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On the Monotonicity of Hilbert functions

Tony J. Puthenpurakal

Abstract – In this paper we show that a large class of one-dimensional Cohen-Macaulay local rings $(A, \mathfrak{m})$ has the property that if M is a maximal Cohen-Macaulay A-module then the Hilbert function of M (with respect to $\mathfrak{m}$) is non-decreasing. Examples include

(1) Complete intersections $A = Q/(f, g)$ where $(Q, \mathfrak{n})$ is regular local of dimension three and $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$.

(2) One dimensional Cohen-Macaulay quotients of a two dimensional Cohen-Macaulay local ring with pseudo-rational singularity.

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1. introduction

Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring with residue field k and let M be a finitely generated A-module. Let $\mu(M)$ denote minimal number of generators of M and let $\ell(M)$ denote its length. Let $\operatorname{codim}(A) = \mu(\mathfrak{m}) - d$ denote the codimension of A.

Let $G(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of A (with respect to $\mathfrak{m}$) and let $G(M) = \bigoplus_{\mathfrak{n} \geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ be the associated graded module of M considered as a $G(A)$-module. The ring $G(A)$ has a unique graded maximal ideal $\mathfrak{M}_G = \bigoplus_{n \geq 1} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Set $\operatorname{depth} G(M) = \operatorname{grade}(\mathfrak{M}_G, G(M))$. Let $e(M)$ denote the multiplicity of M (with respect to $\mathfrak{m}$).

The Hilbert function of M (with respect to $\mathfrak{m}$) is the function

$$H(M, n) = \ell\left(\frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1} M}\right) \quad \text{for all } n \geq 0.$$

Tony J. Puthenpurakal, Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India

E-mail: tputhen@math.iitb.ac.in

A natural question is whether $H(M,n)$ is non-decreasing (when $\dim M > 0$). It is clear that if $\operatorname{depth} G(M) > 0$ then the Hilbert function of M is *non*-decreasing, see Proposition 3.2 of [10]. If A is regular local then all maximal Cohen-Macaulay (= MCM) modules are free. Thus every MCM module of positive dimension over a regular local ring has a non-decreasing Hilbert function. The next case is that of a hypersurface ring i.e., the completion $\widehat{A} = Q/(f)$ where $(Q, \mathfrak{n})$ is regular local and $f \in \mathfrak{n}^2$. In Theorem 1,[10] we prove that if A is a hypersurface ring of positive dimension and if M is a MCM A-module then the Hilbert function of M is non-decreasing. See example 3.3, [10] for an example of a MCM module M over the hypersurface ring $k[[x,y]]/(y^3)$ with $\operatorname{depth} G(M) = 0$.

Let $(A, \mathfrak{m})$ be a strict complete intersection of positive dimension and let M be a maximal Cohen-Macaulay A-module with bounded betti-numbers. In Theorem 1, [9] we prove that the Hilbert function of M is non-decreasing. We also prove an analogous statement for complete intersections of codimension two, see Theorem 2, [9].

In the ring case Elias [1, 2.3], proved that the Hilbert function of a one dimensional Cohen-Macaulay ring is non-decreasing if embedding dimension is three. The first example of a one dimensional Cohen-Macaulay ring A with not monotone increasing Hilbert function was given by Herzog and Waldi; [3, 3d]. Later Orecchia, [8, 3.10], proved that for all $b \geq 5$ there exists a reduced one-dimensional Cohen-Macaulay local ring of embedding dimension b whose Hilbert function is not monotone increasing. Finally in [2] we can find similar example with embedding dimension four. A long standing conjecture in theory of Hilbert functions is that the Hilbert function of a one dimensional complete intersection is non-decreasing. Rossi conjectures that a similar result holds for Gorenstein rings. Recently counterexamples to both these conjectures were found, see [7].

In this paper we construct a large class of one dimensional Cohen-Macaulay local rings $(A, \mathfrak{m})$ with the property that if M is an MCM A-module then the Hilbert function of M is non-decreasing. Recall a Cohen-Macaulay local ring $(B, \mathfrak{n})$ is said to have *minimal multiplicity* if

$$e(B) = 1 + \operatorname{codim}(B).$$

Our result is

Theorem 1.1. *Let $(B, \mathfrak{n})$ be a two dimensional Cohen-Macaulay local ring with minimal multiplicity. Let $(A, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring which is a quotient of B. If M is a maximal Cohen-Macaulay A-module then the Hilbert function of M (with respect to $\mathfrak{m}$) is non-decreasing.*

Remark 1.2. Our main contribution in Theorem 1.1 is to guess the result. The proof is not difficult.

