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Nadler Mappings in Cone *b*-Metric Spaces over Banach Algebras

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Abstract – In this study we first define the concept of Nadler type contraction in the setting of H-cone b-metric space with respect to cone b-metric spaces over Banach algebras. Next we prove the Banach contraction principle for such contractions by means of the notion of spectral radius and a solid cone in underlying Banach algebra. Finally we observe that the main result achieved in this work extends and generalizes the well known results associated with contractions of Nadler type.

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1. Introduction

In 1969, Nadler [1] introduced the notion of multivalued contraction as a set-valued mapping in the setting of metric spaces. Nadler also extended Banach's contraction principle to such contractions. Since then, many people have tried to introduce some generalizations and extensions of multivalued contractions in [2],[3], [4],[5].

On the other hand, Huang and Zhang [6] studied Banach's contraction principle in the setting of a cone metric space defined by using an ordered Banach space-valued metric. Later, many works in the literature were devoted to studying the fixed point theory in the setting of (tvs)-cone metric spaces to generalize the well known fixed point theorems in the usual metric spaces. But, in 2010, Du [7] proved that many fixed point theorems in this setting are equivalent to their counterparts in the usual metric spaces. However, in 2013, Liu and Xu [8] constructed the setting of cone metric spaces over Banach algebras where proper generalizations of

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the well known fixed point theorems can be realized. Note that the main results in [8] are obtained in the setting of cone metric space over a Banach algebra with a normal solid cone. In 2014, taking into account some basic properties of spectral radius, Xu and Radenovic [9] proved that the main results of [8] can be achieved by omitting the assumption of normality of cone in the underlying Banach algebra.

In this work we first give the definition of multivalued contraction in the sense of Wardowski by using the setting of H-cone b-metric with respect to cone b-metric spaces over Banach algebras. Next we aim to extend Nadler's fixed point theorem to such contractions without normality condition for the underlying solid cone.

2. Basic Definitions and Results

We briefly recall some basic terms and results which will be needed to obtain the main result of this work. Let $\mathcal{A}$ be a real Banach algebra where the multiplicative unit and the null vector will be denoted by e and θ, respectively. The spectral radius of $a \in A$ is defined as follows:

$$\rho(a) := \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} .$$

If $\rho(a) < 1$, then $e - a$ is invertible (see [12]) and the inverse of $e - a$ is given by

$$(e-a)^{-1} = \sum_{i=0}^{\infty} a^i. \tag{1}$$

Let P be a subset of $\mathcal{A}$ such that $\{\theta, e\} \subset P$. P is called a cone of $\mathcal{A}$ if the following conditions hold:

(c1) P is closed;

(c2) $\lambda P + \mu P \subset P$ for all non-negative real numbers λ and μ;

(c3) $PP \subset P$ and $P \cap (-P) = \theta$.

It is clear that to each cone P of $\mathcal{A}$ there corresponds a partial ordering $\preceq$ on $\mathcal{A}$ defined by $x \preceq y$ iff $y - x \in P$. By $x \prec y$ we understand that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$ where $\text{int}P$ indicates the interior of P. A cone P with $\text{int}P \neq \emptyset$ is called a solid cone. If there exists a positive real number K such that for all $x, y \in \mathcal{A}$

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K \|y\| , \tag{2}$$

then a cone P is called normal. The least of K's with the above condition is called the normal constant of P.

Definition 2.1. (See [10]) Let $\mathcal{A}$ be an ordered Banach algebra, $s \geq 1$ be a constant and $X \neq \emptyset$. A cone b-metric space over $\mathcal{A}$ is given by a pair (X, d) where d is a mapping $d : X \times X \to \mathcal{A}$ satisfying (cbm1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$

(cbm2) $d(x,y) = d(y,x)$

(cbm3) $d(x,y) \preceq s[d(x,z) + d(z,y)]$

for all $x, y, z \in X$ and for null vector $\theta \in \mathcal{A}$.

EXAMPLE 2.2. Let $\mathcal{A}$ be the usual algebra of all real valued continious functions on $X = [0,1]$ which also have continious derivatives on X. If $\mathcal{A}$ is equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, then $\mathcal{A}$ becomes a Banach algebra with unit $e = 1$. Morever, $P = \{f \in \mathcal{A} | f(t) \geq 0$ for all $t \in X\}$ is a nonnormal cone (see [9]). Consider a mapping $d : X \times X \to \mathcal{A}$ defined by $d(x,y)(t) = \frac{1}{2}|x-y|^2 e^t$ for all $x, y \in X$. It is obvious that (X,d) is a cone b-metric space with $s = 2$ on the Banach algebra $\mathcal{A}$. For more examples in this setting, one can refer to [10], [8] and [9].

