Rend. Sem. Mat. Univ. Padova 1xx (201x) **Rendiconti del Seminario Matematico della Università di Padova**

Wavelet transform of Beurling-Björck type ultradistributions

R. S. Pathak * – Abhishek Singh**

Abstract – Wavelet transform of a distribution in $\mathscr{M}_\omega^{'}$ involving wavelet of infraexponential decay (subexponential decay) is studied. An inversion formula is obtained which is valid in the weak topology of $\mathcal{D}^{'}$. A discussion on extension of the results to ultradistribution space of compact support is also given.

Mathematics Subject Classification (2010). 44A15, 42C40, 46F12.

Keywords. Beurling-Björck ultradistributions, infraexponential decaying wavelets, Wavelet transform.

1. Introduction

Wavelet analysis has been used for intrinsic characterizations of important function and distribution spaces ([10], [11]). Recently, the wavelet transform has been extended to distributions, and inversion formulae have been established in distribution setting by Pathak [13, 14], Pathak *et al* [16, 17, 18] and Pandey [12] using duality arguments.

Wavelets of subexponential decay whose Fourier transform have compact support i.e. band limited wavelets, were investigated by Dziubański and Hernández [7]. Pathak and Singh [17] extended the work of Dziubański and Hernández and studied wavelets with more general decay (infraexponential decay) whose Fourier

*Research supported by grant (No. 2084) from Department of Science and Technology, India.
**Research supported by grant (No. F. 4-2/2006(BSR)/13-663/2012) from University Grants Commission, India, under the Dr. D.S. Kothari Post Doctoral Fellowship.

R. S. Pathak, DST Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University, Varanasi- 221 005, India
E-mail: ramshankarpathak@yahoo.co.in

Abhishek Singh, DST Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University, Varanasi- 221 005, India
E-mail: mathdras@gmail.com (**Corresponding author)

transforms have compact support. The aim of the present paper is to develop the theory of wavelet transform involving these wavelets using ultradistribution theory of Beurling [3] and Björck [4].

Now, we recall definitions and properties of the desired test function and ultradistribution spaces from [4], [6] and [9]. Let $\mathfrak{M}$ be the set of all real-valued functions ω on $\mathbb{R}$ which can be represented as $\omega(x) = \sigma(|x|)$, where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty)$ satisfying the following conditions [9, p. 14]:

(1) (α)
$$0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta), \quad \forall \xi, \eta \in \mathbb{R}$$

(2) (β)
$$\int_{\mathbb{R}} \frac{\sigma(\xi)}{(1 + |\xi|)^2} d\xi < \infty,$$

(γ) there exists real number p and positive real number q such that

$$\sigma(\xi) \geq p + q \log(1 + t), \quad t \geq 0. \tag{3}$$

Let $\omega \in \mathfrak{M}$. We denote by $\mathscr{M}_\omega$ the set of all functions $\psi(t) \in C^\infty(\mathbb{R})$ which satisfy

$$P_{k,\lambda}(\psi) = \sup_{t \in \mathbb{R}} \left\{ e^{\lambda \omega(t)} |D^k \psi(t)| \right\} < \infty \tag{4}$$

for all non-negative integers k and all non-negative real λ. The topology on $\mathscr{M}_\omega$ is defined by the semi-norms $\{P_{k,\lambda}\}$. It can be readily seen that $\mathscr{M}_\omega$ is a vector space. A sequence $\{\psi_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence in $\mathscr{M}_\omega$ if for each non-negative integer m and k, $P_{k,\lambda}(\psi_\mu - \psi_\nu) \to 0$ as $\mu, \nu \to \infty$ independently of each other. The space $\mathscr{M}_\omega$ is a sequentially complete space and therefore it is a complete countably multinormed space and so a Fréchet space. The dual of $\mathscr{M}_\omega$ is denoted by ${\mathscr{M}_\omega}'$; it is a distribution space [6, p. 170]. The Schwartz space $\mathcal{D}(\mathbb{R})$ consisting of $C^\infty-$functions of compact support is a subspace of $\mathscr{M}_\omega(\mathbb{R})$ and $\mathscr{M}_\omega^{'} \subset \mathcal{D}^{'}$.