We now give examples where our result holds.

Example 1.3. Let $(Q, \mathfrak{n})$ be a regular local ring of dimension three. Let $f_1, f_2 \in \mathfrak{n}^2$ be an Q-regular sequence. Assume $f_1 \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$. Let $A = Q/(f_1, f_2)$. Then if M is a maximal Cohen-Macaulay A-module then the Hilbert function of M (with respect to $\mathfrak{m}$) is non-decreasing. The reason for this is that $B = Q/(f_1)$ has minimal multiplicity.

EXAMPLE 1.4. Let $(B, \mathfrak{n})$ be a two dimensional local ring with pseudo-rational singularity. Then B has minimal multiplicity, see [6, 5.4]. In particular if $A = B/P$, P a prime ideal of height one or if $A = B/(x)$ where x is B-regular and if M is a maximal Cohen-Macaulay A-module then the Hilbert function of M (with respect to $\mathfrak{m}$) is non-decreasing.

EXAMPLE 1.5. There is a large class of one dimensional local rings $(R, \mathfrak{m})$ with minimal multiplicity. For examples Arf rings have this property, [5, 2.2]. Let $B = R[X]_{(\mathfrak{m},X)}$. Then B is a two dimensional Cohen-Macaulay local ring with minimal multiplicity.

Here is an overview of the contents of the paper. In Section two we introduce notation and discuss a few preliminary facts that we need. In section three we prove Theorem 1.1.

2. Preliminaries

In this paper all rings are Noetherian and all modules considered are assumed to be finitely generated (unless otherwise stated). Let $(A, \mathfrak{m})$ be a local ring of dimension d with residue field $k = A/\mathfrak{m}$. Let M be an A-module. If m is a non-zero element of M and if j is the largest integer such that $m \in \mathfrak{m}^j M$, then we let m^* denote the image of m in $\mathfrak{m}^j M/\mathfrak{m}^{j+1} M$.

The formal power series

$$H_M(z) = \sum_{n \geq 0} H(M, n) z^n$$

is called the *Hilbert series* of M. It is well known that it is of the form

$$H_M(z) = \frac{h_M(z)}{(1-z)^r}, \text{ where } \ r = \dim M \quad \text{and } h_M(z) \in \mathbb{Z}[z].$$

We call $h_M(z)$ the *h-polynomial* of M. If f is a polynomial we use $f^{(i)}$ to denote its i-th derivative. The integers $e_i(M) = h_M^{(i)}(1)/i!$ for $i \geq 0$ are called the *Hilbert coefficients* of M. The number $e(M) = e_0(M)$ is the *multiplicity* of M.

2.1. **Base change:** Let $\phi\colon (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ be a local ring homomorphism. Assume A' is a faithfully flat A algebra with $\mathfrak{m}A' = \mathfrak{m}'$. Set $\mathfrak{m}' = \mathfrak{m}A'$ and if N is an A-module set $N' = N \otimes_A A'$. In these cases it can be seen that

(1) $\ell_A(N) = \ell_{A'}(N')$.

(2) $H(M,n) = H(M',n)$ for all $n \geq 0$.

(3) $\dim M = \dim M'$ and $\operatorname{depth}_A M = \operatorname{depth}_{A'} M'$.

(4) $\operatorname{depth} G(M) = \operatorname{depth} G(M')$.

The specific base changes we do are the following:

(i) $A' = A[X]_S$ where $S = A[X] \setminus \mathfrak{m}A[X]$. The maximal ideal of A' is $\mathfrak{n} = \mathfrak{m}A'$. The residue field of A' is $K = k(X)$.

(ii) $A' = \widehat{A}$ the completion of A with respect to the maximal ideal.

Thus we can assume that our ring A is complete with infinite residue field.

I: $L_i(M)$

Let $(A,\mathfrak{m})$ be a Noetherian local ring and M a A-module. We simplify a construction from [10].

