

## The maximal number of skew lines on Schur's quartic

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ABSTRACT – Since 1882 it is known that the so-called *Schur's* quartic contains exactly 64 lines. However, it has not yet been established what is the maximum number of pairwise disjoint lines that it can have. The aim of our work is to show in an elementary and self-contained way that the maximum number of pairwise disjoint lines in *Schur's* quartic is 16 (without using *Nikulins's* Theorem ([5]) or *Miyaoka's* upper bound ([4])).

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### 1. Introduction

Since 1882 it is known that the so-called *Schur's* quartic contains exactly 64 lines ([8]). In 1943, *B. Segre* proved that a smooth quartic surface in the complex projective space  $\mathbb{P}^3$  cannot contain more than 64 lines ([9]). Even though a gap was discovered in *Segre's* proof by *Rams-Schütt* in 2015 ([7]), the claim is still correct. For smooth surfaces of degree  $d \geq 5$  in  $\mathbb{P}^3$ , it is not yet known what is the maximum number of lines that such a surface can have. In this direction the articles by *Caporaso-Harris-Mazur* ([3]) and *Boissière-Sarti* ([2]) exhibited lower bounds for these numbers.

A related problem is to determine the maximum number  $r_d$  of pairwise disjoint lines (or skew lines) that a smooth surface of degree  $d$  can have. It is known that  $r_3 = 6$  and  $r_4 = 16$  ([5]). In 1975, *Miyaoka* gave the upper

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bound  $r_d \leq 2d(d-2)$  if  $d \geq 4$  ([4]). Some lower bounds were given by *Rams* ([6]) and *Boissière-Sarti* ([2]). However,  $r_d$  remains unknown for  $d \geq 5$ .

To the best of our knowledge, the maximum number of pairwise disjoint lines on *Schur's* quartic has never been explicitly computed in modern literature. Thus, the aim of our work is to show in an elementary and self-contained way that the maximum number of pairwise disjoint lines in *Schur's* quartic is 16 (without using *Nikulins's* Theorem ([5]) or *Miyaoka's* upper bound ([4])). In order to do that, first we determine the lines on *Schur's* quartic, and we describe a family  $\mathcal{L} = \{\ell_{ij}\}$  consisting of 16 lines (cf. (4)), next fixing the lines  $\ell_{11}, \ell_{12}, \ell_{13}, \ell_{14}$  in the family  $\mathcal{L}$  and studying the elliptic fibration induced by the planes containing these fixed lines in Proposition 3.3 (this technique was used by *Rams-Schütt* in [7]), we get the families of lines  $\mathcal{M}_i$  with  $i \in \{1, 2, 3, 4\}$ , each one consisting of 12 lines. In fact,  $\mathcal{M}_i = \Phi_{\ell_{1i}} \setminus \mathcal{L}$  where  $\Phi_{\ell_{1i}} = \{\text{lines } \ell \text{ on } \textit{Schur's} \textit{ quartic such that } \ell \neq \ell_{1i}, \ell \cap \ell_{1i} \neq \emptyset\}$  (cf. Proposition 3.4). Hence, we obtain the  $64 = 16 + 4 \cdot 12$  lines on *Schur's* quartic. Using this description we verify that the maximum number of pairwise disjoint lines on *Schur's* quartic is less than or equal to 16 (cf. Theorem 3.8). We also study the incidences (in Lemma 3.6) between the lines in  $\mathcal{M}_i$  for  $i \in \{1, 2, 3, 4\}$ .

Note that another way to determine the lines on *Schur's* quartic, appeared in [3] (and was used by *Boissière-Sarti* in [2] to determine lower bounds for  $r_d$ , if  $d = 2 \cdot n + 1$  and  $n \geq 3$ ). The same approach was used by *Bauer-Schmitz* to determine the lines on *Schur's* quartic (Proposition 2.2 in [1]). However, *Bauer* and *Schmitz* did not specify in their paper any family of sixteen skew lines.

## 2. Notations and Preliminary Results

Let us consider  $\mathbb{C}^4$  as a vector space over the field of complex numbers  $\mathbb{C}$ . We denote by  $[v_1, \dots, v_k]$  the subspace of  $\mathbb{C}^4$  generated by the vectors  $v_1, \dots, v_k \in \mathbb{C}^4$ . For any linear subspace  $W$  of  $\mathbb{C}^4$  we denote by  $\mathbb{P}(W)$  the *projectivization* of  $W$ . Let  $R_d = \mathbb{C}[x_0, x_1, x_2, x_3]_d$  be the vector space of homogeneous degree  $d$  polynomials. We will denote by  $\mathcal{Z}(F) \subset \mathbb{P}^3$  the degree  $d$  surface defined by the non-constant polynomial  $F \in R_d$ . Furthermore, if  $p \in \mathcal{Z}(F)$  is a smooth point of  $\mathcal{Z}(F)$  we will denote by  $\mathbb{T}_p \mathcal{Z}(F)$  the tangent plane of  $\mathcal{Z}(F)$  at  $p$ .

