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Cohen–Macaulayness and sequentially Cohen–Macaulayness of monomial ideals

Hassan Noormohammadi – Ahad Rahimi

Abstract – In this paper, we give a characterization for Cohen–Macaulay rings R/I where $I \subset R = K[y_1, \dots, y_n]$ is a monomial ideal which satisfies $\operatorname{bigsize} I = \operatorname{size} I$. Next, we let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be a polynomial ring and $I \subset S$ a monomial ideal. We study the sequentially Cohen–Macaulayness of S/I with respect to $Q = (y_1, \dots, y_n)$. Moreover, if $I \subset R$ is a monomial ideal such that the associated prime ideals of I are in pairwise disjoint sets of variables, a classification of R/I to be sequentially Cohen–Macaulay is given. Finally, we compute $\operatorname{grade}(Q, M)$ where M is a sequentially Cohen–Macaulay S-module with respect to Q.

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1. Introduction

The notions of the size and bigsize of a monomial ideal were introduced by Lyubeznik and Popescu in [9] and [11], respectively. Let K be a field, $I \subset R = K[y_1, \dots, y_n]$ a monomial ideal and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the associated prime ideals of I. According to [9], the size of I is the number $v + (n - h) - 1$, where h is the height of $\sum_{i=1}^{r} \mathfrak{p}_i$ and v is the minimum number e for which there exist integers $i_1 < \cdots < i_e$ such that $\sum_{k=1}^{e} \mathfrak{p}_{i_k} = \sum_{i=1}^{r} \mathfrak{p}_i$. The bigsize of I, is the number $t + (n - h) - 1$, where t is the minimal number e such that for all integers $i_1 < \cdots < i_e$ it follows that

Hassan Noormohammadi, Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
E-mail: h.nuri77@azaruniv.edu

Ahad Rahimi, Department of Mathematics, Razi University, Kermanshah, Iran
E-mail: ahad.rahimi@razi.ac.ir

$\sum_{k=1}^{e} \mathfrak{p}_{i_k} = \sum_{i=1}^{r} \mathfrak{p}_i$. Lyubeznik [9] showed that $\operatorname{depth} R/I \geq \operatorname{size} I$. If $\operatorname{bigsize}(I) = \operatorname{size}(I)$, then $\operatorname{depth} R/I = \operatorname{size} I$ and so I satisfies Stanley's Conjecture by [7]. Fact 2.3 gives an equivalent condition for the ideal I satisfies $\operatorname{bigsize}(I) = \operatorname{size}(I)$. We observe that, if $\operatorname{bigsize}(I) = \operatorname{size}(I)$ then I has no embedded prime ideal and all the associated primes are minimal. In Section 1, we give a classification for all Cohen–Macaulay rings R/I where $I \subset R$ is a monomial ideal such that $\operatorname{bigsize} I = \operatorname{size} I$.

Next, we let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring in the variables $x_1, \dots, x_m, y_1, \dots, y_n$. In other words, $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$ for all i and j. We set $Q = (y_1, \dots, y_n)$. The second author has been studying the algebraic properties of a finitely generated bigraded S-module M and also the local cohomology modules of M with respect to Q, see for instance [12], [13], [14], [15]. In Section 2, we study the sequentially Cohen–Macaulayness of S/I with respect to Q where $I \subset S$ is a monomial ideal. A finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of M by bigraded submodules M, is called a Cohen–Macaulay filtration with respect to Q if each quotient M_i/M_{i-1} is Cohen–Macaulay with respect to Q and $0 \leq \operatorname{cd}(Q, M_1/M_0) < \operatorname{cd}(Q, M_2/M_1) < \cdots < \operatorname{cd}(Q, M_r/M_{r-1})$. Here by "Cohen–Macaulay with respect to Q" we mean $\operatorname{grade}(Q, M) = \operatorname{cd}(Q, M)$ where $\operatorname{cd}(Q, M)$ denotes the cohomological dimension of M with respect to Q which is the largest integer i for which $H^i_Q(M) \neq 0$. If M admits a Cohen–Macaulay filtration with respect to Q, then we say that M is a sequentially Cohen–Macaulay S-module with respect to Q. Ordinary sequentially Cohen–Macaulay results from our definition if we assume $m = 0$.

In [14] it is shown that if M is a finitely generated bigraded Cohen–Macaulay S-module, then M is Cohen–Macaulay with respect to $P = (x_1, \dots, x_m)$ if and only if M is Cohen–Macaulay with respect to Q. Inspired by this fact and on the evidence of all known examples we raised the following question in [10].

Question 1.1. *Let $I \subset S$ be a monomial ideal. Suppose S/I is Cohen–Macaulay.*

(a) If S/I is sequentially Cohen–Macaulay with respect to P, is S/I sequentially Cohen–Macaulay with respect to Q?

(b) Is S/I sequentially Cohen–Macaulay with respect to P and Q?

