Quasibases for nonseparable $p$-Groups

Dedicated to Laszlo Fuchs on his 95th birthday

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Abstract – This paper is an extension of the work developed in [4] on quasibases of abelian $p$-groups and based on the doctoral dissertation of Andrija Vodopivec [5]. We introduce the ideas of a $\delta$-combination and height of an inductive quasibasis and show that the height of a quasibasis is invariant for related inductive quasibases. Moreover, an abelian $p$-group is separable if and only if the heights of all $\delta$-combinations are zero. Finally, we show that an abelian $p$-group is not reduced if and only if there exists a $\delta$-combination with infinite height.

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1. Introduction

We deal with abelian groups and we use all definitions and conventions in [3]. For some few classes of torsion-free groups there is a description by cardinals. For all other torsion-free groups there exists basically only a presentation by generators and relations, unavoidably. In view of the convenient description of (simply presented) torsion groups by Ulm-Kaplansky invariants, the use of generators and relations seems to be disadvantageous for torsion groups. But often groups are considered as extensions, and then things change. An explicit description of a mixed group as an extension of a torsion by a torsion-free group is impossible if the torsion group is given by

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Ulm-Kaplansky invariants. The torsion group has to be presented by generators and relations, the same way as the torsion-free group. Here the concept of a quasibasis [3, 33.5] comes into the game.

Investigating mixed groups we recognized that the concept of a quasibasis was not developed far enough for our needs. In [4] the concept of a quasibasis was reduced to that of an inductive quasibases and p-groups are explicitly described by the corresponding diagonal relation arrays α. In particular, we showed that smallness of α is equivalent to splitting and independent diagonal relation arrays were shown to correspond uniquely to reduced, separable groups.

In this paper we determine a relation array of the generalized Prüfer group $\mathcal{H}_{2\omega+1}$, Theorem 4.3. We define a height of an inductive quasibasis and show that this is an invariant for related inductive quasibases, Theorem 5.8. Further, we define δ-combinations and characterize “separable” by the heights of δ-combinations, Theorem 6.3. Finally we establish a criterion for “nonreduced” in terms of heights, Theorem 6.6.

Our concept, for sure, needs additional development for promising applications in the theory of torsion groups. For more results see [5].

2. Preliminaries

We denote the ring of $p$-adic integers by $\mathbb{Z}_p$. As customary, define the $p$-adic norm of $\lambda \in \mathbb{Z}_p$ by $\|\lambda\| = p^{-n}$ if $\lambda \in p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$. Moreover, $\lambda = \sum_{i \in \mathbb{N}_0} \lambda_i p^i$ will denote the standard representation of a $p$-adic integer $\lambda \in \mathbb{Z}_p$.

We consider subgroups of $\prod_{|I|} \mathbb{Z}_p$, the additive group of all tuples $(\lambda_k | k \in I)$ of $p$-adic integers, where $\lambda_k \in \mathbb{Z}_p$, over some index set $I$. A tuple $0 \neq (\lambda_k | k \in I) \in \prod_{|I|} \mathbb{Z}_p$ is called a zero tuple if for every natural number $n$ the norm of almost all $\lambda_k$ is less than $p^{-n}$. A zero tuple is called normed, if there is at least one unit among the entries $\lambda_k$. The zero tuples (together with the trivial tuple $0$) form a subgroup $\left(\prod_{|I|} \mathbb{Z}_p\right)^*$ of $\prod_{|I|} \mathbb{Z}_p$, which clearly contains $\bigoplus_{|I|} \mathbb{Z}_p$. Moreover, $(\prod_{|I|} \mathbb{Z}_p)^*/\bigoplus_{|I|} \mathbb{Z}_p$ is the maximal divisible subgroup of $\prod_{|I|} \mathbb{Z}_p/\bigoplus_{|I|} \mathbb{Z}_p$.

**Proposition 2.1.** Let $G = \bigoplus_{k \in I} \langle g_i^k | i \in \mathbb{N} \rangle \cong \bigoplus_{|I|} \mathbb{Z}(p^\infty)$, where $p \| g_i^k = 0$, $pg_i^k = g_i^k$ for all $k \in I$, $i \in \mathbb{N}$. Then $D = \langle h_i \in G | i \in \mathbb{N} \rangle \cong \mathbb{Z}(p^\infty)$ is a subgroup of $G$, where $ph_1 = 0$, $ph_{i+1} = h_i$ for all $i \in \mathbb{N}$, if and only if there is a normed zero tuple $(\lambda_k | k \in I)$, such that $h_i = \sum_{k \in I} \lambda_k g_i^k$ for all $i \in \mathbb{N}$.

**Proof.** For each $i \in \mathbb{N}$, the element $0 \neq h_i \in \bigoplus_{k \in I} \langle g_i^k | i \in \mathbb{N} \rangle$ can be written in the form $h_i = \sum_{k \in I} \lambda_i^k g_i^k$, where $0 \leq \lambda_i^k < p^i$, $\lambda_i^k = 0$ for almost all $k \in I$, and $p \| \lambda_i^k$ for at least one $k \in I$, by order considerations. Furthermore, we have for each $i \in \mathbb{N}$,

$$0 = h_i - ph_{i+1} = \sum_{k \in I} \lambda_i^k g_i^k - \sum_{k \in I} \lambda_{i+1}^k p g_i^k = \sum_{k \in I} (\lambda_i^k - \lambda_{i+1}^k) g_i^k.$$
Hence, \((\lambda_i^k - \lambda_{i+1}^k)g_i^k = 0\), i.e., \(p^i \mid (\lambda_i^k - \lambda_{i+1}^k)\) for all \(k \in I\). For each \(k \in I\) let

\[
\lambda_k = \lambda_i^k + \sum_{j \ge i} (\lambda_j^k - \lambda_j^k) \in \mathbb{Z}_p,
\]

where the equation holds for arbitrary \(i \in \mathbb{N}\).

For a fixed \(i \in \mathbb{N}\), \(p^i \mid \lambda_k\) for almost all \(k \in I\), because \(\lambda_i^k = 0\) for almost all \(k\). Therefore \((\lambda_k \mid k \in I)\) is a zero tuple. Moreover, \(p \mid \lambda_k\) for at least one \(k \in I\), because \(p \mid \lambda_i^k\) for at least one \(i \in \mathbb{N}\), i.e., \((\lambda_k \mid k \in I)\) is normed. In particular, \(\lambda_k g_i^k = \lambda_i^k g_i^k\). Thus \(h_i = \sum_{k \in I} \lambda_k g_i^k = \sum_{k \in I} \lambda_k g_i^k\) for all \(i \in \mathbb{N}\).

Conversely, let \((\lambda_k \mid k \in I)\) be a normed zero tuple and \(h_i = \sum_{k \in I} \lambda_k g_i^k\) for each \(i \in \mathbb{N}\). Note \(ph_i = \sum_{k \in I} p\lambda_k g_i^k = 0\) and

\[
ph_{i+1} = \sum_{k \in I} \lambda_k pg_i^{k+1} = \sum_{k \in I} \lambda_k g_i^k = h_i
\]

for all \(i \in \mathbb{N}\). In particular, the order \(o(h_i) = p^i\), because \((\lambda_k \mid k \in I)\) is a normed zero tuple. Hence, \(\langle h_i \rangle \cong \mathbb{Z}(p^\infty)\).

