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NORMALIZERS OF CLASSICAL GROUPS ARISING UNDER EXTENSION OF THE BASE RING

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Abstract – Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. In 1994 and then in 2017, V. A. Koibaev and we described normalizers of subgroups $GL(n,S)$ and $E(n,S)$ in $G = GL(mn,R)$, and showed that they are equal and coincide with the set $\{g \in G : E(n,S)^g \leq GL(n,S)\} = Aut(S/R) \ltimes GL(n,S)$. Moreover, for any proper ideal A of R,

$$N_G(E(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(E(n,S/SA))).$$

In the present paper, we prove similar results about normalizers of classical subgroups, namely, the normalizers of subgroups $EO(n,S), SO(n,S), O(n,S)$ and $GO(n,S)$ in G are equal and coincide with the set $\{g \in G : EO(n,S)^g \leq GO(n,S)\} = Aut(S/R) \ltimes GO(n,S)$. Similarly, the ones of subgroups $Ep(n,S)$, $Sp(n,S)$ and $GSp(n,S)$ are equal and coincide with the set $\{g \in G : Ep(n,S)^g \leq GSp(n,S)\} = Aut(S/R) \ltimes GSp(n,S)$. Moreover, for any proper ideal A of R,

$$N_G(EO(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(EO(n,S/SA)))$$

and

$$N_G(Ep(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(Ep(n,S/SA))).$$

When $R = S$, we obtain the known results of N. A. Vavilov and V. A. Petrov.

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1. Introduction

Let S be a commutative ring. It is well known that subgroups $E(n,S)$ and $SL(n,S)$ are normal in $GL(n,S)$. In [6, 7], initiating a generalization of the results of V. I. Kopeiko and G. Taddei in [8, 9] about normality of symplectic subgroups, N. A. Vavilov and V. A. Petrov described normalizers of orthogonal and symplectic subgroups in $GL(n,S)$. It is shown that the normalizers of subgroups $EO(n,S), SO(n,S), O(n,S)$ and $GO(n,S)$ are equal and coincide with the set $\{g \in GL(n,S) : EO(n,S)^g \leq GO(n,S)\} = GO(n,S)$. Moreover, for any proper ideal A of S,

$$N_{GL(n,S)}(EO(n,S)E(n,S,A)) = \rho_A^{-1}(GO(n,S/A)).$$

Similarly, the normalizers of subgroups $Ep(n,S), Sp(n,S)$ and $GSp(n,S)$ are equal and coincide with the set $\{g \in GL(n,S) : Ep(n,S)^g \leq GSp(n,S)\} = GSp(n,S)$. Moreover, for any proper ideal A of S,

$$N_{GL(n,S)}(Ep(n,S)E(n,S,A)) = \rho_A^{-1}(GSp(n,S/A)).$$

Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. In [1, 2], V. A. Koibaev and we described normalizers of subgroups $GL(n,S)$ and $E(n,S)$ in $G = GL(mn,R)$, and proved that they are equal and coincide with the set $\{g \in G : E(n,S)^g \leq GL(n,S)\} = Aut(S/R) \ltimes GL(n,S)$. Moreover, for any proper ideal A of R, we have

$$N_G(E(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(E(n,S/SA))).$$

In the present paper, we prove similar results about normalizers of classical subgroups, namely,

$$\begin{aligned} N_G(EO(n,S)) &= N_G(SO(n,S)) = N_G(O(n,S)) = N_G(GO(n,S)) \\ &= \{g \in G : EO(n,S)^g \leq GO(n,S)\} = Aut(S/R) \ltimes GO(n,S) \end{aligned}$$

and

$$\begin{aligned} N_G(Ep(n,S)) &= N_G(Sp(n,S)) = N_G(GSp(n,S)) \\ &= \{g \in G : Ep(n,S)^g \leq GSp(n,S)\} = Aut(S/R) \ltimes GSp(n,S). \end{aligned}$$

