On log-growth of solutions of $p$-adic differential equations with $p$-adic exponents

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Abstract – We consider a differential system $x \frac{d}{dx} Y = G Y$, where $G$ is a $m \times m$ matrix whose coefficients are power series which converge and are bounded on the open unit disc $D(0, 1^−)$. Assume that $G(0)$ is a diagonal matrix with $p$-adic integer coefficients. Then there exists a solution matrix of the form $Y = F \exp(G(0) \log x)$ at $x = 0$ if all exponent differences are $p$-adically non-Liouville numbers. We give an example where $F$ is analytic on the $p$-adic open unit disc and has log-growth greater than $m$. Under some conditions, we prove that if a solution matrix at a generic point has log-growth $\delta$, then $F$ has log-growth $\delta$.

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1. Introduction

Let $K$ be a non-archimedean complete discrete valuation field and $K[[x]]_0$ be the ring of power series which converge and are bounded on the open unit disc $D(0, 1^−)$. Let $M$ be a $n \times n$ matrix and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $M$ counting multiplicities. We say that $\alpha_0, \cdots, \alpha_l$ are the distinct eigenvalues of $M$ if they satisfy $\alpha_i \neq \alpha_j$ for all $0 \leq i < j \leq l$ and $\{\alpha_0, \cdots, \alpha_l\} = \{\lambda_1, \cdots, \lambda_n\}$. We consider the differential system of rank $m$

(1) \[ \frac{d}{dx} Y = GY, \quad G \in M_m\left( \frac{1}{x} K[[x]]_0 \right). \]

Write $G = \sum_{i=-1}^{\infty} x^i G_i, G_i \in M_m(K)$. Assume the following conditions.

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(i) A solution matrix $U$ of the differential system (1) at a generic point $t$

at distance 1 from 0 is analytic on the open unit disc $D(t,1^-)$ around $t$.

(ii) $G^{-1}$ is a diagonal matrix, with distinct eigenvalues $\alpha_0, \ldots, \alpha_l$ which

are $p$-adic integers with $\alpha_i - \alpha_j$ not an integer for $i \neq j$.

Then there exists a formal solution matrix $Y$ of the form

$$F \exp(G^{-1}\log x) = Fx$$

$G^{-1}$ is a nilpotent matrix or has rational exponents, then the analog of Dwork’s

theorem holds in \[DGS,\] Chap. V Theorem 2.1 and [Kedlaya, Theorem 18.5.1]. We give an example of a system which does not satisfy the analog of Dwork’s theorem in Example 3.3, even though the eigenvalues of $G^{-1}$ have $p$-adic non-Liouville differences. Therefore we need to add another assumption about eigenvalues of $G^{-1}$. Denote the $p$-adic expansion of $\alpha_j$ by $\sum_{i \geq 0} \alpha_{j,i}p^i$ for $j = 0, \ldots, l$. Assume that there exists a positive integer $\mu$ which satisfies the following conditions:

1. There are no $0 \leq j \leq l$ such that $\alpha_{j,i} = \cdots = \alpha_{j,i+\mu} = p - 1$ for some $i \in \mathbb{N}$.

2. For all $i \in \mathbb{N}$, there exists $0 \leq \mu_i \leq \mu$ such that $\alpha_{j_1,i+\mu_i} \neq \alpha_{j_2,i+\mu_i}$ for all $0 \leq j_1 < j_2 \leq l$.

Note that the condition (2) shows $\alpha_i - \alpha_j$ are $p$-adically non-Liouville numbers for all $i \neq j$. By Christol and Mebkhout theorem, $F$ is analytic on $D(0,1^-)$ in [Kedlaya, Theorem 13.7.1]. Then we prove that if the solution matrix $U$ at $x = t$ has log-growth $\delta$, then $F$ has log-growth $\delta$. According to [Andrè], this result means the highest slope of the log-growth polygon of (1) at $x = 0$ lies above the highest slope of the log-growth polygon of (1) at $x = t$. We thank the referee for his many careful and insightful suggestions.

2. Notations

2.1 – Analytic functions

We refer to the notation in [Kedlaya]. Let $p$ be a prime number and let $K$

be a complete discrete valuation field of mixed characteristics $(0,p)$. Let $| \cdot |$

be the valuation of $K$ normalized by $|p| = 1/p$. The open disc


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\{x; |x-a| < r\} of center $a$ and radius $r$ is denoted by $D(a, r^-)$. Let $K\{x\} = \{\sum_{i=0}^{\infty} a_i x^i \in K[x]; \lim_{i \to \infty} |a_i|_p = 0 (0 < \rho < 1)\}$ be the ring of power series which converge on the open disc $D(0,1^-)$ and $K[[x]]_0 = \{\sum_{i=0}^{\infty} a_i x^i \in K[[x]]; \ \sup_i |a_i| < \infty\}$ the ring of bounded power series which converge on the open disk $D(0,1^-)$. For any positive integer $s$, we set $A^s = \left\{\sum_{j=0}^{\infty} a_j (x-t)^j \in K((t))[x-t]; a_j \in \frac{1}{t^s} K[[t]]_0\right\}$ and $B^s = \frac{1}{t^s} K[[x-t]/t]][[t]]$. The map $\Theta: A^s \to B^s$ is defined by $\Theta \left(\sum_{j=0}^{\infty} a_j (x-t)^j\right) = \sum_{i=-s}^{\infty} t^i \sum_{j=0}^{\infty} a_{j-i}(\frac{x-t}{t})^j$, where $a_j = \sum_{i=-s-j}^{\infty} a_{j-i} t^i$ with $a_{j-i} \in K$. Note $\Theta(x-t) = t^{\omega(x-t)}$.

**Lemma 2.1.** We have the following properties.

(i) If $f \in A^s$ (resp. $B^s$) and $g \in A^u$ (resp. $B^u$), then $fg \in A^{s+u}$ (resp. $B^{s+u}$).

(ii) $\Theta(af + bg) = a\Theta(f) + b\Theta(g)$, $\Theta(f)\Theta(g) = \Theta(fg)$, $a, b \in K$, $f, g \in A^s$.

