cdot \mathbf{Q}_n\}$ *at minimal distance from* $e_{\mathbf{K}}$;

(b) Z *is the unique element of* $\mathfrak{l}$ *such that* $\exp(Z^*)\cdot p_0 \cdot\exp(Z)$ *belongs to* $N_{p_0} = \{z^* \cdot p_0 \cdot z \mid z \in \mathbf{V}\}$.

Proof. Indeed the Mostow fibration of $M'_- = \mathbf{K}/(\mathbf{V} \cdot \mathbf{Q}_n)$ can be taken to have a *hermitian* typical fiber $\exp(\mathfrak{k}_0)$ and correspondingly we obtain a unique decomposition

$$\zeta = u \cdot \exp(X) \cdot \xi \cdot \exp(Y) \quad \text{with} \quad \xi \in \mathbf{Q}_n \text{ and } Y \in \mathfrak{v} \cap \mathfrak{p}_0,$$

The characterization of X coming from the proof of Proposition 4.5 yields (a).

Next we consider $p_\xi = \xi^* \cdot \exp(2X) \cdot \xi = \xi^* \cdot p_0 \cdot \xi$. By Lemma 5.11 and the choice of $\mathfrak{l}$ we know that the element p_ξ of $\{z^* \cdot p_0 \cdot z \mid z \in \mathbf{Q}_n\}$ uniquely decomposes as a product $w^* \cdot \exp(Z^*) \cdot p_0 \cdot \exp(Z) \cdot w$ with $w \in \mathbf{V}_n$ and $Z \in \mathfrak{l}$. This completes the proof. □

6. Application to Dolbeault and *CR* cohomologies

The cohomology groups of the tangential Cauchy-Riemann complex on real-analytic forms on M_0 is the inductive limit of the corresponding Dolbeault cohomology groups of its tubular neighborhoods in M_-. We know by [13] that in some degrees these groups coincide with those computed on tangential smooth forms or on currents. We will employ Andreotti-Grauert theory to compare the tangential *CR* cohomology on M_0 with the corresponding *global* Dolbeault cohomology of M_-. To this aim we will use the Mostow fibration $M_- \to M_0$ to construct a non negative exhaustion fuction for M_-, vanishing on M_0, and having a complex Hessian whose signature reflects the pseudoconvexity/pseudoconcavity of M_0. In this way we prove relations of the *CR* cohomology of M_0 with the Dolbeault cohomololy of the $\mathbf{K}$-orbit M_-, similar to what J.A.Wolf did in [27] for the relationship of the open orbits M_+ of a real form $\mathbf{G}_0$ of a complex semisimple Lie group $\mathbf{G}$ in a flag M of $\mathbf{G}$ with the structure of their Matsuki duals $M_- = M_0$, which in this case are compact complex manifolds.

6.1 – *An Exhaustion Function for M_-*

In [12] H. Grauert noticed that a real-analytic manifold admits a fundamental systems of Stein tubular neighborhoods in any of its complexifications. In fact, a homogeneous analogue of Grauert's theorem is the fact that the complexification $\mathbf{K}$ of a compact Lie group $\mathbf{K}_0$ is Stein, and the isomorphism provided by the Cartan decomposition

$$\mathbf{K}_0 \times \mathfrak{k}_0 \ni (x, X) \longrightarrow x \cdot \exp(iX) \in \mathbf{K}$$

also yields the exhaustion function

$$\mathbf{K} \ni x \cdot \exp(iX) \longrightarrow \|X\|^2 = -\mathcal{k}(X, X) \in \mathbb{R},$$

which is zero on $\mathbf{K}_0$, positive on $\mathbf{K} \setminus \mathbf{K}_0$ and strictly pseudo-convex everywhere. Here and in the following we shall denote by $\mathcal{k}$ both the negative definite invariant form of a faithfull unitary representation of $\mathfrak{k}_0$ and its $\mathbb{C}$-bilinear extension to κ. When κ_0 is semisimple, the adjoint representation is faithful and we may take as $\mathcal{k}$ the Killing form.

We proceed in a similar way to construct an exhaustion function on M_- for the canonical embedding $M_0 \hookrightarrow M_-$ of a $\mathfrak{n}$-reductive $\mathbf{K}_0$-homogeneous compact *CR* manifold M_0. We use the notation of the previous sections.