Suppose that the Fourier transform of $\psi \in \mathscr{M}_\omega$, defined by

$$\hat{\psi}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} \psi(x) dx,$$

satisfies

$$\pi_{k,\lambda}(\psi) = \sup_{\xi \in \mathbb{R}} \left\{ e^{\lambda \omega(\xi)} \left| D^k \hat{\psi}(\xi) \right| \right\} < \infty, \qquad \lambda \geq 0, \ k \in \mathbb{N}_0. \tag{5}$$

Then the space of all functions $\psi \in L^1(\mathbb{R})$ such that $\psi, \hat{\psi} \in C^\infty(\mathbb{R})$ and (4), (5) hold, is denoted by $\mathscr{S}_\omega$ the topology of $\mathscr{S}_\omega$ is defined by the seminorms $P_{k,\lambda}$ and $\pi_{k,\lambda}$ [4, p. 377].

Let K be a compact subset of $\mathbb{R}$. The space $\mathcal{D}_\omega$ (K) is the set of all ψ in $L^1(\mathbb{R})$ such that ψ has support in K and

$$\|\psi\|_\lambda := \int_{\mathbb{R}} |\hat{\psi}(\xi)| e^{\lambda \omega(\xi)} d\xi < \infty, \qquad \forall \lambda > 0. \tag{6}$$

Let $\{K_n\}$ be a sequence of compact set in $\mathbb{R}$ such that $\bigcup_{n=1}^{\infty} K_n = \mathbb{R}$ and K_n is contained in the interval of K_{n+1} for all n. Then $\mathcal{D}_\omega(\mathbb{R}) = \lim\operatorname{ind} \mathcal{D}_\omega(K_n)$. Since $\mathcal{D}_\omega \subset \mathscr{S}_\omega$ and the topology of $\mathcal{D}_\omega$ is stronger than that induced on $\mathcal{D}_\omega$ by $\mathscr{S}_\omega$, it follows that the restriction of any $f \in \mathscr{S}_\omega^{'}$ to $\mathcal{D}_\omega$ is in $\mathcal{D}_\omega^{'}$. The elements of $\mathcal{D}_\omega^{'}$ are called ultradistributions [6].

Now we recall from [5] some definitions and results related to wavelet transform needed in the present investigation.

Let $\psi \in L^2(\mathbb{R})$. Define

$$\psi_{b,a}(t) := \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R},\ b \in \mathbb{R}, a \in \mathbb{R}_0 = \mathbb{R}\backslash\{0\}. \tag{7}$$

The wavelet transform $W(b,a)$ of $f \in L^2(\mathbb{R})$ with respect to the wavelet $\psi_{b,a}(t) \in L^2(\mathbb{R})$ is defined by

$$W(b,a) := \int_{\mathbb{R}} f(t)\overline{\psi_{b,a}(t)}dt \tag{8}$$

and the corresponding wavelet inversion formula is given by

$$\frac{1}{C_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}_0} \frac{1}{\sqrt{|a|}}W(b,a)\psi\left(\frac{x-b}{a}\right)\frac{dbda}{a^2} = f(x), \tag{9}$$

where

$$C_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|}dw < \infty \ \ [5, \text{p. } 9].$$

In the present work we shall investigate properties of the wavelet $\psi_{b,a}(t) \in \mathscr{M}_\omega(\mathbb{R})$. Wavelet transform of $f \in \mathscr{M}_\omega^{'}$ will be studied and the inversion formula (9) will be extended to distribution space $\mathscr{M}_\omega^{'}$. It has been noted by Constantinescu *et al* [6, p. 169] that the Schwartz space $\mathscr{S}(\mathbb{R})$ and Gelfand-Shilov space $S^{\alpha,A}$ are special cases of the space $\mathscr{M}_\omega$. Therefore, some of the results obtained in this paper are more general than those derived in [13] and [16].