**Proof.** (i) Write $f = \sum_{i=0}^{\infty} a_i (x-t)^i \in A^s$ and $g = \sum_{i=0}^{\infty} b_i (x-t)^i \in A^u$ with $a_i \in \frac{1}{t^s} K[[t]]_0$ and $b_i \in \frac{1}{t^u} K[[t]]_0$. Then we have $fg = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} a_i b_{k-i} (x-t)^k$ with $a_i b_{k-i} \in \frac{1}{t^{s+u}} K[[t]]_0$.

(ii) We prove that the coefficients of $t^m (\frac{x-t}{t})^n$ in both sides of $\Theta(f)\Theta(g) = \Theta(fg)$ coincide for all $(n, m)$. Since we have

$$\Theta \left(\sum_{j=0}^{\infty} \left(\sum_{i=-s-j}^{\infty} a_{j-i} t^i\right) (x-t)^j\right) = \sum_{i=-s}^{\infty} t^i \sum_{j=0}^{\infty} a_{j-i} \left(\frac{x-t}{t}\right)^j,$$

the coefficient of $(x-t)^n/t^n$ in $\Theta \left(\sum_{j=0}^{\infty} \left(\sum_{i=-s-j}^{\infty} a_{j-i} t^i\right) (x-t)^j\right)$ is zero for $l > n$. Write $f = \sum_{i=0}^{\infty} f_i (x-t)^i$ and $g = \sum_{i=0}^{\infty} g_i (x-t)^i$. Since the coefficient of $(x-t)^n/t^n$ in $\Theta(f)$ is equal to the coefficient of $(x-t)^n/t^n$ in $\Theta(\sum_{i=0}^{n} f_i (x-t)^i)$, we have only to prove that

$$\Theta(\sum_{i=0}^{n} f_i (x-t)^i)\Theta(\sum_{i=0}^{n} g_i (x-t)^i) = \Theta(\sum_{i=0}^{n} f_i (x-t)^i)\sum_{i=0}^{n} g_i (x-t)^i).$$

Since we have $\Theta(\sum_{i=0}^{n} f_i (x-t)^i) = \sum_{i=0}^{n} \Theta(f_i (x-t)^i)$, it suffices to show

$$\Theta(f_1 (x-t)^i)\Theta(g_j (x-t)^j) = \Theta(f_1 (x-t)^i)g_j (x-t)^j),$$
which we may prove by direct computation. □

We define the derivation \( \frac{d}{dx} \) on \( A^s \) and \( B^s \) as follows:

\[
\frac{d}{dx} : A^s \to A^{s-1}, \quad \sum_{j=0}^{\infty} a_j (x-t)^j \mapsto \sum_{j=0}^{\infty} j a_j (x-t)^{j-1},
\]

\[
\frac{d}{dx} : B^s \to B^{s-1}, \quad \sum_{i=-s}^{\infty} t^i \sum_{j=0}^{\infty} a_{j,-j+i} \frac{(x-t)^j}{t^j} \mapsto \sum_{i=-s-1}^{\infty} t^i \sum_{j=0}^{\infty} j a_{j,-j+i} \frac{(x-t)^{j-1}}{t^{j-1}}.
\]

Lemma 2.2. For \( f \in A^s \), we have \( \Theta(\frac{d}{dx}f) = \frac{d}{dx}(\Theta f) \).

**Proof.** Write \( f = \sum_{j=0}^{\infty} \sum_{i=-s-j}^{\infty} f_{ji} t^i (x-t)^j \). Then we have

\[
\Theta \left( \frac{d}{dx} f \right) = \Theta \left( \sum_{j=0}^{\infty} \sum_{i=-s-j-1}^{\infty} (j+1) f_{j+1,i} t^i (x-t)^j \right)
\]

\[
= \sum_{i=-s-1}^{\infty} t^i \sum_{j=0}^{\infty} (j+1) f_{j+1,-j-1+i} \frac{(x-t)^j}{t^j},
\]

\[
\frac{d}{dx} \Theta(f) = \sum_{i=-s-1}^{\infty} t^i \sum_{j=0}^{\infty} j f_{j,-j+i} \frac{(x-t)^{j-1}}{t^{j-1}}
\]

\[
= \sum_{i=-s-1}^{\infty} t^i \sum_{j=0}^{\infty} (j+1) f_{j+1,-j-1+i} \frac{(x-t)^j}{t^j}.
\]

□

2.2 – The norms of \( A^s \) and \( B^s \)

In this subsection, we define norms on \( A^s \) and \( B^s \). We define the Gauss norm on \( \frac{1}{r}K[[t]]_0 \) to be \( |\sum_{i=-s}^{\infty} a_i t^i| = \sup_i |a_i| \).

Lemma 2.3. Let \( f = \sum_{n=0}^{\infty} a_n (x-t)^n \in A^s \) and \( \Theta(f) = \sum_{i=-s}^{\infty} t^i \sum_{j=0}^{\infty} a_{j,-j+i} \left( \frac{x-t}{t} \right)^j \).

(i) \( \sup_n |a_n|r^n \leq C \) if and only if \( \sup_j |a_{j,-j+i}|r^j \leq C \) for all \( i \).

(ii) If \( \sup_n |a_n|r^n < \infty \), then we have \( \sup_n |a_n|r^n = \sup_i \sup_j |a_{j,-j+i}|r^j \).
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Proof. (i) By the definition of $\Theta$, $a_j = \sum_{i=-s}^{\infty} a_{j,i} t^{i-j}$. If $\sup_n |a_n|^r \leq C$, then we obtain

$$C \geq \left\{ \sum_{i=-s}^{\infty} a_{n,i} t^{i-n} \right\} r^n = \sup_{-s \leq i < \infty} |a_{n,i-n}| r^n \geq |a_{n,i-n}| r^n$$

for all $0 \leq n < \infty$. Thus we see

$$\sup_{0 \leq n < \infty} |a_{n,i-n}| r^n \leq C$$

for all $-s \leq i < \infty$. Assume

$$\sup_j |a_{j,-j+i}| j^j \leq C$$

for all $i$. Since $|a_{j,-j+i}| j^j \leq C$ for all $i$, we have $|a_j| j^j = \sup_{-s \leq i < \infty} |a_{j,-j+i}| j^j \leq C$. Therefore this shows $\sup_n |a_n|^r \leq C$.