An example is given to show that this question has negative answer, see Example 3.5. However, it is shown in the case that $\operatorname{bigsize} I = \operatorname{size} I$, the question has positive answer, see Theorem 3.6. We end this section with the following question

Question 1.2. *Let M be a finitely generated bigraded S-module. If M is sequentially Cohen–Macaulay with respect to Q, is M/PM sequentially Cohen–Macaulay?*

In the following section, we let $I \subset R$ be a monomial ideal and the associated prime ideals of I are in pairwise disjoint sets of variables. It is shown that R/I

is sequentially Cohen–Macaulay if and only if I is an intersection of irreducible monomial ideals such that at most one of the factors is not principal. As a consequence, if $I \subset R$ is an intersection of monomial prime ideals in pairwise disjoint sets of variables, then R/I is sequentially Cohen–Macaulay if and only if I is a product of monomial prime ideals such that at most one of the factors is not principal. In particular, R/I is Cohen–Macaulay if and only if I is a product of principal monomial prime ideals.

There is an algebraic proof [6] as well as a combinatorial proof ([4], [16]) to compute the depth sequentially Cohen–Macaulay monomial ideals. In the final section, we extend this result by computing $\operatorname{grade}(Q, M)$ where M is sequentially Cohen–Macaulay with respect to Q.

2. size, bigsize and Cohen–Macaulayness of monomial ideals

Let $I \subset R = K[y_1, \ldots, y_n]$ be a monomial ideal. Then $I = \bigcap_{i=1}^{s} \mathfrak{q}_i$, where each $\mathfrak{q}_i$ is generated by pure powers of the variables. In other words, each $\mathfrak{q}_i$ is of the form $(y_{i_1}^{\beta_1} \ldots, y_{i_t}^{\beta_t})$. Moreover, an irredundant presentation of this form is unique. As a consequence a monomial ideal is irreducible if and only if it is generated by pure powers of the variables, see [5, Theorem 1.3.1] and [5, Corollary 1.3.2]. Thus for a monomial ideal $I \subset R$ an *irredundant irreducible decomposition* always exists. Let $\mathfrak{q}_i$ be $\mathfrak{p}_i$-primary. Then each $\mathfrak{p}_i$ is a monomial prime ideal and $\operatorname{Ass}(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ where $r \leq s$. Notice that if I is a squarefree monomial ideal, then all the associated prime ideals are minimal and hence $r = s$. In this note, by a *minimal(irredundant) primary decomposition*, we mean $\mathfrak{p}_i \neq \mathfrak{p}_j$ if $\mathfrak{q}_i \neq \mathfrak{q}_j$. For the squarefree case, the irredundant irreducible decomposition is the same as minimal primary decomposition.

Example 2.1. The ideal $I = (y_1^3, y_3^3, y_1^2 y_2^2, y_1 y_2^2 y_3, y_3^2 y_2^2) \subset R = K[y_1, y_2, y_3]$ has the irredundant irreducible decomposition

$$I = (y_1^3, y_2^2, y_3^3) \cap (y_1^2, y_3) \cap (y_1, y_3^2).$$

Hence $\operatorname{Ass}(R/I) = \{(y_1, y_3), (y_1, y_2, y_3)\}$.

Definition 2.2. According to Lyubeznik [9, Proposition 2] the *size* of I, denoted $\operatorname{size} I$, is the number $v + (n - h) - 1$, where h is the height of $\sum_{i=1}^{r} \mathfrak{p}_i$ and v is the minimum number t for which there exist integers $i_1 < \cdots < i_t$ such that

$$\sum_{k=1}^{t} \mathfrak{p}_{i_k} = \sum_{i=1}^{r} \mathfrak{p}_i.$$

Replacing in the previous definition "there exist $i_1 < \cdots < i_t$" by "for all $i_1 < \cdots < i_t$" one obtains the definition of *bigsize* of I, introduced by Popescu [11].

Of course, $\operatorname{bigsize} I \geq \operatorname{size} I$ and in fact the bigsize of I is in general much bigger than the size of I. In Example 2.1, we have $\operatorname{size} I = 0$ and $\operatorname{bigsize} I = 1$.

In this section, we may assume $\sum_{i=1}^{r} \mathfrak{p}_i = \mathfrak{m}$ the graded maximal ideal of R, because each free variable on I increases size and bigsize with 1. In fact, if $Z = \{y_j : y_j \notin \sum_{i=1}^{r} \mathfrak{p}_i\}, T = K[Y \setminus Z]$ and $J = I \cap T$. Then $\operatorname{size} I = \operatorname{size} J + |Z|$ and $\operatorname{bigsize} I = \operatorname{bigsize} J + |Z|$. In this case, $h = n$ and so $\operatorname{size} I = v - 1$.

Fact 2.3. Notice that $\operatorname{bigsize} I = \operatorname{size} I = v - 1$ if and only if v is the largest integer such that $\mathfrak{p}_j \nsubseteq \sum_{i \in A \setminus \{j\}} \mathfrak{p}_i$ for all $j \in [r] = \{1, \ldots, r\}$, where $\emptyset \neq A \subseteq [r]$ with $|A| \leq v$. In particular,

$$\operatorname{bigsize} I = \operatorname{size} I = r - 1 \Longleftrightarrow \mathfrak{p}_j \nsubseteq \sum_{i \in [r] \setminus \{j\}} \mathfrak{p}_i. \tag{1}$$

Observe that if $\operatorname{bigsize} I = \operatorname{size} I$, then all the associated prime ideals $\mathfrak{p}_i$ are minimal.

