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Cyclic non-S-permutable subgroups and non-normal maximal subgroups

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Abstract – A finite group G is said to be a T-group (resp. PT-group, PST-group) if normality (resp. permutability, S-permutability) is a transitive relation. Ballester-Bolinches et al. gave some new characterizations of the soluble T-, PT- and PST-groups. A finite group G is called a T_c-group (resp. PT_c-group, PST_c-group) if each cyclic subnormal subgroup is normal (resp. permutable, S-permutable) in G. The present work defines the NNM_c-, PNM_c-, and SNM_c-groups and presents new characterizations of the wider classes of soluble T_c-, PT_c-, and PST_c-groups.

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1. Introduction

In the present work, all groups are finite. Recall that a subgroup H of a group G is said to be S-permutable (or S-quasinormal) if $HP = PH$ for all Sylow subgroups P of G. Kegel proved that every S-permutable subgroup is subnormal. A group G is a PST-group if S-permutability is a transitive relation (i.e., if H and K are subgroups of G such that H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G). It follows from Kegel's result that PST-groups are exactly those groups in which every subnormal subgroup is S-permutable.
Similarly, groups in which permutability (normality) is transitive relation are called PT-groups (T-groups) and can be identified with groups in which subnormal sub-

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groups are always permutable (normal). Recall that a group G is a PST_c-group if each cyclic subnormal subgroup is S-permutable in G. The classes of PT_c-groups and T_c-groups similarly defined as groups in which cyclic subnormal subgroups are permutable or normal, respectively. Kaplan [9] characterized soluble T-groups by means of their maximal subgroups and some classes of pre-Frattini subgroups. He proved a necessary and sufficient condition for a group G to be a soluble T-group as follows: G is a soluble T-group if and only if every non-normal subgroup of every subgroup H of G is contained in a non-normal maximal subgroup of H. Ballester-Bolinches et al. [3] extended the results from Kaplan [8] and presented new characterizations for soluble PT- and PST-groups. The starting point of their results was the following: let H be a proper permutable (resp. S-permutable) subgroup of a soluble group G. Using Kegel's result, H is subnormal in G and so H is contained in a maximal subgroup of G that is normal in G. Following Ballester-Bolinches et al. [3] a group G is said to be a PNM-group (resp. SNM-group) if every non-permutable (resp. non-S-permutable) subgroup of G is contained in a non-normal maximal subgroup of G. Many interesting results can be obtained using these concepts. For example, they proved that a group G is a soluble PT-group (resp. PST-group) if and only if every subgroup of G is a PNM-group (resp. SNM-group). They also showed that if G is an SNM-group, then the nilpotent residual $G^{\mathfrak{N}}$ is supersoluble if and only if G is supersoluble. Consequently, if G is a group whose non-nilpotent subgroups are SNM-groups, then G is supersoluble. Now, we define that a group G is a PNM_c-groups (resp. SNM_c-groups) if every cyclic non-permutable (resp. non-S-permutable) subgroup is contained in a non-normal maximal subgroup. The aim of this paper is to present new characterizations of the wider classes of soluble T_c-, PT_c-, and PST_c-groups. We begin with the following definition.

Definition 1.1. A group G is called an NNM_c-group (resp. PNM_c-group, SNM_c-group) if every cyclic non-normal (resp. non-permutable, non-S-permutable) subgroup of G is contained in a non-normal maximal subgroup of G.

2. Preliminaries

We first collect results from Ballester-Bolinches et al. [3], as the starting point of our results.

Theorem 2.1. *A group G is a soluble PST-group if and only if every subgroup of G is an SNM-group.*

Lemma 2.2. *Every subgroup of a group G is a PNM-group if and only if every subgroup of G is an SNM-group and all Sylow subgroups of G are Iwasawa groups.*

It can be concluded by applying Theorem 2.1 and Lemma 2.2 that:

COROLLARY 2.3. *A group G is a soluble PT-group if and only if every subgroup of G is a PNM-group.*

Every subgroup of a group G is an NNM-group if and only if every subgroup of G is an SNM-group and all Sylow subgroups are Dedekind; thus, it can be concluded:

COROLLARY 2.4. *A group G is a soluble T-group if and only if every subgroup of G is an NNM-group.*

THEOREM 2.5. *If G is an SNM-group, then the nilpotent residual $G^{\mathfrak{N}}$ is supersoluble if and only if G is supersoluble.*

For the sake of easy reference, theorems from Robinson [9] have been provided. These results provide detailed information on the structure of a soluble PST_c-group.