Moreover, for any proper ideal A of R, we have

$$N_G(EO(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(EO(n,S/SA)))$$

and

$$N_G(Ep(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(Ep(n,S/SA))).$$

2. Basic notation

Let G be an any group. By the commutator of two elements $x, y \in G$ we always understand their left-normed commutator $[x, y] = xyx^{-1}y^{-1}$. By ${}^{y}x = yxy^{-1}$ and $x^y = y^{-1}xy$ we denote the left and the right conjugates of x by y, respectively. We write $H \leq G$ to mean that H is a subgroup of G. For a subset $X \subseteq G$, we denote by $\langle X \rangle$ the subgroup of G generated by X and by $\langle X \rangle^H$ the smallest subgroup of G which contains X and is normalized by H. For two subgroups $F, H \leq G$ we denote by [F, H] the corresponding relative commutator subgroup, generated by all commutators $[f, h], f \in F, h \in H$.

Now, let R be an arbitrary associative ring with 1. When A is an ideal of R, we write $A \trianglelefteq R$. For two natural numbers $m, n, M(m, n, R)$ is the additive group of $m \times n$ matrices with entries in R, in particular, $M(n, R) = M(n, n, R)$ is the matrix ring of degree n over R. As always, R^* is the multiplicative group of R and $GL(n, R) = M(n, R)^*$ is the general linear group of degree n over R. As usual, a_{ij} is the entry of a matrix $a \in GL(n, R)$ at the position $(i, j), i.e., a = (a_{ij}), 1 \leq i, j \leq n$. Next, $a^{-1} = (a'_{ij})$ is the inverse of a and a^t is its transpose.

As usual, e is the identity matrix of degree n and e_{ij} is a standard matrix unit, that is the matrix which has 1 in the position (i, j) and zeros elsewhere. For $\xi \in R$ and $1 \leq i \neq j \leq n, t_{ij}(\xi) = e + \xi e_{ij}$ an elementary transvection. In the sequel, without any special reference we use standard relations among elementary transvections such as the additivity formula $t_{ij}(\xi)t_{ij}(\zeta) = t_{ij}(\xi + \zeta)$ and the Chevalley commutator formula $[t_{ij}(\xi), t_{jh}(\zeta)] = t_{ih}(\xi\zeta)$ for distinct i, j and k.

Now, let $A \trianglelefteq R$ be an ideal in R. We denote by $E(n, A)$ the subgroup in $GL(n, R)$ generated by all elementary transvections of level A

$$E(n, A) = \langle t_{ij}(\xi), \xi \in A, 1 \leq i \neq j \leq n \rangle.$$

In the most important case where $A = R$, the group $E(n, R)$ generated by all elementary transvections is called the elementary group. In the sequel, a major role is played by the relative elementary group $E(n, R, A)$. Recall that $E(n, R, A)$ is the normal closure of $E(n, A)$ in $E(n, R)$

$$E(n, R, A) = \langle t_{ij}(\xi), \xi \in A, 1 \leq i \neq j \leq n \rangle^{E(n,R)}.$$

The canonical projection $\rho_A : R \longrightarrow R/A$ sending any element $\lambda \in R$ to the element $\overline{\lambda} = \lambda + A$, defines the corresponding reduction homomorphism

$$\rho_A : GL(n, R) \longrightarrow GL(n, R/A).$$

The kernel of ρ_A is denoted by $GL(n,R,A)$ and is called the *principal congruence−subgroup* in $GL(n,R)$ of level A.

A key point in reduction modulo an ideal is the following (see, e.g., [3, Corollary I.3]).

Lemma 2.1. *Let $n \geq 3$. Then for any ideal $A \trianglelefteq R$, we have*

$$[E(n,R), E(n,R,A)] = E(n,R,A).$$

In particular, the elementary subgroup $E(n,R)$ is perfect, that is

$$[E(n,R), E(n,R)] = E(n,R).$$

Lemma 2.2. [4, Theorem 1] *Let R be a commutative ring and $n \geq 3$. Then for any ideal $A \trianglelefteq R$, we have $[E(n,R), GL(n,R,A)] = E(n,R,A)$. In particular, $E(n,R,A)$ is normal in $GL(n,R)$.*

Lemma 2.3. [1, Lemma 4] *For any $n \geq 2$, the additive group $\langle E(n,R)\rangle$ coincides with $M(n,R)$.*