Assume that the pair $(\kappa_0, \mathfrak{v})$ is $\mathfrak{n}$-reductive and HNR. We already noticed that the last condition is *natural* if we consider on M_0 *maximal* $\mathbf{K}_0$-invariant *CR* structures. Then, by Corollary 5.12, we have a direct product decomposition

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{v}_n) \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0) \tag{6.1}$$

with $\mathfrak{p}_0 = i\kappa_0$ and $\mathfrak{f}_0 = (\mathfrak{v}+\bar{\mathfrak{v}})^\perp \cap \mathfrak{p}_0$. Moreover, the $\exp(\mathfrak{f}_0)$-term in (6.1) is characterized by

$$\begin{cases} \text{if } \zeta = u \cdot \exp(X) \cdot v, \quad \text{with} \quad u \in \mathbf{K}_0, X \in \mathfrak{f}_0 \text{ and } v \in \mathbf{V}, \text{then} \\ \|X\| = \frac{1}{2}\operatorname{dist}(\zeta^*\zeta, N), \quad \text{for} \quad N = \{v^* \cdot v \mid v \in \mathbf{V}\} \subset \mathcal{P}_0(\kappa). \end{cases} \tag{6.2}$$

This is indeed a consequence of Corollary 5.12 when $\mathfrak{l} = \{0\}$.

By passing to the quotient, the map

$$\mathbf{K}_0 \times \mathfrak{f}_0 \ni (x, X) \longrightarrow \|X\|^2 = \mathcal{k}(X, X) \in \mathbb{R}.$$

defines a smooth exhaustion function (as usual square brackets mean equivalence classes)

$$\phi : M_- \simeq \mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{f}_0 \ni [x, X] \longrightarrow \|X\|^2 \in \mathbb{R}\,. \tag{6.3}$$

We have:

Lemma 6.1. *If* $(\kappa_0, \mathfrak{v})$ *is* $\mathfrak{n}$*-reductive and* HNR*, then the map* ϕ *of* (6.3) *has the properties:*

(1) $\phi \in C^\infty(M_-, \mathbb{R})$ *and* $\phi \geq 0$ *on* M_-*;*

(2) $\phi^{-1}(0) = M_0$ *and* $d\phi \neq 0$ *if* $\phi > 0$ *;*

(3) ϕ *is invariant under the left action of* $\mathbf{K}_0$ *on* M_- *:*

$$\phi(x \cdot p) = \phi(p), \quad \forall p \in M_-, \quad \forall x \in \mathbf{K}_0. \qquad \square$$

Notation 6.1. The *level and sublevel sets* of ϕ will be denoted by

$$\Phi_c = \{p \in M_- \mid \phi(p) = c\} \Subset M_- \quad \text{and} \quad \Omega_c = \{p \in M_- \mid \phi(p) < c\}. \tag{6.4}$$

6.2 – $\mathbf{K}_0$-*Orbits in* M_-

The level sets Φ_c are foliated by $\mathbf{K}_0$-orbits. Since all points of M_- have representatives of the form $x \cdot \exp(X)$ with $x \in \mathbf{K}_0$ and $X \in \mathfrak{k}_0$, then every $\mathbf{K}_0$-orbit intersects the fiber F_0 over the base point p_0 at a point $p_X = [\exp(X)]$, for some $X \in \mathfrak{k}_0$. An $x \in \mathbf{K}_0$ stabilizes p_X if and only if $x \cdot \exp(X)$ is still a representative of p_X, and this, by the equivalence relation defining $\mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{k}_0$, means that $x \in \mathbf{V}_0$ and $\mathrm{Ad}(x)(X) = X$. Indeed the equation $x\exp(X)z = \exp(X)$ with $z \in \mathbf{V}$ implies, by the uniqueness of the Mostow decomposition, that $z = x^{-1} \in \mathbf{V}_0$ and $x\exp(X)x^{-1} = \exp(\mathrm{Ad}(x)(X)) = \exp(X)$, yielding $\mathrm{Ad}(x)(X) = X$.