THEOREM 2.6. *Let G be a soluble PST_c-group with $F = Fit(G)$ and $L = \gamma_\infty(G)$. Then the following hold:*
1) L is an abelian group of odd order.
2) p'-elements of G induce power automorphisms in L_p for all primes p.
3) $F = C_G(L)$.
4) G splits conjugately over L.
5) $F = \overline{Z}(G) \times L$.
6) $\pi(L) \cap \pi(F/L) = \emptyset$.
7) G is supersoluble.
Where $\gamma_\infty(G)$ is the hypercommutator subgroup or the limit of the lower central series, $Fit(G)$ is the Fitting subgroup, and $\pi(G)$ is the set of prime divisors of the group order.

The class of soluble PST_c-groups is neither subgroup nor quotient closed, which is in contrast to the behavior of soluble PST-groups. Robinson [9] proved:

THEOREM 2.7. *If every subgroup of a group G is a PST_c-group, then G is a soluble PST-group.*

THEOREM 2.8. *Let G be a soluble group. If every quotient of G is a PST_c-group, then G is a PST-group.*

3. Main Results

THEOREM 3.1. *(1) Let every non-normal maximal subgroup M of a group G does not have a supplement in G. If every subgroup of G is an SNM_c-group, then G is a soluble PST_c-group.*
(2) If every subgroup of G is a PST_c-group, then every subgroup of G is an SNM_c-group.

Proof. (1) Assume that the theorem is not true and let G be a counterexample of minimal order. Then every proper subgroup of G is a soluble PST_c-group. Using Theorem 2.6(7), every proper subgroup of G is supersoluble and so G is soluble. On the other hand, there exists a cyclic subnormal subgroup H of G which is not S-permutable. Let M be a maximal normal subgroup of G containing H. There exists a non-normal maximal subgroup L of G containing H, since G is an SNM_c-group. It is clear that $G = ML$. Since H is not S-permutable in G, it follows that there exists a Sylow p-subgroup P of G such that P does not permute with H. The choice of the minimality of G implies that H is S-permutable in M and L. Using Corollary 1.3.3 [1], there exist Sylow p-subgroups M_0 of M and L_0 of L where $P_0 = M_0L_0$ is a Sylow p-subgroup of G. Let $g \in G$ such that $P^g = P_0$. Hence H permutes with both M_0 and L_0 and so H permutes with P_0. Let N be a minimal normal subgroup of G contained in M. Since the factor group G/N satisfies the hypothesis and $|G/N| < |G|$, then HN permutes with P. If $(HN)P$ is a proper subgroup of G, then H will permute with P. This is a contradiction. Therefore, $G = P(HN)$ and $g = xy$ such that $x \in P$ and $y \in HN$. Using Lemma 14.3.A [6], H is a normal subgroup of HN. Since $HP^g = P^gH$, it follows that $H^{y^{-1}} = H$ permutes with P, which is contrary to the assumption.
(2) It is clear. □

Lemma 3.2. *Every subgroup of a group G is a PNM_c-group if and only if every subgroup of G is an SNM_c-group and all Sylow subgroups of G are Iwasawa groups.*

Proof. Assume that every subgroup of G is a PNM_c-group. It is clear that every subgroup of G is also an SNM_c-group. Moreover, every Sylow subgroup P of G is a nilpotent PNM_c-group. Let H be a subgroup of P such that H is not permutable in P. If H is cyclic, then there exists a non-normal maximal subgroup M_1 of P such that $H \subseteq M_1$, which is a contradiction. If H is non-cyclic, then $H = M\langle x\rangle$ where M is a maximal subgroup of H of prime index and $x \in H - M$. Either M or $\langle x\rangle$ will not permute in P. If $\langle x\rangle$ does not permute, then there exists a non-normal maximal subgroup M_2 of P such that $\langle x\rangle \subseteq M_2$, which is a contradiction. If M does not permute in P, by the same argument, we have a contradiction. Hence H must be permutable in P. This means that P is an Iwasawa group.
Conversely, assume that every subgroup of G is an SNM_c-group and all Sylow subgroups of G are Iwasawa groups. Let K be a cyclic S-permutable subgroup of a subgroup H of G. Because all Sylow subgroups of H are also Iwasawa groups, we can apply Theorem 2.1.10 [2] to conclude that K is permutable in H. Hence H is a PNM_c-group. Consequently every subgroup of G is a PNM_c-group. □

Corollary 3.3. *(1) Let every non-normal maximal subgroup M of a group G does not have a supplement in G. If every subgroup of G is a PNM_c-group, then G is a soluble PT_c-group.*
(2) If every subgroup of G is a soluble PT_c-group, then every subgroup of G is a

PNM_c-group.