3. Regular representation

Let R be a commutative ring and let S be a ring extension of R, which is a free module of rank m. Suppose that $1 = w_1, \ldots, w_m$ is a basis of S/R. For any $\alpha \in S$ and $1 \leq j \leq m$, there exist $\alpha_{1j}, \ldots, \alpha_{mj} \in R$ such that

$$\alpha w_j = \alpha_{1j} w_1 + \cdots + \alpha_{mj} w_m.$$

We denote $[\alpha] = (\alpha_{ij}) \in M(m,R)$. It is clear that the map $\alpha \mapsto [\alpha]$ is a ring monomorphism from S to $M(m,R)$. Then S is considered as a subring of the matrix ring $M(m,R)$, so $GL(n,S)$ is a subgroup of the general linear group $GL(mn,R)$, and $E(n,S)$ is a subgroup of the elementary subgroup $E(mn,R)$.

Let A be a proper ideal of R. We put $\varphi : R \longrightarrow S/SA, \varphi(r) = \overline{r}$. Clearly, $A \subseteq ker\varphi$. Conversely, for any $r \in ker\varphi$, we have $r \in R \cap SA$, so $rw_1 = r = a_1 w_1 + \ldots + a_n w_n$ for some $a_1, \ldots a_n \in A$, it follows that $r = a_1 \in A$. This proves that $ker\varphi = A$, and hence R/A is considered as a subring of S/SA. Moreover, S/SA is free of rank m as an R/A-module. Now, consider the reduction homomorphism $\rho_A : GL(mn,R) \longrightarrow GL(mn,R/A)$, it is easily seen that $\rho_A(EO(n,S)) = EO(n,S/SA)$ and $\rho_A(Ep(n,S)) = Ep(n,S/SA)$.

4. Normalizers of linear groups

Let $Aut(S/R)$ be the group of all ring automorphisms of the ring S that are identical on R. As in [2], for any $\sigma \in Aut(S/R)$, we define a map σ^0 of the module $M = S^n$ with the basis $e_1, \dots, e_n$. For $x \in M, x = \sum_{i=1}^n x_i e_i, x_i \in S$, we put $\sigma^0(x) = \sum_{i=1}^n \sigma(x_i) e_i$, then $\sigma^0 \in Aut_R(M)$, and the map $Aut(S/R) \longrightarrow Aut_R(M), \sigma \mapsto \sigma^0$ is a group monomorphism. Put $G^0 = \{\sigma^0 : \sigma \in Aut(S/R)\}$, then G^0 is a subgroup of $Aut_R(M)$ isomorphic to $Aut(S/R)$, and $G^0 \cap Aut_S(M) = 1$. We identify the subgroup G^0 with $Aut(S/R)$. The following statements are known in [1].

Lemma 4.1. [1, Corollary 1] *Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. Then for $n \geq 3$, $N_{GL(mn,R)}(E(n,S)) = Aut(S/R) \ltimes GL(n,S)$.*

Lemma 4.2. [1, Theorem 2] *Under the assumptions of Lemma 4.1, let $g \in GL(mn,R)$. If $E(n,S)^g \leq GL(n,S)$, then $g \in N_{GL(mn,R)}(E(n,S))$. Moreover, when $Aut(S/R)$ is solvable, if $E(n,S)^g \leq N_{GL(mn,R)}(E(n,S))$, then $g \in N_{GL(mn,R)}(E(n,S))$.*

The following shows that under some assumptions, we can describe the normalizer $N_{GL(mn,R)}(H)$ via $N_{GL(n,S)}(H)$ for $H \leq GL(n,S)$.

Lemma 4.3. *Let R be a unital subring of a commutative ring S, which is a free R-module of rank m and let $H \leq GL(n,S)$ such that $E(n,S) \subseteq \langle H, + \rangle$. Suppose that $n \geq 3$ and H is normalized by $Aut(S/R)$. Then*