Following [4] the set

\[
Q = \{a_i^k, x_j^u \mid i, j \in \mathbb{N}, k \in I, u \in I_j\} \subset G
\]

is called a **quasibasis** of \(G\), if

(i) \(\{x_j^u \mid j \in \mathbb{N}, u \in I_j\}\) is a basis of the basic subgroup \(B = \bigoplus B_j\), where \(o(x_j^u) = p^i\) for all \(j \in \mathbb{N}, u \in I_j\);

(ii) \(G/B = \bigoplus_{k \in I} A^k\), where \(A^k = \langle a_i^k + B \mid i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty), k \in I, \) and \(pa_i^k + B = a_i^k + B\) for all \(i \in \mathbb{N}, k \in I\), with \(pa_1 + B = 0 + B\);

(iii) \(o(a_i^k) = p^i\) for all \(i \in \mathbb{N}, k \in I\).

Note that

\[
G = \langle a_i^k, x_j^u \mid i, j \in \mathbb{N}, k \in I, u \in I_j\rangle.
\]

By [3, 33.5] every \(p\)-group has a quasibasis with corresponding relations

\[
pa_i^{k+1} = a_i^k - \sum_{j \in \mathbb{N}, u \in I_j} \alpha_{i,j}^{k,u} x_j^u (i \in \mathbb{N}, k \in I, \alpha_{i,j}^{k,u} \in \mathbb{Z}).
\]

Given a quasibasis \(Q = \{a_i^k, x_j^u\}\) the array \(\alpha = (\alpha_{i,j}^{k,u})\) is called a **corresponding relation array**, \(B\) the **corresponding** basic subgroup. Note that we may also assume \(\alpha_{i,j}^{k,u} \in \mathbb{Z}_p\). For \(n \in \mathbb{N}\) let \(p^nQ = \{c_i^k, y_j^u \mid i, j \in \mathbb{N}, k \in I, u \in I_{j+n}\}\), where \(y_j^u = p^nx_j^{u+n}\) and \(c_i^k = p^n a_i^{k+n}\).
LEMMA 2.2. Let \( Q = \{a^k_i, x^n_j\} \) be a quasibasis of \( G \) with relation array \( \alpha = (\alpha_{i,j}^{k,u}) \). Then for any \( n \in \mathbb{N} \) the set \( p^n Q \) is a quasibasis of \( p^n G \) with corresponding array \( (\alpha_{i+n,j+n}^{k,u}) \).

PROOF. Since \( p^n B = \bigoplus_{j \in \mathbb{N}} \bigoplus_{u \in I_{j+n}} (p^n x^n_{j+n}) \) is a basic subgroup of \( p^n G \) and \( o(p^n a^k_{i+n}) = p^i \), the conditions (i) and (iii) hold. Since \( p^n G/p^n B \cong G/B \cong \bigoplus_{|I|} \mathbb{Z}(p^\infty) \) condition (ii) follows. The relations

\[
p^{n+1} a^k_{i+n+1} = p^n a^k_{i+n} - \sum_{j \in \mathbb{N}} \sum_{u \in I_{j+n}} \alpha_{i+n,j+n}^{k,u} p^n x^n_{j+n}
\]

give rise to the indicated array. \( \square \)

3. Inductive Quasibases

A quasibasis \( \{a^k_i, x^n_j\} \) is called an inductive quasibasis, see [4], if the corresponding relations are of the form \( p a^k_{i+1} = a^k_i - b^k_i \) for \( i \in \mathbb{N}, k \in I \), where \( b^k_i \in B_i = \bigoplus_{u \in I} (x^n_u) \), cf. also [1]. Furthermore, a relation array \( \alpha = (\alpha_{i,j}^{k,u}) \) is called diagonal, if \( \alpha_{i,j}^{k,u} = 0 \) for \( i \neq j \). A diagonal array is denoted by \( \alpha = (\alpha_{i}^{k,u}) = (\alpha_{i,i}^{k,u}) \). By [4, Theorem 4 and Corollary 5], every \( p \)-group has an inductive quasibasis, and the corresponding relation array is diagonal. Note that an inductive quasibasis is based on a fixed decomposition \( B = \bigoplus B_i \) of the basic subgroup, and we write \( Q = \{a^k_i, \bigoplus B_i\} \) or \( Q = \{a^k_i, B\} \) to suppress the generators of the basic subgroup.

LEMMA 3.1. Let \( Q = \{a^k_i, \bigoplus B_i\} \) be an inductive quasibasis of \( G \) with corresponding relations \( p a^k_{i+1} = a^k_i - b^k_i \), \( i \in \mathbb{N}, k \in I \). Then \( p^n a^k_{i+n} = a^k_i - \sum_{r = 0}^{n-1} p^r b^k_{i+r} \) for all \( n \in \mathbb{N} \).

PROOF. We induct on \( n \). Clearly, \( p a^k_{i+1} = a^k_i - b^k_i \). By hypothesis

\[
p^{n+1} a^k_{i+n+1} = p^n a^k_{i+n} - p^n b^k_{i+n} = a^k_i - \sum_{r = 0}^{n-1} p^r b^k_{i+r} - p^n b^k_{i+n} = a^k_i - \sum_{r = 0}^{n} p^r b^k_{i+r}. \square
\]

Let \( G, H \) be groups with isomorphic basic subgroups \( B = \bigoplus B_i \subset G \) and \( C = \bigoplus C_i \subset H \), and \( G/B \cong H/C \), i.e., in particular, for all \( i, B_i \cong C_i \) are isomorphic homocyclic groups of exponent \( p^i \). Let, assuming equal index sets, the corresponding quasibases be \( Q = \{a^k_i, x^n_j\}, P = \{c^k_i, y^n_j\} \), and the corresponding relation arrays be \( \alpha = (\alpha_{i,j}^{k,u}), \beta = (\beta_{i,j}^{k,u}) \), respectively. Then the groups \( G, H \), the quasibases \( P, Q \) and the relation arrays \( \alpha, \beta \) are called related, respectively. In particular, if \( G = H \) and \( B = C \) we call the two quasibases \( Q = \{a^k_i, x^n_j\}, P = \{c^k_i, y^n_j\} \) and the two corresponding relation arrays \( \alpha, \beta \) of \( G \) related, respectively. The point for related relation arrays is that the respective index sets are equal. We tacitly assume this setting for those related pairs \( G, H \), or for a single group \( G \) with fixed basic subgroup \( B \).
Let $H = \varphi G$ with isomorphism $\varphi$, then by choice $C = \varphi B$ for some basic subgroup $B \subset G$ the groups $G, H$ are related. In other words, related groups coincide in some invariants that are kept by isomorphism.

Otherwise, let $G, H$ be related with related quasibases $Q = \{a_i^k, x^k_j\}, \{c_i^k, y^k_j\}$ and related relation arrays $\alpha, \beta$. If there is another quasibasis $P'$ of $H$ such that the relation array $\beta'$ corresponding to $P'$ is equal to $\alpha$, then $G \cong H$. This is a consequence of [4, Theorem 1], because the relations given by the relation array of a group are defining. Thus all results on changing the quasibasis, respectively changing the relation array, of a group include statements on isomorphism.

There is a strong relationship between two related inductive quasibases of a group.

**Lemma 3.2.** Let $Q = \{a_i^k, \bigoplus B_i\}$ and $P = \{c_i^k, \bigoplus B_i\}$ be related, inductive quasibases of $G$ with corresponding relations $a_i^k - pa_i^{k+1} = b_i^k$ and $c_i^k - pc_i^{k+1} = d_i^k$. Then for each $k_0 \in I$ there is a normed zero tuple $(\lambda_k | k \in I)$ (depending on $k_0$), such that for all $n \in \mathbb{N}$,

$$d_i^{k_0} - \sum_{k \in I} \lambda_kb_i^k \in p^nB_i$$

for almost all $i \in \mathbb{N}$.