(ii) This is seen by (i). \hfill \Box

We define subsets $A^*[0,1)$ and $B^*[0,1)$ of $A^*$ and $B^*$ respectively to be

$$A^*[0,1) = \left\{ \sum_{n=0}^{\infty} a_n (x-t)^n \in A^* : \lim_{n \to \infty} |a_n|^r = 0 \quad (0 < r < 1) \right\},$$

$$B^*[0,1) = \left\{ \sum_{i=-s}^{\infty} t^i \sum_{j=0}^{\infty} a_{j,-j+i} (x-t)^j j^j \in B^* : \sup_n \sup_j |a_{j,-j+i}| j^j < \infty \right\} \sup_j (for all 0 < r < 1 and i).$$

The map $\Theta$ induces $A^*[0,1) \to B^*[0,1)$, denoted also by $\Theta$. Indeed since $|a_{n,-n+i}| r^n \leq |a_n|^r$, it follows $\lim_{n \to \infty} |a_{n,-n+i}| r^n \leq \lim_{n \to \infty} |a_n|^r$ for all $i$.

By Lemma 2.3(ii), we have $\sup_i \sup_j |a_{j,-j+i}| j^j = \sup_n |a_n|^r < \infty$. For $f = \sum_{j=0}^{\infty} a_j (x-t)^j \in A^*[0,1)$ (resp. $g = \sum_{i=-s}^{\infty} t^i \sum_{j=0}^{\infty} a_{j,-j+i} (x-t)^j j^j \in B^*[0,1)$), we define the norm to be $\|f\|_r = \sup_j |a_j| j^j$ (resp. $\|g\|_r = \sup_i \sup_j |a_{j,-j+i}| j^j$). By Lemma 2.3 (ii), we have the following lemma.

**Lemma 2.4.** For $f \in A^*[0,1)$, we have $|\Theta(f)|_r = \|f\|_r$ for all $0 < r < 1$.

We denote the image of $f$ via the map $A^* \to \frac{1}{r} K[[t]]$, $\sum_{i=0}^{\infty} a_i (x-t)^i \mapsto a_0$ (resp. $B^* \to \frac{1}{r} K[[t]]$, $\sum_{i=-s}^{\infty} t^i \sum_{j=0}^{\infty} a_{j,-j+i} (x-t)^j j^j \mapsto \sum_{i=-s}^{\infty} a_{0,i} t^i$) by $f(t)$, Note $\Theta(f)(t) = f(t)$. We consider formal solutions of differential system over $B^1$. 
Lemma 2.5. Let $G$ be an element of $M_m(B^1)$ and $F$ be any element of $M_m(\frac{1}{t} K[[t]])$. Then there is a unique solution $Y \in Gl_m(B^0)$ of the differential system $Y' = GY$ with $Y(t) = F$.

Proof. Write $G = \sum_{i=-1}^{\infty} t^i G_i$, $Y = \sum_{i=0}^{\infty} t^i Y_i$ and $F = \sum_{i=0}^{\infty} t^i F_i$ with $G_i, Y_i \in M_m(K[[x-t]/t])$ and $F_i \in M_m(K)$. We seek $Y_i$ which satisfies $Y' = GY$ with $Y_i(t) = F_i$. Note $t^j Y'_i \in M_m(K[[x-t]/t])$. Comparing the coefficients of $t^i$ on both sides of $Y' = GY$, we have

$$
t^0 Y'_i = G_{-1} Y_0
$$

$$
t^1 Y'_i = G_{-1} Y_1 + G_0 Y_0
$$

$$
\vdots
$$

$$
t^i Y'_i = \sum_{j=0}^{i} G_{j-1} Y_{i-j}.
$$

We transform $(x-t)/t$ into $X$. Then we obtain the equality

$$
\frac{d}{dX} Y_i = \sum_{j=0}^{i} G_{j-1} Y_{i-j}, \quad Y_i, G_j \in M_m(K[[X]])
$$

for $i = -1, 0, 1 \cdots$. Let $V \in Gl_m(K[[X]])$ be the solution of $\frac{d}{dX} V = G_{-1} V$ with $V(0) = I_m$. We set $Y_i = VW_i$. We set $W_0 = F_0$. Substitute this equality into (2). We have

$$
\frac{d}{dX} W_i = V^{-1} \sum_{j=1}^{i} G_{j-1} Y_{i-j}
$$

Assume that there exists $Y_j$ which satisfies $Y' = GY$ with $Y_j(t) = F_j$ for $j = 0, \cdots, i-1$. Then $W_i$ is determined uniquely by (3) and $W_i(t) = F_i$. Therefore there exist $Y_i = VW_i$ uniquely which satisfies $Y' = GY$ with $Y_i(t) = F_i$. \qed

Let $(R, | \cdot |)$ be a valuation field. We define $|A| = \max_{i,j} |a_{i,j}|$ for all $A = (a_{i,j}) \in M_m(R)$.

2.3 - The map $\tau$ and $\tilde{\tau}$

For $f \in \frac{1}{x} K[[x]]$, we denote the image of $f$ of the map $\frac{1}{x} K[[x]] \to \frac{1}{t} K[[t]]$, $x \mapsto t$ by $f(t)$. The map $\tau : \frac{1}{x} K[[x]]_0 \to A^*[0,1)$ is defined to be $f \mapsto \sum_{i=0}^{\infty} \frac{d^i f}{dx^i}(t) \frac{(x-t)^i}{i!}$. Then $\tau$ satisfies the following properties.
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**Lemma 2.6.** [Christol, p.52] (i) \( \tau(af + bg) = a\tau(f) + b\tau(g), \quad \tau(fg) = \tau(f)\tau(g) \) \((a, b \in K, f, g \in \frac{1}{x}K[[x]]_0)\).

(ii) \( \tau \left( \frac{df}{dx} \right) = \frac{d}{dx}\tau(f) \).