$$N_{GL(mn,R)}(H) = Aut(S/R) \ltimes N_{GL(n,S)}(H).$$

Proof. It is clear that $Aut(S/R) \ltimes N_{GL(n,S)}(H) \leq N_{GL(mn,R)}(H)$. Conversely, let $g \in N_{GL(mn,R)}(H)$. For any $t_{ij}(\xi) \in E(n,S), 1 \leq i \neq j \leq n, \xi \in S$, we write $t_{ij}(\xi)$ in the form $\sum_{h_k \in H} h_k$, then $t_{ij}(\xi)^g = \sum_{h_k \in H} h_k^g \in \langle H, + \rangle$, so $t_{ij}(\xi)^g \in M(n,S)$. It follows that $E(n,S)^g \subseteq M(n,S)$, and hence $E(n,S)^g \leq GL(n,S)$. By Lemma 4.2, $g \in N_{GL(mn,R)}(E(n,S))$, so by Lemma 4.1, there exist $\omega \in Aut(S/R)$ and $h \in GL(n,S)$ such that $g = \omega h$. Then $H^h = H^g \leq H$, and hence $h \in N_{GL(n,S)}(H)$ as desired. □

Lemma 4.4. *The special linear group $SL(n,S)$ is normalized by $Aut(S/R)$.*

Proof. Since $E(n,S)$ is normalized by $Aut(S/R)$, considering $Aut(S/R)$ and S as subsets of $M(m,R)$, we have $\omega^{-1} S \omega = S$ for all $\omega \in Aut(S/R)$, thus $SL(n,S)^\omega \le GL(n,S)$. For $h \in SL(n,S)$ and $\omega \in Aut(S/R)$, we have

$$\begin{aligned} det(\omega^{-1}h\omega) &= \sum_{\sigma\in S_n} sign(\sigma)[\omega^{-1}h_{1\sigma(1)}\omega][\omega^{-1}h_{2\sigma(2)}\omega]\dots[\omega^{-1}h_{n\sigma(n)}\omega] \\ &= \omega^{-1}(\sum_{\sigma\in S_n} sign(\sigma)h_{1\sigma(1)}h_{2\sigma(2)}\dots h_{n\sigma(n)})\omega \\ &= \omega^{-1}det(h)\omega = \omega^{-1}1_S\omega = 1_S, \end{aligned}$$

and hence $\omega^{-1}h\omega \in SL(n,S)$. □

Since $E(n,S) \subseteq SL(n,S)$, by Lemma 4.3, Lemma 4.4 and normality of $SL(n,S)$ in $GL(n,S)$, we get the result about the normalizer of $SL(n,S)$ in $GL(mn,R)$.

Theorem 4.5. *Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. Then for $n \ge 3$, we have*

$$N_{GL(mn,R)}(SL(n,S)) = Aut(S/R) \ltimes GL(n,S).$$

Summarizing results in [1, 2] and Theorem 4.5 we get the following.

Corollary 4.6. *Under the assumptions of Theorem 4.5, for $n \ge 3$, we have*

$$\begin{aligned} N_{GL(mn,R)}(E(n,S)) = N_{GL(mn,R)}(SL(n,S)) &= N_{GL(mn,R)}(GL(n,S)) \\ &= \{g \in GL(mn,R) : E(n,S)^g \le GL(n,S)\} \\ &= Aut(S/R) \ltimes GL(n,S). \end{aligned}$$

5. Normalizers of orthogonal groups

Let S be a commutative ring. In this section, we always assume that $2 \in S^*$. Set $l = [n/2]$, i.e., $n = 2l$ or $n = 2l+1$, depending on the parity of n. It is convenient to index the rows and columns of matrices in $GL(n,S)$ as follows: $1,\dots,l,-l,\dots,-1$ when $n = 2l$ is even, and $1,\dots,l,0,-l,\dots,-1$ when $n = 2l+1$ is odd. Let $f = f_n = sdiag(1,\dots,1)$ of degree n, which has $1's$ alongside the second diagonal, and zeros elsewhere. Under the indexing agreement adopted in the preceding paragraph, the entries of $f = (f_{ij})$ are described as $f_{ij} = \delta_{i,-j}$. The *special orthogonal group* $SO(n,S)$ consists of all matrices g in the general linear group $GL(n,S)$ for which $\det g = 1$ and $gfg^t = f$. The orthogonality condition of $g = (g_{ij})$ is most conveniently expressed in the form $g'_{ij} = g_{-j,-i}$. The *general orthogonal group* $GO(n,S)$

consists of all matrices g in $GL(n,S)$ for which $gfg^t = \lambda f$ for an appropriate multiplier $\lambda = \lambda(g) \in S^*$. Thus the matrix entries of a matrix in the group $GO(n,S)$ satisfy the following condition:

$$\lambda(g)g'_{ij} = g_{-j,-i} \quad \text{for all } i,j = 1,\dots,-1.$$

The subgroup in $GO(n,S)$ consisting of all matrices a such that $\lambda = 1$ is the usual *orthogonal group* $O(n,S)$.