2. Wavelet Transform on $\mathscr{M}_\omega^{'}$

In this section, certain basic properties of the wavelets in $\mathscr{M}_\omega$ and wavelet transform of $f \in \mathscr{M}_\omega^{'}$ are obtained.

Lemma 2.1. *If $\psi \in \mathscr{M}_\omega$, then $\psi(\frac{t-b}{a}) \in \mathscr{M}_\omega$ for arbitrary but fixed $b, a \in \mathbb{R}, a \neq 0$.*

Proof. Let a and b be fixed real numbers. Then for $k = 0, 1, 2, ...,$

$$\sup_{-\infty<t<\infty}\left|e^{\lambda\omega(t)}D^k\psi\left(\frac{t-b}{a}\right)\right| = \sup_{-\infty<t<\infty}\left|e^{\lambda\omega\left(\frac{t-b}{a}\right)}\psi^{(k)}\left(\frac{t-b}{a}\right)\left(\frac{1}{a^k}\right)\right|\left|\frac{e^{\lambda\omega(t)}}{e^{\lambda\omega\left(\frac{t-b}{a}\right)}}\right|.$$

Then by Property (α) we get

$$\sup_{-\infty<t<\infty}\left|e^{\lambda\omega(t)}D^k\psi\left(\frac{t-b}{a}\right)\right| \le \frac{1}{|a|^k}P_{k,\lambda}(\psi)\sup_{-\infty<t<\infty}\left|\frac{e^{\lambda\omega\left(\frac{t-b}{a}+\frac{b}{a}\right)}}{e^{\lambda\omega\left(\frac{t-b}{a}\right)}}\right|, \quad \text{by (4)}$$
$$\le P_{k,\lambda}(\psi)\left(\frac{1}{|a|^k}\right)e^{\lambda\omega\left(\frac{b}{a}\right)} < \infty, \qquad \text{by (1)}$$

for all fixed real numbers b and $a \neq 0$. □

In what follows we shall assume that $\psi \in \mathscr{M}_\omega(\mathbb{R})$ is the basic function generating the wavelet $\psi_{b,a}(t)$ given by (7). Since function $\psi\left(\frac{t-b}{a}\right)$ belongs to $\mathscr{M}_\omega$ for fixed b and $a \neq 0$ as a function of t under conditions of Lemma 2.1, for $f \in \mathscr{M}_\omega'$ the wavelet transform $W(b,a)$ of f is defined by

$$(10)\qquad \begin{aligned} W(b,a) &= \frac{1}{\sqrt{|a|}}\left\langle f(t),\ \overline{\psi\left(\frac{t-b}{a}\right)}\right\rangle, \quad a\neq 0,\ a,b\in\mathbb{R}\\ &= \left\langle f(t),\ \overline{\psi_{b,a}(t)}\right\rangle. \end{aligned}$$

Theorem 2.2. *For real b and $a \neq 0$ let $W(b,a)$ be defined by (10), then under conditions of Lemma 2.1, there exists $m \in \mathbb{N}_0$ such that*

$$|W(b,a)| \le C(m,\psi)\left(|a|^{-m-1/2}\exp(m\omega(b/a))\right), \quad \textit{for some } m\in\mathbb{N}_0.$$

Proof. To very $f \in \mathscr{M}_\omega'$ there exists a non-negative integer m and a constant $C > 0$ such that, for all $\psi \in \mathscr{M}_\omega$,