Thus the $\mathbf{K}_0$-orbit

$$M_X = \{x \cdot p_X = [x \cdot \exp(X)] \mid x \in \mathbf{K}_0\} \tag{6.5}$$

in M_- through p_X can be identified with the homogeneous space $\mathbf{K}_0/\mathbf{V}_X$, where

$$\mathbf{V}_X = \{x \in \mathbf{V}_0 \mid \mathrm{Ad}(x)(X) = X\},$$

is the stabilizer of p_X in $\mathbf{K}_0$. It is a closed Lie subgroup of $\mathbf{K}_0$ with Lie algebra

$$\mathfrak{v}_X = \{Y \in \mathfrak{v}_0 \mid [Y, X] = 0\}.$$

Lemma 6.2. *M_X is a compact $\mathbf{K}_0$-homogeneous CR-manifold with CR-algebra* $(\kappa_0, \mathrm{Ad}(\exp(X))(\mathfrak{v}))$ *at* $p_X = [\exp(X)]$. $\square$

Remark 6.3. In general, M_X may not be diffeomorphic to M_0. Indeed, M_0 is a *minimal* $\mathbf{K}_0$-orbit in M_- and M_X is diffeomorphic (and *CR*-diffeomorphic) to M_0 if and only if M_X and M_0 have the same dimension.

For $X \in \mathfrak{k}_0$, the left translation $M_- \ni p \longrightarrow \exp(X) \cdot p \in M_-$ is a biholomorphism of M_- which transforms M_0 onto a *CR*-diffeomorphic submanifold

$$\tilde{M}_X = \exp(X) \cdot M_0. \tag{6.6}$$

Lemma 6.4. *For* $X \in \mathfrak{k}_0$, *we have*

$$\tilde{M}_X \subset \{\phi \leq \|X\|^2\} = \overline{\Omega}_{\|X\|^2}. \tag{6.7}$$

Proof. Let $\pi : \mathbf{K} \ni \zeta \to [\zeta] \in \mathbf{K}/\mathbf{V} \simeq M_-$ be the canonical projection. Any point of M_0 is $\pi(u)$ for some $u \in \mathbf{K}_0$ and then the points p of $\tilde{M}_X$ have the form $p = \exp(X)\pi(u) = \pi(\exp(X)\cdot u)$. Set $\zeta = \exp(X)\cdot u$. We know that $\phi(p)$ is the square of the half-distance in $\mathcal{P}_0(\mathbf{K})$ from the base point $e_{\mathbf{K}}$ to

$$N_{\zeta^*\zeta} = \{v^*\cdot\zeta^*\cdot\zeta\cdot v \mid v \in \mathbf{V}\}.$$

Since the point $(\zeta^*\zeta)$ belongs to $N_{\zeta^*\cdot\zeta}$ and has distance $2\|X\|$ from $e_{\mathbf{K}}$, (in fact $t \to u^*\cdot\exp(2tX)\cdot u$ is the geodesic joining $e_{\mathbf{K}}$ to $(\zeta^*\cdot\zeta)$), it follows that $\phi(p) \leq \|X\|^2$. □

We summarize:

Proposition 6.5. *Let $c > 0$. Then*

$$\Phi_c = \bigcup_{\substack{X\in\mathfrak{k}_0,\\ \|X\|^2=c}} M_X \ \textit{(disjoint union)}, \quad \tilde{M}_X \subset \{\phi \leq \|X\|^2\}, \quad \forall X \in \mathfrak{k}_0 . \tag{6.8}$$ □

In particular, for $c > 0$, we can draw through each point of Φ_c a translate $\tilde{M}_X$ of M_0, which is CR-diffeomorphic to M_0 and tangent to Φ_c *from inside*, i.e. lying in $\overline{\Omega}_c$. This means that the boundary U_c of Ω_c is at each point *less convex* than M_0.

6.3 – *Application to Dolbeault and CR Cohomologies I*

By Andreotti-Grauert theory (see [4]) we know that for every coherent sheaf $\mathcal{F}$ on an r-pseudoncave complex manifold X we have

$$\mathbf{H}^j(\mathrm{X}, \mathcal{F}) < \infty, \quad \forall < r - \mathrm{hd}(\mathcal{F}),$$

where $\mathrm{hd}(\mathcal{F})$ is the homological dimension of $\mathcal{F}$.

We obtain the following:

Theorem 6.6. *Let M_0 be a compact $\mathfrak{n}$-reductive homogeneous CR manifold, with* $(\kappa_0, \mathfrak{v})$ HNR *and canonical complex embedding $M_0 \hookrightarrow M_-$.*

If M_0 is an r-psudoconvave CR-manifold, then M_- is an r-pseudoconcave complex manifold and for every coherent sheaf $\mathcal{F}$ we have

$$\dim(\mathbf{H}^j(M_0, \mathcal{F}) \simeq \mathbf{H}^j(M_-, \mathcal{F})) < \infty, \quad \forall j < r - \mathrm{hd}(\mathcal{F}). \tag{6.9}$$