A most important tool for the study of the above groups is provided by elementary root unipotent elements, or, in the classical language, elementary orthogonal transvections. An *elementary orthogonal transvection* is one of the matrices $T_{ij}(\xi), \xi \in S, i \neq \pm j$, of the form

$$T_{ij}(\xi) = T_{-j,-i}(\xi) = e + \xi e_{ij} - \xi e_{-j,-i}$$

when $i,j \neq 0$ (from the viewpoint of Chevalley groups, this is a *long root unipotent*), and of the form

$$T_{i0}(\xi) = T_{0,-i}(-\xi) = e + \xi e_{i0} - \xi e_{0,-i} - (\xi^2/2)e_{i,-i}$$

when exactly one of the indices i,j equals 0 (this is a *short root unipotent*). Obviously, the elements of the second kind arise only for an odd n. It is sometimes convenient to assume that $T_{i,-i}(\xi) = e$.

A central role in the present paper is played by the *elementary orthogonal group* $EO(n,S)$, i.e., the subgroup in $SO(n,S)$ generated by all elementary orthogonal transvections of the form $T_{ij}(\xi), \xi \in S, i \neq j$.

The following easy but fundamental fact is well known (see, e.g., [15, Corollary 4.4]).

Lemma 5.1. *Let S be a commutative ring, and let $n \geq 5$; for $n = 5$ we assume additionally that S has no factor fields of 2 elements. Then the elementary orthogonal group $EO(n,S)$ is perfect.*

The following result is well known in [6].

Lemma 5.2. *Let S be a commutative ring and $n \geq 5$. Then*

$$\begin{aligned} N_{GL(n,S)}(EO(n,S)) &= N_{GL(n,S)}(SO(n,S)) = N_{GL(n,S)}(GO(n,S)) \\ &= \{g \in GL(n,S) : EO(n,S)^g \leq GO(n,S)\} = GO(n,S). \end{aligned}$$

Lemma 5.3. *Let S be a commutative ring and $n \geq 5$. Then*

$$\langle EO(n,S), +\rangle = \langle SO(n,S), +\rangle = \langle O(n,S), +\rangle = \langle GO(n,S), +\rangle = M(n,S).$$

Proof. Put $K = \langle EO(n, S), +\rangle$. It suffices to show that $K = M(n, S)$. Indeed, let $\xi \in S$ and nonzero indices $j \neq \pm i$. Take $h \neq 0, h \neq \pm i$, we have

$$t_{i,-i}(\xi) = T_{ih}(\xi)T_{h,-i}(1) - T_{ih}(\xi) + T_{h,-i}(-1) \in K.$$

When $n = 2l$, we take an index k such that all six indices $\pm i, \pm j, \pm k$ are pairwise distinct, then

$$t_{ij}(\xi) = T_{i,k}(-\xi) + T_{kj}(1) - T_{ik}(-\xi)T_{kj}(1) \in K.$$

When $n = 2l + 1$, we have

$$t_{ij}(\xi) = T_{i0}(1) + T_{0j}(-\xi) - T_{i0}(1)T_{0j}(-\xi) \in K$$

and

$$t_{i0}(\xi) = T_{ij}(-\xi) + T_{j0}(1) - T_{ij}(-\xi)T_{j0}(1) + t_{i,-j}(\xi/2) - e \in K.$$

Therefore, $E(n, S) \subseteq K$, so by Lemma 2.3, $K = M(n, S)$ as desired. □

Lemma 5.4. *Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. Then subgroups $EO(n, S), SO(n, S), O(n, S)$ and $GO(n, S)$ are normalized by $Aut(S/R)$.*

Proof. Reminding $\xi^\omega \in S$ for all $\omega \in Aut(S/R), \xi \in S$, we have $T_{ij}(\xi)^\omega = T_{ij}(\xi^\omega) \in EO(n, S)$, thus $EO(n, S)^{Aut(S/R)} \leq EO(n, S)$.