$$\begin{aligned} |W(b,a)| &= \left|\frac{1}{\sqrt{|a|}}\left\langle f(t),\ \overline{\psi\left(\frac{t-b}{a}\right)}\right\rangle\right|\\ &\le \frac{C}{\sqrt{|a|}}\max_{0\le k\le m}\sup_{b,t\in\mathbb{R}}\left|e^{m\omega\left(\frac{t}{a}\right)}D_t^k\overline{\psi\left(\frac{t-b}{a}\right)}\right|\\ &= \frac{C}{\sqrt{|a|}}\max_{0\le k\le m}\sup_{b,t\in\mathbb{R}}\left|e^{m\omega\left(\frac{t-b}{a}\right)}\overline{\psi^{(k)}\left(\frac{t-b}{a}\right)}\left(\frac{1}{a^k}\right)\right|\left|\frac{e^{m\omega\left(\frac{t}{a}\right)}}{e^{m\omega\left(\frac{t-b}{a}\right)}}\right|\\ &= \frac{C}{\sqrt{|a|}}\left[\sup_{b,t\in\mathbb{R}}\left|\frac{e^{m\omega\left(\frac{t}{a}\right)}}{e^{m\omega\left(\frac{t-b}{a}\right)}}\right|\right]\left(\frac{1}{|a|^m}\right)\max_{0\le k\le m}P_{k,m}(\psi)\\ &\le C\max_{0\le k\le m}P_{k,m}(\psi)\frac{e^{m\omega(b/a)}}{|a|^{m+1/2}} \end{aligned}$$

by using property (α), as in the Lemma 2.1. This gives the required result. □

3. Inversion of the wavelet transform on $\mathscr{M}_\omega'$

In order to study properties of the wavelet transform of $f \in \mathscr{M}_\omega'$ we obtain an appropriate structure formula for $f \in \mathscr{M}_\omega'$.

Assume that $f \in \mathscr{M}_\omega'$ then as in the proof of the above theorem there exists a non-negative integer m and constant $C > 0$ such that for all $\phi \in \mathscr{M}_\omega$,

$$|\langle f, \phi\rangle| \leq C \max_{0\leq k\leq m} \sup_{t\in\mathbb{R}} \left| e^{m\omega(t)} D_t^k \phi(t)\right|. \tag{11}$$

Then following [8, p. 112] we have following two cases:
Case I. For $t > 0$,

$$\begin{aligned} e^{m\omega(t)}\left|D_t^k\phi(t)\right| &= e^{m\omega(t)}\left|\int_t^\infty \frac{d}{dz}\left(D_z^k\phi(z)\right)dz\right| \\ &\leq \max_{0\leq k\leq m}\int_t^\infty e^{m\omega(z)}\left|\frac{d}{dz}\left(D_z^k\phi(z)\right)dz\right| \\ &\leq \max_{0\leq k\leq m+1}\int_{-\infty}^\infty \left|e^{m\omega(z)}D_z^k\phi(z)\right|dz \\ &\leq \max_{0\leq k\leq m+1}\int_{-\infty}^\infty \left|e^{-\omega(z)}e^{(m+1)\omega(z)}D_z^k\phi(z)\right|dz \\ &\leq \max_{0\leq k\leq m+1}\left\|e^{-\omega(z)}\right\|_2\left\|e^{(m+1)\omega(z)}D_z^k\phi(z)\right\|_2. \end{aligned}$$

Case II. For $t < 0$,

$$\begin{aligned} e^{m\omega(t)}\left|D_t^k\phi(t)\right| &= e^{m\omega(t)}\left|\int_{-\infty}^t \frac{d}{dz}\left(D_z^k\phi(z)\right)dz\right| \\ &\leq \max_{0\leq k\leq m}\int_{-\infty}^t e^{m\omega(z)}\left|\frac{d}{dz}\left(D_z^k\phi(z)\right)dz\right| \\ &\leq \max_{0\leq k\leq m+1}\int_{-\infty}^\infty \left|e^{m\omega(z)}D_z^k\phi(z)\right|dz \\ &\leq \max_{0\leq k\leq m+1}\int_{-\infty}^\infty \left|e^{-\omega(z)}e^{(m+1)\omega(z)}D_z^k\phi(z)\right|dz \\ &\leq \max_{0\leq k\leq m+1}\left\|e^{-\omega(z)}\right\|_2\left\|e^{(m+1)\omega(z)}D_z^k\phi(z)\right\|_2. \end{aligned}$$

Therefore, $\exists\, C > 0$, such that

$$|\langle f,\ \phi\rangle| \leq C \max_{0\leq k\leq m+1}\left\|e^{(m+1)\omega(t)}D_t^k\phi(t)\right\|_2, \qquad \phi\in\mathscr{M}_\omega. \tag{12}$$