Remark 2.4. Suppose $\operatorname{size} I = \operatorname{bigsize} I$ where $I \subset R$ is a monomial ideal. We observed that the ideal I has no embedded prime ideal, and so all the associated prime ideals are minimal. Thus if $I = \bigcap_{i=1}^{r} \mathfrak{q}_i$ is an irredundant irreducible decomposition of I, then $\sqrt{I} = \bigcap_{i=1}^{r} \mathfrak{p}_i$ is an irredundant irreducible decomposition of $\sqrt{I}$ where $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ for $i = 1, \ldots, r$. It follows that $\operatorname{Ass}(R/I) = \operatorname{Ass}(R/\sqrt{I})$ and hence $\operatorname{size} I = \operatorname{size} \sqrt{I}$. Note that $\operatorname{size} I$ is not equal to $\operatorname{size} \sqrt{I}$ in general. Consider the ideal $I = (y_1^2, y_1 y_2) \subset K[y_1, y_2]$. As $\operatorname{Ass}(R/I) = \{(y_1), (y_1, y_2)\}$ and $\operatorname{Ass}(R/\sqrt{I}) = \{(y_1)\}$, we have $0 = \operatorname{size} I \neq \operatorname{bigsize} I = 1$ and $\operatorname{size} \sqrt{I} = 1$.

The following example shows that if all the associated prime ideals are minimal, then the equality $\operatorname{size} I = \operatorname{bigsize} I$ may not hold.

Example 2.5. Let $I = \bigcap_{i=1}^{3} \mathfrak{q}_i$ be an ideal of $R = K[y_1, y_2, y_3, y_4]$ such that $\mathfrak{q}_1 = (y_1, y_2^2, y_3^3)$, $\mathfrak{q}_2 = (y_3^2, y_4^2)$ and $\mathfrak{q}_3 = (y_2^3, y_4)$. Thus

$$\operatorname{Ass}(R/I) = \{(y_1, y_2, y_3), (y_3, y_4), (y_2, y_4)\},$$

and so all the associated prime ideals are minimal. On the other hand,

$$\operatorname{size} I = \underbrace{2}_{v} + (\underbrace{4}_{n} - \underbrace{4}_{h}) - 1 = 1, \quad \operatorname{bigsize} I = \underbrace{3}_{v} + (\underbrace{4}_{n} - \underbrace{4}_{h}) - 1 = 2.$$

In the following, we give a classification for R/I to be Cohen–Macaulay when $\operatorname{bigsize} I = \operatorname{size} I$. We first recall the following result from [7, Theorem 1.2].

Lemma 2.6. *Let $I \subset R$ be a monomial ideal. Assume that* $\operatorname{bigsize} I = \operatorname{size} I$. *Then*

$$\operatorname{depth} R/I = \operatorname{size} I.$$

For the proof of our main result we need the following.

LEMMA 2.7. *Let $I \subset R$ be a monomial ideal and $I = \bigcap_{i=1}^{r} \mathfrak{q}_i$ an irredundant irreducible decomposition of I. Assume that* $\operatorname{bigsize} I = \operatorname{size} I$. *Then for each $F \subset [r]$ we have* $\operatorname{bigsize} I_F = \operatorname{size} I_F$ *where $I_F = \bigcap_{i \in F} \mathfrak{q}_i$.*

PROOF. Put $\operatorname{Ass}(R/I_F) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ where $t \le r$. Here we consider two cases. First suppose $t \ge v$. It follows that $\operatorname{bigsize} I_F = \operatorname{size} I_F = v - 1$. Now let $t < v$. By Fact 2.3

$$\mathfrak{p}_j \nsubseteq \sum_{i \in A \setminus \{j\}} \mathfrak{p}_i \quad \text{for all } j \in [t],$$

where $\emptyset \neq A \subset [t]$ with $|A| \le t$. In particular, $\operatorname{bigsize} I_F = \operatorname{size} I_F = t - 1$, as desired. □

THEOREM 2.8. *Let $I \subset R$ be a monomial ideal and $I = \bigcap_{i=1}^{r} \mathfrak{q}_i$ an irredundant irreducible decomposition of I with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$. Assume that* $\operatorname{bigsize} I = \operatorname{size} I$. *Then the following statements are equivalent*

(a) R/I is Cohen–Macaulay;

(b) $R/\sqrt{I}$ is Cohen–Macaulay;

(c) $\mathfrak{p}_i$ differs with $\mathfrak{p}_j$ only in one variable for all $i \neq j$ with $i, j \in [r]$;

(d) For each subset $F \subseteq [r]$, $\dfrac{R}{\bigcap_{i \in F} \mathfrak{q}_i}$ is Cohen–Macaulay.

PROOF. $(a) \Leftrightarrow (b)$: By Lemma 2.6, we have

$$\operatorname{depth} R/I = \operatorname{size} I = \operatorname{size} \sqrt{I} = \operatorname{depth} R/\sqrt{I}.$$

Remark 2.4 provides the second equality. On the other hand, $\dim R/I = \dim R/\sqrt{I}$. Thus the assertion follows.