**Proof.** Since $G/B = \bigoplus_{k \in I} (a_i^k + B \mid i \in \mathbb{N}) = \bigoplus_{k \in I} (c_i^k + B \mid i \in \mathbb{N}) \cong \bigoplus_{\mid I \mid} \mathbb{Z}(p^\infty)$ by Proposition 2.1, there is, for a fixed $k_0 \in I$, a normed zero tuple $(\lambda_k | k \in I)$ (depending on $k_0$), such that $c_i^{k_0} = \sum_{k \in I} \lambda_k a_i^k + b_n, b_n \in B$ for all $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$,

$$d_i^{k_0} - \sum_{k \in I} \lambda_kb_i^k = c_i^{k_0} - pc_i^{k_0} + \sum_{k \in I} \lambda_k(a_i^k - pa_i^{k+1}) = b_n - pb_{n+1} \in B_n,$$

because $Q$ and $P$ are inductive and related. The elements $b_n \in B$ are of the form $b_n = \sum_{i \in \mathbb{N}} b_{n,i}$, where $b_{n,i} \in B_i$. Thus for each $n \in \mathbb{N}$

$$b_n - pb_{n+1} = \sum_{i \in \mathbb{N}} (b_{n,i} - pb_{n+1,i}) \in B_n,$$

i.e., $b_{n,i} - pb_{n+1,i} = 0$ for all $i \in \mathbb{N}$ with $i \neq n$. Consequently, for all $n \in \mathbb{N}$,

$$b_{n,i} = pb_{n+1,i} = p^2b_{n+2,i} = ... = 0, \text{ if } i < n,$$

$$b_{n,i} = pb_{n+1,i} = p^2b_{n+2,i} = ... = p^{i-n}b_{i,i}, \text{ if } i \geq n,$$

and the first part of the following sum is 0, hence

$$b_n = b_{n,1} + ... + b_{n,n-1} + b_{n,n} + b_{n,n+1} + b_{n,n+2} + ...$$

$$= b_{n,n} + pb_{n+1,n+1} + p^2b_{n+2,n+2} + ... + p^rb_{n+r,n+r} + ...$$
This is a finite sum, thus we have the equality $p^r b_{n+r,n+r} = 0$ for all $n \in \mathbb{N}$, or $p^n \mid b_{n+r,n+r}$ for almost all $r \in \mathbb{N}_0$. This implies for all $n \in \mathbb{N}$,

$$
\sum_{k \in I} \lambda_k b_{n+r}^k = b_{n+r} - pb_{n+r+1}
$$

$$
= \sum_{m \in \mathbb{N}_0} p^m b_{n+r+m,n+r+m} - \sum_{m \in \mathbb{N}_0} p^{m+1} b_{n+r+m+1,n+r+m+1}
$$

$$
= b_{n+r,n+r} \in p^n \mathcal{B}_{n+r}
$$

for almost all $r \in \mathbb{N}_0$, as claimed. \(\square\)

Let $Q = \{a_i^k, x_j^u\}$ be an inductive quasibasis of $G$ with corresponding relations

$$
(1) \quad a_i^k - pa_i^{k+1} = \sum_{u \in I_i} \alpha_{k,u}^i x_i^u = b_i^k \in B_i.
$$

We write the corresponding diagonal relation array $\alpha = (\alpha_i^k,u)$ in the following form

$$
(2) \quad \alpha = (\alpha_k)_{k \in I}, \quad \alpha^k = \text{diag}(\alpha_1^k, \alpha_2^k, \ldots) \quad \text{and} \quad \alpha_i^k = (\alpha_i^k,u)_{u \in I_i} \quad \text{with} \ i \in \mathbb{N}, \ \alpha_i^k,u \in \mathbb{Z},
$$

where $\alpha_i^k \in \mathbb{Z}^{\mid I_i \mid}$ is a tuple, with only finitely many nonzero entries, and $\alpha$ can be considered as a tuple of (infinite) diagonal matrices $\alpha^k$.

Two related diagonal relation arrays $\alpha = (\alpha_i^k,u)$ and $\beta = (\beta_i^k,u)$, i.e., with equal index sets, are called almost equal, if for each $k$ the equation $\alpha_i^k = \beta_i^k$ holds for almost all $i \in \mathbb{N}$.

The following proposition shows that a group allows a whole class of almost equal relation arrays, and, moreover, that almost equal relation arrays of groups imply isomorphism.

**Proposition 3.3.** Let $Q = \{a_i^k, x_j^u\}$ be an inductive quasibasis of $G$ with relation array $\alpha = (\alpha_i^k,u)$ and let $\beta$ be an array almost equal to $\alpha$. Then there is an inductive quasisbasis $P = \{c_i^k, x_j^u\}$ of $G$ with relation array $\beta$ and $c_i^k = a_i^k$ for each $k \in I$, and for almost all $i \in \mathbb{N}$.

Related groups $G, H$ with almost equal (related) relation arrays are isomorphic.

**Proof.** Since $\alpha$ and $\beta$ are almost equal we need to show that we can make finitely many changes for each $k$. Thus it suffices to construct a new inductive quasibasis $P$ of $G$ which differs from $Q$ only for one fixed $k$ and a fixed $i_k$. Let $\beta = (\beta_i^k)$ be given by

$$
\beta_i^k = \begin{cases} 
(\alpha_i^k,u)_{u \in I_i}, & \text{if } i \neq i_k \\
(z_i^k,u)_{u \in I_i}, & \text{if } i = i_k
\end{cases}
$$
where \((z^{k,u} \mid u \in I_k) \in \mathbb{Z}^{I_k}\) is an arbitrary tuple with only finitely many nonzero entries. We show that \(P = \{c^k_i, x^u_j \mid i, j \in \mathbb{N}, k \in I, u \in I_j\} \subset G\) with

\[
\ell^k_i = \begin{cases} a^k_i + p^i k-i \sum_{u \in I_k} (z^{k,u} - \alpha^{k,u}_{ik})x^u_i, & \text{if } i \leq i_k \\ 0, & \text{if } i > i_k \end{cases}
\]

for all \(k \in I\), is an inductive quasibasis of \(G\). The conditions (i) and (iii) of the definition of a quasibasis are obviously satisfied. Since \(c^k + B = a^k + B\) for all \(k \in I, i \in \mathbb{N}\), condition (ii) is also satisfied. Furthermore, for all \(k \in I\),

\[
p_{c_{i+1}}^k = p(a^k_{i+1} + p^{i-k-i-1} \sum_{u \in I_k} (z^{k,u} - \alpha^{k,u}_{ik})x^u_i)
\]

\[
= a^k_i - \sum_{u \in I_k} \alpha^{k,u}_{ik}x^u_i + p^{i-k-i} \sum_{u \in I_k} (z^{k,u} - \alpha^{k,u}_{ik})x^u_i = c_i^k - \sum_{u \in I_i} \alpha^{k,u}_{i}x^u_i, \text{ if } i < i_k,
\]

\[
p^k_{a_{i+1}}^k = p^k_{a_{i+1}} = a^k_k - \sum_{u \in I_k} \alpha^{k,u}_{ik}x^u_k
\]

\[
= c_k^k - \sum_{u \in I_k} (z^{k,u} - \alpha^{k,u}_{ik})x^u_k - \sum_{u \in I_k} \alpha^{k,u}_{ik}x^u_k = c_k^k - \sum_{u \in I_k} z^{k,u}x^u_k,
\]

\[
p_{c_{i+1}}^k = p\sum_{u \in I_i} a^k_i - \sum_{u \in I_i} \alpha^{k,u}_{i}x^u_i = c_i^k - \sum_{u \in I_i} \alpha^{k,u}_{i}x^u_i, \text{ if } i > i_k.
\]

Hence \(\beta\) is the desired relation array.

By the argument above and because relation arrays provide defining relations, see [4, Theorem 1], the groups \(G, H\) are isomorphic.

4. Construction of a Quasibasis for \(H_{2\omega+1}\)

Well known examples for nonseparable reduced \(p\)-groups are the generalized Prüfer groups \(H_{\sigma}\) for ordinals \(\sigma\). For a definition see [3, Section 81]. By [3, 83.1] all generalized Prüfer groups are simply presented.