Now, we show that the above L^2-norm exists finitely. Indeed, for some $\mu > 0$, we have

$$\begin{aligned}\max_{0\le k\le m+1}&\left(\int_{-\infty}^{\infty}|e^{(m+1)\omega(t)}D_t^k\phi(t)|^2dt\right)^{\frac{1}{2}}\\ &=\max_{0\le k\le m+1}\left(\int_{-\infty}^{\infty}|e^{-\mu\omega(t)}e^{(m+1+\mu)\omega(t)}D_t^k\phi(t)|^2dt\right)^{\frac{1}{2}}\\ &\le\max_{0\le k\le m+1}P_{k,m+1+\mu}(\phi)\left(\int_{-\infty}^{\infty}e^{-2\mu[p+q\log(1+t)]}dt\right)^{\frac{1}{2}}, \qquad \text{by (3)}\end{aligned}$$

we can choose μ large so that the last integral is finite. Thus

$$\max_{0\le k\le m+1}\left\|e^{(m+1)\omega(t)}D_t^k\phi(t)\right\|_2<\infty.$$

Now, applying Hahn-Banach theorem and Riesz representation theorem to (12) we get g_k belonging to the space $L^2(\mathbb{R})$ such that

$$\begin{aligned}|\langle f,\phi\rangle| &= \sum_{k=0}^{m+1}\left\langle g_k(t), e^{(m+1)\omega(t)}D_t^k\phi(t)\right\rangle\\ &=\sum_{k=0}^{m+1}\left\langle D_t^k\left(e^{(m+1)\omega(t)}g_k(t)\right),\ \phi(t)\right\rangle.\end{aligned}$$

Therefore desired structure formula is

$$f=\sum_{k=0}^{m+1}D_t^k\left(e^{(m+1)\omega(t)}g_k(t)\right), \tag{13}$$

where $g_k\in L^2(\mathbb{R})$ and $k=0,1,2,3,....$

THEOREM 3.1. *Let $f\in\mathscr{M}_\omega'$, $\psi\in\mathscr{M}_\omega$ and $W(b,a)$ be defined by (10). Then*

$$D_b^rW(b,a)=\frac{\partial^rW}{\partial b^r}=\left\langle f(t),\ \frac{\partial^r}{\partial b^r}\frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)\right\rangle,$$

and

$$D_a^rW(b,a)=\frac{\partial^rW}{\partial a^r}=\left\langle f(t),\ \frac{\partial^r}{\partial a^r}\frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)\right\rangle.$$

PROOF. Using the structure formula for f as given in (13) and following [12] we have

$$\begin{aligned}
W(b,a) &= \langle f(t),\ \psi_{b,a}(t)\rangle \\
&= \left\langle \sum_{k=0}^{m+1} D_t^k \left(e^{(m+1)\omega(t)} g_k(t)\right),\ \psi_{b,a}(t) \right\rangle \\
&= \left\langle g_k(t),\ \sum_{k=0}^{m+1} \left(e^{(m+1)\omega(t)}\right)(-1)^k D_t^k \left(\frac{\psi\left(\frac{t-b}{a}\right)}{\sqrt{|a|}}\right)\right\rangle.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{\partial^r W}{\partial b^r}(b,a) &= \sum_{k=0}^{m+1} \int_{-\infty}^{\infty} g_k(t) e^{(m+1)\omega(t)} \frac{\partial^r}{\partial b^r}(-1)^k \left[D_t^k \overline{\psi_{b,a}(t)}\right] dt \\
&= \sum_{k=0}^{m+1} \int_{-\infty}^{\infty} g_k(t) e^{(m+1)\omega(t)} (-1)^{k+r} D_t^{k+r} \overline{\psi_{b,a}(t)} dt \\
&= \sum_{k=0}^{m+1} \int_{-\infty}^{\infty} g_k(t) e^{(m+1)\omega(t)} (-1)^{k} D_t^{k} \frac{\partial^r}{\partial b^r} \overline{\psi_{b,a}(t)} dt \\
&= \sum_{k=0}^{m+1} \left\langle D_t^k \left(e^{(m+1)\omega(t)} g_k(t)\right), \frac{\partial^r}{\partial b^r}\psi_{b,a}(t) \right\rangle \\
&= \left\langle f(t), \frac{\partial^r}{\partial b^r}\psi_{b,a}(t) \right\rangle \qquad \text{[by structure formula (13)]}.
\end{aligned}$$