$(a) \Rightarrow (c)$: Suppose R/I is Cohen–Macaulay. It follows that R/I is unmixed and hence $\dim R/I = \dim R/\mathfrak{p}_i = n - \operatorname{height} \mathfrak{p}_i$ for all $i \in [r]$. On the other hand, $\operatorname{depth} R/I = \operatorname{size} I = v - 1$ by Lemma 2.6. Thus

$$n - \operatorname{height} \mathfrak{p}_i = v - 1 \quad \text{for all} \quad i \in [r]. \tag{2}$$

Let $A \subset [r]$ with $|A| = v$. Note that

$$\begin{aligned} n &= \operatorname{height} \Big(\sum_{i \in A} \mathfrak{p}_i \Big) \\ &= \operatorname{height} \mathfrak{p}_j + \operatorname{height} \Big(\sum_{i \in A \setminus \{j\}} \big(\mathfrak{p}_i \setminus \{ y_{k_j} : y_{k_j} \in \mathfrak{p}_j \} \big) \Big). \end{aligned}$$

We set

$$\mathfrak{c}_j = \sum_{i \in A \setminus \{j\}} \big(\mathfrak{p}_i \setminus \{ y_{k_j} : y_{k_j} \in \mathfrak{p}_j \} \big).$$

Thus height $\mathfrak{c}_j = v-1$ by (2). It follows that each $\mathfrak{p}_i$ differs with $\mathfrak{p}_j$ only in one variable for all $i \neq j$.

$(c) \Rightarrow (a)$: Let $\mathfrak{p}_i$ differs with $\mathfrak{p}_j$ only in one variable and $\mathfrak{c}_j$ and A be as above. It follows that R/I is unmixed and height $\mathfrak{c}_j = v-1$. Using these facts we have,

$$\begin{aligned}\dim R/I &= n - \operatorname{height} \mathfrak{p}_j \\ &= \operatorname{height} \big(\sum_{i\in A} \mathfrak{p}_i\big) - \operatorname{height} \mathfrak{p}_j \\ &= \operatorname{height} \mathfrak{p}_j + \operatorname{height} \mathfrak{c}_j - \operatorname{height} \mathfrak{p}_j \\ &= v-1 \\ &= \operatorname{size} I \\ &= \operatorname{depth} R/I,\end{aligned}$$

as desired.

$(c) \Rightarrow (d)$: Lemma 2.7 and the equivalence (a) and (c) yield the desired conclusion.

The implication $(d) \Rightarrow (a)$ is trivial. □

In particular, if $\operatorname{size} I = \operatorname{bigsize} I = r-1$ which is equivalent to say $\mathfrak{p}_j \nsubseteq \sum_{i\in[r]\setminus\{j\}} \mathfrak{p}_i$ by (1), then we have the following

Corollary 2.9. *Let $I \subset R$ be a monomial ideal and $I = \bigcap_{i=1}^r \mathfrak{q}_i$ an irredundant irreducible decomposition of I. Assume that $\mathfrak{p}_j \nsubseteq \sum_{i\in[r]\setminus\{j\}} \mathfrak{p}_i$ for all $j \in [r]$. Then R/I is Cohen–Macaulay if and only if $\sqrt{I} = \mathfrak{q} + L$ where $\mathfrak{q}$ is a monomial prime ideal and L is a product of principal monomial prime ideals.*

Proof. Suppose R/I is Cohen–Macaulay. By Theorem 2.8, each $\mathfrak{p}_i$ differs with $\mathfrak{p}_j$ only in one variable for all $i \neq j$. Our assumption implies that each $\mathfrak{p}_i$ is of the form $(z_1, z_2, \dots, z_t, w_i)$ where $z_1, z_2, \dots, z_t, w_i \in \{y_1, \dots, y_n\}$. Note that

$$\sqrt{I} = \bigcap_{i=1}^{r} \mathfrak{p}_i = (z_1, z_2, \dots, z_t, \prod_{i=1}^{r} w_i).$$

We set $\mathfrak{q} = (z_1, z_2, \dots, z_t)$. Hence the assertion follows.

For the converse, we suppose $\sqrt{I} = \mathfrak{q} + L$. It follows that $R/\sqrt{I}$ is Cohen–Macaulay. Hence by Theorem 2.8, R/I is Cohen–Macaulay as well. □

In particular, we have the following classification of all Cohen–Macaulay rings R/I where I is an intersection of monomial prime ideals in pairwise disjoint sets of variables.

Corollary 2.10. *If I is an intersection of monomial prime ideals in pairwise disjoint sets of variables, then R/I is Cohen–Macaulay if and only if I is a product of principal monomial prime ideals.*

3. Sequentially Cohen–Macaulayness of monomial ideals with respect to P, Q and $P+Q$

Let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over K. In other words, $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$ for all i and j. We set $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Let M be a finitely generated bigraded S-module. A filtration $\mathcal{D}$: $0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_t = M$ of bigraded submodules of M is called the dimension filtration of M with respect to Q if D_{i-1} is the largest bigraded submodule of D_i for which $\operatorname{cd}(Q, D_{i-1}) < \operatorname{cd}(Q, D_i)$ for all $i = 1, \dots, t$. We recall the following facts from [10].

Fact 3.1. Let $\mathcal{D} : 0 = D_0 \subsetneq D_1 \subsetneq \dots \subsetneq D_t = M$ be the dimension filtration of M with respect to Q. Then

(a) $D_i = \bigcap_{\mathfrak{p}_j \not\in B_{i,Q}} N_j$ for $i = 1, \dots, t-1$ where $0 = \bigcap_{j=1}^{s} N_j$ is an irredundant primary decomposition of 0 in M with N_j is $\mathfrak{p}_j$-primary for $j = 1, \dots, s$ and

$$B_{i,Q} = \{\mathfrak{p} \in \operatorname{Ass}(M) : \operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, D_i)\}.$$

(b) $\operatorname{Ass}(M/D_i) = \operatorname{Ass}(M) \setminus \operatorname{Ass}(D_i)$ for $i = 1, \dots, t$.