2.2. Set $L_0(M) = \bigoplus_{n\geq 0} M/\mathfrak{m}^{n+1}M$. Let $\mathcal{R} = A[\mathfrak{m}u]$ be the *Rees-algebra* of $\mathfrak{m}$. Let $\mathcal{S} = A[u]$. Then $\mathcal{R}$ is a subring of $\mathcal{S}$. Set $M[u] = M \otimes_A \mathcal{S}$ an $\mathcal{S}$-module and so an $\mathcal{R}$-module. Let $\mathcal{R}(M) = \bigoplus_{n\geq 0} \mathfrak{m}^n M$ be the Rees-module of M with respect to $\mathfrak{m}$. We have the following exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{R}(M) \to M[u] \to L_0(M)(-1) \to 0.$$

Thus $L_0(M)(-1)$ (and so $L_0(M)$) is an $\mathcal{R}$-module. We note that $L_0(M)$ is *not* a finitely generated $\mathcal{R}$-module. Also note that $L_0(M) = M \otimes_A L_0(A)$.

2.3. For $i \geq 1$ set

$$L_i(M) = \operatorname{Tor}_i^A(M, L_0(A)) = \bigoplus_{n\geq 0} \operatorname{Tor}_i^A(M, A/\mathfrak{m}^{n+1}).$$

We assert that $L_i(M)$ is a finitely generated $\mathcal{R}$-module for $i \geq 1$. It is sufficient to prove it for $i = 1$. We tensor the exact sequence $0 \to \mathcal{R} \to \mathcal{S} \to L_0(A)(-1) \to 0$ with M to obtain a sequence of $\mathcal{R}$-modules

$$0 \to L_1(M)(-1) \to \mathcal{R} \otimes_A M \to M[u] \to L_0(M)(-1) \to 0.$$

Thus $L_1(M)(-1)$ is a $\mathcal{R}$-submodule of $\mathcal{R} \otimes_A M$. The latter module is a finitely generated $\mathcal{R}$-module. It follows that $L_1(M)$ is a finitely generated $\mathcal{R}$-module.

2.4. Now assume that A is Cohen-Macaulay of dimension $d \geq 1$. Set $N = \operatorname{Syz}_1^A(M)$ and $F = A^{\mu(M)}$. We tensor the exact sequence

$$0 \to N \to F \to M \to 0,$$

with $L_0(A)$ to obtain an exact sequence of $\mathcal{R}$-modules

$$0 \to L_1(M) \to L_0(N) \to L_0(F) \to L_0(M) \to 0.$$

It is elementary to see that the function $n \to \ell(\operatorname{Tor}_1^A(M, A/\mathfrak{m}^{n+1}))$ is polynomial of degree $\leq d-1$. By [4, Corollary II] if M is non-free then it is polynomial of degree $d-1$. Thus $\dim L_1(M) = d$ if M is non-free.

II: *Superficial sequences.*

2.5. An element $x \in \mathfrak{m}$ is said to be *superficial* for M if there exists an integer $c > 0$ such that

$$(\mathfrak{m}^n M \colon {}_M x) \cap \mathfrak{m}^c M = \mathfrak{m}^{n-1} M \quad \text{for all} \quad n > c.$$

Superficial elements always exist if k is infinite [12, p. 7]. A sequence $x_1, x_2, \ldots, x_r$ in a local ring $(A, \mathfrak{m})$ is said to be a *superficial sequence* for M if x_1 is superficial for M and x_i is superficial for $M/(x_1, \ldots, x_{i-1})M$ for $2 \leq i \leq r$.

We need the following:

PROPOSITION 2.6. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension d and let M be a Cohen-Macaulay A-module of dimension r. Let $x_1, \ldots, x_c$ be an M-superficial sequence with $c \leq r$. Assume $x_1^*, \cdots, x_c^*$ is a $G(M)$-regular sequence. Let $\mathcal{R} = A[\mathfrak{m}u]$ be the Rees algebra of $\mathfrak{m}$. Set $X_i = x_i u \in \mathcal{R}_1$. Then $X_1, \ldots, X_c$ is a $L_0(M)$-regular sequence.*

PROOF. We prove the result by induction. First consider the case when $c = 1$. Then the result follows from [10, 2.2(3)]. We now assume that $c \geq 2$ and the result holds for all Cohen-Macaulay A-modules and sequences of length $c - 1$. By $c = 1$ result we get that X_1 is $L_0(M)$-regular. Let $N = M/x_1M$. As x_1^* is $G(M)$-regular we get $G(M)/x_1^*G(M) \cong G(N)$. So $x_2^*, \ldots, x_c^*$ is a $G(N)$-regular sequence. Now also note that $L_0(M)/X_1L_0(M) = L_0(N)$. Thus the result follows. □

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. We also give an example which shows that it is possible for $\operatorname{depth} G(M)$ to be zero.