Let $g \in GO(n, S)$, there exists $\lambda = \lambda(g) \in S^*$ such that $gfg^t = \lambda f$. Then for $\omega \in Aut(S/R)$, we have

$$\begin{aligned}(\omega^{-1}g\omega)f(\omega^{-1}g\omega)^t &= (\omega^{-1}g\omega)f(\omega^{-1}g^t\omega) = \omega^{-1}g(\omega f\omega^{-1})g^t\omega \\ &= \omega^{-1}(gfg^t)\omega = \omega^{-1}\lambda f\omega = \lambda_1 f,\end{aligned}$$

where $\lambda_1 \in S$, therefore $GO(n, S)^{Aut(S/R)} \leq GO(n, S)$. Note that $\lambda_1 = 1$ if $\lambda = 1$, so $O(n, S)^{Aut(S/R)} \leq O(n, S)$, and hence by Lemma 4.4, $SO(n, S)^{Aut(S/R)} \leq SO(n, S)$. □

Now, we get the main result in this section.

Theorem 5.5. *Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. Then for $n \geq 5$, we have*

$$\begin{aligned}N_{GL(mn,R)}(EO(n, S)) &= N_{GL(mn,R)}(SO(n, S)) = N_{GL(mn,R)}(O(n, S)) \\ &= N_{GL(mn,R)}(GO(n, S)) \\ &= \{g \in GL(mn, R) : EO(n, S)^g \leq GO(n, S)\} \\ &= Aut(S/R) \ltimes GO(n, S).\end{aligned}$$

Proof. Let $H \in \{EO(n,S), SO(n,S), O(n,S), GO(n,S)\}$. Then by the above lemmas (5.4, 5.3, 5.2, 4.3), we get $N_{GL(mn,R)}(H) = Aut(S/R) \ltimes GO(n,S)$.

Now, put $G = GL(mn,R)$ and $K = \{g \in G : EO(n,S)^g \leq GO(n,S)\}$. Clearly, $N_G(EO(n,S)) \leq K$. Conversely, let $g \in K$, we have $EO(n,S)^g \leq GO(n,S)$, so by Lemma 5.3, $M(n,S)^g \subseteq M(n,S)$, and hence $E(n,S)^g \leq GL(n,S)$. By Corollary 4.6, $g = \omega h$ for some $\omega \in Aut(S/R)$ and $h \in GL(n,S)$. Now, we have

$$EO(n,S)^h = (EO(n,S)^\omega)^h = EO(n,S)^g \leq GO(n,S),$$

so $h \in GO(n,S)$ by Lemma 5.2, and hence $g \in N_{GL(n,S)}(EO(n,S))$. □

The following result provides a sufficient condition for an element to belong to the normalizer of $EO(n,S)$.

Corollary 5.6. *Under the assumptions of Theorem 5.5 and let $n \geq 5$; for $n = 5$ we assume additionally that S has no factor fields of 2 elements. Suppose that $Aut(S/R)$ is solvable. If $EO(n,S)^g \leq N_{GL(mn,R)}(EO(n,S))$, then $g \in N_{GL(mn,R)}(EO(n,S))$.*

Proof. By Theorem 5.5, if $Aut(S/R)$ is solvable, then mth commutator subgroup of $N_{GL(mn,R)}(EO(n,S))$ is contained in $GO(n,S)$ for some $m \in \mathbb{N}$. Since $EO(n,S)$ is perfect, the inclusion $EO(n,S)^g \leq N_{GL(mn,R)}(EO(n,S))$ implies that $EO(n,S)^g \leq GO(n,S)$, and hence $g \in N_{GL(mn,R)}(EO(n,S))$. □

The relation between the normalizer of $EO(n,S)E(mn,R,A)$ in $GL(mn,R)$ and the one of $EO(n,S/SA)$ in $GL(mn,R/A)$ is stated in the following.