(c) $\operatorname{grade}(Q, M/D_{i-1}) = \operatorname{cd}(Q, D_i)$ for $i = 1, \dots, t$ if and only if M is sequentially Cohen–Macaulay with respect to Q.

Fact 3.2. The following statements hold.

(a) The exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated S-modules yields $\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, M'), \operatorname{cd}(Q, M'')\}$, see [2, Proposition 4.4].

(b) $\operatorname{cd}(Q, M) = \max\{\operatorname{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Ass}(M)\} = \max\{\operatorname{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}(M)\}$, see [2, Corollary 4.6].

(c) $\operatorname{grade}(Q, M) \leq \dim M - \operatorname{cd}(P, M)$, and the equality holds if M is Cohen–Macaulay, see [14, Formula 5].

(d) $\operatorname{cd}(P, M) = \dim M/QM$ and $\operatorname{cd}(Q, M) = \dim M/PM$, see [14, Formula 3].

A finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of M by bigraded submodules M is called a *Cohen–Macaulay filtration with respect to* Q if each quotient M_i/M_{i-1} is Cohen–Macaulay with respect to Q and $0 \leq \operatorname{cd}(Q, M_1/M_0) < \operatorname{cd}(Q, M_2/M_1) < \cdots < \operatorname{cd}(Q, M_r/M_{r-1})$. If M admits a Cohen–Macaulay filtration with respect to Q, then we say M is *sequentially Cohen–Macaulay with respect to* Q. Ordinary sequentially Cohen–Macaulay introduced by Stanley results from our definition if we assume $P = 0$. Note that if M is sequentially Cohen–Macaulay with respect to Q, then the filtration $\mathcal{F}$ is uniquely determined and it is just the dimension filtration of M with respect to Q, that is, $\mathcal{F} = \mathcal{D}$, see [15].

Remark 3.3. Let $I \subset S$ be a monomial ideal and $I = \bigcap_{i=1}^{r} \mathfrak{q}_i$ an irredundant irreducible decomposition of I where $\mathfrak{q}_i$ are $\mathfrak{p}_i$-primary monomial ideals. As before, we may write $\mathfrak{q}_i = \mathfrak{q}_i^x + \mathfrak{q}_i^y$ where $\mathfrak{q}_i^x = (x_{i_1}^{\alpha_1}, \dots, x_{i_k}^{\alpha_k})$ and $\mathfrak{q}_i^y = (y_{i_1}^{\beta_1} \dots, y_{i_s}^{\beta_s})$ are monomial ideals in $K[x_1, \dots, x_m]$ and $K[y_1, \dots, y_n]$, respectively. We set $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i = \mathfrak{p}_i^x + \mathfrak{p}_i^y$ for all i where $\mathfrak{p}_i^x = \sqrt{\mathfrak{q}_i^x}$ and $\mathfrak{p}_i^y = \sqrt{\mathfrak{q}_i^y}$. The ideal I has the irredundant irreducible decomposition

$$I = (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{a_1}) \cap \cdots \cap (\mathfrak{q}_{a_{t-1}+1} \cap \cdots \cap \mathfrak{q}_{a_t})$$

where

$$\operatorname{height} \mathfrak{p}_{a_{i-1}+1}^y = \cdots = \operatorname{height} \mathfrak{p}_{a_i}^y = d_i^y \quad \text{for} \quad i \in \{1, \dots, t\};$$

assuming $a_0 = 0$ and $d_1^y < d_2^y < \cdots < d_t^y$. By Fact 3.1(a), S/I has the dimension filtration $\mathcal{F}$: $0 = I_0/I \subsetneq I_1/I \subsetneq \cdots \subsetneq I_t/I = S/I$ with respect to Q where

$$I_0 = I,$$

$$I_1 = (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{a_1}) \cap \cdots \cap (\mathfrak{q}_{a_{t-2}+1} \cap \cdots \cap \mathfrak{q}_{a_{t-1}}),$$

$$\vdots$$

$$I_{t-2} = (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{a_1}) \cap (\mathfrak{q}_{a_1+1} \cap \cdots \cap \mathfrak{q}_{a_2}),$$

$$I_{t-1} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{a_1} \quad \text{and}$$

$$I_t = S.$$

Here I_{t-1} is the unmixed component of S/I with respect to Q. Observe that

$$\operatorname{cd}(Q, I_i/I_{i-1}) = \operatorname{cd}(Q, I_i/I) = n - d_{t-i+1}^y, \tag{3}$$

by Fact 3.2(b) and Fact 3.1(b).

In [14] it is shown that if M is a finitely generated bigraded Cohen–Macaulay S-module, then M is Cohen–Macaulay with respect to P if and only if M is Cohen–Macaulay with respect to Q. Inspired by this fact and on the evidence of all known examples we raised the following question in [10].