In [4] for the generalized Prüfer group \(H_{\omega+n}\) of length \(\omega + n\) for natural \(n\) an inductive quasibasis was given with a corresponding relation array. Our next goal is to determine an inductive quasibasis of the generalized Prüfer group \(H_{2\omega+1}\) of length \(2\omega + 1\). Note that the Ulm-Kaplansky-invariants of \(H_{2\omega+1}\) are

\[
f_{\sigma}(H_{2\omega+1}) = \begin{cases} \mathbb{N}_0, & \text{for } 0 \leq \sigma < \omega, \\ 1, & \text{for } \omega \leq \sigma \leq 2\omega. \end{cases}
\]

We construct a simply presented group \(G\) that has the same Ulm-Kaplansky-invariants as \(H_{2\omega+1}\). Hence \(G \cong H_{2\omega+1}\), because simply presented groups with equal Ulm-Kaplansky-invariants are
isomorphic, see [3, 83.3]. Further we use the presentation of this group $G$ to obtain an inductive quasibasis.

We begin by developing some notation. Let $H = \langle h_0^0 \rangle \oplus \bigoplus_{k \in \mathbb{N}, i \in \mathbb{N}_0} \langle h_i^k \rangle$ be a free abelian group and $L = \langle p g_0^0, p^k g_0^k - h_0^0, p^i h_0^k - h_0^k | i, k \in \mathbb{N} \rangle$ a subgroup of $H$. We denote $G = H/L = \langle g_0^0, g_i^k | k \in \mathbb{N}, i \in \mathbb{N}_0 \rangle$ with $g_0^0 = h_0^0 + L$ and $g_i^k = h_i^k + L$. The group $G$ is given by the relations

\[ pg_0^0 = 0, p^k g_i^k = g_0^0 \text{ and } p^i g_i^k = g_0^k \text{ for } i, k \in \mathbb{N}. \]

In particular, $G$ is simply presented. It is straightforward to show that the following hold for the groups $L$ and $G$ as described above.

(i) Every $l \in L$ has the form $l = \lambda_0^0 h_0^0 + \sum_{k \in \mathbb{N}, i \in \mathbb{N}_0} \lambda_i^k h_i^k$, where $\lambda_i^k \in p^i \mathbb{Z}$ for $i, k \in \mathbb{N}$. Moreover, $h_0^0 \not\in L$ and for $l = \sum_{k \in \mathbb{N}} \lambda_i^k h_i^k$, $\lambda_0^k \in p^k \mathbb{Z}$ for all $k \in \mathbb{N}$.

(ii) For $r \in \mathbb{N}$ each $g \in p^r G$ has the form $g = \sum_{k \in \mathbb{N}} (m_0^k g_0^k + \sum_{i > r} \mu_i^k g_i^k)$ with $\mu_0^k \in \mathbb{Z}$ and $\lambda_i^k \in p^i \mathbb{Z}$ for $i > r$.

We determine a basic subgroup of $G$ and construct an inductive quasibasis. Let $x_i^k = g_i^k - pg_{i+1}^k \in G$ for all $i, k \in \mathbb{N}$ and let $B = \langle x_i^k | k, i \in \mathbb{N} \rangle$.

**Lemma 4.1.** The subgroup $B$ of $G$ defined above is a direct sum, $B = \bigoplus_{i,k \in \mathbb{N}} \langle x_i^k \rangle$, with $o(x_i^k) = p^i$ for all $i, k \in \mathbb{N}$. Moreover, $B$ is a basic subgroup of $G$.

**Proof.** Similar to the arguments for the generalized Prüfer group $\mathcal{H}_{\omega+1}$, see [3, Section 35, Example], it is easy to verify that $\{ x_i^k | k, i \in \mathbb{N} \}$ is a $p$-independent system of $G$ with each $b \in B$ of the form $b = \sum_{i,k \in \mathbb{N}} \lambda_i^k x_i^k = \sum_{i,k \in \mathbb{N}} (\lambda_i^k - p \lambda_i^{k-1}) g_i^k$, where $0 \leq \lambda_i^k < p^i$ and agreeing $\lambda_0^k = 0$. Moreover, $G/B$ is divisible with a decomposition into the $\mathbb{Z}(p^\infty)$ summands given by $\langle \bar{g}_i^1 | i \in \mathbb{N}_0 \rangle$ and $\langle p^{k-1} \bar{g}_i^k - p^{k+1} \bar{g}_i^k | i \in \mathbb{N}_0 \rangle$ for $k \in \mathbb{N}$ and where $\bar{g} = g + B$. \qed

**Lemma 4.2.** $G \cong \mathcal{H}_{2\omega+1}$.

**Proof.** Since $B = \bigoplus_{i,k \in \mathbb{N}} \langle x_i^k \rangle$ and $p^r G = \langle g_0^0, g_i^k | pg_0^0 = 0, p^k g_0^k = g_0^k, p^i g_i^k = g_0^k | k \in \mathbb{N} \rangle$, we have $p^r G = \mathcal{H}_{\omega+1}$, see [3, Section 83, Example 3]. So the the Ulm-Kaplansky invariants of the simply presented group $G$ are equal to those of $\mathcal{H}_{2\omega+1}$. Thus $G \cong \mathcal{H}_{2\omega+1}$ by the consideration above. \qed

Now we use the presentation of $G$ to obtain an inductive quasibasis of $\mathcal{H}_{2\omega+1}$. In Lemma 4.1 we defined $B$, the basic subgroup of $G$, by $B = \bigoplus_{i,k \in \mathbb{N}} \langle x_i^k \rangle$. This shows that condition (i) of an inductive quasibasis holds. Now we show conditions (ii) and (iii). Define the generators $a_i^k$ as follows $a_i^0 = p^2 g_i^1$ and $a_i^k = -p^k g_i^k + p^{k+1} g_i^{k+1}$ for $i, k \in \mathbb{N}$. In particular, $a_i^0 - pa_{i+1}^0 = p^2 g_i^1 - p^3 g_{i+1}^1 = p^2 x_i^1 \in B_i$ and for all $i, k \in \mathbb{N}$

\[ a_i^k - pa_{i+1}^k = -p^k g_i^k + p^{k+1} g_i^{k+1} + p^{k+1} g_{i+1}^k - p^{k+2} g_{i+1}^{k+1} = -p^k x_i^k + p^{k+1} x_{i+1}^{k+1} \in B_i. \]
Define $A^k$ by $A^k = \langle a^k_i + B \mid i \in \mathbb{N} \rangle \subseteq G/B$ for all $k \in \mathbb{N}_0$. Note that the subgroups $A^k$ are precisely the $\mathbb{Z}(p^\infty)$ summands given in Lemma 4.1 by the generators $(g^1_i \mid i \in \mathbb{N}_0)$ and $(p^{k-1}g^1_i - p^k g^{k+1}_i \mid i \in \mathbb{N}_0)$. Hence condition (ii) holds. Finally we show condition (iii), that $o(a_i^k) = p^s$. This follows from $p^s a_i^0 = p^{s+2} g^1_i = p^s g^{k+1}_i = 0$, $p^s a_i^k = p^{s}(-p^k g^{k+1}_i + p^{k+1} g^{k+2}_i) = -p^k g^{k+1}_i + p^{k+1} g^{k+2}_i = -g^{k+1}_i + g^{k+2}_i = 0$ and the defining relations for the groups $G$ and $L$. We summarize the above results in the following theorem.