Similarly result for differentiation with respect to a can be proved. □

In order to derive inversion formula for the wavelet transform of $f \in \mathscr{M}_\omega'$ we define function $g_\nu(t)$ as follows [1]:

$$g_\nu(t) = \begin{cases} g(t), & -\nu \le t \le \nu \\ 0, & elsewhere. \end{cases}$$

Also define $f_\nu \in \mathscr{M}_\omega'$ by

$$\langle f_\nu,\ \phi\rangle = \sum_{k=0}^{m+1} \left\langle g_\nu(t), \left(e^{(m+1)\omega(t)} D_t^k \phi(t)\right)\right\rangle,\ \phi \in \mathcal{D}_\omega, \tag{14}$$

then $g_\nu \to g$ in $L^2(\mathbb{R})$ as $\nu \to \infty$ therefore, $< f_\nu, \phi > \longrightarrow < f, \phi >$ as $\nu \to \infty$.

Theorem 3.2. *Assume that the wavelet transform $W(b,a)$ of $f \in \mathscr{M}_\omega'$ with respect to $\psi \in \mathscr{M}_\omega$ is defined by (10). Then*

$$\lim_{\substack{N\to\infty \\ R\to\infty}} \left\langle \frac{1}{C_\psi} \int_{-R}^{R} \int_{-N}^{N} W(b,a)\psi_{b,a}(x) \frac{dbda}{a^2},\ \phi(x) \right\rangle = \langle f, \phi\rangle, \tag{15}$$

for each $\phi \in \mathcal{D}_\omega$ and $b, a \in \mathbb{R}$, $a \neq 0$ where $\psi_{b,a}(x)$ is given by (7).

PROOF. Using the structure formula for f_ν as given in (14) we have

$$(Wf_\nu)(b,a) = \left\langle f_\nu(t),\ \overline{\psi_{b,a}(t)} \right\rangle$$

$$= \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} D_t^k \overline{\psi_{b,a}(t)} dt. \tag{16}$$

We wish to derive the inversion formula

$$\frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Wf_\nu)(b,a)\psi_{b,a}(x) \frac{dbda}{a^2} = f_\nu,$$

interpreting convergence in the weak topology of $\mathcal{D}'_\omega$, i.e.

$$J \equiv \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Wf_\nu)(b,a)\psi_{b,a}(x) \frac{dbda}{a^2},\ \phi(x) \right\rangle = \langle f_\nu, \phi \rangle\,,\ \forall \phi \in \mathcal{D}_\omega.$$

From (16) we have

$$J = \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} D_t^k \overline{\psi_{b,a}(t)} \psi_{b,a}(x) \frac{dtdbda}{a^2},\ \phi(x) \right\rangle$$

$$= \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} (-1)^k D_b^k \overline{\psi_{b,a}(t)} \right\} \psi_{b,a}(x) dt \right] \frac{dbda}{a^2},\ \phi(x) \right\rangle$$

as $D_t\psi_{b,a}(t) = -D_b\psi_{b,a}(t)$. Therefore, by integration by parts with respect to b we have

$$J = \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} \overline{\psi_{b,a}(t)} D_b^k \psi_{b,a}(x) \frac{dtdbda}{a^2},\ \phi(x) \right\rangle$$

$$= \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} \overline{\psi_{b,a}(t)} (-1)^k D_x^k \psi_{b,a}(x) \frac{dtdbda}{a^2},\ \phi(x) \right\rangle.$$

Hence, by distributional differentiation,

$$J = \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} \overline{\psi_{b,a}(t)} \psi_{b,a}(x) \frac{dtdbda}{a^2},\ D_x^k \phi(x) \right\rangle \tag{17}$$