The map \( \tilde{\tau} : \frac{1}{x}K\{x\} \to B^*[0,1) \) is defined to be \( \sum_{n=-s}^{\infty} a_n x^n \mapsto \sum_{i=-s}^{\infty} t^i a_i \tau(x^i) \). We say set of \( p \)-adic integers \( \lambda_0, \cdots, \lambda_l \) are prepared set if they satisfy the following conditions:

\[
\lambda_i \in \mathbb{Z} \Leftrightarrow \lambda_i = 0, \quad \lambda_i - \lambda_j \in \mathbb{Z} \Leftrightarrow \lambda_i = \lambda_j.
\]

Let \( \alpha_0, \cdots, \alpha_l \) be a prepared set. The map \( \tilde{\tau} \) is extended to the map from \( \bigoplus_{i=0}^{l} x^{\alpha_i} \frac{1}{x^s}K\{x\} \) to \( \bigoplus_{i=0}^{l} x^{\alpha_i} \frac{1}{x^{s+1}}K\{x\} \) by the action

\[
\tilde{\tau}(x^{\alpha_j}g_{j,i}x^i) \mapsto \sum_{j=0}^{l} \sum_{i=-s}^{\infty} x^{\alpha_j}(\alpha_j + i + 1)g_{j,i+1}x^i.
\]

Then \( \tilde{\tau} \) satisfies the same properties as \( \tau \).

**Lemma 2.7.** (i) \( \tilde{\tau}(af + bg) = a\tilde{\tau}(f) + b\tilde{\tau}(g), \quad (a, b \in K, f, g \in \frac{1}{x}K\{x\}) \).

(ii) \( \tilde{\tau}(fg) = \tilde{\tau}(f)\tilde{\tau}(g), \quad (f \in \frac{1}{x}K\{x\}, g \in \bigoplus_{i=0}^{l} x^{\alpha_i} \frac{1}{x^s}K\{x\}) \).

(iii) \( \tilde{\tau} \left( \frac{df}{dx} \right) = \frac{d}{dx}\tilde{\tau}(f), \quad (g \in \bigoplus_{i=0}^{l} x^{\alpha_i} \frac{1}{x}K\{x\}) \).

(iv) \( \tilde{\tau}|_{\frac{1}{x}K[[x]]_0} = \Theta \circ \tau \).

**Proof.** (i) We can prove easily.
(ii) Write \( f = \sum_{i=-s}^{\infty} f_i x^i \) and \( g = \sum_{j=0}^{l} \sum_{i=-s}^{\infty} x^{\alpha_i} g_{j,i} x^i \). Then we have

\[
\hat{\tau}(f)\hat{\tau}(g) = \sum_{i=-s}^{\infty} t^i \tau(f_i) \frac{\tau(x^i)}{t^i} \sum_{j=0}^{l} \sum_{i=-s}^{\infty} \tau(g_{j,i}) \frac{\tau(x^i)}{t^i}
\]

\[
= \sum_{j=0}^{l} \sum_{i=-2s}^{\infty} t^i \sum_{u+k=i}^{u \geq -s, k \geq -s} \tau(f_u) \frac{\tau(x^u)}{t^u} \tau(g_{j,k}) \frac{\tau(x^k)}{t^k}
\]

\[
= \sum_{j=0}^{l} \sum_{i=-2s}^{\infty} t^i \sum_{u+k=i}^{u \geq -s, k \geq -s} \tau(f_u g_{j,k}) \frac{\tau(x^i)}{t^i} = \hat{\tau}(fg).
\]

(iii) We prove \( \frac{d}{dx} \hat{\tau}(x^\alpha) = \hat{\tau} \left( \frac{d}{dx} (x^\alpha) \right) \).

\[
\frac{d}{dx} \hat{\tau}(x^\alpha) = \frac{d}{dx} x^\alpha \sum_{j=0}^{\infty} \binom{\alpha}{j} \frac{(x-t)^j}{t^j} = x^\alpha \sum_{j=0}^{\infty} \binom{\alpha}{j} \frac{(x-t)^j}{t^j}
\]

\[
= \alpha t^\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j-1} \frac{(x-t)^j}{t^j} = \alpha \frac{t^\alpha}{t} \sum_{j=0}^{\infty} \binom{\alpha}{j} x^j \frac{(x-t)^j}{t^j} = \alpha \frac{t^\alpha}{t} \left( 1 + \frac{x-t}{t} \right)^{\alpha-1} \frac{t^\alpha}{t^j} \left( 1 + \frac{x-t}{t} \right)^\alpha
\]

\[
= \frac{\alpha t^\alpha}{t} \left( 1 + \frac{x-t}{t} \right)^{\alpha-1}.
\]

For \( g = \sum_{i=-s}^{\infty} g_i x^i \in \frac{1}{x^s} K\{x\} \), we have

\[
\frac{d}{dx} \hat{\tau}(g) = \frac{d}{dx} \sum_{i=-s}^{\infty} g_i x^i \sum_{i=-s}^{\infty} \frac{\tau(x^i)}{t^i} = \sum_{i=-s}^{\infty} g_i x^i \frac{d}{dx} \sum_{i=-s}^{\infty} \frac{\tau(x^i)}{t^i} = \sum_{i=-s}^{\infty} g_i x^i \frac{\tau(x^{i-1})}{t^{i-1}}
\]

\[
= \hat{\tau} \left( \sum_{i=-s}^{\infty} g_i x^{i-1} \right) = \hat{\tau} \left( \sum_{i=-s-1}^{\infty} (i+1) g_{i+1} x^i \right) = \sum_{i=-s}^{\infty} t^i (i+1) g_{i+1} \frac{\tau(x^i)}{t^i}.
\]

By using (i) and (ii), we can prove the case \( g \in \bigoplus_{i=0}^{l} K\{x\} \).