**Theorem 4.3.** $Q = \{a_i^k, x_i^k \mid i \in \mathbb{N}, k \in \mathbb{N}_0\}$ with $a_i^k$ and $x_i^k$ as defined above is an inductive quasibasis of $G = \mathcal{H}_{2\omega+1}$ with the corresponding relation array $\alpha^k = \text{diag}(\alpha_i^k, \alpha_i^k, \ldots)$, where $\alpha_i^0 = (p^2, 0, \ldots)$ and $\alpha_i^k = (0, \ldots, 0, -p^k, p^{k+1}, 0, \ldots)$ for all $i, k \in \mathbb{N}$.

**5. Invariance of Height for Quasibases**

Let $B = \bigoplus_{i \in \mathbb{N}} B_i$ be a basic subgroup of $G$, and let $B^\Pi = \prod_{i \in \mathbb{N}} B_i$. The elements $\delta \in B^\Pi$ are written in the form $\delta = (b_1, b_2, \ldots)$, where $b_i \in B_i$ for all $i \in \mathbb{N}$. Let $h(\delta)$ denote the height of $\delta = \delta + B$ in $B^\Pi/B$. If $h^B(b_i)$ denotes the height of $b_i$ in $B$ then it is easy to see that $h(\delta) = \liminf_{i \to \infty} (h^B(b_i))$. Note that $h^B(b_i) = \infty$ if and only if $b_i = 0$ and $h^B(b_i) \in \{0, 1, \ldots, i-1\}$ for $b_i \neq 0$.

Let $B^\Pi_0 = \{\delta \in B^\Pi \mid h(\delta) = 0\}$. Note that $B^\Pi_0 = \bar{B}$, the completion of $B$ in the $p$-adic topology, cf. also [2]. Then $B \subseteq B^\Pi_0 \subseteq B^\Pi$ and $B^\Pi_0/B$ is the first Ulm subgroup of $B^\Pi/B$. Clearly, $\delta = (b_i \mid i) \in B^\Pi_0$ if and only if $\lim_{i \to \infty} (h^B(b_i)) = \infty$.

**Lemma 5.1.** $B^\Pi_0/B = p^\omega(B^\Pi/B)$ is the maximal divisible subgroup of $B^\Pi/B$. In particular, $h(\delta) \in \mathbb{N}_0 \cup \{\infty\}$ for $\delta \in B^\Pi/B$. Moreover, $\text{tor}(B^\Pi/B) \subseteq B^\Pi_0/B$.

**Proof.** We show that $B^\Pi_0/B$ is divisible. If $\delta = (b_i \mid i)$ and $\delta + B \in B^\Pi_0/B$, then there is a $j \in \mathbb{N}$ such that $h(b_i) \geq 1$ for all $i \geq j$, i.e., $\delta + B = (pc_i \mid i) + B = p(c_i \mid i) + B$, where $b_i = pc_i$ for all $i \geq j$. Consequently $B^\Pi_0/B = p(B^\Pi_0/B)$, as desired. In particular, $h(\delta) \in \mathbb{N}_0 \cup \{\infty\}$ for $\delta \in B^\Pi/B$. Moreover, it follows from $i = h(b_i) + n_i$, if the element $b_i \in B_i$ is of order $p^{n_i}$, that the torsion subgroup of $B^\Pi/B$ is contained in $B^\Pi_0/B$. \[ \square \]

Recall the following rules for heights [3, Section 37].

**Lemma 5.2.** The following hold for $\delta, \delta_1, \delta_2 \in B^\Pi$ and $\lambda \in \mathbb{Z}_p$ with $\|\lambda\| = p^{-n}$.

(i) $h(\delta_1 + \delta_2) \geq \min\{h(\delta_1), h(\delta_2)\}$,

(ii) $h(\delta_1 + \delta_2) = h(\delta_1)$, if $h(\delta_1) < h(\delta_2)$,

(iii) $h(\lambda \delta) = h(\delta) + n$. 

Proof. (i) and (ii) are obvious, cf. [3, Section 37]. Condition (iii) follows from $h(\lambda \overline{d}) = h(p^a \lambda \overline{d}) = h(\lambda \overline{d}) + n = h(\overline{d}) + n$, where $\lambda' \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$ with $\lambda = p^a \lambda'$. \hfill $\square$

Notation 5.3. Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of $G$ with corresponding relations $a_i^k - p a_{i+1}^k = b_i^k \in B_i$. Define $\delta^k = \delta^k(Q) = (b_1^k, b_2^k, \ldots) \in B^{\Pi}$, then the $Q$-tuple $\Delta(Q) = (\delta^k(Q) \mid k \in I)$ describes the corresponding relations of $G$. An important property of these relations can be formulated by the height function $h$ given by $h(Q) = \min\{h(\delta^k(Q)) \mid k \in I\} \in \mathbb{N}_0 \cup \{\infty\}$. We will refer to it as the height of $Q$ in $G$. For a zero tuple $(\lambda_k \mid k \in I) \neq 0$ and the $Q$-tuple $\Delta(Q)$ we define the sum $\delta = \sum_{k \in I} \lambda_k \delta^k(Q) = (\sum_{k \in I} \lambda_k b_i^k \mid i \in \mathbb{N}) \in B^{\Pi}$ and call it a $Q$-combination. Note that $\delta$ is a well defined element of $B^{\Pi}$, because the sum in each component is finite. A $Q$-combination is called normed if the zero tuple $(\lambda_k \mid k \in I)$ is normed.

If $\alpha = (\alpha_i^k,u)$ is the diagonal relation array corresponding to the inductive quasibasis $Q$, then we write the relation array as in the Equations (1) and (2). Thus

$$
\delta^k(Q) = (b_1^k, b_2^k, \ldots) = \left(\sum_{\alpha \in I_i} \alpha_i^k, u x_i^k \mid i \in \mathbb{N}\right),
$$

and $\Delta(Q) = (\delta^k \mid k)$ is the $Q$-tuple. In particular, the heights $h(\delta^k(Q))$ are precisely determined by the $p$-powers dividing the entries $\alpha_i^k,u$. Hence also $h(Q) = \min\{h(\delta^k(Q)) \mid k \in I\}$ can be read off the entries $\alpha_i^k,u$.

We now determine the heights of some quasibases that have been studied previously.

Example 5.4. Let $Q = \{a_i^k, x_i^k\}$ where $|I| = |I_i| = 1$ for all $i \in \mathbb{N}$, i.e., $B = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}(p^i)$, and $G/B \cong \mathbb{Z}(p^\infty)$. Thus $\Delta(Q) = (b_1^1, b_2^1, \ldots)$, i.e., $h(Q) = \liminf_{i \to \infty}(h^B(b_i))$. We now give the heights of three quasibases that appeared in [4, Section 2 and 5] together with their relation arrays.

$$
\begin{align*}
H_{\omega+1} : \quad & \alpha = \text{diag}(p^n, p^n, \ldots), \quad \delta = (x_1, p^n x_2, \ldots), \quad h(\overline{d}) = n. \\
B : \quad & \alpha = \text{diag}(1, 0, 1, 0, \ldots), \quad \delta = (x_1, 0, x_3, 0, \ldots), \quad h(\overline{d}) = 1. \\
\mathbb{Z}(p^\infty) \oplus B : \quad & \alpha = \text{diag}(1, p, p^2, p^3, \ldots), \quad \delta = (x_1, px_2, p^2 x_3, \ldots), \quad h(\overline{d}) = \infty.
\end{align*}
$$

In the next example we consider the generalized Prüfer group $G \cong \mathcal{H}_{2\omega+1}$ and determine the $Q$-tuple $\Delta(Q)$, the heights $h(\delta^k(Q))$, and the height $h(Q)$ of the quasibasis $Q = \{a_i^k, x_i^k\}$ for the generalized Prüfer group $\mathcal{H}_{2\omega+1}$.