In particular,

$$\dim(\mathbf{H}^{p,j}(M_0) \simeq \mathbf{H}^{p,j}(M_-)) < \infty, \quad \forall j < r. \tag{6.10}$$

Here we used the notation $\mathbf{H}^{p,j}$ for the $\bar{\partial}$ and $\bar{\partial}_{M_0}$-cohomologies on forms of type $(p, *)$. Because of the validity of the Poincaré lemma in degree j, for $0 < j < r$ (see [25]), they coincide with the Čech cohomology with coefficients in the sheaf of germs of *CR* or holomorphic p-forms. Moreover, in this range, the tangential Cauchy-Riemann complexes on currents, C^∞-smooth forms and real-analytic forms on M_0 have isomorphic finite dimensional cohomology groups.

Proof. By the *HNR* assumption, the exhaustion function ϕ in (6.3) is well defined. Then to verify (6.9) we can apply Andreotti-Grauert's theory, after showing that, for $c > 0$, each subdomain $\Omega_c = \{\phi < c\}$ is r-pseudoconcave. To this aim, we prove that the complex Hessian of ϕ admits at least r negative eigenvalues on the analytic tangent to $\Phi_c = \partial\Omega_c$. By exploiting the $\mathbf{K}_0$-invariance of ϕ, we can, without any loss of generality, restrict our consideration to points $p_0 = [\exp(X)] \in \Phi_c$, with $\|X\|^2 = c \in \mathbb{R}$. We may consider $(0,1)$-vector fields which are tangent to the submanifold $\tilde{M}_X$, defined in (6.6) and that are also tangent to $\partial\Omega_c$ at p_0, because $\tilde{M}_X$ is tangent to Φ_c at p_0. By Lemma 6.4, $\tilde{M}_X$ is contained in $\overline{\Omega}_c = \{\phi \leq \|X\|^2\}$. Since $\tilde{M}_X$ is *CR*-diffeomorphic to M_0, it is r-pseudoconcave. Being $\tilde{M}_X \subset \overline{\Omega}_c$, the restriction of the complex Hessian of ϕ to the analytic tangent to $\tilde{M}_X$ at p_0 has at least as many negative eigenvalues as the Levi form of $\tilde{M}_X$ in the codirection $Jd\phi([\exp(X)])$, which, by the assumption, are at least r. This completes the proof. □

6.4 – *Application to Dolbeault and CR cohomologies II*

In this section we want to exploit the amount of pseudo-convexity of the exhaustion function ϕ. We keep the assumption that $(\kappa_0, \mathfrak{v})$ is $\mathfrak{n}$-reductive and HNR and set $\mathfrak{q} = \{Z \in \kappa \mid [Z, \mathfrak{v}_n] \subset \mathfrak{v}_n\}$ for the maximal parabolic subalgebra in $\mathfrak{P}_0(\mathfrak{v})$. We recall that $\mathfrak{v}_n = \mathfrak{q}_n$ is the nilradical of $\mathfrak{q}$. Let $\mathbf{Q}$ be the parabolic subgroup of $\mathbf{K}$ with $\mathrm{Lie}(\mathbf{Q}) = \mathfrak{q}$ and $\mathbf{Q}_r$ its conjugation-invariant reductive factor. Let $\varpi : \mathbf{K} \to M_- = \mathbf{K}/\mathbf{V}$ be the quotient map. The image of $\mathbf{Q}_r$ by ϖ is a $\mathbf{Q}_r$-homogeneous complex submanifold Q_- of M_-.

Lemma 6.7. *For every $X \in \mathfrak{k}_0$, the CR manifold $\tilde{M}_X$ and the complex manifold Q_- are transversal at p_X and their analytic tangent spaces at p_X are orthogonal for the complex Hessian of* ϕ.

Proof. The pull-backs of $T^{0,1}_{p_X}\tilde{M}_X$ and $T^{0,1}_{p_X}Q_-$ to the base point p_0 by the bi-holomorphic map $p \to \exp(X) \cdot p$ are, respectively, $\mathfrak{v}_n$ and $\mathfrak{q}_r/(\mathfrak{v} \cap \bar{\mathfrak{v}})$. This is a consequence of the fact that $X \in \mathfrak{q}_r$. The statement follows from the fact that $\mathfrak{q}_r = \bar{\mathfrak{q}}_r$ and $[\mathfrak{q}_r, \mathfrak{v}_n] \subset \mathfrak{v}_n$, $[\mathfrak{q}_r, \bar{\mathfrak{v}}_n] \subset \bar{\mathfrak{v}}_n$. □