## 2.1 – Generalities on lines in a smooth surface

Let  $\Phi(S)$  denote the set of all the lines in the smooth surface  $S \subseteq \mathbb{P}^3$ , and set

$$\Phi_\ell = \left\{ m \in \Phi(S) \mid m \cap \ell \neq \emptyset, m \neq \ell \right\} \quad \text{for each line } \ell \subset \mathbb{P}^3.$$

LEMMA 2.1. *Let  $\ell_1$  and  $\ell_2$  be two distinct lines in  $S$ . Then, we have that:*

- (a) *If  $\ell_1 \cap \ell_2 = \{p\}$  then  $\langle \ell_1, \ell_2 \rangle = \mathbb{T}_p S$ . Moreover, any line in  $\Phi_{\ell_1} \cap \Phi_{\ell_2}$  lies in the plane  $\langle \ell_1, \ell_2 \rangle$ .*
- (b) *If  $\ell_1$  and  $\ell_2$  are disjoint, then any two distinct lines in  $\Phi_{\ell_1} \cap \Phi_{\ell_2}$  (if they exist) are disjoint.*

PROOF. If  $\ell_1 \cap \ell_2 = \{p\}$  then  $p \in \ell_i \subset S$ . Thus  $\ell_i \subset \mathbb{T}_p S$  for  $i = 1, 2$  and we conclude that  $\langle \ell_1, \ell_2 \rangle = \mathbb{T}_p S$ . Furthermore, if  $\ell \in \Phi_{\ell_1} \cap \Phi_{\ell_2}$  then we have two possibilities:

- $\ell$  meets  $\ell_1$  and  $\ell_2$  at two distinct points (that belong to the plane  $\langle \ell_1, \ell_2 \rangle = \mathbb{T}_p S$ ) and then we have  $\ell \subset \langle \ell_1, \ell_2 \rangle$ , or
- $\ell$  passes through the point  $p$ , and then  $\ell \subset \langle \ell_1, \ell_2 \rangle$ .

In order to prove the second statement, we will assume that  $\ell$  and  $m$  belong to  $\Phi_{\ell_1} \cap \Phi_{\ell_2}$  and  $\ell_1 \cap \ell_2 = \emptyset$ . Note that,  $\ell_1$  and  $\ell_2$  belong to  $\Phi_\ell \cap \Phi_m$ . Let us suppose that  $\ell \cap m \neq \emptyset$ , then (a) above implies that  $\ell_i \subset \langle \ell, m \rangle$  for  $i = 1, 2$ , so  $\ell_1$  and  $\ell_2$  are concurrent, a contradiction.  $\square$

PROPOSITION 2.2. *If  $\ell_1, \dots, \ell_n$  are  $n$  coplanar lines on  $S$  and  $S$  has degree  $n \geq 2$ , then  $\Phi(S) = \Phi_{\ell_1} \cup \dots \cup \Phi_{\ell_n}$ .*

PROOF. Let  $\pi = \mathcal{Z}(L)$  be the plane containing the lines  $\ell_1, \dots, \ell_n$ . Thus,  $L \in \mathcal{I}(\ell_i)$  and we can find  $L_i \in R_1$  such that  $\ell_i = \mathcal{Z}(L, L_i)$  for each  $i$ ,  $1 \leq i \leq n$ . Furthermore, if  $S = \mathcal{Z}(F)$  then  $\bigcup_{i=1}^n \ell_i = \mathcal{Z}(L, L_1 \cdots L_n) \subseteq \mathcal{Z}(L, F) = \pi \cap S$ . By Hilbert's Nullstellensatz we get that  $\langle L, F \rangle \subseteq \langle L, G \rangle$  with  $G = L_1 \cdots L_n$ . Now, since  $\pi$  does not contain the surface  $S$  and  $F, G$  are homogeneous of the same degree, we conclude that  $\langle L, F \rangle = \langle L, G \rangle$ . Therefore,  $\ell_1 \cup \dots \cup \ell_n = \pi \cap S$ . Now, let us consider  $\ell \in \Phi(S)$ . We have either that  $\ell \subset \pi$  or  $\ell \cap \pi = \{p\}$ . In the first case,  $\ell \in \Phi_{\ell_i}$  for each  $i$ ,  $1 \leq i \leq n$ . In the second case, note that  $p \in \pi \cap S = \ell_1 \cup \dots \cup \ell_n$  (since  $\ell \subset S$ ), thus  $\ell \in \Phi_{\ell_i}$  for some  $i$ ,  $1 \leq i \leq n$ .  $\square$

### 3. Lines on Schur's Quartic

We are interested in *Schur's* quartic surface  $\mathcal{S}$  defined by:

$$(1) \quad f = x_0^4 - x_0x_1^3 - x_2^4 + x_2x_3^3.$$

Note that  $\mathcal{S}$  is a smooth surface in  $\mathbb{P}^3$ . For any subset  $X$  of  $\Phi(\mathcal{S})$  we denote by  $r(X)$  the maximum number of pairwise disjoint lines in  $X$ . For simplicity, we will use the notation  $r(\mathcal{S})$  instead of  $r(\Phi(\mathcal{S}))$ .