Example 5.5. By Theorem 4.3 the generalized Prüfer group $\mathcal{H}_{2\omega+1}$ has the quasibasis $Q = \{a_i^k, x_i^k \mid i \in \mathbb{N}, k \in \mathbb{N}_0\}$ and the corresponding relation array is $\alpha = (\alpha^k)_{k \in I}$ with $\alpha^k = \text{diag}(\alpha_i^k, \alpha_i^k, \ldots)$, where

$$
\begin{align*}
\alpha_i^0 = (p^2, 0, \ldots) \quad \text{and} \quad \alpha_i^k = (0, \ldots, 0, -p^k, p^{k+1}, 0, \ldots) \quad \text{for all} \quad i, k \in \mathbb{N}.
\end{align*}
$$
Thus

\[ \Delta(Q) = (\delta^k \mid k \in \mathbb{N}_0), \] where \( \delta^0 = (p^2 x_i^1 \mid i) \) and \( \delta^k = (-p^k(x_i^k - px_i^{k+1}) \mid i) \) for \( k \in \mathbb{N} \).

So the heights of the \( \tilde{\delta}^k \) are \( h(\delta^0) = 2 \) and \( h(\delta^k) = k \) for \( k \in \mathbb{N} \), hence \( h(Q) = h(\tilde{\delta}^1) = 1 \).

Our next objective is to show that the height is invariant for related inductive quasibases. Let \( Q \) be a quasibasis of a group \( G \) with \( Q \)-tuple \( \Delta(Q) = (\delta^k(Q) \mid k) \). We begin with Proposition 5.6 by showing that for some fixed \( k_0 \in I \) we may switch to a related quasibasis of \( G \) such that only the entry \( \delta^{k_0}(Q) \) is changed and this in a quite arbitrary way.

**Proposition 5.6.** Let \( Q = \{a_i^k, \bigoplus B_i\} \) be an inductive quasibasis of \( G \) with \( Q \)-tuple \( \Delta(Q) = (\delta^k(Q) \mid k) \). Let \( k_0 \in I \) be fixed and let \( \delta = \sum_{k \in I} \lambda_k \delta^k(Q) \) be a (normed) \( Q \)-combination with \( p \mid \lambda_{k_0} \). Then there is an inductive quasibasis \( P = \{c_i^k, \bigoplus B_i\} \) of \( G \), related to \( Q \), with \( P \)-tuple \( \Delta(P) = (\delta^k(P) \mid k) \) such that

\[
\delta^k(P) = \begin{cases} 
\delta, & \text{if } k = k_0, \\
\delta^k(Q), & \text{if } k \neq k_0.
\end{cases}
\]

**Proof.** For

\[
c_i^k = \begin{cases} 
\sum_{l \in I} \lambda_l a_i^l, & \text{if } k = k_0, \\
 a_i^k, & \text{if } k \neq k_0,
\end{cases}
\]

we show that \( P = \{c_i^k, \bigoplus B_i\} \) is an inductive quasibasis of \( G \). The conditions (i) and (iii) in the definition are obvious and it remains to show (ii). Since \( \lambda_{k_0} a_i^{k_0} = c_i^{k_0} - \sum_{k \in I \setminus \{k_0\}} \lambda_k c_i^k \in (c_i^k \mid k \in I, i \in \mathbb{N}) \) and \( p \mid \lambda_{k_0} \), we get \( \langle a_i^k \mid k \in I, i \in \mathbb{N} \rangle = \langle c_i^k \mid k \in I, i \in \mathbb{N} \rangle \). Hence

\[
G/B = \bigoplus_{k \in I} \langle a_i^k + B \mid i \in \mathbb{N} \rangle = \sum_{k \in I} \langle c_i^k + B \mid i \in \mathbb{N} \rangle.
\]

Define \( C^k = \langle c_i^k + B \mid i \in \mathbb{N} \rangle \). We now prove that \( \sum_{k \in I} C^k \) is a direct sum. Since \( C^k \cong \mathbb{Z}(p^\infty) \) for all \( k \in I \), we may write an arbitrary element \( c \in \sum_{k \in I} C^k \) in the form \( c = \sum_{k \in I} \mu_k c_i^k + B \), \( \mu_k \in \mathbb{Z} \) for some \( i \in \mathbb{N} \). Then

\[
c = \sum_{l \in I} \mu_{k_0} a_l^l + \sum_{k \in I \setminus \{k_0\}} \mu_k a_i^k + B = \mu_{k_0} a_i^{k_0} + \sum_{k \in I \setminus \{k_0\}} (\mu_{k_0} \lambda_k + \mu_k) a_i^k + B.
\]

If \( c = 0 \in G/B \), then \( p^j \mid \mu_{k_0} \lambda_k \) and \( p^j \mid (\mu_{k_0} \lambda_k + \mu_k) \) for all \( k \in I \setminus \{k_0\} \), because \( Q \) is a quasibasis. Thus, \( p^j \mid \mu_k \) for all \( k \in I \), because \( p \nmid \lambda_{k_0} \). This shows that \( c = 0 \in G/B \) implies
Thus, \( \mu_k c_i^k \in B \), and the sum \( \sum_{k \in I} C^k \) is direct. Consequently, \( P \) is a quasibasis of \( G \). Moreover, it is inductive, because \( c_i^k - pc_{i+1}^k \in B_i \) for all \( k \in I \).

In particular, \( \delta^k (P) = \delta^k (Q) \) for \( k \neq k_0 \), and

\[
\delta^{k_0} (P) = \left( c_i^{k_0} - pc_{i+1}^{k_0} \mid i \in \mathbb{N} \right) = \left( \sum_{i \in I} \lambda_i a_i^k - p \sum_{i \in I} \lambda_i a_i^{k+1} \mid i \in \mathbb{N} \right) = \left( \sum_{i \in I} \lambda_i b_i^k \mid i \in \mathbb{N} \right) = \sum_{k \in I} \lambda_k \delta^k (Q) = \delta.
\]

An inductive quasibasis \( Q \) of \( G \) is called \textit{normed}, if \( h(\delta^k) = h(Q) \) for every \( k \in I \). Now we show that the group \( G \) has a normed, related inductive quasibasis \( P \) with \( h(P) = h(Q) \).

**Lemma 5.7.** For every inductive quasibasis \( Q \) of \( G \) there is a normed, related inductive quasibasis \( P \) of \( G \) with \( h(P) = h(Q) \).

**Proof.** Let \( Q = \{ a_i^k, \bigoplus B_i \mid k \in I \} \) be an inductive quasibasis of \( G \). Recall the notation in Equation (1). Let \( \hat{\delta}^k = \delta^k (Q) \) for all \( k \in I \). To construct \( P \), we choose some \( k_0 \in I \) with \( h(\hat{\delta}^{k_0}) = h(Q) \) and define the subset \( J = \{ k \in I \mid h(\hat{\delta}^k) \neq h(Q) \} \subset I \). We use the idea of Proposition 5.6 and show that \( P = \{ c_i^k, \bigoplus B_i \} \) with

\[
c_i^k = \begin{cases} a_i^k, & \text{for } k \in I \setminus J, \\ a_i^k + a_i^{k_0}, & \text{for } k \in J, \end{cases}
\]

is a normed, related inductive quasibasis of \( G \) with \( h(P) = h(Q) \). The set \( P \) clearly satisfies the conditions (i) and (iii) of the definition. Condition (ii) is also satisfied from the following. Note that for \( k \in J \)

\[
c_i^k - pc_{i+1}^k = a_i^k - pa_{i+1}^k + a_i^{k_0} - pa_{i+1}^{k_0} = b_i^k + b_i^{k_0} \in B_i.
\]

Thus

\[
\bigoplus_{k \in I} (c_i^k + B \mid i \in \mathbb{N}) = \left( \bigoplus_{k \in J} (a_i^k + a_i^{k_0} + B \mid i \in \mathbb{N}) \right) \oplus \left( \bigoplus_{k \in I \setminus J} (a_i^k + B \mid i \in \mathbb{N}) \right)
\]

\[
= \bigoplus_{k \in I} (a_i^k + B \mid i \in \mathbb{N}).
\]

Hence, \( P \) is an inductive quasibasis of \( G \) which is related to \( Q \) and given by

\[
\delta^k (P) = \begin{cases} \delta^k, & \text{for } k \in I \setminus J, \\ \delta^k + \delta^{k_0}, & \text{for } k \in J. \end{cases}
\]

By Lemma 5.2,

\[
h(\hat{\delta}^k (P)) = h(\hat{\delta}^k + \hat{\delta}^{k_0}) = h(\hat{\delta}^{k_0}) = h(Q)
\]
for all \( k \in J \), because \( h(\delta^{k_0}) < h(\delta^k) \). Thus \( h(\delta^k) = h(Q) \) for all \( k \in I \) and hence, \( P \) is normed with \( h(P) = h(Q) \).