Theorem 6.8. *Let M_0 be a compact $\mathfrak{n}$-reductive homogeneous CR manifold of type (n,k), with* $(\kappa_0, \mathfrak{v})$ HNR *and canonical complex embedding $M_0 \hookrightarrow M_-$.*

If M_0 is an r-psudoconvave CR-manifold, then M_- is $n-r$-pseudoconvex complex manifold and for every coherent sheaf $\mathcal{F}$ we have

$$\dim\left(\mathbf{H}^j(M_0, \mathcal{F}) \simeq \mathbf{H}^j(M_-, \mathcal{F})\right) < \infty, \quad \forall j > n-r. \tag{6.11}$$

In particular,

$$\dim\left(\mathbf{H}^{p,j}(M_0) \simeq \mathbf{H}^{p,j}(M_-)\right) < \infty, \quad \forall j > n-r. \tag{6.12}$$

Proof. By [13, Theorem 2.1], under the r-pseudoconcavity assumption, the tangential *CR* cohomology groups on M_0 are the inductive limits of the corresponding groups of sheaf and Dolbeault cohomology of the tubular neighborhoods of M_0 in M_-. While computing the Levi form of ϕ, it suffices to note that its restriction to Q_- is strictly pseudo-convex, since it is the exhaustion function associated to the canonical *CR*-embedding $M_0 \cap N_- \hookrightarrow N_-$ of a totally real $(\mathbf{K}_0 \cap \mathbf{Q}_r)$-homogeneous manifold. Indeed, by [5, Theorem 4.1], the distance from the totally geodesic submanifold $N' = \{\zeta^*\zeta \mid \zeta \in \mathbf{V} \cap \mathbf{Q}_r\}$ in the negatively curved space $\mathcal{M}' = \mathbf{Q}_r/(\mathbf{Q}_r \cap \mathbf{K}_0)$ is strictly convex on $\mathcal{M}' \setminus N'$, and $\phi|_{Q_-}$ pulls back on $\mathbf{Q}_r$ to the composition of $\zeta \to \zeta^*\zeta$ with the square of the distance from N'.

Hence, for $X \neq 0$, the complex Hessian of ϕ restricts to a Hermitian symmetric form having, by Lemma 6.7, at least $r + k - 1$ positive eigenvalues on the analytic tangent of Φ_c at p_X.

The thesis is then a consequence of the isomorphisms proved in [4, §20]. □

Example 6.9. Fix integers $1 \leq p < q \leq n$ and consider the *real* action of $\mathbf{SL}_{n+1}(\mathbb{C})$ on the Cartesian product $\mathcal{G}r_p(\mathbb{C}^{n+1}) \times \mathcal{G}r_q(\mathbb{C}^{n+1})$ of the Grassmannians of p and q planes, described by

$$a \cdot (\ell_p, \ell_q) = (a(\ell_p), \bar{a}(\ell_q)), \quad \forall\, a \in \mathbf{SL}_{n+1}(\mathbb{C}),\ \ell_p \in \mathcal{G}r_p(\mathbb{C}^{n+1}),\ \ell_q \in \mathcal{G}r_q(\mathbb{C}^{n+1}).$$

The orbits of the real form $\mathbf{G}_0 = \mathbf{SL}_{n+1}(\mathbb{C})$ are parametrized by the dimension of the intersection $\ell_p \cap \overline{\ell}_q$: with $k_0 = \max\{0, p + q - n - 1\}$ we have the orbits

$$M_+(k) = \{(\ell_p, \ell_q) \in \mathcal{G}r_p(\mathbb{C}^{n+1}) \times \mathcal{G}r_q(\mathbb{C}^{n+1}) \mid \dim_{\mathbb{C}}(\ell_p \cap \overline{\ell}_q) = k\}, \quad k_0 \leq k \leq p.$$

The complexification $\mathbf{K} = \mathbf{SL}_{n+1}(\mathbb{C})$ of the compact form $\mathbf{K}_0 = \mathbf{SU}(n + 1)$ acts on $\mathcal{G}r_p(\mathbb{C}^{n+1}) \times \mathcal{G}r_q(\mathbb{C}^{n+1})$ by

$$a \cdot (\ell_p, \ell_q) = (a(\ell_p), {}^T a^{-1}(\ell_q)), \quad \forall a \in \mathbf{SL}_{n+1}(\mathbb{C}),\ \ell_p \in \mathcal{G}r_p(\mathbb{C}^{n+1}),\ \ell_q \in \mathcal{G}r_q(\mathbb{C}^{n+1}).$$