(iv) A direct computation proves the case \( f = 1/x \). By using (ii) and Lemma
2.7, we may assume \( f \in K[[x]]_0 \). Write \( f = \sum_{i=0}^{\infty} f_i x^i \).

\[
\Theta(\tau(f)) = \Theta \left( \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{d^j f}{dx^j} \right)^t (x-t)^j \right) = \Theta \left( \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} \left( \frac{j+i}{j} \right) f_{j+i} t^i \right) (x-t)^j \right)
\]

\[
= \sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} \left( \frac{j+i}{j} \right) f_i \left( \frac{x-t}{t} \right)^j = \sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} \left( \frac{i}{j} \right) f_i \left( \frac{x-t}{t} \right)^j,
\]

\[
\tilde{\tau}(f) = \sum_{i=0}^{\infty} t^i f_i \frac{\tau(x)^i}{i!} = \sum_{i=0}^{\infty} t^i f_i \left( 1 + \frac{x-t}{t} \right)^i.
\]

For example, we have \( \tau(x) = t + (x - t) = x \) and \( \tilde{\tau}(x) = t^{\frac{1}{t}}(x-t) = x \). Thus if \( \tau(f) \) and \( \tilde{\tau}(f) \) see as elements of \( K(t)[x-t] \) for \( f \in K[x] \), then we obtain \( \tau(f) = \tilde{\tau}(f) \). We can prove \( \tilde{\tau}(fg) = \tilde{\tau}(f)\tilde{\tau}(g) \) with \( f \in \bigoplus_{i=0}^{\infty} x^{\alpha_i} \frac{1}{x^i} K\{x\} \) and \( g \in K\{x\} \).

2.4 - Log-growth

We denote the \( \rho \)-Gauss norm on \( K\{x\} \) by \( |\sum_i c_i x^i|_\rho = \sup_i |c_i| \rho^i \). For \( f \in K\{x\} \) (resp. \( f \in A^0[0,1) \)), we say \( f \) has log-growth \( \delta \) if \( \lim_{x \to 1^-} |f|_r(-\log r)^\delta < \infty \). It is known that \( f = \sum_j c_j x^j \in K\{x\} \) (resp. \( f = \sum_j c_j (x-t)^j \in A^0[0,1) \)) has log-growth \( \delta \) if and only if there exists \( C > 0 \) which satisfies \( |c_i| \leq C i^\delta \) for all \( i \). For a matrix \( (f_{i,j}) \in (K\{x\})^{s \times t} \), we say \( (f_{i,j}) \) has log-growth \( \delta \) if \( f_{i,j} \) has log-growth \( \delta \) for all \( i,j \).

3. The proof of the main theorem.

We consider a differential system of rank \( m \)

\[
(4) \quad \frac{d}{dx} Y = GY, \quad G \in M_m \left( \frac{1}{x} K[[x]]_0 \right).
\]

Write \( G = \sum_{j=-1}^{\infty} G_j x^j \) \( (G_j \in M_m(K)) \). Let \( \alpha_0, \ldots, \alpha_l \in \mathbb{Z}_p \) be the distinct eigenvalues of \( G_{-1} \). Assume that the eigenvalues \( \alpha_0, \ldots, \alpha_l \) are prepared set. Then there exists the solution matrix \( Y \) of the form

\[
F \exp(G_{-1} \log x) = F x^{G_{-1}} \text{ with } F = \sum_{i=0}^{\infty} F_i x^i \in M_m(K[[x]]) \text{ and } F_0 = I_m
\]

by the formal Fuchsian theory. Denote the \( p \)-adic expansion of \( \alpha_i \) by \( \sum_{j=0}^{\infty} \alpha_{i,j} p^j \) for \( i = 0, \ldots, l \). We assume that there exists a positive integer
Theorem 3.1. Assume that the solution matrix \( U = \sum_{i=0}^{\infty} U_i (x - t)^i \in \text{Gl}_m(K[[t]])) \) of \( U' = \tau(G)U \) with \( U_0 = I_m \) satisfies \( \sup_r |U_i| r^l (-\log r)^\delta \leq C \) for all \( 0 < r < 1 \) and some \( C > 0 \). In particular \( U \) is analytic on \( D(t,1^-) \). Then we have the inequality \( \sup_{n \leq p^s} |F_n| \leq C p^{1+2\mu+\delta} (l+1)\delta (\log p)^{-\delta} \). In particular if \( U \) has log-growth \( \delta \), then \( F \) has log-growth \( \delta \).

**Proof.** By the recurrence equation with respect to the coefficients of \( U \) and the assumption, we see \( U \in \text{Gl}_m(A^0[0,1)) \). Indeed since \( U_i \) satisfies \( (i+1)U_{i+1} = \frac{d}{dt}U_i + U_i G(t) \) in [DGS, III. 5 p.93], \( U_i \) is an element of \( \frac{1}{r}K[[t]] \). Let \( Z = \Theta(U) \). We have \( Z = \sum_{i=0}^{\infty} t^i Z_i \in \text{M}_m(B^0[0,1)) \) with \( Z_i \in M_m(K[[x - t]/t]] \). By Lemma 2.7 (iii) and Lemma 2.2, \( Z \) is the solution of the differential system

\[
Z' = \tilde{\tau}(G)Z, \quad Z(t) = I_m.
\]

Then \( \tilde{\tau}(Y) = \sum_{i=0}^{\infty} t^i \tilde{\tau}(x^{G^{-1}}) F_n \tilde{\tau}(x^{G^{-1}}) \) is also a solution of (5). Therefore there exists a constant matrix \( C_1 \) such that \( \tilde{\tau}(Y) = ZC_1 \) by Lemma 2.5. By substituting \( t \) for \( x \) on both sides, we see \( C_1 = \sum_{i=0}^{\infty} t^i F_n t^{G^{-1}}. \) Comparing the coefficients of \( t^n \) on the both sides of \( \tilde{\tau}(Y) = ZC_1 \) we have the equality.

\[
\frac{\tau(x^n)}{t^n} F_n \tau(x^{G^{-1}})(t^{G^{-1}})^{-1} = \sum_{i=0}^{n} Z_{n-i} F_i.
\]

Note \( \tau(x^{G^{-1}})(t^{G^{-1}})^{-1} \in M_m(K[[x - t]/t]]) \). The equality (6) shows inductively that \( Z_{n} \) is the form \( \sum_{i=0}^{n} \sum_{j=0}^{i} Z_{ij} \frac{(x - t)^i}{t^n} \) with \( Z_{ij} \in M_m(K) \). Comparing the coefficients of \( (x - t)^n/t^n \) on the both sides of (6), we have \( F_n \tau(x^{G^{-1}})(t^{G^{-1}})^{-1} = \sum_{j=0}^{n} Z_{nj} (1 + (x - t)/t)^{\alpha_j} \).