Next, consider the disjoint lines  $L = \mathcal{Z}(x_0, x_1)$ ,  $M = \mathcal{Z}(x_2, x_3)$  and let  $\xi = \frac{-1+\sqrt{3}i}{2}$  be a primitive third root of unity. After a straightforward verification, we have that  $L \cap \mathcal{S}$  is equal to

$$(2) \quad \{p_1 = [0 : 0 : 0 : 1], p_2 = [0 : 0 : 1 : 1], p_3 = [0 : 0 : \xi : 1], p_4 = [0 : 0 : \xi^2 : 1]\},$$

and  $M \cap \mathcal{S}$  is equal to

$$(3) \quad \{q_1 = [0 : 1 : 0 : 0], q_2 = [1 : 1 : 0 : 0], q_3 = [\xi : 1 : 0 : 0], q_4 = [\xi^2 : 1 : 0 : 0]\}.$$

LEMMA 3.1. *In the above notation. Let*

$$(4) \quad \mathcal{L} = \left\{ \ell_{ij} \mid \ell_{ij} \text{ is the line that passes through the points } p_i \text{ and } q_j \right\}.$$

Then

- (a)  $\#\mathcal{L} = 16$  and  $\mathcal{L} \subset \Phi(\mathcal{S})$ ,
- (b)  $\mathcal{L} = \Phi_L = \Phi_M$ ,
- (c)  $r(\mathcal{L}) = 4$ .

PROOF. Keeping in mind that  $L = \mathcal{Z}(x_0, x_1)$  and  $M = \mathcal{Z}(x_2, x_3)$  are disjoint lines, it follows from the definition of  $\mathcal{L}$  that  $\#\mathcal{L} = 16$ . It is not difficult to check that  $\mathcal{L} \subset \Phi(\mathcal{S})$ .

Note that, for any line  $\ell \subset \mathcal{S}$  we have that  $\ell \cap L \subset \mathcal{S} \cap L$  and  $\ell \cap M \subset \mathcal{S} \cap M$ . So, if  $\ell \in \Phi_L \cap \Phi_M$  then according to (2) and (3), we have that  $\ell \in \mathcal{L}$ . In fact,  $\mathcal{L} = \Phi_L \cap \Phi_M$ .

Now, we want to show that  $\Phi_L = \Phi_L \cap \Phi_M$ . Let us suppose that there exists a line  $\ell \in \Phi_L \setminus \Phi_M$ . So, since  $L = \{[0 : 0 : u : v] \in \mathbb{P}^3 \mid (0, 0) \neq (u, v) \in \mathbb{C}^2\}$  then either

- (i)  $\ell = \mathbb{P}([(0, 0, 0, 1), (b, c, 1, 0)])$ ,
- (ii)  $\ell = \mathbb{P}([(0, 0, 1, a), (b, c, 0, 1)])$ .

- In case (i), we have that  $[b : c : 1 : u] \in \ell$  (for any  $u \in \mathbb{C}$ ) and it does not belong to  $\mathcal{S}$ , if we consider  $u \in \mathbb{C}$  such that  $u^3 + b(b^3 - c^3) - 1 \neq 0$ .
- In case (ii), the point  $[b : c : u : 1 + au] \in \ell$  does not belong to  $\mathcal{S}$ , if we choose  $u \in \mathbb{C}$  such that  $(a^3 - 1)u^4 + 3a^2u^3 + 3au^2 + u + b(b^3 - c^3) \neq 0$ .

Therefore,  $\Phi_L = \Phi_L \cap \Phi_M = \mathcal{L}$ . In the same way can show that  $\Phi_M = \mathcal{L}$ .

Finally, note that  $\ell_{ij}, \ell_{kt} \in \mathcal{L}$  are disjoint if and only if  $i \neq k$  and  $j \neq t$ . Thus, any set of four mutually disjoint lines in  $\mathcal{L} = \{\ell_{ij}\}_{1 \leq i, j \leq 4}$  (after reordering if necessary) will be equal to  $\{\ell_{1i}, \ell_{2j}, \ell_{3k}, \ell_{4t}\}$  with  $\{i, j, k, t\} = \{1, 2, 3, 4\}$  (i.e. the second indexes of these lines are pairwise distinct). Therefore,  $r(\mathcal{L}) = 4$ .  $\square$

LEMMA 3.2. For each point  $x$  in (2) or (3), let

$$\mathcal{L}_x = \left\{ \ell \in \mathcal{L} \mid \ell \text{ passes through the point } x \right\}.$$

Then  $\mathcal{L}_x$  consists of four coplanar lines lying in a plane that we will denote by  $H_x$  as illustrated in the following figures