Letting $s = 1$ in Definition 2.1, we obtain the definition of cone metric space over Banach algebras introduced in [8]. Consequently, we note that the class of cone b-metric spaces over Banach algebras contains those of metric spaces and cone metric spaces as special cases. From now on we always suppose that $\mathcal{A}$ is a real Banach algebra with the multiplicative unit e, P is a solid cone of $\mathcal{A}$ and $\preceq$ stands for the partial ordering induced by P.

DEFINITION 2.3. (See [8, 10]) Let $\{x_n\}$ be a sequence in a cone b-metric space (X,d) on $\mathcal{A}$. Then

(i) We say that $\{x_n\}$ is convergent to $x \in X$ if to each $c \gg \theta$ there corresponds a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is called Cauchy if for each $c \gg \theta$ there is a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n \geq n_0$.

(iii) A cone b-metric space (X,d) is called complete if every Cauchy sequence in X converges to an element x of X.

LEMMA 2.4. *(See [9]) Let $u \in \mathcal{A}$. If $\theta \preceq u \ll c$ for each $c \gg \theta$, then $u = \theta$.*

DEFINITION 2.5. (see [9]) A sequence $\{u_n\}$ in P is said to be a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

LEMMA 2.6. *(See [9]) If $\{u_n\}$ and $\{v_n\}$ are two c-sequences in P, then $\{\alpha u_n + \beta v_n\}$ is also a c-sequence for positive real numbers α and β.*

LEMMA 2.7. *(See [9]) Let $k \in P$. If $\{u_n\}$ is a c-sequence in P, then $\{k u_n\}$ is also a c-sequence in P.*

LEMMA 2.8. *(See [9]) Let $\{u_n\}$ be a sequence in P. Then the following items are equivalent:*

(i) $\{u_n\}$ *is a c-sequence.*

(ii) *For each* $c \gg \theta$ *there is* $n_0 \in \mathbb{N}$ *such that* $u_n \prec c$ *whenever* $n \geq n_0$*.*

(iii) *For each* $c \gg \theta$ *there is* $n_1 \in \mathbb{N}$ *such that* $u_n \preceq c$ *whenever* $n \geq n_1$*.*

Lemma 2.9. *(See* [9, 10]*) If* (X, d) *is a complete cone b-metric space over* $\mathcal{A}$ *and* $\{x_n\} \subset X$ *is a sequence that converges to* $x \in X$*, then the following assertions are true:*

(i) $\{d(x_n, x)\}$ *is a c-sequence.*

(ii) $\{d(x_n, x_{n+m})\}$ *is a c-sequence for all* $m \in \mathbb{N}$*.*

Lemma 2.10. *(See* [10]*) Let* $h \in P$ *with* $\rho(h) < 1$*. Then* $\{u_n\}$ *with* $u_n = h^n$ *is a c-sequence.*

Definition 2.11. (see [2, 3]) Let (X, d) be a cone metric space over a Banach space E with solid cone P and let $\mathcal{N}(X)$ be a collection of nonempty subsets of X. A mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \to E$ is called an H-cone metric over a Banach space E with respect to (X, d) if for any $A_1, A_2 \in \mathcal{N}(X)$ the following conditions hold:

(H1) $H(A_1, A_2) = 0 \Rightarrow A_1 = A_2$.

(H2) $H(A_1, A_2) = H(A_2, A_1)$.

(H3) for all $\varepsilon \in E$ with $\varepsilon \gg \theta$ and for all $x \in A_1$, there exists at least one $y \in A_2$ such that $d(x, y) \preceq H(A_1, A_2) + \varepsilon$.

(H4) one of the following holds:

(i) for all $\varepsilon \in E$ with $\varepsilon \gg \theta$ there is at least one $x \in A_1$ such that $H(A_1, A_2) \preceq d(x, y) + \varepsilon$ for all $y \in A_2$.

(ii) for all $\varepsilon \in E$ with $\varepsilon \gg \theta$ there is at least one $x \in A_2$ such that $H(A_1, A_2) \preceq d(x, y) + \varepsilon$ for all $y \in A_1$.

In [2], Wardowski shows that an H-cone metric is a cone metric over underlying Banach space. We refer to [2, 3] for examples and more results. In the sequel, inspired by H-cone metric in the sense of Wardowski, we consider the notion of H-cone b-metric, which will be necessary for our main result:

Definition 2.12. Let (X, d) be a cone b-metric space over $\mathcal{A}$. A mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \to \mathcal{A}$ is called an H-cone b-metric over $\mathcal{A}$ with respect to the cone b-metric d if it holds the conditions (H1-H4) given above.