The integrand

$$D_x^k \phi(x) \psi_{b,a}(x) \overline{\psi_{b,a}(t)} g_\nu(t) \frac{e^{(m+1)\omega(t)}}{a^2}$$

is absolutely integrable with respect to x and t in the x,t-plane and so Fubini's theorem is applicable with respect to integration by x and t. Therefore (17) yields

$$J = \frac{1}{C_\psi}\sum_{k=0}^{m+1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_x^k\phi(x)\psi_{b,a}(x)\overline{\psi_{b,a}(t)}g_\nu(t)e^{(m+1)\omega(t)}\frac{dxdtdbda}{a^2}$$

$$= \frac{1}{C_\psi}\sum_{k=0}^{m+1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\overline{W_\psi\left\{D_x^k\phi(x)\right\}(b,a)\psi_{b,a}(t)\frac{dbda}{a^2}}\right]g_\nu(t)e^{(m+1)\omega(t)}dt$$

$$= \sum_{k=0}^{m+1}\int_{-\infty}^{\infty}\overline{D_t^k\phi(t)}g_\nu(t)e^{(m+1)\omega(t)}dt \qquad \text{[by inversion formula (9)]}$$

$$= \sum_{k=0}^{m+1}\left\langle g_\nu(t),\ (-1)^k e^{(m+1)\omega(t)}D_t^k\phi(t)\right\rangle$$

$$= \left\langle \sum_{k=0}^{m+1}D_t^k\left(e^{(m+1)\omega(t)}g_\nu(t)\right),\ \phi(t)\right\rangle$$

$$= \langle f_\nu,\ \phi\rangle \qquad \text{[by structure formula (14)]}$$

$$\longrightarrow \langle f,\ \phi\rangle \text{ as } \nu\to\infty.$$

This completes the proof of the theorem. □

4. Wavelet transform on $\mathscr{S}_\omega'$

In this section we assume that wavelets are of infraexponential decay so that their Fourier transforms are of compact support. To deal with such wavelets we suppose that $\psi \in \mathscr{S}_\omega(\mathbb{R})$, then $\psi_{b,a} \in \mathscr{S}_\omega$ for fixed $a,b\in\mathbb{R}$, $a\neq 0$. Now, we extend the wavelet transform in Fourier space defined by

$$W(b,a) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ib\omega}\hat{f}(\omega)\overline{\hat{\psi}(a\omega)}d\omega. \tag{18}$$

Assume that $\hat{f}(\omega)\in\mathcal{D}_\omega'(\mathbb{R})$ and $\overline{\hat{\psi}(\omega)}\in\mathscr{S}_\omega(\mathbb{R})$ is of compact support, then $\overline{\hat{\psi}(a\omega)}\hat{f}(\omega)\in\mathscr{E}_\omega'(\mathbb{R})$ [2, pp. 121-127]. Now, we define generalized wavelet transform of $f\in\mathcal{Z}_\omega'(\mathbb{R})$ [2, pp. 127] as generalized Fourier transform of $\hat{f}(\cdot)\overline{\hat{\psi}(a\cdot)}$:

$$\begin{aligned} W(b,a) &= \frac{1}{2\pi}\left\langle \hat{f}(\omega),\ \overline{\hat{\psi}(a\omega)}e^{ib\omega}\right\rangle = \frac{1}{2\pi}\left\langle \hat{f}(\omega)\overline{\hat{\psi}(a\omega)},\ e^{ib\omega}\right\rangle \\ &= \frac{1}{2\pi}\left\langle \frac{1}{a}\hat{f}(u/a)\overline{\hat{\psi}(u)},\ e^{i\zeta u/a}\right\rangle = \frac{1}{2\pi a}\left\langle g_a(u),\ e^{i\zeta u/a}\right\rangle. \end{aligned} \tag{19}$$

where $g_a(u) = \hat{f}(u/a)\overline{\hat{\psi}(u)},\ a\neq 0$.

Assume that $\text{supp}\hat{\psi}(u) = [-\beta,\beta],\ \alpha>0$. Then $\text{supp} g_a(u) = [-\beta,\beta],\ \beta>0$.