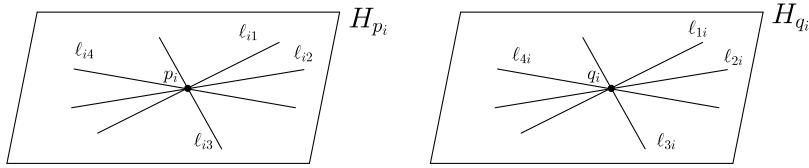


Figure 1

PROOF. From the definition of  $\mathcal{L}$  in (4) we easily conclude that  $\#(\mathcal{L}_x) = 4$ . After a straightforward verification, we have that:

(5)

$x$	$H_x$	$x$	$H_x$
$p_1$	$\mathcal{Z}(x_2)$	$q_1$	$\mathcal{Z}(x_0)$
$p_2$	$\mathcal{Z}(x_2 - x_3)$	$q_2$	$\mathcal{Z}(x_0 - x_1)$
$p_3$	$\mathcal{Z}(x_2 - \xi x_3)$	$q_3$	$\mathcal{Z}(x_0 - \xi x_1)$
$p_4$	$\mathcal{Z}(x_2 - \xi^2 x_3)$	$q_4$	$\mathcal{Z}(x_0 - \xi^2 x_1)$

$\square$

Note that, if  $\ell \in \Phi(\mathcal{S})$  and  $\pi$  is a plane containing  $\ell$ , then  $\pi \cap \mathcal{S} = \ell \cup C$  where  $C$  is a plane cubic curve. In what follows, the curve  $C$  will be called *the residual curve* to the line  $\ell$  in the plane  $\pi$ .

PROPOSITION 3.3. *In the above notation. For each line  $\ell_{1i} \in \mathcal{L}_{p_1}$  there exist exactly six planes containing it such that the residual curve is singular. In fact, such singular plane curves are union of three concurrent lines and these planes are given by:*

(6)

line	Planes containing the line where the residual curve is singular
$\ell_{11}$	$H_{p_1}, H_{q_1}$ and $H_1(a) = \mathcal{Z}(x_2 - ax_0)$ with $a \in \mathcal{U}$
$\ell_{12}$	$H_{p_1}, H_{q_2}$ and $H_2(a) = \mathcal{Z}(x_2 - \frac{a}{\sqrt{3}}(x_0 - x_1))$ with $a \in \mathcal{U}$
$\ell_{13}$	$H_{p_1}, H_{q_3}$ and $H_3(a) = \mathcal{Z}(x_2 - \frac{a}{\sqrt{3}}(x_0 - \xi x_1))$ with $a \in \mathcal{U}$
$\ell_{14}$	$H_{p_1}, H_{q_4}$ and $H_4(a) = \mathcal{Z}(x_2 - \frac{a}{\sqrt{3}}(x_0 - \xi^2 x_1))$ with $a \in \mathcal{U}$

where  $\mathcal{U} = \{1, -1, i, -i\}$ .

In the next figure we illustrate the lines in the plane  $H_i(a)$  (cf. Lemma 3.6 for the definition of  $m_i(a, b)$  with  $b \in \{1, \xi, \xi^2\}$ ).

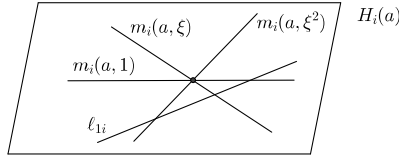


Figure 2

PROOF. Since  $\ell_{11} = \mathcal{Z}(x_0, x_2)$  the planes containing this line are given by  $H_1 = \mathcal{Z}(x_0)$  and  $H_1(a) = \mathcal{Z}(x_2 - ax_0)$  with  $a \in \mathbb{C}$ . According to (5)  $H_1 = H_{q_1}$  and by a direct computation, we verify that  $H_1 \cap \mathcal{S} = \ell_{11} \cup C_1$  with  $C_1 = \mathcal{Z}(x_0, x_2^3 - x_3^3)$  which is union of three lines passing through the point  $[0 : 1 : 0 : 0]$ . Furthermore  $H_1(a) \cap \mathcal{S} = \ell_{11} \cup C_1(a)$  where  $C_1(a) = \mathcal{Z}(x_2 - ax_0, x_1^3 + (a^4 - 1)x_0^3 - ax_3^3)$  is singular if and only if  $a = 0$  (in this case  $H_1(0) = H_{p_1}$ , see (5)) or  $a \in \mathcal{U}$ . In fact, for each  $a \in \mathcal{U}$  the curve  $C_1(a)$  is the union of three lines passing through the point  $[1 : 0 : a : 0]$  and  $C_1(0) = \ell_{12} \cup \ell_{13} \cup \ell_{14}$  (the three lines pass through  $p_1$ ).

An analogous computation can be used to find such planes for the other lines in  $\mathcal{L}_{p_1}$ .  $\square$

In the next Proposition we study some sets that will allow us to stratify  $\Phi(\mathcal{S})$  (cf. Corollary 3.5).