Consider the polarity $\mathcal{G}r_h(\mathbb{C}^{n+1}) \ni \ell_h \to \ell_h^0 \in \mathcal{G}r_{n+1-h}(\mathbb{C}^{n+1})$ defined by the symmetric bilinear form

$$b(v, w) = ({}^T w) \cdot v = \textstyle\sum_{i=0}^n v_i w_i.$$

Then the orbits of $\mathbf{K}$ in $\mathcal{G}r_p(\mathbb{C}^{n+1}) \times \mathcal{G}r_q(\mathbb{C}^{n+1})$ are parametrized by:

$$M_-(k) = \{(\ell_p, \ell_q) \in \mathcal{G}r_p(\mathbb{C}^{n+1}) \times \mathcal{G}r_q(\mathbb{C}^{n+1}) \mid \dim_{\mathbb{C}}(\ell_p \cap \ell_q^0) = p - k\}, \quad k_0 \leq k \leq p.$$

The manifolds $M_+(k)$ and $M_-(k)$ are Matsuki-dual to each other. In fact, since $\mathbf{SU}(n + 1)$ preserves Hermitian orthogonality in $\mathbb{C}^{n+1}$ and $\overline{\ell}_q$ and ℓ_q^0 are Hermitian orthogonal in $\mathbb{C}^{n+1}$, the pair (ℓ_p, ℓ_q) belongs to $M_0(k) = M_+(k) \cap M_-(k)$ iff

$$\ell_p = (\ell_p \cap \overline{\ell}_q) \oplus (\ell_p \cap \ell_q^0), \quad \text{and either } \dim(\ell_p \cap \overline{\ell}_q) = k, \text{ or } \dim(\ell_p \cap \ell_q^0) = p - k.$$

Set $n_1 = p - k$, $n_2 = k$, $n_3 = n + 1 + k - p - q$, $n_4 = q - k$. Then, taking as base point, with obvious notation, $p_0 = (\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}, \mathbb{C}^{n_2} \oplus \mathbb{C}^{n_4})$, the stabilizer of p_0 in $\mathbf{K} = \mathbf{SL}_{n+1}(\mathbb{C})$ has Lie algebra

$$\mathfrak{v} = \left\{ \begin{pmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ 0 & Z_{2,2} & 0 & Z_{2,4} \\ 0 & 0 & Z_{3,3} & Z_{3,4} \\ 0 & 0 & 0 & Z_{4,4}. \end{pmatrix} \middle|\, Z_{i,j} \in \mathbb{C}^{n_i \times n_j} \right\} \cap \mathfrak{sl}_{n+1}(\mathbb{C}).$$

Indeed, in the block matrix $Z = (Z_{i,j})_{1\leq i,j\leq 4}$ se have $Z_{3,1} = 0$, $Z_{3,2} = 0$, $Z_{4,1} = 0$, $Z_{4,2} = 0$ because $Z(\langle e_1,\dots,e_p\rangle) \subset \langle e_1,\dots,e_p\rangle$. Moreover, the inclusion ${}^TZ(\mathbb{C}^{n_2}\oplus\mathbb{C}^{n_4}) \subset \mathbb{C}^{n_2}\oplus\mathbb{C}^{n_4}$ is equivalent to

$$\begin{pmatrix} {}^TZ_{1,1} & {}^TZ_{2,1} & 0 & 0 \\ {}^TZ_{1,2} & {}^TZ_{2,2} & 0 & 0 \\ {}^TZ_{1,3} & {}^TZ_{2,3} & {}^TZ_{3,3} & {}^TZ_{4,3} \\ {}^TZ_{1,4} & {}^TZ_{2,4} & {}^TZ_{3,4} & {}^TZ_{4,4} \end{pmatrix}\begin{pmatrix} 0 \\ X_2 \\ 0 \\ X_4 \end{pmatrix} = \begin{pmatrix} 0 \\ Y_2 \\ 0 \\ Y_4 \end{pmatrix}, \quad \forall X_2 \in \mathbb{C}^{n_2},\ X_4 \in \mathbb{C}^{n_4},$$