PROOF OF THEOREM 1.1. We may assume that the residue field of A is infinite. Let $N = \operatorname{Syz}_1^B(M)$. Then N is a maximal Cohen-Macaulay B-module. As B has minimal multiplicity it follows that N also has minimal multiplicity (this is well-known; for instance see [11, Theorem 14]). So $G(N)$ is Cohen-Macaulay and $\deg h_N(z) \leq 1$, see [11, Theorem 16]. Set $r = \mu(M)$, $h_B(z) = 1 + hz$ and as $e(N) = re(A)$ we write $h_N(z) = r + c + (rh - c)z$ (here c can be negative). Let $\mathcal{R} = B[\mathfrak{n}u]$ be the Rees algebra of B with respect to $\mathfrak{n}$.

Set $F = B^r$. The exact sequence $0 \to N \to F \to M \to 0$ induces an exact sequence

$$0 \to L_1(M) \to L_0(N) \xrightarrow{\phi} L_0(F) \to L_0(M) \to 0 \tag{3.0.1}$$

of $\mathcal{R}$-modules. Set $K = \operatorname{image}(\phi)$. Let x_1, x_2 be an $N \oplus B$-superficial sequence. Then x_1^*, x_2^* is a $G(N) \oplus G(B)$-regular sequence. Set $X_i = x_i u \in \mathcal{R}_1$. Then by 2.6 it follows that X_1, X_2 is a $L_0(N) \oplus L_0(F)$-regular sequence. As K is a submodule of $L_0(F)$ it follows that X_1 is K-regular. As $L_1(M)$ is a submoudle of $L_0(N)$ we get that X_1 is $L_1(M)$-regular. So the exact sequence $0 \to L_1(M) \to L_0(N) \to K \to 0$ induces the exact sequence

$$0 \to \frac{L_1(M)}{X_1 L_1(M)} \to \frac{L_0(N)}{X_1 L_0(N)} \to \frac{K}{X_1 K} \to 0.$$

Since X_2 is $L_0(N)/X_1L_0(N)$-regular it follows that X_2 is $L_1(M)/X_1L_1(M)$-regular. It follows that X_1, X_2 is also a $L_1(M)$-regular sequence. As $\dim L_1(M) = 2$ (see 2.4) it follows that $L_1(M)$ is a Cohen-Macaulay $\mathcal{R}$-module. Let the Hilbert series of $L_1(M)$ be $l(z)/(1-z)^2$. Then the coefficients of $l(z)$ are non-negative.

Let $l(z) = l_0 + l_1 z + \cdots + l_m z^m$ and let $h_M(z) = h_0 + h_1 z + \cdots + h_p z^p$. By (3.0.1) we get

$$\begin{aligned}
(1-z)l(z) &= h_N(z) - h_F(z) + (1-z)h_M(z), \\
&= r + c + (rh - c)z - r(1 + hz) + (1-z)h_M(z), \\
&= c(1-z) + (1-z)h_M(z).
\end{aligned}$$

It follows that

$$l(z) = c + h_M(z).$$

It follows that $m = p$ and $h_i = l_i$ for $i \geq 1$. In particular $h_i \geq 0$ for $i \geq 1$. Also $h_0 = \mu(M) > 0$. Thus $h_M(z)$ has non-negative coefficients. It follows that the Hilbert function of M is non-decreasing. □

We now give an example which shows that it is possible for $\operatorname{depth} G(M)$ to be zero.

Example 3.1. Let K be a field and let $A = K[[t^6, t^7, t^{15}]]$. It can be verified that

$$A \cong \frac{K[[X, Y, Z]]}{(Y^3 - XZ, X^5 - Z^2)}$$

and that

$$G(A) \cong \frac{K[X, Y, Z]}{(XZ, Y^6, Y^3 Z, Z^2)}$$

Note that ZY^2 annihilates (X, Y, Z). So $\operatorname{depth} G(A) = 0$.

Set $B = K[[X, Y, Z]]/(Y^3 - XZ)$. Then B is a two-dimensional Cohen-Macaulay ring with minimal multiplicity and A is a one-dimensional Cohen-Macaulay quotient of B. Set $M = A$.

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