Proof. (1) If every subgroup of G is a PNM_c-group, then every subgroup of G is an SNM_c-group according to Lemma 3.2 and so G is a soluble PST_c-group. This implies that every cyclic subnormal subgroup H of G is S-permutable in G. Applying Theorem 2.1.10 [2], we see that H is permutable in G, since all Sylow subgroups of G are Iwasawa groups. Thus G is a soluble PT_c-group.
2) It is clear. □

Lemma 3.4. *Every subgroup of a group G is an NNM_c-group if and only if every subgroup of G is an SNM_c-group and all Sylow subgroups of G are Dedekind groups.*

Proof. Let every subgroup of G be an NNM_c-group. It is clear that G is an SNM_c-group. Let H be a non-normal subgroup of P where $P \in Syl(G)$. If H is cyclic, then there exists a non-normal maximal subgroup M_1 of P such that $H \subseteq M_1$, which is a contradiction. If H is non-cyclic, then $H = M\langle x\rangle$ where M is a maximal subgroup of H of prime index and $x \in H - M$. Either M or $\langle x\rangle$ is not normal in P, since H is not normal in P. If $\langle x\rangle$ is not normal in P, then there exists a non-normal maximal subgroup M_2 of P such that $\langle x\rangle \subseteq M_2$, which is a contradiction. If M is not normal in P, we have a similar contradiction. Thus P is a Dedekind group.
Conversely, let every subgroup of G be an SNM_c-group and every Sylow subgroup of G be a Dedekind group. Let K be an S-permutable subgroup of H such that $H \leq G$. Applying Theorem 2.1.10 [2], we see that K is normal in H, since all Sylow subgroups of H are also Dedekind groups. Hence H is an NNM_c-group. The above argument implies that every subgroup of G is an NNM_c-group. □

Corollary 3.5. *(1) Let every non-normal maximal subgroup M of a group G does not have a supplement in G. If every subgroup of G is an NNM_c-group, then G is a soluble T_c-group.*
(2) If every subgroup of G is a soluble T_c-group, then every subgroup is an NNM_c-group.

Proof. (1) If every subgroup of G is an NNM_c-group, then every subgroup of G is an SNM_c-group and all Sylow subgroups of G are Dedekind groups. Thus G is a soluble PT_c-group. This implies that every cyclic subnormal subgroup H of G is permutable in G. Applying Theorem 2.1.10 [2], we see that H is normal in G, since all Sylow subgroups of G are Dedekind groups. Thus G is a soluble T_c-group.
(2) It is clear. □

Theorem 3.6. *Let G and each quotient group of G/N be an SNM_c-group and G be not factorized by the nilpotent residual $G^{\mathfrak{N}}$. Then $G^{\mathfrak{N}}$ is supersoluble if and only if G is supersoluble.*

Proof. The sufficiency of the condition is evident; we need only prove the necessity of the condition. We use induction on the order of G. Let N be a minimal normal subgroup of G. Then $G^{\mathfrak{N}}N/N$ is the nilpotent residual of G/N according to Proposition 2.2.8 (1) [4]. Moreover, $G^{\mathfrak{N}}N/N$ is supersoluble and according to the hypothesis, G/N is an SNM_c-group. By induction, G/N is supersoluble. Since the class of all supersoluble groups is a saturated formation, we can suppose that G has an unique minimal normal subgroup N and $\Phi(G) = 1$. This means that $N = C_G(N)$ in addition $G = MN$, $M \cap N = 1$ and $Core_G(M) = 1$. Let p be the prime dividing $|N|$. Then N has the structure of a semisimple $KG^{\mathfrak{N}}$-module where K is the field of p elements. Therefore, N is a direct product of the minimal normal subgroups of $G^{\mathfrak{N}}$. Let A be a minimal normal subgroup of $G^{\mathfrak{N}}$ contained in N. Then A has order p because $G^{\mathfrak{N}}$ is supersoluble. If $AM^{\mathfrak{N}} = \langle a \rangle M^{\mathfrak{N}}$ is not S-permutable in G, then there exists a non-normal maximal subgroup L of G containing $AM^{\mathfrak{N}}$. Since $A \leq L \cap N$, it follows that N is contained in L. In particular, $G^{\mathfrak{N}}$ is contained in L and L is normal in G. This contradiction shows that $AM^{\mathfrak{N}}$ is S-permutable in G. It implies that $AM^{\mathfrak{N}}$ is subnormal in G and so N normalizes $AM^{\mathfrak{N}}$ according to Lemma 14.3.A [6]. It follows that $[M^{\mathfrak{N}}, N] \leq AM^{\mathfrak{N}} \cap N = A$, which holds for every minimal normal subgroup of $G^{\mathfrak{N}}$ contained in N.
If $A = N$, then N is of prime order and G is supersoluble. Hence N is a direct product of at least two minimal normal subgroups of $G^{\mathfrak{N}}$. In this case, $M^{\mathfrak{N}}$ centralizes N and $M^{\mathfrak{N}} = 1$. Therefore, every subgroup of N is S-permutable in G. According to Lemma 2.1.3 [2], it follows that N is of prime order. Hence G is supersoluble. This establishes the theorem. □

4. References

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