Proposition 2.13. *If a mapping* $H : \mathcal{N}(X) \times \mathcal{N}(X) \to \mathcal{A}$ *is an* H*-cone b-metric with respect to a cone b-metric space* (X, d) *over* $\mathcal{A}$*, then* $(\mathcal{N}(X), H)$ *is a cone b-metric space over* $\mathcal{A}$*.*

PROOF. Let $\{\varepsilon_n\}$ be a sequence in $intP$ such that $\varepsilon_n \to \theta$ as $n \to \infty$. For (cbm1), we first take any $A_1, A_2 \in \mathcal{N}(X)$ and $x_0 \in A_1$. By (H3), to each $n \in \mathbb{N}$, there corresponds $y_n \in A_2$ such

$$\theta \preceq d(x_0, y_n) \preceq H(A_1, A_2) + \varepsilon_n \tag{3}$$

implying that $H(A_1, A_2) + \varepsilon_n \in P$. Since P is closed, $H(A_1, A_2) \in P$, that is, $\theta \preceq H(A_1, A_2)$. Now, we assume that $A_1 = A_2$. Then, it is clear from (H4) that $H(A_1, A_2) \preceq \varepsilon_n$ for any $n \in \mathbb{N}$. Therefore, since $\varepsilon_n \to \theta$ as $n \to \infty$, we obtain $H(A_1, A_2) = \theta$. By the definition of H, (cbm2) holds.

For (cbm3), let $A_1, A_2, A_3 \in \mathcal{N}(X)$ such that the condition (i) of (H4) holds for A_1 and A_2. Then, for each $n \in \mathbb{N}$, there is $x_n \in A_1$ such that

$$H(A_1, A_2) \preceq d(x_n, y) + \varepsilon_n \text{ for all } y \in A_2. \tag{4}$$

Morever, it is clear from (H3) that to the sequence $\{x_n\} \subset A_1$ there corresponds a sequence $\{z_n\} \subset A_3$ such that

$$d(x_n, z_n) \preceq H(A_1, A_3) + \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{5}$$

Similarly, by (H3), there is a sequence $\{y_n\} \subset A_2$ such that

$$d(z_n, y_n) \preceq H(A_3, A_2) + \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{6}$$

Consequently, using (4), (5) and (6), we get

$$\begin{aligned} H(A_1, A_2) &\preceq d(x_n, y_n) + \varepsilon_n \\ &\preceq s\,[d(x_n, z_n) + d(z_n, y_n)] + \varepsilon_n \\ &\preceq s\,[H(A_1, A_3) + H(A_3, A_2)] + (2s+1)\varepsilon_n. \end{aligned} \tag{7}$$

Finally, it implies that $H(A_1, A_2) \preceq s\,[H(A_1, A_3) + H(A_3, A_2)]$. For the case (ii) in (H4), one can prove this by a similar method. □

As a corollary of the above proposition, we observe that the class of H-cone b-metric spaces w.r.t cone b-metric contains that of H-cone metric spaces in the sense of Wardowski.

3. Main Results

LEMMA 3.1. *Let $k \in P$ such that $\rho(k) < 1$. Then*

$$\sum_{i=p}^{n} k^i \preceq k^p (e - k)^{-1} \tag{8}$$

for all $p \in \mathbb{N}$.

Proof. Since $\rho(k) < 1$, $e - k$ is invertible. Thus, using (1) with the properties of cone P, one can obtain (8). □

In the following theorem, we suppose that (X, d) is a complete cone b-metric space over $\mathcal{A}$, $\mathcal{N}(X)$ is a collection of nonempty closed subsets of X, and $(\mathcal{N}(X), H)$ is an H-cone b-metric space over $\mathcal{A}$ with respect to (X, d).

Theorem 3.2. *Let $T : X \to \mathcal{N}(X)$ be a set-valued mapping. If there is $k \in P$ with $\rho(sk) \in [0, 1)$ such that*

$$H(Tx, Ty) \preceq kd(x, y) \text{ for all } x, y \in X, \tag{9}$$

then there is at least one $x \in X$ such that $x \in Tx$.