PROPOSITION 3.4. *In the above notation. Let  $\mathcal{M}_i = \Phi_{\ell_{1i}} \setminus \mathcal{L}$  and  $\Phi_i = \Phi_{\ell_{1i}} \setminus \mathcal{L}_{\mathcal{P}_1}$  for each  $\ell_{1i} \in \mathcal{L}_{\mathcal{P}_1}$ . Then*

- (a)  $\Phi_i \cap \Phi_j = \emptyset$  and  $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$  for all  $i \neq j$  with  $i, j \in \{1, 2, 3, 4\}$ .
- (b)  $\#(\Phi_i) = 15$  and  $\#(\mathcal{M}_i) = 12$  for each  $i \in \{1, 2, 3, 4\}$ .
- (c)  $r(\Phi_i) = 5$  and  $r(\mathcal{M}_i) = 4$  for each  $i \in \{1, 2, 3, 4\}$ .

PROOF. If we consider a line  $\ell \in \Phi_{\ell_{1i}} \cap \Phi_{\ell_{1j}}$  with  $i \neq j$ , then it follows from Lemma 2.1 that  $\ell \subset \langle \ell_{1i}, \ell_{1j} \rangle = H_{\mathcal{P}_1}$ . Since  $H_{\mathcal{P}_1} \cap \mathcal{S} = \ell_{11} \cup \ell_{12} \cup \ell_{13} \cup \ell_{14}$ , we conclude that  $\Phi_{\ell_{1i}} \cap \Phi_{\ell_{1j}} = \mathcal{L}_{\mathcal{P}_1} \setminus \{\ell_{1i}, \ell_{1j}\}$ . Therefore,  $\Phi_i \cap \Phi_j = \emptyset$  for all  $i \neq j$ . Keeping in mind that  $\mathcal{M}_i \subset \Phi_i$  for each  $i$ , we get that  $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$  for all  $i \neq j$ .

Now, each line in  $\Phi_{\ell_{1i}}$  determines a plane containing  $\ell_{1i}$ , so it follows from Proposition 3.3 that there exist exactly six such planes:  $H_{\mathcal{P}_1}$ ,  $H_{\mathcal{Q}_i}$  and  $H_i(a)$  with  $a \in \mathcal{U}$  containing  $\ell_{1i}$  (see (6)). Therefore,  $\#(\Phi_{\ell_{1i}}) = 6 \cdot 3 = 18$ ,  $\#(\Phi_i) = 5 \cdot 3 = 15$  and  $\#(\mathcal{M}_i) = 4 \cdot 3 = 12$  for each  $i \in \{1, 2, 3, 4\}$ .

Finally, let  $\mathcal{P}_i = \{H_{\mathcal{P}_1}, H_{\mathcal{Q}_i}, H_i(a)\}_{a \in \mathcal{U}}$  be the set of planes containing the line  $\ell_{1i}$ . If we consider the lines  $\ell$  and  $m$  both distinct from  $\ell_{1i}$ , which are contained in different planes of  $\mathcal{P}_i$ , then it follows from Lemma 2.1 that  $\ell \cap m = \emptyset$ . Now, since the lines in  $\Phi_i$  and  $\mathcal{M}_i$  are distributed among the five planes of  $\mathcal{P}_i \setminus \{H_{\mathcal{P}_1}\}$  and these four planes of  $\mathcal{P}_i \setminus \{H_{\mathcal{P}_1}, H_{\mathcal{Q}_i}\}$ , respectively. Then  $r(\Phi_i) = 5$  and  $r(\mathcal{M}_i) = 4$  for each  $i \in \{1, 2, 3, 4\}$ .  $\square$

COROLLARY 3.5. *In the above notation.  $\Phi(\mathcal{S})$ , can be expressed as a union of pairwise disjoint sets, as follows:*

$$\Phi(\mathcal{S}) = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4 \cup \mathcal{L} \quad \text{and} \quad \Phi(\mathcal{S}) = \Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4 \cup \mathcal{L}_{\mathcal{P}_1}.$$

*In particular, we have that  $\#(\Phi(\mathcal{S})) = 4 \cdot 12 + 16 = 4 \cdot 15 + 4 = 64$ .*

PROOF. Since  $\ell_{11}, \ell_{12}, \ell_{13}$  and  $\ell_{14}$  are coplanar lines in  $\mathcal{S}$ , it follows from Proposition 2.2 that  $\Phi(\mathcal{S}) = \Phi_{\ell_{11}} \cup \Phi_{\ell_{12}} \cup \Phi_{\ell_{13}} \cup \Phi_{\ell_{14}}$ . Keeping in mind that  $\mathcal{M}_i = \Phi_{\ell_{1i}} \setminus \mathcal{L}$  and  $\Phi_{\ell_{1i}} = \Phi_i \cup (\mathcal{L}_{\mathcal{P}_1} \setminus \{\ell_{1i}\})$  for each  $i \in \{1, 2, 3, 4\}$ , we get the first and second equality. Finally, the statements in Proposition 3.4 allow us to finish the proof.  $\square$