Since \( |(1 + (x - t)/t)^{\alpha_j}| \leq 1 \) for all \( 0 < r < 1 \) and \( j \), we have the inequality \( |F_n| = |\sum_{j=0}^{n} Z_{nj} (1 + (x - t)/t)^{\alpha_j} t^{G^{-1}}(\tau(x^{G^{-1}}))^{-1}| \leq \sup_{j} |Z_{nj}| \).

Let \( s \) be the least integer which satisfies \( n \leq p^s \). In the lemma below, it follows \( \sup_{j} |Z_{nj}| \leq |Z_n| r^{-(l+1)p^s} \) for all \( 0 < r < 1 \), \( X = (x - t)/t \) and for some \( s + \mu \leq \nu \leq s + 2\mu \). By Lemma 2.4 and the assumption, we have \( |Z_n| r \leq |Z| r = |U| r \leq C (-\log r)^{-\delta} \). Therefore we have \( |F_n| \leq C (-\log r)^{-\delta} r^{-(l+1)p^s} \) for all \( 0 < r < 1 \). The desired inequality is given by setting \( r = p^{-1/(l+1)p^s} \).
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**Lemma 3.2.** Let $\alpha_0, \cdots, \alpha_l \in \mathbb{Z}_p$ be a prepared set, which satisfies conditions (1) and (2) in Section 1. Let $n$ be a positive integer and $s$ be the least integer which satisfies $n \leq p^s$. Let $f = \sum_{i=0}^{n} \sum_{j=0}^{l} f_{ij} X^i (1 + X)^{\alpha_i} \in K[[X]]_0$ be a power series. Then we have $|f|_r \geq \sup_{i,j} |f_{ij}|_r (l+1)^s p^s$ for all $0 < r < 1$.

**Proof.** Applying the assumption (2) for $i = s+\mu$, there exists $s+\mu \leq \nu \leq s+2\mu$ such that $\alpha_{j_1, \nu} \neq \alpha_{j_2, \nu}$ for all $0 \leq j_1 < j_2 \leq l$. The coefficient of $X^k$ of $f$ is $\sum_{j=0}^{l} \sum_{i=0}^{n} f_{ij} (\alpha_j - i)$, where $(\alpha_k) := 0$ for $k < 0$. Write $f = \sum_{i=0}^{\infty} f_i X^i$ with $f_i \in K$. We consider the system of linear equations $f_k = \sum_{j=0}^{l} \sum_{i=0}^{n} f_{ij} (\alpha_j - i)$ in $(n+1)(l+1)$ variables $f_{ij}$. We define the $(n+1)(l+1) \times (n+1)(l+1)$ matrix $C$ to be

$$C = \begin{pmatrix}
A_n(\alpha_0, 0) & \cdots & A_n(\alpha_l, 0) \\
A_n(\alpha_0, p^r) & \cdots & A_n(\alpha_l, p^r) \\
\vdots & & \vdots \\
A_n(\alpha_0, lp^r) & \cdots & A_n(\alpha_l, lp^r)
\end{pmatrix},$$

where $A_n(\alpha, k) = \left(\binom{\alpha}{k+i-j}\right)_{1 \leq i,j \leq n+1}$. We will prove $\det C \neq O \mod p$. Using elementary column operations, we can transform $A_n(\alpha, k)$ into $B_n(\alpha, k) = \left(\binom{\alpha+j-1}{k+i-j}\right)_{1 \leq i,j \leq n}$. Therefore we see

$$\det C = \det \begin{pmatrix}
B_n(\alpha_0, 0) & \cdots & B_n(\alpha_l, 0) \\
B_n(\alpha_0, p^r) & \cdots & B_n(\alpha_l, p^r) \\
\vdots & & \vdots \\
B_n(\alpha_0, lp^r) & \cdots & B_n(\alpha_l, lp^r)
\end{pmatrix}.$$

By the Lucas formula and the definition of $\nu$ ($\nu \geq s+\mu$), we have $(\alpha_j, \nu \nu) \equiv (\alpha_j, \nu \nu) \mod p$ for $0 \leq l_1, j \leq l$ and $0 \leq i_1, i_2 \leq n$ (We need the assumption (1) in Section 1.). Therefore it follows that $B_n(\alpha_j, \nu p^r) \equiv (\nu p^r) A_n(\alpha_j, 0) \mod p$. By direct computation, we have $\det(B_n(\alpha, 0)) = 1$. 


This shows
\[
\begin{vmatrix}
B_n(\alpha_0,0) & \cdots & B_n(\alpha_l,0) \\
(\alpha_0,\nu)B_n(\alpha_0,0) & \cdots & (\alpha_l,\nu)B_n(\alpha_l,0) \\
\vdots & & \vdots \\
(\alpha_0,\nu)I_{n+1} & \cdots & (\alpha_l,\nu)I_{n+1}
\end{vmatrix}
\]
\[
= \det
\begin{pmatrix}
B_n(\alpha_0,0) & 0 & \cdots & 0 \\
0 & B_n(\alpha_1,0) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_n(\alpha_l,0)
\end{pmatrix}
\]
\[
= \prod_{i=1}^l i!^{n-1} \prod_{0 \leq j_1 < j_2 \leq l} (\alpha_{j_2,\nu} - \alpha_{j_1,\nu})^{n+1} \text{ mod } p
\]
(We use the Vandermonde’s determinant).