Question 3.4. *Let $I \subset S$ be a monomial ideal. Suppose S/I is Cohen–Macaulay.*

(a) If S/I is sequentially Cohen–Macaulay with respect to P, is S/I sequentially Cohen–Macaulay with respect to Q?

(b) Is S/I sequentially Cohen–Macaulay with respect to P and Q?

The following example shows that the answer is negative.

EXAMPLE 3.5. Let $S = K[x_1, x_2, y_1, y_2, y_3, y_4]$ be the standard bigraded polynomial ring. We set $R = S/I$ where $I = (y_2y_4, y_1y_4, y_2y_3, y_1y_3, x_1y_3, x_2y_2)$, $P = (x_1, x_2)$ and $Q = (y_1, y_2, y_3, y_4)$. The ideal I has the minimal primary decomposition $I = \bigcap_{i=1}^{4} \mathfrak{p}_i$ where $\mathfrak{p}_1 = (x_1, y_1, y_2)$, $\mathfrak{p}_2 = (x_2, y_3, y_4)$, $\mathfrak{p}_3 = (y_1, y_2, y_3)$ and $\mathfrak{p}_4 = (y_2, y_3, y_4)$. The ring R has dimension 3 and by using CoCoA [3] depth 3. Hence R is Cohen–Macaulay.

We first show that R is sequentially Cohen–Macaulay with respect to P. By Fact 3.1(a), R has the dimension filtration $\mathcal{F}$: $0 = J_0/I \subsetneq J_1/I \subsetneq J_2/I = S/I$ with respect to P where $J_0 = I$, $J_1 = \mathfrak{p}_3 \cap \mathfrak{p}_4$ and $J_2 = S$. By Fact 3.2(c) and Fact 3.1(b) we have $\operatorname{grade}(P, S/I) = \operatorname{cd}(P, J_1/I) = 1$. One has $\operatorname{grade}(P, S/J_1) = \operatorname{cd}(P, S/I) = 2$. Thus, R is sequentially Cohen–Macaulay with respect to P by Fact 3.1(c).

Next we show that R is not sequentially Cohen–Macaulay with respect to Q. By Fact 3.1(a), R has the dimension filtration $\mathcal{F}$: $0 = I_0/I \subsetneq I_1/I \subsetneq I_2/I = S/I$ with respect to Q where $I_0 = I$, $I_1 = \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $I_2 = S$. Observe that $\operatorname{grade}(Q, S/I) = \operatorname{cd}(Q, I_1/I) = 1$ by Fact 3.2(c) and Fact 3.1(b). Hence $1 = \operatorname{grade}(Q, S/I_1) \neq \operatorname{cd}(Q, S/I) = 2$. Thus, R is not sequentially Cohen–Macaulay with respect to Q by Fact 3.1(c).

However, we show that Question 3.4 has positive answer in the following special case. Notice that in Example 3.5, $\operatorname{size} I = 1$ and $\operatorname{bigsize} I = 3$.

THEOREM 3.6. *Let $I \subset S$ be a monomial ideal such that* $\operatorname{bigsize} I = \operatorname{size} I$. *If S/I is Cohen–Macaulay, then S/I is sequentially Cohen-Macaulay with respect to P and Q.*

PROOF. We show that S/I is sequentially Cohen–Macaulay with respect to Q. The argument for P is similar. By Fact 3.1(c) we only need to show $\operatorname{grade}(Q, S/I_{i-1}) = \operatorname{cd}(Q, I_i/I)$ for $i = 1, \ldots, t$ where I_i described in Remark 3.3. By Theorem 2.8, S/I_{i-1} is Cohen–Macaulay for all $i = 1, \ldots, t$. Thus we have

$$\begin{aligned} \operatorname{grade}(Q, S/I_{i-1}) &= \dim S/I_{i-1} - \operatorname{cd}(P, S/I_{i-1}) \\ &= m + n - (d^x_{t-i+1} + d^y_{t-i+1}) - (m - d^x_{t-i+1}) \\ &= n - d^y_{t-i+1} \\ &= \operatorname{cd}(Q, I_i/I_{i-1}). \end{aligned}$$

Fact 3.2(c) explains the first step in this sequence. For the second step, in Remark 3.3 we set

$$\operatorname{height} \mathfrak{p}^x_{a_{i-1}+1} = \cdots = \operatorname{height} \mathfrak{p}^x_{a_i} = d^x_i \quad \text{for} \quad i \in \{1, \ldots, t\}.$$

Since S/I is Cohen–Macaulay, it follows that $d^x_t < \cdots < d^x_2 < d^x_1$ and $d^x_i + d^y_i = \operatorname{height} \mathfrak{p}_i$. The fourth step follows from (3) and the remaining steps are standard. □

Remark 3.7. The following example shows that the converse of Theorem 3.6 does not hold in general. Let $S = K[x_1, x_2, y_1, y_2]$ be the polynomial ring. We set $P = (x_1, x_2)$, $Q = (y_1, y_2)$, $\mathfrak{p}_1 = (x_1, y_1)$, $\mathfrak{p}_2 = (x_2, y_2)$ and $R = S/I$ where $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$. One has $\operatorname{cd}(Q, R) = \operatorname{cd}(P, R) = 1$ and $\operatorname{grade}(Q, R) = \operatorname{grade}(P, R) = 1$. Thus R is Cohen–Macaulay with respect to P and Q, and hence sequentially Cohen–Macaulay with respect to P and Q. Moreover, $\operatorname{bigsize} I = \operatorname{size} I = 1$. On the other hand, $\dim R = 2$, and $\operatorname{depth} R = 1$ by Lemma 2.6. Hence R is not Cohen–Macaulay.