LEMMA 3.6. *The set of lines  $\mathcal{M}_k = \Phi_{\ell_{1k}} \setminus \mathcal{L}$  with  $\ell_{1k} \in \mathcal{L}_{\mathcal{P}_1}$  are given by*

$$(7) \quad \mathcal{M}_1 = \left\{ m_1(a, b) = \mathbb{P}\left[[(1, 0, a, 0), (0, a^3b, 0, 1)]\right] \right\}_{(a,b) \in \mathcal{U} \times \mathcal{V}}.$$

$$(8) \quad \mathcal{M}_k = \left\{ m_k(a, b) = \mathbb{P}\left( [(\sqrt{3}a^3b, 0, b, -2\xi^{k-2}), (0, \sqrt{3}a^3b, -b\xi^{2-k}, -1)] \right) \right\}_{(a,b) \in \mathcal{U} \times \mathcal{V}}$$

for  $k \in \{2, 3, 4\}$ , where  $\mathcal{U} = \{1, -1, i, -i\}$  and  $\mathcal{V} = \{1, \xi, \xi^2\}$ . Moreover,

$$(a) \quad m_k(a, b) \cap m_k(c, d) = \emptyset \iff a \neq c \text{ for all } k \in \{1, 2, 3, 4\}.$$

$$(b) \quad m_1(a, b) \cap m_k(c, d) = \emptyset \iff \frac{b}{d} \neq \frac{c^2(\sqrt{3}c-3a)}{\sqrt{3}c^3-3a^3} \text{ for all } k \in \{2, 3, 4\}.$$

$$(c) \quad m_2(a, b) \cap m_k(c, d) = \emptyset \iff \begin{cases} \frac{b}{d} \neq \frac{c^2(\sqrt{-3}a^3c+3)}{a^2(\sqrt{-3}ac^3-3)} & \text{if } k = 3. \\ \frac{b}{d} \neq \frac{c^2(\sqrt{-3}a^3c-3)}{a^2(\sqrt{-3}ac^3+3)} & \text{if } k = 4. \end{cases}$$

$$(d) \quad m_3(a, b) \cap m_4(c, d) = \emptyset \iff \frac{b}{d} \neq \frac{c^2(\sqrt{-3}a^3c+3)}{a^2(\sqrt{-3}ac^3-3)}.$$

PROOF. Since the 16 lines in  $\mathcal{L}$  are distributed among the planes  $H_{p_k}$  and  $H_{q_k}$  with  $k \in \{1, 2, 3, 4\}$  (cf. Lemma 3.2 and (5)), we have from (6) that the lines in  $\mathcal{M}_k$  are contained in the plane  $H_k(a)$  with  $a \in \mathcal{U}$  for each  $k$ ,  $k \in \{1, 2, 3, 4\}$ . Now, after a straightforward verification, for each  $a \in \mathcal{U}$  and  $k \in \{1, 2, 3, 4\}$ , we get that  $H_k(a) \cap \mathcal{S} = m_k(a, 1) \cup m_k(a, \xi) \cup m_k(a, \xi^2) \cup \ell_{1k}$ .

In order to study the intersection between the lines in  $\mathcal{M}_k$ , we use the description of these lines given in (7) and (8). In fact, using (7) we have:

$$m_1(a, b) \cap m_1(c, d) = \emptyset \iff \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & a^3b & 0 & 1 \\ 1 & 0 & c & 0 \\ 0 & c^3d & 0 & 1 \end{vmatrix} = (a-c)(c^3d-a^3b) \neq 0 \iff a \neq c.$$

Statements (b), (c) and (d) can be verified via a straightforward computation.  $\square$

LEMMA 3.7. *For each  $\ell \in \mathcal{L}$ , there exists a projective linear transformation  $\mathbb{T} : \mathbb{P}^3 \rightarrow \mathbb{P}^3$  such that  $\mathbb{T}(\mathcal{S}) = \mathcal{S}$  and  $\mathbb{T}(\ell) \in \mathcal{L}_{p_1}$ .*

PROOF. Fix  $\ell \in \mathcal{L}$ . Of course, if  $\ell \in \mathcal{L}_{p_1}$  then we can take  $\mathbb{T}$  equal to the identity in  $\mathbb{P}^3$ . So, we will assume that  $\ell \notin \mathcal{L}_{p_1}$ . Thus,  $\ell = \ell_{ij}$  with  $i \neq 1$  ( $i, j \in \{1, 2, 3, 4\}$ ). We have two possibilities:

**Case 1:**  $i = j$ . Hence,  $\ell = \mathbb{P}([(0, 0, \xi^r, 1), (\xi^r, 1, 0, 0)])$  for some  $r \in \{0, 1, 2\}$ . In this case, we set  $s \neq r$  in  $\{0, 1, 2\}$  and consider the linear isomorphism  $T : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  determined by:

$$\begin{aligned} T(0, 0, \xi^r, 1) &= \frac{1-\xi^{r-s}}{\xi^s-r} (0, 0, 0, 1), & T(\xi^r, 1, 0, 0) &= \frac{1-\xi^{r-s}}{\xi^s-r} (0, 1, 0, 0), \\ T(0, 0, 0, 1) &= \frac{\xi^{r-s}}{1-\xi^{s-r}} (0, 0, \xi^s, 1), & T(0, 1, 0, 0) &= \frac{\xi^{r-s}}{1-\xi^{s-r}} (\xi^s, 1, 0, 0). \end{aligned}$$



Let  $\mathbb{T}$  be the projective linear transformation determined by the automorphism  $T$  above. From the definition of  $T$ , we have that  $\mathbb{T}([0 : 0 : \xi^r : 1]) = p_1$ ,  $\mathbb{T}(p_1) = [0 : 0 : \xi^s : 1]$ ,  $\mathbb{T}([0 : 0 : \xi^s : 1]) = [0 : 0 : \xi^r : 1]$  and  $\mathbb{T}([0 : 0 : \xi^t : 1]) = [0 : 0 : \xi^t : 1]$  for  $t$  such that  $\{0, 1, 2\} = \{r, s, t\}$ . The same equalities hold if we replace the first and second coordinates with the third and fourth coordinates, respectively. In particular,  $\mathbb{T}([\xi^r : 1 : 0 : 0]) = q_1$ . Thus,  $\mathbb{T}(\{p_1, p_2, p_3, p_4\}) = \{p_1, p_2, p_3, p_4\}$  and it can be verified that  $\mathbb{T}$  also leaves invariant the set  $\{q_1, q_2, q_3, q_4\}$ .

Now,  $\mathbb{T}$  preserves lines and it leaves invariant the sets of points  $\{p_i\}$  and  $\{q_i\}$ , which belong to the lines  $L = \mathcal{Z}(x_0, x_1)$  and  $M = \mathcal{Z}(x_2, x_3)$ , respectively. We have that  $\mathbb{T}(L) = L$ ,  $\mathbb{T}(M) = M$  and also that  $\mathbb{T}(\ell) = \ell_{11}$  (keep in mind that  $\mathbb{T}([0 : 0 : \xi^r : 1]) = p_1$  and  $\mathbb{T}([\xi^r : 1 : 0 : 0]) = q_1$ ).

To verify that  $\mathbb{T}(\mathcal{S}) = \mathcal{S}$ , set  $\varphi(u, v) = u^4 - uv^3$ , so  $\mathcal{S} = \mathcal{Z}(f)$  with  $f = \varphi(x_0, x_1) - \varphi(x_2, x_3)$ . Note that,  $\mathbb{T}(L) = L$  and  $\mathbb{T}(M) = M$  implies that  $\langle T_\bullet(x_i), T_\bullet(x_{i+1}) \rangle = \langle x_i, x_{i+1} \rangle$  for  $i = 0, 2$ , thus  $T_\bullet \varphi(x_i, x_{i+1}) \in \mathbb{C}[x_i, x_{i+1}]$  for  $i = 0, 2$ . Furthermore  $\{p_1, p_2, p_3, p_4\} = L \cap \mathcal{S} = \mathcal{Z}(x_0, x_1, \varphi(x_2, x_3))$  and  $\mathbb{T}(L \cap \mathcal{S}) = \mathbb{T}(L) \cap \mathbb{T}(\mathcal{S}) = L \cap \mathbb{T}(\mathcal{S}) = \mathcal{Z}(x_0, x_1, T_\bullet(f))$ . Now, having in mind that  $\mathbb{T}$  leaves invariant the set  $\{p_i\}$ , we get the equality  $\mathcal{Z}(x_0, x_1, \varphi(x_2, x_3)) = \mathcal{Z}(x_0, x_1, T_\bullet f) = \mathcal{Z}(x_0, x_1, T_\bullet \varphi(x_2, x_3))$  (because we know that  $T_\bullet \varphi(x_i, x_{i+1}) \in \mathbb{C}[x_i, x_{i+1}]$  for  $i = 0, 2$ ). Now it follows from *Hilbert's Nullstellensatz* that  $\langle x_0, x_1, \varphi(x_2, x_3) \rangle = \langle x_0, x_1, T_\bullet \varphi(x_2, x_3) \rangle$ . Thereby,  $T_\bullet \varphi(x_2, x_3) = \lambda \varphi(x_2, x_3)$  for some non-zero  $\lambda \in \mathbb{C}$ . Then, it also holds that  $T_\bullet \varphi(x_0, x_1) = \lambda \varphi(x_0, x_1)$ . Thus  $T_\bullet f = T_\bullet \varphi(x_0, x_1) + T_\bullet \varphi(x_2, x_3) = \lambda \varphi(x_0, x_1) + \lambda \varphi(x_2, x_3) = \lambda f$  for some  $\lambda \neq 0$  in  $\mathbb{C}$ . Therefore,  $\mathbb{T}(\mathcal{S}) = \mathcal{S}$ .