Theorem 5.7. *Under the assumptions of Corollary 5.6, for any proper ideal A of R, we have*

$$N_{GL(mn,R)}(EO(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(EO(n,S/SA))).$$

Proof. Since $\rho_A(EO(n,S) = EO(n,S/SA)$, it is easily seen that

$$N_{GL(mn,R)}(EO(n,S)E(mn,R,A)) \leq H,$$

where $H = \rho_A^{-1}(N_{GL(mn,R/A)}(EO(n,S/SA)))$. By normality of $E(mn,R,A)$ in $GL(mn,R)$, to prove the inverse inclusion, it suffices to show that

$$EO(n,S)^H \leq EO(n,S)E(mn,R,A).$$

Clearly,

$$EO(n,S)^H \le EO(n,S)GL(mn,R,A).$$

By Lemma 2.2, we also have

$$[EO(n,S)GL(mn,R,A), EO(n,S)] \le EO(n,S)E(mn,R,A).$$

Therefore,

$$[EO(n,S)^H, EO(n,S)] \le EO(n,S)E(mn,R,A).$$

On the other hand,

$$EO(n,S)^H \le E(mn,R).$$

Thus by Lemma 2,

$$[EO(n,S)^H, GL(mn,R,A)] \le E(mn,R,A).$$

Now, by the perfect property of $EO(n,S)$, we get

$$\begin{aligned} EO(n,S)^H &= [EO(n,S)^H, EO(n,S)^H] \\ &\le [EO(n,S)^H, EO(n,S)GL(mn,R,A)] \\ &\le [EO(n,S)^H, EO(n,S)] \cdot [EO(n,S)^H, GL(mn,R,A)]^{EO(n,S)} \\ &\le EO(n,S)E(mn,R,A) \cdot E(mn,R,A)^{EO(n,S)} = EO(n,S)E(mn,R,A) \end{aligned}$$

as desired. □

6. Normalizers of symplectic groups

Let S be a commutative ring and let $n = 2l$. It will be convenient to index the rows and columns of matrices in $GL(n,S)$ by $1 \ldots l, -l, \ldots, -1$. We denote the sign of the index i by ϵ_i. Thus $\epsilon_i = +1$ if $i = 1, \ldots, l$, and $\epsilon_i = -1$ otherwise. Let $F = Fn = sdiag(1, \ldots, 1, -1, \ldots, -1)$ (where the number of 1's as well as the number of -1's equals l) denote the matrix that has $1, \ldots, 1, -1, \ldots, -1$ on the second diagonal (the one in the North-East to South-West direction). In other words, the entries of the matrix $F = (F_{ij})$ are determined by the condition $F_{ij} = \epsilon_i \delta_{i,-j}$. The *symplectic group* $Sp(2l,S)$ consists of all matrices g in the special linear group $SL(2l,S)$ such that $gFg^t = F$. In other words, a matrix $g = (g_{ij}) \in SL(2l,S)$ belongs to $Sp(2l,S)$ if and only if,

$$g'_{ij} = \epsilon_i \epsilon_j g_{-j,-i} \quad \text{for all } i,j = 1, \ldots, -1.$$

We consider the *general symplectic group* $GSp(2l, S)$ consisting of all matrices g in $GL(2l, S)$ satisfying the relation $gFg^t = \lambda F$ for an appropriate multiplier $\lambda = \lambda(g) \in R^*$. Thus the entries of a matrix in the group $GSp(2l, S)$ satisfy the following condition:

$$\lambda(g)g'_{ij} = \epsilon_i \epsilon_j g_{-j,-i} \quad \text{for all } i, j = 1, \dots, -1.$$

As usual, by an *elementary symplectic transvection* we understand one of the matrices $T'_{ij}(\xi), \xi \in S, i \neq j$, of the form

$$T'_{ij}(\xi) = T'_{-j,-i}(\xi) = e + \xi e_{ij} - \epsilon_i \epsilon_j \xi e_{-j,-i}, \quad i \neq \pm j$$

or of the form

$$T'_{i,-i}(\xi) = e + \xi e_{i,-i}, \quad j = -i.$$

The *elementary symplectic group* $Ep(n, S)$ generated by all elementary symplectic transvections.