and this yields $Z_{2,1} = 0$, $Z_{2,3} = 0$, $Z_{4,3} = 0$. The compact *CR* manifold $M_0(k)$ has *CR* dimension equal to $\nu = (n_1n_2+n_1n_3+n_1n_4+n_2n_4+n_3n_4)$ and *CR*-codimension $d = 2n_2n_3$. The case $k = k_0$, where $n_3 = 0$, is the one where $\mathfrak{v}$ is parabolic, and $M_0(k_0) = M_-(k_0)$ is a complex flag manifold. In general, $(\kappa_0, \mathfrak{v}_n)$ is HNR because

$$\mathfrak{v}_n = \left\{ \begin{pmatrix} 0 & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ 0 & 0 & 0 & Z_{2,4} \\ 0 & 0 & 0 & Z_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| Z_{i,j} \in \mathbb{C}^{n_i\times n_j} \right\} \cap \mathfrak{sl}_{n+1}(\mathbb{C})$$

is the nilpotent radical of

$$\mathfrak{q} = \left\{ \begin{pmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ 0 & Z_{2,2} & Z_{2,3} & Z_{2,4} \\ 0 & Z_{3,2} & Z_{3,3} & Z_{3,4} \\ 0 & 0 & 0 & Z_{3,4} \end{pmatrix} \middle| Z_{i,j} \in \mathbb{C}^{n_i\times n_j} \right\} \cap \mathfrak{sl}_{n+1}(\mathbb{C})$$

Then

$$\mathfrak{f}_0 = \mathfrak{m}_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{2,3} & 0 \\ 0 & -Z^*_{2,3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| Z_{2,3} \in \mathbb{C}^{n_2\times n_3} \right\} \simeq \mathbb{C}^{n_2\times n_3}. \tag{6.13}$$

The *CR* algebra $(\kappa_0, \mathfrak{v})$ is weakly degenerate when $k < p$ and strictly nondegenerate, according to [22], when $k = p$. The *vector valued Levi form* is

$$(Z_{1,2}, Z_{1,3}, Z_{1,4}, Z_{2,4}, Z_{3,4}) \to Z^*_{1,2}Z_{1,3} + Z_{2,4}Z^*_{3,4}$$

and hence all the nonzero *scalar Levi form* have a Witt index equal to $\mu = (n_1+n_4) = (p-k)+(q-k) = p+q-2k$. The complex manifold $M_-(k)$ has dimension $N = n_1n_2+n_1n_3+n_1n_4+n_2n_3+n_2n_4+n_3n_4$ and, according to Theorems 6.6 and 6.8 is μ-pseudoconcave and $(\nu-\mu)$-pseudoconvex.

6.5 – *Application to Dolbeault and CR cohomologies III*

In this section we extend Theorem 6.6 to the case where we do not assume that $(\kappa_0, \mathfrak{v})$ is HNR. To this aim we utilize an r-pseudoconcave exhausting functions which is only continuous (see [9, 10, 16, 28]). Namely, we will consider the function

$$\phi([\zeta]) = \mathrm{dist}^2(\zeta^*\zeta, N), \quad \text{for } \zeta \in \mathbf{K}, \tag{6.14}$$

where $N = \{v^*v \mid v \in \mathbf{V}\}$ as in (4.5) and $[\zeta]$ is the element of $M_- = \mathbf{K}/\mathbf{V}$ corresponding to $\zeta \in \mathbf{K}$.

We recall that a continuous function ϕ, defined on a complex ν-dimensional manifold M_-, is said to be *weakly r-pseudoconcave* if, for every point $p \in M_-$, we can find a coordinate neighborhood (U, z), centered at p, such that, for every $(\nu - r + 1)$-dimensional linear subspace ℓ of $\mathbb{C}^\nu$, for every coordinate ball $B \Subset U$ and ψ plurisubharmonic on a neighborhood of $\bar{B}$, with $\phi \geq \psi$ on $\ell \cap \partial B$ we also have $\phi \geq \psi$ on $\ell \cap B$.

We say that ϕ is *strictly r-pseudoconcave* if, for each $p \in M_-$, we can find an open coordinate neighborhood (U, p) centered in p and an $\epsilon > 0$ such that $\phi + \epsilon|z|^2$ is weakly r-pseudoconcave in U.

By Bungart's approximation theorem ([9, Theorem 5.2]) strictly r-pseudoconcave functions can be uniformly approximated on compacts by piece-wise smooth strictly r-pseudoconcave functions. Thus (see e.g. [2, Chapter IV]) we can still apply the Andreotti-Grauert theory when we have a strictly-r-pseudoconcave exhaustion function which is only continuous.

Our application relies then on the following lemmas.