We are now ready to prove the main result in this section on the invariance of height for related inductive quasibases.

**Theorem 5.8.** Related inductive quasibases of \( G \) have the same height.

**Proof.** By Lemma 5.7 it suffices to show that \( h(Q) = h(P) \) for two normed, related inductive quasibases \( Q = \{a_i^*, \bigoplus B_i\} \) and \( P = \{c_i^*, \bigoplus B_i\} \) of \( G \). Assume \( h(Q) > h(P) \) and let \( h = h(P) \). Also, let \( b_i^k = a_i^k - p^k_i t_i + 1 \) and \( d_i^k = c_i^k - p^k_i t_i + 1 \) for all \( i \in \mathbb{N}, k \in I \). By Lemma 3.2, let \( k_0 \in I \) and let \( (\lambda_k \mid k \in I) \) be a normed zero tuple such that

\[
\delta_k^k - \sum_{k \in I} \lambda_k b_i^k \in p^n B_i,
\]

for each \( n \in \mathbb{N} \) and almost all \( i \in \mathbb{N} \). Let \( \delta_k^{k_0} = \delta_k^{k_0}(P) = (d_i^{k_0} \mid i \in \mathbb{N}) \) and \( \delta = \sum_{k \in I} \lambda_k \delta_k^k(Q) = (\sum_{k \in I} \lambda_k b_i^k \mid i \in \mathbb{N}) \). From (3) it follows that \( h(\delta_k^{k_0} - \delta) = \infty \). By Lemma 5.2 we get

\[
h(\delta) = h(\delta_k^{k_0}) = h.
\]

Thus the set \( J = \{k \in I \mid p^{k+1} \mid \lambda_k\} \) must be finite and nonempty. Write \( \delta = \delta_1 + \delta_2 \) where \( \delta_1 = \sum_{k \in J} \lambda_k \delta_k^k(Q) \) and \( \delta_2 = \sum_{k \in I \setminus J} \lambda_k \delta_k^k(Q) \). Thus,

\[
h(\delta) = h(\delta_1 + \delta_2) = \min\{h(\delta_1), h(\delta_2)\} > h,
\]

because of \( h(\lambda_k \delta_k(Q)) \geq h(\delta_k^k(Q)) = h(Q) > h \) for all \( k \in J \). On the other hand \( h(\delta_2) > h \), because \( \delta_2 \in p^{k+1} B_i \). Moreover, \( h(\delta_1), h(\delta_2) > h \), and

\[
h = h(\delta) = h(\delta_1 + \delta_2) = \min\{h(\delta_1), h(\delta_2)\} > h,
\]

a contradiction. Hence, \( h(Q) = h(P) \).

**6. Quasibases of Reduced Groups**

In this section we characterize separable and nonreduced groups in terms of the height of an inductive quasibasis. We begin with a lemma which describes the Ulm subgroup.

**Lemma 6.1.** Let \( Q = \{a_i^*, \bigoplus B_i\} \) be an inductive quasibasis of \( G \). Let \( 0 \neq g \in G \) be of order \( p^j \). Then \( g \in p^i G \) if and only if there is a normed zero tuple \( (\lambda_k \mid k \in I) \), such that \( g = \sum_{k \in I} \lambda_k a_i^k + b, b \in B \), and there is a natural number \( n \) such that

\[
g = p^n \sum_{k \in I} \lambda_k a_{j+n}^k = p^{n+1} \sum_{k \in I} \lambda_k a_{j+n+1}^k = p^{n+2} \sum_{k \in I} \lambda_k a_{j+n+2}^k = ...
\]

In particular, \( h(\sum_{k \in I} \lambda_k \delta_k) \geq j \).
PROOF. Since $G/B = \bigoplus_{k \in I} (\lambda k^n + B \mid k \in I) \cong \bigoplus_{i \in \mathbb{N}} \mathbb{Z}(p^\infty)$, there is, by Proposition 2.1, a normed zero tuple $(\lambda_k \mid k \in I)$ such that $g \in \sum_{k \in I} \lambda_k a^k_j + B$ and the set $\{\sum_{k \in I} \lambda_k a^k_j + B \mid i \in \mathbb{N}\}$ generates a $\mathbb{Z}(p^\infty)$. Now let $g = \sum_{k \in I} \lambda_k a^k_j + b \in p^nG$ with $b \in \bigoplus_{i < l} B_i$ for some $l \in \mathbb{N}$. Then by Lemma 3.1

$$g = \sum_{k \in I} \lambda_k a^k_j + b = \sum_{k \in I} \lambda_k \left( p^n a^k_{j+n} + \sum_{r=0}^{n-1} p^r b^k_{j+r} \right) + b \in p^nG,$$

for all $n \in \mathbb{N}$. Hence for all $n \geq l$

$$g - p^n \sum_{k \in I} \lambda_k a^k_{j+n} = \sum_{r=0}^{n-1} \sum_{k \in I} \lambda_k p^r b^k_{j+r} + b = \sum_{r=j}^{n+j-1} \sum_{k \in I} \lambda_k p^{r-j} b^k_r + b = \sum_{r=j}^{l-1} \sum_{k \in I} \lambda_k p^{r-j} b^k_r + b + \sum_{r=j}^{n+j-1} \sum_{k \in I} \lambda_k p^{r-j} b^k_r \in p^nB.$$

Let $b_r = \sum_{k \in I} \lambda_k p^{r-j} b^k_r \in B_r$. In view of height and order considerations we conclude that $\sum_{r=j}^{l-1} b_r + b = 0$. Thus $\sum_{r=j}^{n+j-1} b_r \in p^nB$ for all $n \geq l$, i.e., $b_r = 0$ for all $r \geq l$. It follows that $g = p^n \sum_{k \in I} \lambda_k a^k_{j+n}$ for all $n \geq l$. Consequently $g \in p^nG$ has the indicated form. Moreover, $b_r = \sum_{k \in I} \lambda_k p^{r-j} b^k_r = 0$ implies that $p^j \mid \sum_{k \in I} \lambda_k b^k_r$, for all $r \geq l$, and $h(\sum_{k \in I} \lambda_k 3^k) \geq j$. \hfill \qed

COROLLARY 6.2. Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of $G$. If $j > h(\delta_{\lambda_0})$, then $a_j^{k_0} \notin p^nG$.

PROOF. If $a_j^{k_0} \in p^nG$, then by Lemma 6.1 there is a normed zero tuple $(\lambda_k \mid k \in I)$, such that $a_j^{k_0} = \sum_{k \in I} \lambda_k a^k_j + b$ and $a_j^{k_0} = p^n \sum_{k \in I} \lambda_k a^k_{j+n}$ for all $j$. Thus $b = 0$, $p^j \mid \lambda_k$ for all $k \neq k_0$, and consequently $a_j^{k_0} = p^n a^k_{j+n}$ for all $n$.