**Case 2:  $i \neq j$ .** Now,  $\ell = \mathbb{P}([(0, 0, \xi^a, 1), (\beta, 1, 0, 0)])$  where  $a \in \{0, 1, 2\}$  and  $\beta \in \{0, 1, \xi, \xi^2\}$  with  $\xi^a \neq \beta$ . So,  $\{0, 1, \xi, \xi^2\} \setminus \{\xi^a, 0\} = \{\xi^b, \xi^c\}$  with  $0 \leq b < c \leq 2$  and  $a \notin \{b, c\}$ . Now consider the linear isomorphism  $T : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  determined by

$$\begin{aligned} T(0, 0, \xi^a, 1) &= \frac{\xi^a - \xi^c}{\xi^b} (0, 0, 0, 1), & T(\xi^a, 1, 0, 0) &= \frac{\xi^a - \xi^c}{\xi^b} (0, 1, 0, 0), \\ T(0, 0, 0, 1) &= \frac{\xi^c}{\xi^a - \xi^b} (0, 0, \xi^a, 1), & T(0, 1, 0, 0) &= \frac{\xi^c}{\xi^a - \xi^b} (\xi^a, 1, 0, 0). \end{aligned}$$

In this case, it is also verified that the linear projective transformation  $\mathbb{T}$  obtained from  $T$  leaves invariant the set of points  $\{p_i\}$  and  $\{q_i\}$  (in fact,  $\mathbb{T}([0 : 0 : \xi^a : 1]) = p_1$  which implies that  $\mathbb{T}(\ell) \in \mathcal{L}_{p_1}$ ), and it leaves invariant the lines  $L$  and  $M$  as well. Thus, the same argument used in case 1 allows us to conclude that  $\mathbb{T}(\mathcal{S}) = \mathcal{S}$ .  $\square$

**THEOREM 3.8.** *Let  $C \subseteq \Phi(\mathcal{S})$  be a set of skew lines. Then  $\#(C) \leq 16$ .*

**PROOF.** We consider two cases.

- $C \cap \mathcal{L} = \emptyset$ .

In this case, it follows from Corollary 3.5 that  $C = \bigcup_{i=1}^4 C \cap \mathcal{M}_i$ . Therefore,  $\#(C) = \sum_{i=1}^4 \#(C \cap \mathcal{M}_i) \leq 4 \cdot 4 = 16$ , since  $r(\mathcal{M}_i) = 4$  for any  $i \in \{1, 2, 3, 4\}$  (cf. (c) in Proposition 3.4).

- $C \cap \mathcal{L} \neq \emptyset$ .

If  $\ell_{1j} \in C \cap \mathcal{L}$  for some  $j \in \{1, 2, 3, 4\}$ , then  $C \cap \mathcal{L}_{\mathbb{P}^1} = \{\ell_{1j}\}$  and  $C \cap \Phi_j = \emptyset$ . Thus, again from Corollary 3.5 and Proposition 3.4, we get:

$$C = \bigcup_{k \neq j} C \cap \Phi_k \cup \{\ell_{1j}\} \implies \#(C) = \sum_{k \neq j} \#(C \cap \Phi_k) + 1 \leq 3 \cdot 5 + 1 = 16,$$

since  $r(\Phi_k) = 5$  for any  $k \in \{1, 2, 3, 4\}$ . Otherwise, take an  $\ell \in C \cap \mathcal{L}$  and use Lemma 3.7 to obtain a projective linear transformation  $\mathbb{T} : \mathbb{P}^3 \rightarrow \mathbb{P}^3$  such that  $\mathbb{T}(\mathcal{S}) = \mathcal{S}$  and  $\mathbb{T}(\ell) = \ell_{1i} \in \mathcal{L}_{\mathbb{P}^1}$  for some  $i \in \{1, 2, 3, 4\}$ . In this case, the above arguments implies that  $\#(\mathbb{T}(C)) \leq 16$ . So,  $\#C = \#(\mathbb{T}(C)) \leq 16$ .  $\square$

**COROLLARY 3.9.** *If  $\mathcal{S}$  is Schur's quartic surface, then  $r(\mathcal{S}) = 16$ .*

**PROOF.** From Theorem 3.8 we have that  $r(\mathcal{S}) \leq 16$ . Now, let us consider  $C = \bigcup_{k=1}^4 C_k$  where  $C_k \subset \mathcal{M}_k$  is given by (see (7) and (8) for the definition of  $m_k(a, b)$ )

$$C_1 = \left\{ m_1(1, 1), m_1(-1, 1), m_1(i, \xi), m_1(-i, \xi) \right\}$$

and

$$C_k = \left\{ m_k(1, \xi), m_k(-1, \xi), m_k(i, 1), m_k(-i, 1) \right\} \text{ for } k \in \{2, 3, 4\}.$$

Using 1) to 4) from Lemma 3.6, we get that any two distinct lines in  $C$  are disjoint. Therefore,  $r(C) = 16$ .  $\square$

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