Theorem 4.1. *If* $\overline{\hat{\psi}(u)}\hat{f}(u/a) \in \mathscr{E}'_\omega(\mathbb{R})$*, its wavelet transform is a* C^∞ *function in* $\mathbb{R}$ *given by*

$$W(b,a) = \frac{1}{2\pi a}\left\langle g_\alpha(u),\ e^{ibu/a}\right\rangle. \tag{20}$$

Moreover, $W(b,a)$ *can be extended to the complex space* $\mathbb{C}$ *as an entire analytic function given by*

$$W(\zeta,a) = \frac{1}{2\pi a}\left\langle g_\alpha(u),\ e^{i\zeta u/a}\right\rangle. \tag{21}$$

Proof. In (20), g_a is a distribution with compact support while $e^{ibu/a}$ is a C^∞-function of u. Thus, the right-hand side of (20) is well defined. Also, by [2, Theorem 4.6, p. 124] it follows that the right-hand side of (20) is a C^∞-function of $b \in \mathbb{R}$ and $W(b,a)$ can be extended to the complex plane as an entire analytic function given by (21).

Further proof is very similar to that given in [2, p. 124] in the case of Fourier transform of distributions. □

Remark 4.2. Relation (21) can be used to study Paley-Wiener-Schwartz theorem for wavelet transform of ultradistribution of compact support.

References

[1] N.I. Akhiezer – I.M. Glazman, *Theory of linear operators in Hilbert space*, Frederick Ungar Publishing Company, New York, 1966.

[2] José Barros-Neto, *An Introduction to the Theory of Distributions*, Marcel Dekker, INC., New York, 1973.

[3] A. Beurling, *Quasi-analyticity and general distributions*, Lectures 4 and 5, A.M.S. Summer Institute, Stanford, 1961.

[4] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Math. **6** (1966), pp. 351-407.

[5] C. K. Chui, *An Introduction to Wavelets*, Academic Press, New York, 1992.

[6] F. Constantinescu – W. Thalheimer, *Ultradistributions and quantum fields: Fourier-Laplace Transforms and boundary values of analytic functions*, Rep. Math. Phys. **16** (2) (1979), pp. 167-180.

[7] J. Dziubański – E. Hernández, *Band-limited wavelets with sub-exponential decay*, Canad. Math. Bull. **41** (1998), pp. 398-403.

[8] I. M. Gel'fand – G. E. Shilov, *Generalized Functions, Vol. 2*, Academic Press, New York, London, 1968.

[9] Hans-Jürgen Schmeisser – Hans Triebel, *Topics in Fourier Analysis and Function Spaces*, John Wiley and Sons, Chichester, New York, Brisbane, Toronto, Singapore, 1987.

[10] E. Hernández – G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, New York, 1996.

[11] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, 1992.

[12] J. N. Pandey, *Wavelet transforms of Schwartz distributions*, J. Comput. Anal. Appl. **13** (1) (2011), pp. 47-83.

[13] R. S. Pathak, *The Wavelet transform of distributions*, Tôhoku Math. J. **56** (2004), pp. 411-421.

[14] R. S. Pathak, *The Wavelet Transform*, Atlantis Press/World Scientific, Amsterdam, Paris, 2009

[15] R. S. Pathak, *Integral Transforms of Generalized Functions and Their Applications*, Gordon and Breach Science Publishers, Amsterdam, 1997.

[16] R. S. Pathak – S. K. Singh, *The Wavelet transform on spaces of type S*, Proc. Roy. Soc. (Edinburgh) **136A** (2006), pp. 837-850.

[17] R. S. Pathak – S. K. Singh, *Infraexponential decay of wavelets*, Proc. Nat. Acad. Sci. India Sect. A **78** (2008), pp. 155-162.

[18] R. S. Pathak – G. Pandey, *Wavelet transform on spaces of type W*, Rocky Mt. J. Math. **39** (2) (2009), pp. 619-631.

[19] A. H. Zemanian, *Generalized Integral Transformations*, Interscience Publishers, New York, 1996.

Received submission date; revised revision date