We see \( \det C \not\equiv 0 \) mod \( p \) by the assumption of \( \nu \). Note
\[
\begin{pmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{(n+1)(l+1)-1}
\end{pmatrix}
= C
\begin{pmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{(n+1)(l+1)-1}
\end{pmatrix}, \quad f_j = \begin{pmatrix}
f_{0j} \\
f_{1j} \\
\vdots \\
f_{nj}
\end{pmatrix}.
\]
Since \( |\det C| = 1 \), we see \( |C^{-1}| = 1 \). Thus we have
\[
\begin{vmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{(n+1)(l+1)-1}
\end{vmatrix} \geq |C^{-1}|
\begin{vmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{(n+1)(l+1)-1}
\end{vmatrix} = \begin{vmatrix}
f_{0j} \\
f_{1j} \\
\vdots \\
f_{nj}
\end{vmatrix}.
\]
This shows the inequality \( \sup_{i,j} |f_{ij}| \leq \sup_{0 \leq i \leq (l+1)p^s} |f_i| \). It follows that
\[
\sup_{i,j} |f_{ij}|^{r(l+1)p^s} \leq \sup_{0 \leq i \leq (l+1)p^s} |f_i|^{r(l+1)p^s} \leq |f|^r.
\]

In the proof of Lemma 3.2, by considering \( f = \sum_{i=0}^n \sum_{j=0}^l f_{ij}X^i(1 + X)^{\alpha_j - \alpha_0} \), we can replace the condition (1) of introduction with

(1') There are no \( 1 \leq j \leq l \) such that \( \alpha'_{j,i} = \cdots = \alpha'_{j,i+\mu} = p - 1 \) for some \( i \in \mathbb{N} \),

(2') For all \( i \in \mathbb{N} \), there exists \( 0 \leq \mu_i \leq \mu \) such that \( \alpha'_{j_1,i+\mu_i} \neq \alpha'_{j_2,i+\mu_i} \) for all \( 0 \leq j_1 < j_2 \leq l \).
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where $\alpha' := \alpha - \alpha_0 = \sum_{k=0}^{\infty} \alpha'_k b^k$. For $f \in K\{x\}$ (resp. $f \in A^0[0, 1]$), we say that $f$ has exactly log-growth $\delta$ if $f$ has log-growth $\delta$ and does not have log-growth $\varepsilon$ for all $\varepsilon < \delta$. We give examples which does not satisfy the analog of the Dwork’s theorem.

**Example 3.3.** Let $\alpha$ be a non-zero $p$-adic integer which is not a negative integer. Consider the differential equation

$$ (7) \quad xY' + NY = 0, \quad N = - \left( 0 \quad 1 \quad x \right). $$

We prove that the solution matrix of the system (7) at $x = t$ has exactly log-growth 1.

**Proof.** For $\sum_{j=0}^{\infty} a_j (x-t)^j$ with $a_j \in K(t)$, $\int \sum_{j=0}^{\infty} a_j (x-t)^j dx$ is defined as $\sum_{j=0}^{\infty} a_j (x-t)^{j+1}$. Since a formal solution of $y' = \left( \frac{x}{x} + \frac{1}{1-x} \right) y$ is $y = x^\alpha/(1-x)$, we see that

$$ V = \left( \begin{array}{ccc} 1 & \int \tau(x^\alpha/(1-x)) dx & 0 \\ 0 & \tau(x^\alpha/(1-x)) & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 \\ 0 & (1-t)/t^\alpha \end{array} \right) \in Gl_2(K(t)[[x-t]]) $$

is the solution of system (7) at $x = t$ and satisfies $V(t) = I_m$. We prove that $\int \tau(x^\alpha/(1-x)) dx (1-t)/t^\alpha$ has exactly log-growth 1. To prove this, we show that the $p$-adic value of the coefficient of $(x-t)^{s-1}$ in $\tau(x^\alpha/(1-x))(1-t)/t^\alpha$ is 1 for all $s \in \mathbb{N}$. We compute

$$ \tau(x^\alpha/(1-x))(1-t)/t^\alpha = \sum_{i=0}^{\infty} \frac{(x-t)^i}{(1-t)^i} \sum_{j=0}^{\infty} \frac{\alpha_j}{j !} \frac{(x-t)^j}{j !} = \sum_{k=0}^{\infty} \sum_{i+j=k} \frac{\alpha_j}{j !} \frac{(x-t)^k}{j !}, $$

$$ 1 \geq \sum_{i+j=k} \left| \frac{\alpha_j}{j !} \frac{(x-t)^k}{j !} \right| = \max_{i+j=k} \left| \left( \frac{\alpha_j}{j !} \frac{(x-t)^k}{j !} \right) \right| \geq \left| \frac{\alpha_i}{j !} \frac{(x-t)^k}{j !} \right| = 1. $$

We regarded $\sum_{i=0}^{\infty} t^i$ as $\frac{1}{1-t}$. \hfill \Box

The system (7) has a regular singular point at $x = 0$ whose exponent are $0, \alpha$, has a solution matrix $Y = \begin{pmatrix} 1 & \sum_{i=1}^{\infty} \frac{x^i}{\alpha+i} \\ 0 & \sum_{i=0}^{\infty} x^i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x^\alpha \end{pmatrix}$ at $x = 0$. Note that the log-growth of $U = \begin{pmatrix} 1 & \sum_{i=1}^{\infty} \frac{x^i}{\alpha+i} \\ 0 & \sum_{i=0}^{\infty} x^i \end{pmatrix}$ is equal to the log-growth of $y_\alpha := \sum_{i=1}^{\infty} \frac{x^i}{\alpha+i}$. Write $-\alpha = \sum_{j=0}^{\infty} \alpha_j p^j$ with $0 \leq \alpha_j \leq p - 1$. In this case, conditions (1) and (2) in Section 1 are as follows:
(1) There exists $\mu \in \mathbb{N}$ such that if $\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i+\mu-1} = 0$, then $\alpha_{i+\mu} \neq 0$.

(2) There exists $\mu \in \mathbb{N}$ such that if $\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i+\mu-1} = p-1$, then $\alpha_{i+\mu} \neq p-1$.

If $\alpha$ satisfies conditions (1) and (2), then $y_\alpha$ has log-growth 1 by Theorem 3.1. In fact $y_\alpha$ has exactly log-growth 1.