We end this section with the following question

Question 3.8. *Let M be a finitely generated bigraded S-module. If M is sequentially Cohen–Macaulay with respect to Q, is M/PM sequentially Cohen–Macaulay?*

4. Sequentially Cohen–Macaulayness of monomial ideals

In the following, our aim is to classify all rings R/I for a special class of monomial ideal I for which R/I to be sequentially Cohen–Macaulay.

Proposition 4.1. *Let $I \subset R$ be a monomial ideal and $I = \bigcap_{i=1}^{s} \mathfrak{q}_i$ an irredundant irreducible decomposition of I where the associated prime ideals of I are in pairwise disjoint sets of variables. Then R/I is sequentially Cohen–Macaulay if and only if I is an intersection of irreducible monomial ideals such that at most one of the factors is not principal.*

Proof. (⇒): Suppose R/I is sequentially Cohen–Macaulay. By Fact 3.1(c) we have

$$\operatorname{depth} R/I_{i-1} = \dim I_i/I = n - d_{t-i+1},$$

for all $i = 1, \dots, t$ where t and I_i described in Remark 3.3 with setting $P = 0$ and $d_i^y = d_i$. The second equality follows from (3). Let $\mathfrak{p}_1, \dots, \mathfrak{p}_{b_1}$ and $\mathfrak{p}_{b_1+1}, \dots, \mathfrak{p}_{b_2}$ with $b_i \leq a_i$ for $i = 1, 2$ be the distinct monomial prime ideals of height d_1 and d_2, respectively. For $i = t, t-1$, by using Lemma 2.6 we have

$$b_1 + (n - b_1 d_1) - 1 = n - d_1 \quad \text{and} \quad b_2 + (n - b_1 d_1 - (b_2 - b_1) d_2) - 1 = n - d_2. \tag{4}$$

Thus

$$b_1 - 1 = d_1(b_1 - 1) \quad \text{and} \tag{5}$$

$$b_2 - b_1 d_1 - 1 = d_2(b_2 - b_1 - 1). \tag{6}$$

We claim that $d_1 = 1$, $b_2 - b_1 = 1$ and $t \leq 2$. This completes the proof. To show the first claim, suppose $d_1 > 1$. Thus $b_1 = 1$ by (5). Hence $b_2 - d_1 - 1 = d_2(b_2 - 2)$

by (6). This yields $d_2 < 1$, a contradiction. Therefore, $d_1 = 1$. For the second claim, we observe that $b_2 - b_1 - 1 = d_2(b_2 - b_1 - 1)$ by (6). If $b_2 - b_1 - 1 > 0$, then $d_2 = 1$, a contradiction. Thus $b_2 - b_1 = 1$. Finally we show that $t \leq 2$. Suppose $t > 2$. Let $\mathfrak{p}_{b_2+1}, \dots, \mathfrak{p}_{b_3}$ with $b_3 \leq a_3$ be the distinct monomial prime ideals of height d_3. For $i = t - 2$, by using Lemma 2.6 we have

$$b_3 + (n - b_1 d_1 - (b_2 - b_1)d_2 - (b_3 - b_2)d_3) - 1 = n - d_3.$$

Thus

$$b_3 - b_1 - d_2 - 1 = d_3(b_3 - b_1 - 2).$$

As $d_2 \geq 2$, we have $d_3 < 1$, a contradiction.

($\Leftarrow$): The assertion follows by replacing $d_1 = 1$ and $b_2 - b_1 = 1$ in (4). □

Corollary 4.2. *Let $I \subset R$ be the intersection of monomial prime ideals in pairwise disjoint sets of variables. Then R/I is sequentially Cohen–Macaulay if and only if I is a product of monomial prime ideals such that at most one of the factors is not principal. In particular, R/I is Cohen–Macaulay if and only if I is a product of principal monomial prime ideals.*

Proof. The first statement follows from Proposition 4.1. To show the second statement, suppose R/I is Cohen–Macaulay. It follows from the proof of Proposition 4.1 that $b_1 = b_2$ and $t = 1$. Therefore, the conclusion follows. The converse of the second statement is obvious. □

5. Compute grade(Q, M) where M is sequentially Cohen–Macaulay with respect to Q

In this section, we compute grade(Q, M) where M is sequentially Cohen–Macaulay with respect to Q. Here M is a finitely generated bigraded S-module and as usual $R = K[y_1, \dots, y_n]$. We recall the following fact from [15].