Proof. Let $\delta \in intP$ and $\{\delta_n\}$ be a sequence in $\mathcal{A}$ such that for all $n \in \mathbb{N}$

$$\theta \ll \delta_n \text{ and } \delta_n \preceq k^{2n}\delta. \tag{10}$$

Let $x_1 \in Tx_0$ for an arbitrary but fixed $x_0 \in X$. The case $x_0 = x_1$ implies that x_0 is a fixed point of T. Let $x_0 \neq x_1$. Thus, from (H3), there is $x_2 \in Tx_1$ such that

$$d(x_2, x_1) \preceq H(Tx_1, Tx_0) + \delta_1. \tag{11}$$

Hence we can obtain inductively a sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ holding

$$d(x_{n+1}, x_n) \preceq H(Tx_n, Tx_{n-1}) + \delta_n \tag{12}$$

for all $n > 1$. Now, using (9),(10) and (12), we can write the following inequalities:

$$\begin{aligned} d(x_{n+1}, x_n) &\preceq H(Tx_n, Tx_{n-1}) + \delta_n \preceq kd(x_n, x_{n-1}) + \delta_n \\ &\preceq kH(Tx_{n-1}, Tx_{n-2}) + \delta_{n-1}) + \delta_n \\ &\preceq k^2 d(x_{n-1}, x_n) + k\delta_{n-1} + \delta_n \\ &\preceq \cdots \preceq k^{n-1}(H(Tx_1, Tx_0) + \delta_1) + \sum_{i=2}^{n} k^{n-i}\delta_i \\ &\preceq k^n d(x_1, x_0) + \sum_{i=1}^{n} k^{n-i}k^{2i}\delta \\ &\preceq k^n d(x_1, x_0) + k^n \left(\sum_{i=1}^{n} k^i\right)\delta \end{aligned} \tag{13}$$

Thus, using Lemma 3.1, we have

$$d(x_{n+1}, x_n) \preceq k^n d(x_1, x_0) + k^n k(e - k)^{-1}\delta = k^n\lambda \tag{14}$$

where $\lambda = d(x_1, x_0) + k(e - k)^{-1}\delta$. On the other hand, for $m, n \in \mathbb{N}$ with $m > n$, by (cmb3), we get

$$d(x_m, x_n) \preceq s^{m-n-1}d(x_{m-1}, x_m) + \sum_{j=n}^{m-2} s^{j-n+1}d(x_{j+1}, x_j). \tag{15}$$

Since $s \geq 1$ and the properties of P imply $s^{m-n-1}d(x_{m-1}, x_m) \preceq s^{m-n}d(x_{m-1}, x_m)$, we get from (15) that

$$d(x_m, x_n) \preceq \sum_{j=n}^{m-1} s^{j-n+1} d(x_{j+1}, x_j). \tag{16}$$

By substituting (14) to (16), and using $\rho(sk) = s\rho(k) < 1$ together with Lemma 3.1, we have

$$\begin{aligned} d(x_m, x_n) &\preceq \sum_{j=n}^{m-1} s^{j-n+1} k^j \lambda \\ &\preceq s^{-n+1} \left(\sum_{j=n}^{m-1} (sk)^j \right) \lambda \\ &\preceq s^{-n+1}(sk)^n (e - sk)^{-1} \lambda \\ &\preceq sk^n (e - sk)^{-1} \lambda. \end{aligned} \tag{17}$$

Let $h_n = sk^n(e - sk)^{-1}\lambda$. Thus, according to Lemma 2.7 and Lemma 2.10, one can conclude that the sequence $\{h_n\}$ is a c-sequence. So, for each $c \in \mathcal{A}$ with $c \gg \theta$ we have $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$

$$d(x_m, x_n) \preceq h_n \ll c,$$

which implies that $\{x_n\}$ is a Cauchy sequence. Taking into account completeness of X, we see that $\{x_n\}$ converges to $x \in X$. Thus, by Lemma 2.9, $\{d(x_n, x)\}$ is a c-sequence. Therefore, for each $\varepsilon \in intP$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll \frac{\varepsilon}{3s}$ for $n \geq n_0$. Further, since $x_n \in Tx_{n-1}$ for $n \geq 1$, we obtain from (H3) that for all $n \in \mathbb{N}$ there is $y_n \in Tx$ such that

$$d(x_n, y_n) \preceq H(Tx_{n-1}, Tx) + \delta_n \preceq kd(x_{n-1}, x) + k^{2n}\delta. \tag{18}$$

Since $\left\{kd(x_{n-1}, x) + k^{2n}\delta\right\}$ is a c-sequence according to Lemma 2.6, Lemma 2.7, Lemma 2.9 and Lemma 2.10, for each $\varepsilon \in intP$ there is $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$d(x_n, y_n) \preceq kd(x_{n-1}, x) + k^{2n}\delta \ll \frac{2\varepsilon}{3s}. \tag{19}$$