**Proof.** For any $s \in \mathbb{N}$, if $i = \sum_{j=0}^{s} \alpha_j p^j$, then the $p$-adic value of the coefficient of $x^i$ in $y_\alpha$ is $|1/(i + \alpha)| \geq p^{s+1}$. This shows that $y_\alpha$ does not have log-growth $\delta$ for all $\delta < 1$.

We prove that if $\alpha$ does not satisfy condition (1), then $y_\alpha$ does not have log-growth 1.

**Proof.** Assume that $y_\alpha$ has log-growth 1. Then there exists positive real number $C$ such that $|1/(i + \alpha)| \leq Ci$. By the assumption, for any positive integer $s$, there exist a positive integer $n_s$ such that $\alpha_{n_s} = \alpha_{n_s+1} = \cdots = \alpha_{n_s+s} = 0$. Then the $p$-adic value of the coefficient of $x^{n_s}$ in $y_\alpha$ satisfies $|1/(\sum_{j=0}^{n_s-1} \alpha_j p^j + \alpha)| \geq p^{n_s+s}$. Since $\sum_{j=0}^{n_s-1} \alpha_j p^j < p^{n_s}$, we have

$$C \geq \frac{|1/(\sum_{j=0}^{n_s-1} \alpha_j p^j + \alpha)|}{\sum_{j=0}^{n_s-1} \alpha_j p^j} \geq \frac{p^{n_s+s}}{p^n} = p^s.$$

This is a contradiction. \(\square\)

In particular, if $\alpha$ does not satisfy condition (1), then the inequality of Theorem 3.1 does not hold. Let $k$ be a positive integer greater than 1. We set $\alpha = -\sum_{i=1}^{\infty} p^k$. We prove that $y_\alpha$ has exactly log-growth $k$.

**Proof.** If $n = \sum_{i=1}^{s} p^k i$, then we have $|1/(\alpha + n)| = p^{k+s}$, which shows $n^k / 2^k < |1/(\alpha + n)| < n^k$. Indeed, we can compute

$$|1/(\alpha + n)| = p^{k+s} = (p^{k+s})^k < n^k,$$

$$s \leq p^{(k-1)k-1} \rightarrow sp^{k-1} \leq p^{k} \rightarrow n \leq sp^{k-1} + p^{k} \leq 2p^{k} \rightarrow (n/2)^k \leq p^{k+s}.$$

On the other hand, if $\sum_{i=1}^{s} p^k i < n < \sum_{i=1}^{s} p^k$, then $|1/(\alpha + n)| \leq p^{k-1} < n^k$. This shows the assertion. \(\square\)

By using (7), $U$ satisfies the relation

$$U^{-1}x \frac{d}{dx} U + U^{-1} NU = -\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} = N(0).$$
If $\alpha \in \mathbb{Q} - \mathbb{Z}$, then $\alpha$ satisfies condition (1) and (2). Naturally we would like to consider the case when $\alpha$ is a $p$-adic integer and is non-trivial algebraic number over $\mathbb{Q}$. We prove that $y_\alpha$ has log-growth $n$ with $n = \left[ \mathbb{Q}(\alpha); \mathbb{Q} \right]$.

**Proof.** We refer to [DGS, VI, Proposition 1.1]. Let $g(X) = A(X - \alpha)(X - \alpha_2) \cdots (X - \alpha_n) \in \mathbb{Z}[X]$ be the irreducible polynomial for $\alpha$ over $\mathbb{Q}$, with $\alpha_2, \cdots, \alpha_n \in \mathbb{Q}^{alg}$. Let $\varepsilon = \min_{2 \leq j \leq n} |\alpha - \alpha_j|$. Let $s \in \mathbb{N}$ such that $|s - \alpha| < \varepsilon$. Then $|s - \alpha_j| = |\alpha - \alpha_j|$, so that $|g(s)| = |s - \alpha||g'(\alpha)|$. Thus we have

$$\frac{1}{|s - \alpha|} = \frac{|g'(\alpha)|}{|g(s)|} \leq |g(s)|_\infty = O(s^n),$$

where $|\cdot|_\infty$ denotes the usual absolute value on $\mathbb{R}$. If $|s - \alpha| \geq \varepsilon$, then $\frac{1}{|s - \alpha|} \leq \frac{1}{\varepsilon}$. This shows that $y_\alpha$ has log-growth $n$. \qed

In this case, we expect that $y_\alpha$ does not have log-growth 1, but that it has log-growth $\delta$ for all $\delta > 1$. (We could not prove it).

**Remark 3.4.** Let $N = \sum_{i=0}^\infty N_i x^i \in M_m(K[[x]])$ and $\alpha_0, \cdots, \alpha_l \in \mathbb{Z}_p$ be the distinct eigenvalues of $N_0$. Write the $p$-adic expansion $\alpha_i = \sum_{j=0}^\infty \alpha_{ij} p^j$ of $\alpha_i$ for $i = 0, \cdots, l$. Assume the following conditions.

(i) $U = \sum_{i=0}^\infty U_i x^i \in \text{GL}_m(K\{x\})$ satisfies $U^{-1}NU + U^{-1}x \frac{d}{dx}U = N_0$ and $U_0 = I_m$.

(ii) $\alpha_0, \cdots, \alpha_l$ are a prepared set and satisfy

$$\max_{0 \leq i,j \leq l} \sup_{0 \leq s < \infty} \left| \sum_{k=s}^{\infty} (\alpha_{ik} - \alpha_{jk})p^{k-s} \right|^{-1} < \infty$$

Recently Kedlaya proved that $U$ has log-growth $m - 1$ under these conditions in his errata of PDE. We seem that his result is very precise. If $\alpha_0, \cdots, \alpha_l$ satisfy conditions (1) and (2) of introduction, then they satisfy (8). Kedlaya assumes that solution matrix at $x = 0$ is analytic on $D(0, 1^-)$. On the other hand we assume that solution matrix at $x = t$ is analytic on $D(t, 1^-)$. Moreover we prove that if solution matrix at $x = t$ has log-growth $\delta$, then solution matrix at $x = 0$ has log-growth $\delta$. 


References