Fact 5.1. If M is sequentially Cohen–Macaulay with respect to Q with the Cohen–Macaulay filtration $\mathcal{F} : 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$, then one observes that

$$\operatorname{grade}(Q, M_i) = \operatorname{grade}(Q, M_1) \quad \text{for} \quad i = 1, \dots, r.$$

Lemma 5.2. *Let M be sequentially Cohen–Macaulay with respect to Q with the Cohen–Macaulay filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$. Then for $i = 1, \dots, r$, we have*

$$\operatorname{Ass}(M_i/M_{i-1}) = \{\mathfrak{p} \in \operatorname{Ass}(M_i) : \operatorname{cd}(Q, S/\mathfrak{p}) = \operatorname{cd}(Q, M_i)\}.$$

In particular,

$$\operatorname{Ass}(M) = \bigcup_{i=1}^{r} \operatorname{Ass}(M_i/M_{i-1}).$$

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(M_i/M_{i-1})$. Since M_i/M_{i-1} is Cohen–Macaulay with respect to Q, it follows that $\operatorname{cd}(Q, S/\mathfrak{p}) = \operatorname{cd}(Q, M_i/M_{i-1}) = \operatorname{cd}(Q, M_i)$. Thus we only need to show that $\mathfrak{p} \in \operatorname{Ass}(M_i)$. As we always have $\operatorname{Ass}(M_i/M_{i-1}) \subset \operatorname{Ass}(M_i) \cup \operatorname{Supp}(M_{i-1})$, it suffices to show that $\mathfrak{p} \notin \operatorname{Supp}(M_{i-1})$. Assume $\mathfrak{p} \in \operatorname{Supp}(M_{i-1})$. Fact 3.2(b) implies that $\operatorname{cd}(Q, S/\mathfrak{p}) \leq \operatorname{cd}(Q, M_{i-1}) < \operatorname{cd}(Q, M_i)$, a contradiction. Thus $\mathfrak{p} \notin \operatorname{Supp}(M_{i-1})$ and hence $\mathfrak{p} \in \operatorname{Ass}(M_i)$.

Now let $\mathfrak{p} \in \operatorname{Ass}(M_i)$ such that $\operatorname{cd}(Q, S/\mathfrak{p}) = \operatorname{cd}(Q, M_i)$. The exact sequence $0 \to M_{i-1} \to M_i \to M_i/M_{i-1}$ yields $\operatorname{Ass}(M_i) \subset \operatorname{Ass}(M_{i-1}) \cup \operatorname{Ass}(M_i/M_{i-1})$. A similar argument as above shows that $\mathfrak{p} \notin \operatorname{Ass}(M_{i-1})$. Hence $\mathfrak{p} \in \operatorname{Ass}(M_i/M_{i-1})$. □

Proposition 5.3. *Suppose that the maximal height of an associated prime of M in R is d and $|K| = \infty$. Then*

$$\operatorname{grade}(Q, M) \leq n - d.$$

In particular, if M is sequentially Cohen-Macaulay with respect to Q, then

$$\operatorname{grade}(Q, M) = n - d.$$

Proof. By [8, Proposition 1.7] we have $\operatorname{grade}(Q, M) \leq \operatorname{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. Let $\mathfrak{q} \in \operatorname{Ass}(M)$ has maximal height d in R. Thus by using Fact 3.2(d) we have

$$\operatorname{grade}(Q, M) \leq \operatorname{cd}(Q, S/\mathfrak{q}) = \dim S/(P + \mathfrak{q}) = \dim S/(P + \mathfrak{q}^y) = n - d.$$

Now let M be sequentially Cohen–Macaulay with respect to Q. Observe that

$$\begin{aligned} \operatorname{grade}(Q, M) &= \operatorname{grade}(Q, M_1) \\ &= \operatorname{cd}(Q, M_1) \\ &= \operatorname{cd}(Q, S/\mathfrak{p}) \quad \text{for all } \mathfrak{p} \in \operatorname{Ass}(M_1) \\ &= n - d. \end{aligned}$$

Fact 5.1 provides the first step in this sequence. The second step follows from the definition. [8, Corollary 1.11] explains the third step. The final step follows from the definition and Lemma 5.2. □

As a consequence we have the following known result. For a combinatorial proof see [4, Theorem 4]. See also ([6], [16]).

Corollary 5.4. *Let $J \subset R$ be a monomial ideal with $|K| = \infty$. Suppose that the maximal height of an associated prime of J is d. Then*

$$\operatorname{depth} R/J \leq n - d \quad \textit{and} \quad \operatorname{pd} R/J \geq d.$$

In particular, if R/J is sequentially Cohen-Macaulay, then

$$\operatorname{depth} R/J = n - d \quad \textit{and} \quad \operatorname{pd} R/J = d.$$

We end this section with the following.

Proposition 5.5. *Let $I \subset S$ be a monomial ideal such that S/I is Cohen–Macaulay. Suppose that the maximal height of an associated prime of I in R is d. Then*

$$\operatorname{grade}(Q, S/I) = n - d.$$

Proof. Since S/I is Cohen–Macaulay, it follows that $d_t^x < \cdots < d_2^x < d_1^x$ where

$$\operatorname{height} \mathfrak{p}^x_{a_{i-1}+1} = \cdots = \operatorname{height} \mathfrak{p}^x_{a_i} = d_i^x \quad \text{for} \quad i \in \{1, \ldots, t\};$$

and $d_i^x + d_i^y = \operatorname{height} \mathfrak{p}_i$, see Remark 3.3. By Fact 3.2(c) we have

$$\begin{aligned} \operatorname{grade}(Q, S/I) &= \dim S/I - \operatorname{cd}(P, S/I) \\ &= m + n - (d_t^x + d_t^y) - (m - d_t^x) \\ &= n - d_t^y, \end{aligned}$$

as desired. □

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