Now, let $j > h = h(\delta_{\lambda_0})$. So there are infinitely many $i \in \mathbb{N}$ where $p^{h+1} \mid b_i^{k_0} = a_i^{k_0} - pa_i^{k_0} \in B_{j+n}$. Hence we may select an $n$ with $p^{h+1} \mid b_i^{k_0}$ for all $j$. We assume $a_j^{k_0} \in p^nG$ and apply Lemma 3.1, i.e.,

$$\sum_{r=0}^{n} p^r b_{j+r}^{k_0} = a_j^{k_0} - p^{n+1} a_{j+n+1}^{k_0} = 0.$$

Using $B = \bigoplus B_i$, we get $p^r b_{j+r}^{k_0} = 0$ for all $0 \leq r \leq n$. In particular, $p^n b_{j+n}^{k_0} = 0$ implies $p^j \mid b_{j+n}^{k_0}$. But $j > h$ further implies that $p^{h+1} \mid b_{j+n}^{k_0}$, a contradiction. Thus $a_j^{k_0} \notin p^nG$. \hfill \qed

Our next theorem describes separable groups in terms of the height of a quasibasis.

THEOREM 6.3. Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of the reduced group $G$. Then $G$ is separable if and only if $h(\delta) = 0$ for all $Q$-combinations $\delta$. 

PROOF. Suppose $G$ is separable and let $\delta = \sum_{k \in I} \lambda_k \delta^k(Q)$ be a Q-combination. If $h(\delta) > 0$, then there is a $j \in \mathbb{N}$, such that $\sum_{k \in I} \lambda_k b_i^k \in pB_i$ for all $i \geq j$. By this fact combined with Lemma 3.1 we get for all $n \in \mathbb{N}$

$$\sum_{k \in I} \lambda_k p^{j-1} a_j^k = \sum_{k \in I} \lambda_k p^{j-1} \left( p^n a_{j+n}^k + \sum_{r=0}^{n-1} p^r b_{j+r}^k \right)$$

$$= p^{j+n-1} \sum_{k \in I} \lambda_k a_{j+n}^k + \sum_{r=0}^{n-1} p^{j+r-1} \sum_{k \in I} \lambda_k b_{j+r}^k = p^{j+n-1} \sum_{k \in I} \lambda_k a_{j+n}^k.$$ 

This holds for all $n$, thus $\sum_{k \in I} \lambda_k p^{j-1} a_j^k \neq 0$ has infinite height, a contradiction, by [3, 65.1].

Conversely, assume that $G$ is nonseparable, i.e., there is a $0 \neq g \in G$ of infinite height and order $p^i$. By Lemma 6.1 there is a normed zero tuple $(\lambda_k | k \in I)$ with $h(\sum_{k \in I} \lambda_k \delta^k) \geq j > 0$, contradicting the hypothesis that $h(\delta) = 0$ for all normed Q-combinations $\delta$. □

The following lemma shows, that a Q-combination $\delta$ with $h(\delta) = \infty$ allows to find a divisible subgroup of $G$.

**Lemma 6.4.** Let $Q = \{a_i^k, \bigoplus B_i \}$ be an inductive quasibasis of $G$. If $\delta = \sum_{k \in I} \lambda_k \delta^k(Q)$ is a Q-combination with $h(\delta) = \infty$, then there is a strictly increasing sequence $(n_i)_{i \in \mathbb{N}}$ of natural numbers, such that $\langle d_i \in G | i \in \mathbb{N} \rangle \cong \mathbb{Z}(p^\infty)$ with $pd_1 = 0$ and $d_i = pd_{i+1} = p^{n_i} \sum_{k \in I} \lambda_k a_{i+n_i}^k$ for $i \in \mathbb{N}$.

**Proof.** Since $h(\delta) = \infty$, there is a strictly increasing sequence $(n_i)_{i \in \mathbb{N}}$ of natural numbers, such that $p^i \mid \sum_{k \in I} \lambda_k b_i^k$ for all $n \geq n_i$. Define $d_i = p^{n_i} \sum_{k \in I} \lambda_k a_{i+n_i}^k$, for all $i \in \mathbb{N}$. Clearly $pd_1 = p^{n_1+1} \sum_{k \in I} \lambda_k a_{1+n_1}^k = 0$. Moreover, for all $i \in \mathbb{N}$

$$pd_{i+1} = p^{n_{i+1}+1} \sum_{k \in I} \lambda_k a_{i+n_{i+1}+1}^k$$

$$= p^{n_i+(n_{i+1}-n_i+1)} \sum_{k \in I} \lambda_k a_{i+n_i+(n_{i+1}-n_i+1)}^k$$

$$= p^{n_i} \sum_{k \in I} \lambda_k a_{i+n_i}^k - \sum_{r=0}^{n_{i+1}-n_i} p^{n_i+r} \sum_{k \in I} \lambda_k b_{i+n_i+r}^k = d_i,$$

by Lemma 3.1 and the height condition above. Since $(\lambda_k | k \in I) \neq 0$, we have $d_i \neq 0$, for almost all $i \in \mathbb{N}$. Hence $\langle d_i \in G | i \in \mathbb{N} \rangle \cong \mathbb{Z}(p^\infty)$. □

**Lemma 6.5.** Let $Q = \{a_i^k, \bigoplus B_i \}$ be an inductive quasibasis of $G$. If $D$ is a divisible subgroup of $G$ of rank 1, then there is a Q-combination $\delta = \sum_{k \in I} \lambda_k \delta^k(Q)$ such that $h(\delta) = \infty$, and $D \subset \langle \sum_{k \in I} \lambda_k a_i^k | i \in \mathbb{N} \rangle$. 
Proof. Let \( D = \langle g_i \in G \mid i \in \mathbb{N} \rangle \cong \mathbb{Z}(p^\infty) \) be a divisible subgroup of \( G \) with \( pg_1 = 0 \), \( pg_{i+1} = g_i \neq 0 \) for \( i \in \mathbb{N} \). Since \( \mathbb{Z}(p^\infty) \cong \langle g_i + B \mid i \in \mathbb{N} \rangle \subset G/B \), there is a normed zero tuple \((\lambda_k \mid k \in I)\), such that \( g_i = \sum_{k \in I} \lambda_k a^k_i + B \), for all \( i \in \mathbb{N} \), by Proposition 2.1. Hence by Lemma 6.1

\[ g_i = p^n \sum_{k \in I} \lambda_k a^k_{i+n} \text{ for almost all } n \in \mathbb{N} \text{ and all } i \in \mathbb{N}. \]

Thus for each \( i \in \mathbb{N} \) and for almost all \( n \in \mathbb{N} \)

\[ 0 = g_i - pg_{i+1} = p^n \sum_{k \in I} \lambda_k (a^k_{i+n} - pa^k_{i+n+1}) = p^n \sum_{k \in I} \lambda_k b^k_{i+n}, \]
i.e., \( p^i \mid \sum_{k \in I} \lambda_k b^k_{i+n} \). Consequently, \( h(\sum_{k \in I} \lambda_k \delta^k) = \infty \), and, in particular, \( D = \langle g_i \in G \mid i \in \mathbb{N} \rangle \subset \langle \sum_{k \in I} \lambda_k a^k_i \mid i \in \mathbb{N} \rangle \).

The Lemmata 6.4 and 6.5 lead to the main result in this section.

**Theorem 6.6.** Let \( Q = \{a^i \oplus B_i\} \) be an inductive quasibasis of \( G \). Then \( G \) is not reduced if and only if there is a \( Q \)-combination \( \delta \) with \( h(\delta) = \infty \).

The Theorems 6.3 and 6.6 characterize also the reduced nonseparable groups.

**References**