The following easy but fundamental fact is well known (see, e.g., [15, Corollary 4.4]).

Lemma 6.1. *Let S be a commutative ring, and let $n = 2l \geq 4$; for $n = 4$ we assume additionally that S has no factor fields of 2 elements. Then the elementary symplectic group $Ep(n, S)$ is perfect.*

Lemma 6.2. [7, Theorem 3] *Let S be a commutative ring and $n = 2l \geq 4$. Then*

$$\begin{aligned} N_{GL(n,S)}(Ep(n,S)) &= N_{GL(n,S)}(Sp(n,S)) = N_{GL(n,S)}(GSp(n,S)) \\ &= \{g \in GL(n,S) : Ep(n,S)^g \leq GSp(n,S)\} = GSp(n,S). \end{aligned}$$

Lemma 6.3. *Let S be a commutative ring and $n = 2l \geq 4$. Then*

$$\langle Ep(n,S), + \rangle = \langle Sp(n,S), + \rangle = \langle GSp(n,S), + \rangle = M(n,S).$$

Proof. For any $\xi \in S, 1 \leq i \neq j \leq n$, if $j \neq -i$, then

$$t_{ij}(\xi) = T'_{i,-j}(-1) + T'_{-j,j}(\xi) - T'_{i,-j}(-1)T'_{-j,j}(\xi)$$

and $t_{i,-i}(\xi) = T'_{i,-i}(\xi)$. Now the proof is completed by using Lemma 2.3. □

Lemma 6.4. *Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. Then subgroups $Ep(n, S), Sp(n, S)$ and $GSp(n, S)$ are normalized by $Aut(S/R)$.*

Proof. Reminding $\xi^\omega \in S$ for all $\omega \in Aut(S/R), \xi \in S$, we have $T'_{ij}(\xi)^\omega = T'_{ij}(\xi^\omega) \in Ep(n,S)$, thus $Ep(n,S)^{Aut(S/R)} \leq Ep(n,S)$.

Let $g \in GSp(n,S)$, there exists $\lambda = \lambda(g) \in S^*$ such that $gFg^t = \lambda F$. Then for $\omega \in Aut(S/R)$, we have

$$\begin{aligned}(\omega^{-1}g\omega)F(\omega^{-1}g\omega)^t &= (\omega^{-1}g\omega)F(\omega^{-1}g^t\omega) = \omega^{-1}g(\omega F\omega^{-1})g^t\omega \\ &= \omega^{-1}(gFg^t)\omega = \omega^{-1}\lambda F\omega = \lambda_1 F,\end{aligned}$$

where $\lambda_1 \in S$, therefore $GSp(n,S)^{Aut(S/R)} \leq GSp(n,S)$. Note that $\lambda_1 = 1$ if $\lambda = 1$, so $Sp(n,S)^{Aut(S/R)} \leq Sp(n,S)$. □

Now, as in the previous section, we obtain similar results about the normalizer of symplectic subgroups.

Theorem 6.5. *Let R be a unital subring of a commutative ring S, which is a free R-module of rank m. Then for $n = 2l \geq 4$, we have*

$$\begin{aligned}N_{GL(mn,R)}(Ep(n,S)) = N_{GL(mn,R)}(Sp(n,S)) &= N_{GL(mn,R)}(GSp(n,S)) \\ &= \{g \in GL(mn,R) : Ep(n,S)^g \leq GSp(n,S)\} \\ &= Aut(S/R) \ltimes GSp(n,S).\end{aligned}$$

Corollary 6.6. *Under the assumptions of Theorem 6.5 and let $n = 2l \geq 4$; for $n = 4$ we assume additionally that S has no factor fields of 2 elements. Suppose that $Aut(S/R)$ is solvable. If $Ep(n,S)^g \leq N_{GL(mn,R)}(Ep(n,S))$, then $g \in N_{GL(mn,R)}(Ep(n,S))$.*

Theorem 6.7. *Under the assumptions of Corollary 6.6, for any proper ideal A of R, we have*

$$N_{GL(mn,R)}(Ep(n,S)E(mn,R,A)) = \rho_A^{-1}(N_{GL(mn,R/A)}(Ep(n,S/SA))).$$

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