Lemma 6.10. *Let ϕ be a continuous exhaustion function on M_- and assume that, for all $c > 0$ and $p_0 \in \Phi_c = \{p \in M_- \mid \phi(p) = c\}$ there is a germ of CR generic r-pseudoconcave CR submanifold $M_0(p_0)$ of M_- through p_0 with $M_0(p_0) \subset \{\phi_p \leq c\}$. Then ϕ is weakly r-pseudoconcave.*

Proof. We argue by contradiction, assuming that, for every coordinate neighborhood (U, z) centered at a point $p_0 \in M_,$ we can find a $(\nu - r + 1)$-dimensional linear subspace ℓ of $\mathbb{C}^\nu$ and a plurisubharmonic ψ, defined on a neighborhood of the closure $\bar{B}$ of a coordinate ball in U, and a point $p_1 \in \ell \cap B$ where $\phi(p_1) < \psi(p_1)$, while $\phi(p) \geq \psi(p)$ for all $p \in \partial B \cap \ell$. Clearly the same condition is satisfied by any linear $(\nu - r + 1)$-plane sufficiently close to ℓ, so that we can assume that ℓ intersects $M_0(p_1)$ transversally. The intersection $M_0(p_1) \cap \ell$ is then a 1-pseudoconcave CR submanifold of ℓ, but the restriction of ψ to $\ell \cap M_0(p_1) \cap \bar{B}$ contradicts then the maximum principle, since takes at the interior point p_1 a value larger than the supremum of the values taken on the boundary $\ell \cap M_0(p_1) \cap \bar{B}$ (see e.g. [15]). The contradiction proves that ϕ is weakly r-pseudoconcave. □

Lemma 6.11. *The exhaustion function ϕ defined by* (6.14) *is strictly r-pseudoconcave on $M_- \setminus M_0$.*

Proof. By Proposition 4.8, there is $c_0 > 0$ such that ϕ is strictly r-pseudoconcave when $0 < \phi(p) \leq c_0^2$, since, by [16, Lemma 2.6], for a smooth function the notion of strict r-pseudoconcavity coincides with the requirement about the signature of its complex Hessian.

For $\zeta \in \mathbf{K}$, we can consider the function $\phi_\zeta(p) = \phi(\zeta^{-1} \cdot p)$, which is continuous and weakly r-pseudoconcave on $M_- \setminus (\zeta \cdot M_0)$ and strictly r-pseudoconcave when it takes positive values smaller than c_0^2. Let $p_0 \in M_-$ with $\phi(p_0) > c_0^2$ and fix a relatively compact coordinate neighborhood (U, z) in M_-, centered at p_0. We can assume, for a fixed δ with $0 < 2\delta < c_0$, that $U \subset \{p \mid |\phi(p) - \phi(p_0)| < \delta^2\}$. We observe that

$\phi(p) = \inf_{\phi([\zeta])=\phi(p_0)-\delta}(\sqrt{\delta} + \sqrt{\phi_\zeta(p)})^2$. The functions $p \to \eta_\zeta(p) = (\sqrt{\delta} + \sqrt{\phi_\zeta(p)})^2$, when $\phi(\zeta) = \phi(p_0) - \delta^2$, are uniformly strictly r-pseudoconcave on a neighgorhood of $\bar{U}$. Thus, for a small $\epsilon > 0$, the functions $\eta_\zeta + \epsilon|z - z_0|^2$, for $\phi(\zeta) = \phi(p_0) - \delta^2$, are still r-pseudoconcave on U. Passing to the infimum, we deduce, by using [10, Proposition 2.2. (ii)] that $\phi + \epsilon|z - z_0|^2$ is weakly r-pseudoconcave on U. The proof is complete. □

¿From this and the remarks at the beginning of this subsection, we obtain:

Theorem 6.12. *Let M_0 be a compact* n*-reductive homogeneous CR manifold, with canonical complex embedding $M_0 \hookrightarrow M_-$.*

If M_0 is an r-psudoconvave CR-manifold, then M_- is an r-pseudoconcave complex manifold and for every coherent sheaf $\mathcal{F}$ we have

$$\dim(\mathbf{H}^j(M_0, \mathcal{F}) \simeq \mathbf{H}^j(M_-, \mathcal{F})) < \infty, \quad \forall j < r - \mathrm{hd}(\mathcal{F}). \tag{6.15}$$

In particular,

$$\dim(\mathbf{H}^{p,j}(M_0) \simeq \mathbf{H}^{p,j}(M_-)) < \infty, \quad \forall j < r. \tag{6.16}$$

□

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