Hence it follows that for $n \geq \max\{n_0, n_1\}$ we have

$$d(y_n, x) \preceq s\left[d(y_n, x_n) + d(x_n, x)\right] \ll \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon, \tag{20}$$

which shows that $y_n \to x$ as $n \to \infty$. Finally, since Tx is closed, we see $x \in Tx$. $\square$

Example 3.3. Let (X,d) be a complete metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Consider the Pompeiu-Hausdorf metric $H : CB(X)\times CB(X) \to \mathbb{R}$ given by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},\ \ A,B\in CB(X), \tag{21}$$

where $d(b,A) = \inf\{d(b,a)|a\in A\}$. It is clear that $(H, CB(X))$ is an H- cone metric over $\mathcal{A}=\mathbb{R}$ with respect to (X,d). Now suppose that $T : X \to CB(X)$ is a mapping satisfying

$$H(Tx,Ty) \preceq kd(x,y) \text{ for all } x,y\in X, \tag{22}$$

where k is a real constant with $0<k<1$. Finally, since $\rho(k)=k<1$, T holds the conditions of Theorem 3.2.

Example 3.4. Let (X,d) be the cone metric space with Banach algebra mentioned in Example 2.2. Then it is clear that (X,d) is a complete cone metric space. Let $\mathcal{N}(X)=\{[0,x]|x\in X\}$. Now we consider a mapping $H : \mathcal{N}(X)\times\mathcal{N}(X)\to\mathcal{A}$ defined by $H([0,x],[0,y])(t)=\frac{1}{2}|x-y|e^t$. It is easily seen that $(\mathcal{N}(X),H)$ is an H-cone metric over $\mathcal{A}$. Now let us define $T : X\to\mathcal{N}(X)$ by $Tx=[0,cos\frac{x}{4}], x\in X$. T holds the generalized contraction condition (9) with $k\in P$ defined by $k(t)=e^t\lambda$ where $\lambda\in[\frac{1}{4},\frac{1}{e})$. Indeed,

$$H(Tx,Ty)(t) = \frac{1}{2}\left|cos\frac{x}{4}-cos\frac{y}{4}\right|e^t \preceq \frac{1}{2}\left|\frac{x}{4}-\frac{y}{4}\right|e^t \preceq k(t)d(x,y)(t). \tag{23}$$

Morever $\rho(k)=e\lambda<1$ implies that T has a fixed point.

Example 3.5. Consider the Banach algebra $\mathcal{A}=\mathbb{R}^2$ endowed with the pointwise multiplication and the usual norm. Let $P=\left\{(x,y)\in\mathbb{R}^2|x,y\geq 0\right\}$ and $X=\mathbb{R}^2$ and $p\in\mathbb{R}$ with $p>1$. Then, using advantage of the inequality $(a+b)^p\leq 2^p(a^p+b^p)$ for all $a,b\geq 0$ and the properties of the cone P, one can show that a mapping $d : X\times X\to\mathcal{A}$ defined by $d((x_1,y_1),(x_2,y_2))=(|x_1-x_2|^p, |y_1-y_2|^p)$ is a cone b-metric with $s=2^p$ over $\mathcal{A}$. Let $a\otimes b$ be a closed subset of X defined by $a\otimes b := \{(x,y)\in X|0\leq x\leq a, 0\leq y\leq b\}$ for $a,b\geq 0$. Now consider $\mathcal{N}(X)=\{a\otimes b|a,b\geq 0\}$. Then it is clear that a mapping $H : \mathcal{N}(X)\times\mathcal{N}(X)\to\mathcal{A}$ defined by $H(a_1\otimes b_1, a_2\otimes b_2)=(|a_1-a_2|^p, |b_1-b_2|^p)$ is H-cone b-metric with respect to (X,d) over $\mathcal{A}$. Let $T : X\to\mathcal{N}(X)$ given by $T(x,y)=\left|\frac{cosx}{4}\right|\otimes\left|\frac{cosy}{16}\right|$. Then, by using the basic properties of $|\cdot|$, one can show that

$$H(T(x_1,y_1),T(x_2,y_2) \preceq kd\left((x_1,y_1),(x_2,y_2)\right), \tag{24}$$

where $k=\left(\frac{1}{s^2},\frac{1}{s^4}\right)\in P$. Since T holds the conditions of Theorem 3.6, it has a fixed point.

4. Conclusion

In this paper we provide a significant generalization of Nadler's fixed point theorem for multivalued contractions in the setting of H-cone b-metric in the sense of Wardowski by means of spectral radius and a solid cone in underlying Banach algebra.

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